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Complete orbits for twist maps on the plane

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Abstract. Twist maps $(\theta_1, r_1) = f(\theta, r)$ are considered in this paper, with no assumption on the periodicity of the map in θ. Under appropriate assumptions, the existence of infinitely many bounded (in r) complete orbits is proven. In particular, our results apply to the class of maps

$$\theta_1 = \theta + r, \quad r_1 = r + \lambda(\sin \omega_1(\theta + r) + \sin \omega_2(\theta + r)),$$

where $\lambda > 0$ and no arithmetic condition has to be imposed on ω_1/ω_2.

1. *Introduction*

The mathematical investigation of twist maps has a long tradition: originating in the work of Poincaré, a major relevance of these maps lies in their relation to continuous-time dynamical systems models. From the very beginning, the main focus of the theory has been on maps $f = f(\theta, r) : \mathbb{R} \times [a, b] \to \mathbb{R}^2$ that are periodic in the θ-variable. In this case, f should properly be viewed as the lift of the actual map to the universal covering of the cylinder. The topic of periodic twist maps turned out to be very fruitful, leading, in particular, to the development of KAM theory and Aubry–Mather theory for Hamiltonian systems; see [**2, 4**] for an overview and many relevant references. There are also some papers dealing with twist maps which are quasi-periodic in the angle (see, for example, [**5, 7, 11**]). In this paper we consider general twist maps $f = f(\theta, r)$ on the plane without imposing any assumption of periodicity or almost-periodicity in the θ-variable. To illustrate the type of results that will be obtained, let us remain, for the moment, in the quasi-periodic context and consider the family of maps

$$\theta_1 = \theta + r, \quad r_1 = r + \lambda(\sin \omega_1(\theta + r) + \sin \omega_2(\theta + r)), \tag{1.1}$$

where $\lambda, \omega_1, \omega_2 > 0$. The KAM method is applicable when λ is small and ω_1/ω_2 satisfies a diophantine condition. It gives the existence of invariant curves and, as a consequence, the

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boundedness of all orbits. In contrast, our results apply to arbitrary parameters λ, ω_1 and ω_2, and lead to a more modest conclusion: the existence of infinitely many complete orbits $(\theta_n, r_n)_{n\in\mathbb{Z}}$ which are bounded and which have finite upper and lower rotation numbers,

$$\sup_{n\in\mathbb{Z}} |r_n| < \infty, \quad -\infty < \liminf_{|n|\to\infty} \frac{\theta_n}{n} \le \limsup_{|n|\to\infty} \frac{\theta_n}{n} < \infty.$$

Our original motivation for studying aperiodic twist maps was the so-called Littlewood problem for oscillatory differential equations. The basic question there is to decide on the boundedness or unboundedness of solutions to

$$\ddot{x} + F(x) = p(t), \tag{1.2}$$

where F is a nonlinear function satisfying $F(x) \to \pm\infty$ as $x \to \pm\infty$, and the forcing p is bounded. Since the first definitive answer given by Morris [**8**] for $F(x) = 2x^3$ and periodic p (in which case all solutions are bounded), the method of proof has been to describe the dynamics of the differential equation by means of a certain twist map f; this is followed by the application of a suitable invariant curve theorem. After Morris's result came many others dealing with the periodic case, and also some treating the quasi-periodic case with diophantine frequencies [**5**]. If the forcing p is only bounded, Littlewood himself already noticed in [**6**] that unbounded motions could appear even for the nonlinearity $F(x) = 2x^3$. For such p, however, bounded and unbounded motions must coexist, as was recently shown in [**9, 10**]. In the aperiodic case, it is still possible to associate a twist map with equation (1.2). Assume that we are given a solution satisfying $x(\tau) = 0$ and $\dot{x}(\tau) = v$ with $v > 0$ large enough. We compute the next positive zero $\tau_1 > \tau$ such that $x(\tau_1) = 0$ and $\dot{x}(\tau_1) = v_1 > 0$, and consider the map $(\tau, v) \mapsto (\tau_1, v_1)$. The role of the angle θ is played by the variable τ, and if the forcing p is not periodic, then τ will also not be periodic. It therefore seems reasonable to expect that the study of general twist maps should be helpful in understanding the dynamics of the non-autonomous equation (1.2). The results in this paper are not directly useful for (1.2), so we preferred to consider the quasi-periodic map (1.1) as a first application. We plan to give some further applications, including ones related to the Littlewood problem, in a follow-up to this paper.

We end this introduction with an outline of the main contents. In §2 we consider a Lagrangian formulation and work only with the sequence of angles $(\theta_n)_{n\in\mathbb{Z}}$. These satisfy the equations

$$\partial_2 h(\theta_{n-1}, \theta_n) + \partial_1 h(\theta_n, \theta_{n+1}) = 0, \quad n \in \mathbb{Z},$$

where h is a generating function. Assuming that h grows quadratically, we prove the existence of solutions with bounded upper and lower rotation numbers. This is reminiscent of Aubry–Mather theory, but the map h does not have any periodicity property, which leads to some complications. The rest of the paper is devoted to maps of the form

$$\theta_1 = \theta + r + F(\theta, r), \quad r_1 = r + G(\theta, r). \tag{1.3}$$

In §3 the theorem for quadratic generating functions is applied to (1.3), in the case where (1.3) leaves invariant the boundary of the strip on which it is defined. In §4 this hypothesis is dropped; it is then not enough to assume that f is exact symplectic,

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in the sense that $r_1\,d\theta_1 - r\,d\theta = d\hat{h}$ for some C^2 function $\hat{h} = \hat{h}(\theta, r)$. Even if f is periodic in θ, for f to be exact on the cylinder one additionally needs that $\hat{h}$ is periodic in θ. With this observation in mind, we introduce a natural generalized notion of 'exact symplectic' in Definition 4.1. Under the sole (technical) restriction that $|\partial_r F| \le 1/43$, it is then shown in Theorem 4.3 that (1.3) admits infinitely many complete orbits. In addition, the r-component of these orbits can be controlled, as is the case for the upper and lower rotation numbers of the orbits. Section 4 also contains an easy application of Theorem 4.3 to (1.1), leading to the results mentioned at the beginning of this introduction.

2. *Complete orbits for quadratic generating functions*

The main result of this section is about the existence of a complete orbit under the assumption that the generating function $h = h(\theta, \theta')$ on the plane grows quadratically. Although no periodicity condition is imposed on h, we will use θ and θ' to denote the variables.

THEOREM 2.1. *Let $\Delta > \delta > 0$. Suppose that $h : \Omega = \{(\theta, \theta') \in \mathbb{R}^2 : \delta \le \theta' - \theta \le \Delta\} \to \mathbb{R}$ is C^1 and satisfies*

$$\underline{\alpha}(\theta' - \theta)^2 \le h(\theta, \theta') \le \overline{\alpha}(\theta' - \theta)^2, \quad (\theta, \theta') \in \Omega \tag{2.1}$$

for some constants $\overline{\alpha} \ge \underline{\alpha} > 0$ such that $\overline{\alpha} < (3/2)\,\underline{\alpha}$. Then there is a constant $\sigma_{**} \ge 1$, depending only on $\overline{\alpha}/\underline{\alpha} \in [1, 3/2[$, that has the following property. If

$$\sigma_{**}\delta < \sigma_{**}^{-1}\Delta, \tag{2.2}$$

then there exists $(\theta_n^*)_{n\in\mathbb{Z}}$ such that $|\theta_0^*| \le \Delta$, $\delta \le \theta_{n+1}^* - \theta_n^* \le \Delta$ for all $n \in \mathbb{Z}$, and

$$\partial_2 h(\theta_{n-1}^*, \theta_n^*) + \partial_1 h(\theta_n^*, \theta_{n+1}^*) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Moreover,

$$\delta \le \liminf_{n\to\infty} \frac{\theta_n^*}{n} \le \limsup_{n\to\infty} \frac{\theta_n^*}{n} \le \Delta \quad \text{and} \quad \delta \le \liminf_{n\to-\infty} \frac{\theta_n^*}{n} \le \limsup_{n\to-\infty} \frac{\theta_n^*}{n} \le \Delta. \tag{2.3}$$

Before proving Theorem 2.1, we look at an example.

Example 2.2. Consider

$$h(\theta, \theta') = \tfrac{1}{2}(\theta - \theta')^2 - V(\theta)$$

for some potential $V \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\|V\|_\infty \le 1/42$. Defining $\overline{\alpha} = 1/2 + \|V\|_\infty$ and $\underline{\alpha} = 1/2 - \|V\|_\infty$, we have $\overline{\alpha}/\underline{\alpha} \in [1, 11/10]$. Corollary 2.8 below shows that we can take $\sigma_{**} = 10$ in (2.2). Thus, by choosing $\delta = 1$ and $\Delta = 101$ for instance, it is found that the difference equation

$$\theta_{n+1} - 2\theta_n + \theta_{n-1} = -V'(\theta_n), \quad n \in \mathbb{Z}$$

(discretization of a Newtonian equation) has a solution $(\theta_n^*)_{n\in\mathbb{Z}} \subset \mathbb{R}$ such that (2.3) is satisfied for $\delta = 1$ and $\Delta = 101$, with $|\theta_0^*| \le 101$ and $1 \le \theta_{n+1}^* - \theta_n^* \le 101$ for all $n \in \mathbb{Z}$.

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The proof of Theorem 2.1 is split into several parts. First we need to construct $(\theta_n^*)_{n\in\mathbb{Z}}$ on finite segments $-N \le n \le N$. To do this, for fixed $A > 0$, $N \in \mathbb{N}$ and $\Delta > \delta > 0$, put

$$\Sigma^{(N)} = \{\Theta = (\theta_n)_{-N\le n\le N} : \theta_{\pm N} = \pm A,\ \delta \le \theta_{n+1} - \theta_n \le \Delta \text{ for } n = -N, \dots, N-1\}.$$

Since later $A = A_N$ will be chosen to depend on N, the dependence of $\Sigma^{(N)}$ on A is suppressed in our notation.

LEMMA 2.3. *If $\delta \le A/N \le \Delta$, then $\Sigma^{(N)} \ne \emptyset$, and $\Sigma^{(N)} \subset \mathbb{R}^{2N+1}$ is compact.*

Proof. Defining $\hat{\Theta}$ by $\hat{\theta}_n = -A + A/N(n+N)$, we see that $\hat{\Theta} \in \Sigma^{(N)}$ because $\hat{\theta}_{n+1} - \hat{\theta}_n = A/N \in [\delta, \Delta]$; in particular, $\Sigma^{(N)} \ne \emptyset$. Also, $\Sigma^{(N)} \subset \mathbb{R}^{2N+1}$ is closed and bounded; for the latter, we have

$$(N+n)\delta - A \le \theta_n \le (N+n)\Delta - A \quad \text{for } -N \le n \le N,$$

which follows from $\theta_{-N} = -A$ and $\delta \le \theta_{n+1} - \theta_n \le \Delta$. Hence $\Sigma^{(N)} \subset [-A, 2N\Delta - A]^{2N+1}$. □

If $\Theta = (\theta_n)_{-N\le n\le N} \in \Sigma^{(N)}$, then by the assumptions of Theorem 2.1, $h(\theta_n, \theta_{n+1})$ is defined for $-N \le n \le N-1$. Put

$$S(\Theta) = \sum_{n=-N}^{N-1} h(\theta_n, \theta_{n+1}), \qquad \Theta = (\theta_n)_{-N\le n\le N} \in \Sigma^{(N)}. \tag{2.4}$$

Since $S : \Sigma^{(N)} \to \mathbb{R}$ is continuous, there exists a minimizer, i.e.

$$S(\Theta^{(N)}) = \min_{\Theta\in\Sigma^{(N)}} S(\Theta), \tag{2.5}$$

for a suitable $\Theta^{(N)} = (\theta_n^{(N)})_{-N\le n\le N} \in \Sigma^{(N)}$, which henceforth we consider to be fixed.

First we need to derive some N-independent estimates for the minimizers, along the lines of [**9,** Lemmas 6.1 and 6.2].

LEMMA 2.4. *Suppose that $\overline{\alpha} < (3/2)\,\underline{\alpha}$. Then there exists a constant $\sigma_* = \sigma_*(\overline{\alpha}/\underline{\alpha}) \ge 1$ such that for all $N \in \mathbb{N}$,*

$$\sigma_*^{-1}(\theta_n^{(N)} - \theta_{n-1}^{(N)}) \le \theta_{n+1}^{(N)} - \theta_n^{(N)} \le \sigma_*(\theta_n^{(N)} - \theta_{n-1}^{(N)}) \quad \text{for } -N+1 \le n \le N-1.$$

Proof. First we derive the upper bound. Write $\theta_n^{(N)} - \theta_{n-1}^{(N)} = L$ and $\theta_{n+1}^{(N)} - \theta_n^{(N)} = \sigma L$ for $L, \sigma > 0$. Consider

$$\tilde{\Theta} = (\tilde{\theta}_k)_{-N\le k\le N} = (\theta_{-N}^{(N)}, \dots, \theta_{n-1}^{(N)}, s, \theta_{n+1}^{(N)}, \dots, \theta_N^{(N)}),$$

where $s = (\theta_{n+1}^{(N)} + \theta_{n-1}^{(N)})/2$. Then $\tilde{\theta}_{\pm N} = \theta_{\pm N}^{(N)} = \pm A$ and $s - \theta_{n-1}^{(N)} = \theta_{n+1}^{(N)} - s = (\theta_{n+1}^{(N)} - \theta_{n-1}^{(N)})/2 \in [\delta, \Delta]$, because $\theta_{n+1}^{(N)} - \theta_n^{(N)}, \theta_n^{(N)} - \theta_{n-1}^{(N)} \in [\delta, \Delta]$. By $\tilde{\Theta} \in \Sigma^{(N)}$ in conjunction with (2.5), we have $S(\Theta^{(N)}) \le S(\tilde{\Theta})$. Using the definition of S from (2.4), this relation can be rewritten as

$$h(\theta_{n-1}^{(N)}, \theta_n^{(N)}) + h(\theta_n^{(N)}, \theta_{n+1}^{(N)}) \le h(\theta_{n-1}^{(N)}, s) + h(s, \theta_{n+1}^{(N)}),$$

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since all the other terms cancel. Therefore assumption (2.1) leads to

$$\begin{aligned}\underline{\alpha}(1+\sigma^2)L^2 &= \underline{\alpha}(\theta_n^{(N)}-\theta_{n-1}^{(N)})^2+\underline{\alpha}(\theta_{n+1}^{(N)}-\theta_n^{(N)})^2 \le \overline{\alpha}(s-\theta_{n-1}^{(N)})^2+\overline{\alpha}(\theta_{n+1}^{(N)}-s)^2\\ &= \tfrac{1}{2}\overline{\alpha}(\theta_{n+1}^{(N)}-\theta_{n-1}^{(N)})^2 = \tfrac{1}{2}\overline{\alpha}(1+\sigma)^2L^2. \end{aligned} \tag{2.6}$$

The function $\varphi(\sigma)=2(1+\sigma^2)/(1+\sigma)^2 : [1,\infty[\to [1,2[$ is strictly increasing. As $q=\overline{\alpha}/\underline{\alpha}\in[1,3/2[$, there is a unique $\sigma_*\in[1,\infty[$ such that $\varphi(\sigma_*)=q$; explicitly, we have

$$\sigma_* = (2-q)^{-1}(q+2\sqrt{q-1}), \quad q=\overline{\alpha}/\underline{\alpha}. \tag{2.7}$$

Since (2.6) says that $\varphi(\sigma)\le q$, we must have $\sigma\le\sigma_*$, proving the upper bound.

For the lower bound, consider the function

$$h_0(\theta,\theta') = h(-\theta',-\theta), \quad (\theta,\theta')\in\Omega.$$

Since $\delta\le\theta'-\theta\le\Delta$ if and only if $\delta\le-\theta-(-\theta')\le\Delta$, h_0 is well-defined and satisfies (2.1). Put

$$S_0(\Phi)=\sum_{n=-N}^{N-1} h_0(\varphi_n,\varphi_{n+1}), \quad \Phi=(\varphi_n)_{-N\le n\le N}\in\Sigma^{(N)}.$$

Then S_0 has a minimizer, i.e. $S_0(\Phi^{(N)})=\min_{\Phi\in\Sigma^{(N)}} S_0(\Phi)$ for some $\Phi^{(N)}=(\varphi_n^{(N)})_{-N\le n\le N}\in\Sigma^{(N)}$. According to the upper bound, for every such minimizer we have

$$\varphi_{n+1}^{(N)}-\varphi_n^{(N)} \le \sigma_*(\varphi_n^{(N)}-\varphi_{n-1}^{(N)}), \quad -N+1\le n\le N-1. \tag{2.8}$$

By definition of h_0, a minimizer is obtained by taking $\varphi_n^{(N)}=-\theta_{-n}^{(N)}$ for $-N\le n\le N$. Using this minimizer in (2.8) and replacing $-n$ by n, the lower bound is obtained. □

For $N\in\mathbb{N}$, define

$$\delta^{(N)}=\min_{-N\le n\le N-1}(\theta_{n+1}^{(N)}-\theta_n^{(N)}) \quad\text{and}\quad \Delta^{(N)}=\max_{-N\le n\le N-1}(\theta_{n+1}^{(N)}-\theta_n^{(N)}). \tag{2.9}$$

Then $\delta\le\delta^{(N)}\le\Delta^{(N)}\le\Delta$ holds, since $\Theta^{(N)}=(\theta_n^{(N)})_{-N\le n\le N}\in\Sigma^{(N)}$.

LEMMA 2.5. *Suppose that $\overline{\alpha}<(3/2)\,\underline{\alpha}$. Then there exists a constant $\sigma_{**}=\sigma_{**}(\overline{\alpha}/\underline{\alpha})\ge 1$ such that for all $N\in\mathbb{N}$,*

$$\Delta^{(N)}\le\sigma_{**}\delta^{(N)}.$$

Proof. Put

$$\sigma_{**}=5\sigma_*. \tag{2.10}$$

If $\delta^{(N)}\ge\Delta/(2\sigma_*)$, then $\Delta\ge\Delta^{(N)}$ yields $\delta^{(N)}\ge\Delta^{(N)}/(2\sigma_*)\ge\Delta^{(N)}/\sigma_{**}$. Therefore we can assume that $\delta^{(N)}\le\Delta/(2\sigma_*)$. Similarly, if $\Delta^{(N)}\le 2\delta$, then $\Delta^{(N)}\le 2\delta^{(N)}\le\sigma_{**}\delta^{(N)}$, recalling that $\sigma_*\ge 1$. Thus we only have to consider the cases where

$$\delta^{(N)}\le\frac{1}{2\sigma_*}\Delta \quad\text{and}\quad \Delta^{(N)}\ge 2\delta. \tag{2.11}$$

Let $-N\le m, n\le N-1$ be such that

$$\delta^{(N)}=\theta_{m+1}^{(N)}-\theta_m^{(N)} \quad\text{and}\quad \Delta^{(N)}=\theta_{n+1}^{(N)}-\theta_n^{(N)}.$$

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Without loss of generality, we can suppose that $m \le n$, since the argument is similar for $m \ge n$. We may further restrict ourselves to $m+2 \le n$. In fact, if $m = n$, then $\Delta^{(N)} = \delta^{(N)}$; if $m+1 = n$, then by Lemma 2.4, $\Delta^{(N)} = \theta_{n+1}^{(N)} - \theta_n^{(N)} \le \sigma_*(\theta_n^{(N)} - \theta_{n-1}^{(N)}) = \sigma_*\delta^{(N)} \le \sigma_{**}\delta^{(N)}$. Hence $m+2 \le n$ can be assumed. Now put

$$\tilde{\Theta} = (\tilde{\theta}_k)_{-N \le k \le N} = (\theta_{-N}^{(N)}, \dots, \theta_m^{(N)}, \theta_{m+2}^{(N)}, \dots, \theta_n^{(N)}, s, \theta_{n+1}^{(N)}, \dots, \theta_N^{(N)}),$$

where $s = (\theta_{n+1}^{(N)} + \theta_n^{(N)})/2$; that is, we remove $\theta_{m+1}^{(N)}$ from $\Theta^{(N)}$, then shift the block $(\theta_{m+2}^{(N)}, \dots, \theta_n^{(N)})$ one place to the left, and finally insert s for $\theta_n^{(N)}$.

First we check that $\tilde{\Theta} \in \Sigma^{(N)}$. To begin with, $\tilde{\theta}_N = \theta_N^{(N)} = A$. Also, since $m+1 \ge -N+1$, $\theta_{-N}^{(N)}$ is not removed, which means that $\tilde{\theta}_{-N} = \theta_{-N}^{(N)} = -A$. Furthermore,

$$\theta_{m+2}^{(N)} - \theta_m^{(N)} = (\theta_{m+2}^{(N)} - \theta_{m+1}^{(N)}) + (\theta_{m+1}^{(N)} - \theta_m^{(N)}),$$

so that $\theta_{m+2}^{(N)} - \theta_m^{(N)} \ge 2\delta > \delta$. In addition, by Lemma 2.4 and (2.11), recalling that $\sigma_* \ge 1$, we have

$$\theta_{m+2}^{(N)} - \theta_m^{(N)} \le (\sigma_* + 1)\,(\theta_{m+1}^{(N)} - \theta_m^{(N)}) \le 2\sigma_*\delta^{(N)} \le \Delta. \tag{2.12}$$

The next observation is that $\gamma := s - \theta_n^{(N)} = \theta_{n+1}^{(N)} - s = (\theta_{n+1}^{(N)} - \theta_n^{(N)})/2 = \Delta^{(N)}/2$. In particular, $\gamma = \Delta^{(N)}/2 \le \Delta/2 < \Delta$, and also $\gamma = \Delta^{(N)}/2 \ge \delta$ by (2.11). To summarize the preceding arguments, we have shown that $\tilde{\Theta} \in \Sigma^{(N)}$. Hence (2.5) gives $S(\Theta^{(N)}) \le S(\tilde{\Theta})$, so that by the definition of S in (2.4),

$$\begin{aligned} &h(\theta_m^{(N)}, \theta_{m+1}^{(N)}) + h(\theta_{m+1}^{(N)}, \theta_{m+2}^{(N)}) + h(\theta_n^{(N)}, \theta_{n+1}^{(N)}) \\ &\quad \le h(\theta_m^{(N)}, \theta_{m+2}^{(N)}) + h(\theta_n^{(N)}, s) + h(s, \theta_{n+1}^{(N)}). \end{aligned}$$

As $h \ge 0$, we can drop two terms on the left-hand side and keep only $h(\theta_n^{(N)}, \theta_{n+1}^{(N)})$. Using (2.1), (2.12) and the condition $\overline{\alpha} < (3/2)\,\underline{\alpha}$, we get

$$\begin{aligned} \underline{\alpha}(\Delta^{(N)})^2 = \underline{\alpha}(\theta_{n+1}^{(N)} - \theta_n^{(N)})^2 &\le h(\theta_n^{(N)}, \theta_{n+1}^{(N)}) \\ &\le h(\theta_m^{(N)}, \theta_{m+2}^{(N)}) + h(\theta_n^{(N)}, s) + h(s, \theta_{n+1}^{(N)}) \\ &\le \overline{\alpha}(\theta_{m+2}^{(N)} - \theta_m^{(N)})^2 + 2\overline{\alpha}\gamma^2 \le 4\overline{\alpha}\sigma_*^2(\delta^{(N)})^2 + \tfrac{1}{2}\overline{\alpha}(\Delta^{(N)})^2 \\ &\le 4\overline{\alpha}\sigma_*^2(\delta^{(N)})^2 + \tfrac{3}{4}\underline{\alpha}(\Delta^{(N)})^2. \end{aligned}$$

Consequently,

$$(\Delta^{(N)})^2 \le 16\left(\frac{\overline{\alpha}}{\underline{\alpha}}\right)\sigma_*^2(\delta^{(N)})^2 \le 24\sigma_*^2(\delta^{(N)})^2,$$

which yields the claim. □

COROLLARY 2.6. *Suppose that the assumptions of Lemma 2.5 are satisfied. If*

$$\sigma_{**}\delta < \frac{A}{N} < \sigma_{**}^{-1}\Delta, \tag{2.13}$$

then for all $N \in \mathbb{N}$ and $-N \le n \le N-1$,

$$\delta < \delta^{(N)} \le \theta_{n+1}^{(N)} - \theta_n^{(N)} \le \Delta^{(N)} < \Delta.$$

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Proof. If $\Delta^{(N)} = \Delta$, then by (2.9) and Lemma 2.5,

$$2A = \theta_N^{(N)} - \theta_{-N}^{(N)} = \sum_{n=-N}^{N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)}) \geq 2N\delta^{(N)} \geq \frac{2N\Delta^{(N)}}{\sigma_{**}} = \frac{2N\Delta}{\sigma_{**}},$$

contradicting (2.13). Similarly, if $\delta^{(N)} = \delta$, then also

$$2A = \sum_{n=-N}^{N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)}) \leq 2N\Delta^{(N)} \leq 2\sigma_{**}N\delta^{(N)} = 2\sigma_{**}N\delta,$$

which contradicts (2.13). □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. We define $\sigma_{**} \geq 1$ as in Lemma 2.5, and suppose that $\sigma_{**}\delta < \sigma_{**}^{-1}\Delta$ is satisfied. For $N \in \mathbb{N}$, we take $A = A_N = (\sigma_{**}^{-1}\Delta + \sigma_{**}\delta)N/2$; then $\sigma_{**}\delta < A_N/N < \sigma_{**}^{-1}\Delta$ holds for every $N \in \mathbb{N}$. In particular, $\delta \leq \sigma_{**}\delta < A_N/N < \sigma_{**}^{-1}\Delta \leq \Delta$. Therefore all the preceding results apply, and we obtain from Corollary 2.6 that

$$\delta < \theta_{n+1}^{(N)} - \theta_n^{(N)} < \Delta$$

for $N \in \mathbb{N}$ and $-N \leq n \leq N-1$. Fix n such that $-N+1 \leq n \leq N-1$. Then $\Theta(\varepsilon) = (\theta_k(\varepsilon))_{-N\leq k\leq N} \in \Sigma^{(N)}$ for all $\varepsilon \in \mathbb{R}$ with $|\varepsilon| \geq 0$ sufficiently small, where

$$\Theta(\varepsilon) = (\theta_{-N}^{(N)}, \ldots, \theta_{n-1}^{(N)}, \theta_n^{(N)} + \varepsilon, \theta_{n+1}^{(N)}, \ldots, \theta_N^{(N)}).$$

Recalling from (2.5) that

$$S(\Theta(0)) = S(\Theta^{(N)}) = \min_{\Theta\in\Sigma^{(N)}} S(\Theta) \leq S(\Theta(\varepsilon)),$$

differentiating $S(\Theta(\varepsilon))$ with respect to ε gives

$$0 = \frac{d}{d\varepsilon} S(\Theta(\varepsilon))\Big|_{\varepsilon=0} = \partial_2 h(\theta_{n-1}^{(N)}, \theta_n^{(N)}) + \partial_1 h(\theta_n^{(N)}, \theta_{n+1}^{(N)}), \quad -N+1 \leq n \leq N-1.$$

We intend to pass to the limit $N \to \infty$ next, but before doing so, we need to normalize the $\Theta^{(N)} = (\theta_n^{(N)})_{-N\leq n\leq N}$ appropriately. Denoting by $n_0(N)$ the largest index n for which $\theta_n^{(N)} \leq 0$, we know $-N \leq n_0(N) \leq N-1$ because $\theta_{\pm N}^{(N)} = \pm A_N$; also $\theta_{n_0(N)+1}^{(N)} > 0$. As $\delta \leq \theta_{n_0(N)+1}^{(N)} - \theta_{n_0(N)}^{(N)} \leq \Delta$, this gives $|\theta_{n_0(N)}^{(N)}| = -\theta_{n_0(N)}^{(N)} \leq \Delta$. In addition,

$$A_N - \Delta \leq \theta_N^{(N)} - \theta_{n_0(N)}^{(N)} = \sum_{n=n_0(N)}^{N-1} (\theta_{n+1}^{(N)} - \theta_n^{(N)}) \leq (N - n_0(N))\Delta$$

implies that

$$N - n_0(N) \geq \frac{A_N - \Delta}{\Delta} \to \infty \quad \text{as } N \to \infty, \tag{2.14}$$

and we also have $-N - n_0(N) \to -\infty$ as $N \to \infty$. Defining

$$\Phi^{(N)} = (\varphi_k^{(N)})_{-N-n_0(N)\leq k\leq N-n_0(N)}, \quad \varphi_k^{(N)} = \theta_{n_0(N)+k}^{(N)},$$

we find $|\varphi_0^{(N)}| \leq \Delta$ and

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$$\delta < \varphi_{k+1}^{(N)} - \varphi_k^{(N)} < \Delta, \tag{2.15}$$

for $N \in \mathbb{N}$ and $-N - n_0(N) \le k \le N - n_0(N) - 1$. As above, this allows us to deduce that

$$0 = \partial_2 h(\varphi_{k-1}^{(N)}, \varphi_k^{(N)}) + \partial_1 h(\varphi_k^{(N)}, \varphi_{k+1}^{(N)}), \quad -N - n_0(N) + 1 \le k \le N - n_0(N) - 1, \tag{2.16}$$

since $\Phi^{(N)}$ is a minimizer of the appropriately index-shifted functional S in the appropriately index-shifted set $\Sigma^{(N)}$. Using $|\varphi_0^{(N)}| \le \Delta$ in conjunction with (2.15), we obtain

$$|\varphi_k^{(N)}| \le k\Delta \quad \text{for } 0 \le k \le N - n_0(N), \qquad |\varphi_k^{(N)}| \le |k|\Delta \quad \text{for } -N - n_0(N) \le k \le 0. \tag{2.17}$$

Furthermore, we extend $\Phi^{(N)}$ to all indices $k \in \mathbb{Z}$ by defining $\varphi_k^{(N)} = 0$ for $k < -N - n_0(N)$ and for $k > N - n_0(N)$. Hence if we fix $k \in \mathbb{Z}$, then the sequence $(\varphi_k^{(N)})_{N \in \mathbb{N}}$ is bounded, by (2.17). We may therefore employ a diagonal sequence argument to find a subsequence of $N \in \mathbb{N}$, indexed by $N' \to \infty$, such that for all $k \in \mathbb{Z}$ the limit

$$\theta_k^* = \lim_{N' \to \infty} \varphi_k^{(N')} \tag{2.18}$$

does exist. If $k \in \mathbb{Z}$ is fixed, then by (2.14) there is $M_k \in \mathbb{N}$ such that $N' \ge M_k$ implies that $-N' - n_0(N') + 1 \le k \le N' - n_0(N') - 1$. Thus (2.16) yields

$$\partial_2 h(\varphi_{k-1}^{(N')}, \varphi_k^{(N')}) + \partial_1 h(\varphi_k^{(N')}, \varphi_{k+1}^{(N')}) = 0 \quad \text{for } N' \ge M_k,$$

so that

$$\partial_2 h(\theta_{k-1}^*, \theta_k^*) + \partial_1 h(\theta_k^*, \theta_{k+1}^*) = 0$$

is obtained in the limit $N' \to \infty$. In addition, since $|\varphi_0^{(N)}| \le \Delta$, we have $|\theta_0^*| \le \Delta$ as well by (2.18). Concerning (2.3), we first note that

$$\delta \le \theta_{k+1}^* - \theta_k^* \le \Delta, \quad k \in \mathbb{Z}$$

follows from (2.15) and (2.18). Hence if $n \ge 1$, then $\theta_n^* - \theta_0^* = \sum_{k=0}^{n-1} (\theta_{k+1}^* - \theta_k^*)$ shows that $n\delta + \theta_0^* \le \theta_n^* \le n\Delta + \theta_0^*$, thus implying the first part of (2.3). Similarly, if $n \le -1$, then we write $\theta_0^* - \theta_n^* = \sum_{k=n}^{-1} (\theta_{k+1}^* - \theta_k^*)$ and get $(-n)\delta \le \theta_0^* - \theta_n^* \le (-n)\Delta$. This gives the second part of (2.3) and completes the proof of Theorem 2.1. □

Remark 2.7. Readers familiar with Aubry–Mather theory will notice that the above proof is inspired by it. In Aubry–Mather theory, the generating function h satisfies the additional condition

$$h(\theta + 1, \theta' + 1) = h(\theta, \theta').$$

As a consequence, global minimals are invariant under shifts of the indices and under integer translations. That is, if $(\theta_n^*)_{n \in \mathbb{Z}}$ is minimal, then so are $(\theta_{n+1}^*)_{n \in \mathbb{Z}}$ and $(\theta_n^* + 1)_{n \in \mathbb{Z}}$. The second invariance fails if the periodicity is lost. This fact prevents a straightforward extension of Aubry–Mather theory to our present situation.

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COROLLARY 2.8. *Under the assumptions of Theorem 2.1, an explicit expression for* σ_{**} *is*

$$\sigma_{**} = 5(2-q)^{-1}(q + 2\sqrt{q-1}), \quad q = \overline{\alpha}/\underline{\alpha}.$$

In particular, one can take $\sigma_{**} = 10$ *in (2.2), if* $\overline{\alpha}/\underline{\alpha} \in [1, 11/10]$.

Proof. The form of σ_{**} is due to (2.10) and (2.7). The last statement follows from the fact that $\sigma_{**} = \sigma_{**}(q)$ is increasing as a function of $q \in [1, 2[$, and $\sigma_{**}(11/10) \leq 10$. Hence if $10\delta < \Delta/10$ holds and $q = \overline{\alpha}/\underline{\alpha} \in [1, 11/10]$, then we obtain $\sigma_{**}(q)\delta \leq 10\delta < \Delta/10 \leq \Delta/\sigma_{**}(q)$, and (2.2) is satisfied. □

Remark 2.9. With regard to the form of $\sigma_{**} = \sigma_{**}(q)$ as obtained in Corollary 2.8, it is not completely satisfactory that we get $\lim_{q \searrow 1} \sigma_{**}(q) = 5$, not $\lim_{q \searrow 1} \sigma_{**}(q) = 1$ as would be expected from (2.1).

3. *Non-periodic twist maps: a special case*

In this section we are going to apply Theorem 2.1 to twist maps that leave invariant the boundary of the strip where they are defined. This additional assumption will be removed in §4.

Definition 3.1. Let $f = f(\theta, r) : \mathbb{R} \times [a, b] \to \mathbb{R}^2$ be a C^1 map.

(a) We say that f is *symplectic* if there exists a C^2 function $\hat{h} : \mathbb{R} \times [a, b] \to \mathbb{R}$ such that

$$r_1\, d\theta_1 - r\, d\theta = d\hat{h}, \quad \text{where } (\theta_1, r_1) = f(\theta, r).$$

(b) A sequence $(\theta_n, r_n)_{n\in\mathbb{Z}} \subset \mathbb{R} \times [a, b]$ is called a *complete orbit for* f if $(\theta_{n+1}, r_{n+1}) = f(\theta_n, r_n)$ for $n \in \mathbb{Z}$.

Remark 3.2. In Theorem 3.3 we will need to impose the following three kinds of assumptions.

(a) The map $f : (\theta, r) \mapsto (\theta_1, r_1) = (\theta + r + F(\theta, r), r + G(\theta, r))$ is close to the integrable map $(\theta, r) \mapsto (\theta + r, r)$, where 'close' only requires that $|\partial_r F| \leq 1/22$.

(b) The boundaries $\mathbb{R} \times \{a\}$ and $\mathbb{R} \times \{b\}$ of the strip $\mathbb{R} \times [a, b]$ enjoy some invariance properties, as expressed by (3.1) below.

(c) The strip $\mathbb{R} \times [a, b]$ is sufficiently large in the sense that $b/a > 100$ is sufficiently large. Note that, in the present context, the size of the strip should be measured in terms of b/a rather than $b - a$, since (as opposed to periodic twist maps) both r and θ can be scaled. Putting

$$\Theta = \lambda\theta \quad \text{and} \quad R = \lambda r$$

for some $\lambda > 0$, the map f becomes

$$\Theta_1 = \Theta + R + \mathcal{F}(\Theta, R), \quad R_1 = R + \mathcal{G}(\Theta, R)$$

on the strip $(\Theta, R) \in \mathbb{R} \times [\lambda a, \lambda b]$, where

$$\mathcal{F}(\Theta, R) = \lambda F(\Theta/\lambda, R/\lambda), \quad \mathcal{G}(\Theta, R) = \lambda G(\Theta/\lambda, R/\lambda).$$

Both of the conditions $|\partial_r F| \leq 1/22$ and $b/a > 100$ are invariant under this scaling.

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THEOREM 3.3. *Let $b > a > 0$ be such that*

$$10a < \tfrac{1}{10}b.$$

Suppose that $f : \mathbb{R} \times [a, b] \to \mathbb{R}^2$ is a symplectic C^1 map given by

$$\theta_1 = \theta + r + F(\theta, r), \quad r_1 = r + G(\theta, r),$$

where $(\theta_1, r_1) = f(\theta, r)$. For $F, G : \mathbb{R} \times [a, b] \to \mathbb{R}$ we assume that $F, G \in C^1$,

$$F(\theta, a) = F(\theta, b) = G(\theta, a) = G(\theta, b) = 0 \quad \text{for } \theta \in \mathbb{R}, \tag{3.1}$$

and

$$|\partial_r F(\theta, r)| \le \tfrac{1}{22} \quad \text{for } \theta \in \mathbb{R},\ r \in [a, b]. \tag{3.2}$$

Then there exists a complete orbit $(\theta_n, r_n)_{n\in\mathbb{Z}}$ for f such that $|\theta_0| < b$ and $a < \inf_{n\in\mathbb{N}}(\theta_{n+1} - \theta_n) \le \sup_{n\in\mathbb{N}}(\theta_{n+1} - \theta_n) < b$. Moreover,

$$a < \liminf_{n\to\pm\infty} \frac{\theta_n}{n} \le \limsup_{n\to\pm\infty} \frac{\theta_n}{n} < b. \tag{3.3}$$

Proof. First we need to construct the generating function h for f and discuss its properties. Although this method is standard (see [**1, 3**]), we include some details adapted to the present set-up. For any fixed $\theta \in \mathbb{R}$, the function $[a, b] \ni r \mapsto \theta_1(\theta, r) = \theta + r + F(\theta, r) \in \mathbb{R}$ satisfies $\partial_r\theta_1 = 1 + \partial_r F \ge 21/22$, by (3.2). Hence the inverse function

$$[\theta + a, \theta + b] \ni \theta_1 \mapsto r(\theta, \theta_1) \in [a, b] \tag{3.4}$$

is well-defined, strictly increasing, and onto; note that $\theta_1(\theta, a) = \theta + a$ and $\theta_1(\theta, b) = \theta + b$ by (3.1). With $\hat{h}$ as in Definition 3.1, let

$$h(\theta, \theta_1) = \hat{h}(\theta, r(\theta, \theta_1)) + C,$$
$$(\theta, \theta_1) \in \Omega_0 := \{(\theta, \theta_1) \in \mathbb{R}^2 : \theta \in \mathbb{R},\ \theta + a \le \theta_1 \le \theta + b\},$$

where $C = a^2/2 - \hat{h}(0, a)$ is a normalizing constant. Now, the relation $r_1\, d\theta_1 - r\, d\theta = d\hat{h}$ can be seen to be equivalent to

$$\partial_\theta\hat{h} = r_1\partial_\theta\theta_1 - r \quad \text{and} \quad \partial_r\hat{h} = r_1\partial_r\theta_1. \tag{3.5}$$

Since $\partial_\theta r = -(1 + \partial_\theta F)\,(1 + \partial_r F)^{-1} = -(\partial_\theta\theta_1)\,(\partial_r\theta_1)^{-1}$ and $\partial_{\theta_1} r = (\partial_r\theta_1)^{-1}$, this results in

$$\partial_\theta h = \partial_\theta\hat{h} + (\partial_r\hat{h})\,(\partial_\theta r) = r_1\partial_\theta\theta_1 - r - r_1(\partial_r\theta_1)\,(\partial_\theta\theta_1)\,(\partial_r\theta_1)^{-1} = -r \tag{3.6}$$

and

$$\partial_{\theta_1} h = (\partial_r\hat{h})\,(\partial_{\theta_1} r) = r_1(\partial_r\theta_1)\,(\partial_r\theta_1)^{-1} = r_1, \tag{3.7}$$

i.e. the usual relations for the generating function h. We also observe that by (3.6) and (3.2),

$$|\partial^2_{\theta\theta_1} h(\theta, \theta_1) + 1| = |1 - \partial_{\theta_1} r(\theta, \theta_1)| = |1 - (\partial_r\theta_1)^{-1}(\theta, \theta_1)| = |1 - (1 + \partial_r F(\theta, r))^{-1}|$$
$$= (1 + \partial_r F(\theta, r))^{-1}|\partial_r F(\theta, r)| \le \tfrac{1}{21}, \quad (\theta, \theta_1) \in \Omega_0. \tag{3.8}$$

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Put

$$V(\theta, \theta_1) = h(\theta, \theta_1) - \tfrac{1}{2}(\theta_1 - \theta)^2, \quad (\theta, \theta_1) \in \Omega_0. \tag{3.9}$$

Then the relations

$$\partial_\theta V = -r + \theta_1 - \theta = F \quad \text{and} \quad \partial_{\theta_1} V = r_1 - \theta_1 + \theta = G - F \tag{3.10}$$

are obtained. In addition, $r(0, a) = a$ yields

$$V(0, a) = h(0, a) - \frac{a^2}{2} = \hat{h}(0, a) + C - \frac{a^2}{2} = 0$$

by the definition of C. Moreover, (3.10) and (3.1) lead to

$$\partial_\theta V(\theta, \theta + a) = F(\theta, r(\theta, \theta + a)) = F(\theta, a) = 0, \quad \theta \in \mathbb{R}, \tag{3.11}$$

and in the same way,

$$\partial_{\theta_1} V(\theta, \theta + a) = G(\theta, a) - F(\theta, a) = 0, \quad \theta \in \mathbb{R}. \tag{3.12}$$

Noting that

$$V(\theta, \theta_1) = V(0, a) + \int_\gamma dV = \int_\gamma F\, d\theta + (G - F)\, d\theta_1$$

for any path γ in the simply connected Ω_0 which connects $(0, a) \in \Omega_0$ to $(\theta, \theta_1) \in \Omega_0$, it follows from (3.11) and (3.12) that

$$V(\theta, \theta + a) = 0 \quad \text{for all } \theta \in \mathbb{R}, \tag{3.13}$$

since γ may be chosen to be part of the lower boundary $\{\theta_1 = \theta + a : \theta \in \mathbb{R}\}$ of Ω_0. By (3.8),

$$|\partial^2_{\theta\theta_1} V(\theta, \theta_1)| = |\partial^2_{\theta\theta_1} h(\theta, \theta_1) + 1| \le \tfrac{1}{21}, \quad (\theta, \theta_1) \in \Omega_0. \tag{3.14}$$

From (3.13), (3.12) and (3.14), Lemma 3.5 below shows that

$$|V(\theta, \theta_1)| \le \tfrac{1}{42}(\theta_1 - \theta)^2, \quad (\theta, \theta_1) \in \Omega_0.$$

Thus, by (3.9),

$$\tfrac{10}{21}(\theta_1 - \theta)^2 \le h(\theta, \theta_1) \le \tfrac{11}{21}(\theta_1 - \theta)^2, \quad (\theta, \theta_1) \in \Omega_0. \tag{3.15}$$

Fix $\varepsilon^* > 0$ sufficiently small so that

$$10(a + \varepsilon^*) < \tfrac{1}{10}(b - \varepsilon^*), \tag{3.16}$$

and put

$$\delta = a + \varepsilon^*, \quad \Delta = b - \varepsilon^*, \quad \underline{\alpha} = \tfrac{10}{21}, \quad \text{and} \quad \overline{\alpha} = \tfrac{11}{21}.$$

Then $\Delta > \delta > 0$ by (3.16), and

$$\Omega = \{(\theta, \theta_1) \in \mathbb{R}^2 : \delta \le \theta_1 - \theta \le \Delta\} \subset \Omega_0.$$

Also, $\overline{\alpha}/\underline{\alpha} = 11/10$; so (3.15) and (3.16) together with Corollary 2.8 show that Theorem 2.1 applies. Hence there exists $(\theta_n)_{n\in\mathbb{Z}}$ such that $|\theta_0| \le \Delta$, $\delta \le \theta_{n+1} - \theta_n \le \Delta$ for $n \in \mathbb{Z}$, and

$$\partial_{\theta_1} h(\theta_{n-1}, \theta_n) + \partial_\theta h(\theta_n, \theta_{n+1}) = 0, \quad n \in \mathbb{Z}. \tag{3.17}$$

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In addition,

$$\delta \le \liminf_{n\to\pm\infty} \frac{\theta_n}{n} \le \limsup_{n\to\pm\infty} \frac{\theta_n}{n} \le \Delta$$

implies that (3.3) is satisfied. Putting $r_n = -\partial_\theta h(\theta_n, \theta_{n+1})$ in accordance with (3.6), it follows that $(\theta_n, r_n)_{n\in\mathbb{Z}}$ is a complete orbit for f. In fact, by (3.17) and (3.7),

$$r_{n+1} = -\partial_\theta h(\theta_{n+1}, \theta_{n+2}) = \partial_{\theta_1} h(\theta_n, \theta_{n+1}) = r_n + G(\theta_n, r_n).$$

Moreover, $r_n = -\partial_\theta h(\theta_n, \theta_{n+1}) = r(\theta_n, \theta_{n+1})$, by (3.6), is equivalent to

$$\theta_n + r_n + F(\theta_n, r_n) = \theta_1(\theta_n, r_n) = \theta_{n+1}.$$

Hence $f(\theta_n, r_n) = (\theta_n + r_n + F(\theta_n, r_n), r_n + G(\theta_n, r_n)) = (\theta_{n+1}, r_{n+1})$ is obtained. It remains to verify that $r_n \in [a, b]$ for $n \in \mathbb{Z}$; owing to $\theta_n + a \le \theta_{n+1} \le \theta_n + b$, it is a consequence of (3.4) that $r_n = r(\theta_n, \theta_{n+1}) \in [a, b]$. □

Remark 3.4. Observe that, because of the conclusion

$$a < \inf_{n\in\mathbb{N}} (\theta_{n+1} - \theta_n) \le \sup_{n\in\mathbb{N}} (\theta_{n+1} - \theta_n) < b$$

in Theorem 3.3, the bounded orbit lies in the interior of the strip $\mathbb{R} \times [a, b]$ and does not accumulate on its boundary.

The following technical lemma was used in the preceding proof.

LEMMA 3.5. *Let $\Delta > \delta > 0$. Suppose that $V : \Omega = \{(\theta, \theta_1) \in \mathbb{R}^2 : \delta \le \theta_1 - \theta \le \Delta\} \to \mathbb{R}$ is C^2 and satisfies*

$$V(\theta, \theta + \delta) = \partial_{\theta_1} V(\theta, \theta + \delta) = 0 \quad \text{and} \quad |\partial^2_{\theta\theta_1} V(\theta, \theta_1)| \le \varepsilon$$

for $\theta \in \mathbb{R}$ and $(\theta, \theta_1) \in \Omega$, respectively. Then

$$|V(\theta, \theta_1)| \le \frac{\varepsilon}{2}(\theta_1 - \theta)^2, \quad (\theta, \theta_1) \in \Omega. \tag{3.18}$$

Proof. Note that this is a consequence of d'Alembert's formula. Since

$$\begin{aligned} -\int_{\theta+\delta}^{\theta_1} d\eta \int_{\theta}^{\eta-\delta} d\xi\ \partial^2_{\theta\theta_1} V(\xi, \eta) &= -\int_{\theta+\delta}^{\theta_1} d\eta\ (\partial_{\theta_1} V(\eta - \delta, \eta) - \partial_{\theta_1} V(\theta, \eta)) \\ &= \int_{\theta+\delta}^{\theta_1} d\eta\ \partial_{\theta_1} V(\theta, \eta) \\ &= V(\theta, \theta_1) - V(\theta, \theta + \delta) = V(\theta, \theta_1), \end{aligned}$$

it follows that $|V(\theta, \theta_1)| \le \varepsilon \int_{\theta+\delta}^{\theta_1} d\eta \int_{\theta}^{\eta-\delta} d\xi = \varepsilon(\theta_1 - \theta - \delta)^2/2 \le \varepsilon(\theta_1 - \theta)^2/2$, which is (3.18). □

4. *Existence of infinitely many complete orbits*

In this section, the assumption (3.1) on the invariance of the boundaries will be dropped. This has to be compensated for by imposing a stronger and more qualitative hypothesis on the generating function $\hat{h}$.

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Definition 4.1. Let $f = f(\theta, r) : \mathbb{R} \times [a, \infty[\to \mathbb{R}^2$ be a C^1 map and suppose that $c : [a, \infty[\to \mathbb{R}$ is a function. We say that f is *c-exact symplectic* if there exists a C^2 function $\hat{h} : \mathbb{R} \times [a, \infty[\to \mathbb{R}$ such that

$$r_1\, d\theta_1 - r\, d\theta = d\hat{h} \quad \text{where } (\theta_1, r_1) = f(\theta, r),$$

and

$$\sup_{\theta_0 \in \mathbb{R}} \sup_{r \in [a, \infty[} \min_{\theta \in [\theta_0,\, \theta_0 + T]} |\hat{h}(\theta, r) - c(r)| < \infty$$

for some $T > 0$.

Before turning to our next theorem, we give some motivation for Definition 4.1, which may be viewed as a proper extension of the notion of exact symplectic maps from the periodic case. This is illustrated by the following.

LEMMA 4.2. *Let $f = f(\theta, r) : (\mathbb{R}/T\mathbb{Z}) \times [a, \infty[\to \mathbb{R}^2$ be a symplectic C^1 map. Then f is exact symplectic on the cylinder if and only if the condition from Definition 4.1 holds.*

Proof. First suppose that f is exact symplectic on the cylinder. Then $r_1\, d\theta_1 - r\, d\theta = d\hat{h}$ for a C^2 function $\hat{h}$ which is T-periodic in θ. Defining $c(r) = (1/T) \int_0^T \hat{h}(\theta, r)\, d\theta$ as the average, it follows that

$$\hat{h}(\theta, r) = c(r) + \hat{V}(\theta, r),$$

where $\int_0^T \hat{V}(\theta, r)\, d\theta = 0$ for $r \in [a, \infty[$. Fix $\theta_0 \in \mathbb{R}$ and $r \in [a, \infty[$. Since $\hat{V}(\cdot, r)$ is T-periodic, $\int_{\theta_0}^{\theta_0 + T} \hat{V}(\theta, r)\, d\theta = 0$ too. Therefore $\hat{V}(\theta_*, r) = 0$ for some $\theta_* \in [\theta_0, \theta_0 + T]$ and hence

$$\min_{\theta \in [\theta_0, \theta_0 + T]} |\hat{h}(\theta, r) - c(r)| = \min_{\theta \in [\theta_0, \theta_0 + T]} |\hat{V}(\theta, r)| = 0$$

for every $\theta_0 \in \mathbb{R}$ and $r \in [a, \infty[$. Conversely, suppose that

$$M = \sup_{\theta_0 \in \mathbb{R}} \sup_{r \in [a, \infty[} \min_{\theta \in [\theta_0, \theta_0 + T]} |\hat{h}(\theta, r) - c(r)| < \infty.$$

We need to show that $\hat{h}(\cdot, r)$ is T-periodic for $r \in [a, \infty[$. Writing $(\theta_1, r_1) = f(\theta, r)$, the functions $\theta \mapsto \theta_1(\theta, r) - \theta$ and $\theta \mapsto r_1(\theta, r)$ are T-periodic by assumption, for every $r \in [a, \infty[$. The relation $d\hat{h} = r_1 d\theta_1 - r d\theta$ implies that $\partial_\theta \hat{h} = r_1 \partial_\theta \theta_1 - r$ and $\partial_r \hat{h} = r_1 \partial_r \theta_1$. In particular, every $\partial_\theta \hat{h}(\cdot, r)$ is T-periodic. From $M < \infty$ it then follows that $\hat{h}(\cdot, r)$ is bounded. In fact, if $\theta_0 \in \mathbb{R}$ is fixed, then there is $\theta_* \in [\theta_0, \theta_0 + T]$ such that $|\hat{h}(\theta_*, r) - c(r)| \le M$. Thus

$$|\hat{h}(\theta_0, r)| = \left| \hat{h}(\theta_*, r) - \int_{\theta_0}^{\theta_*} \partial_\theta \hat{h}(\theta, r)\, d\theta \right| \le M + c(r) + \|\partial_\theta \hat{h}(\cdot, r)\|_\infty T, \quad \theta_0 \in \mathbb{R}.$$

Since $\hat{h}(\cdot, r)$ is bounded and $\partial_\theta \hat{h}(\cdot, r)$ is T-periodic, $\hat{h}(\cdot, r)$ is also T-periodic. □

In [**11**], the author used a notion of exact symplectic for some quasiperiodic maps. In general, if we assume that the map and its derivatives of first order are almost periodic, uniformly in r, then the natural definition of exact symplectic would be to require that $\hat{h}$ and its first derivatives are also almost periodic, uniformly in r. With some more effort one can then prove a result similar to Lemma 4.2, showing that in this case too, this notion is equivalent to the one introduced by Definition 4.1.

The main result of this section is the following.

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THEOREM 4.3. *Suppose that $f : \mathbb{R} \times [a, \infty[\to \mathbb{R}^2$ is c-exact symplectic with $c(r) = r^2/2$. Let f be given by*

$$\theta_1 = \theta + r + F(\theta, r), \quad r_1 = r + G(\theta, r),$$

where $(\theta_1, r_1) = f(\theta, r)$. For $F, G : \mathbb{R} \times [a, \infty[\to \mathbb{R}$ we assume that $F, G \in C^1$ and

$$\|F\|_\infty < \infty, \quad \|G\|_\infty < \infty \quad \text{and} \quad |\partial_r F(\theta, r)| \le \tfrac{1}{43} \tag{4.1}$$

for $\theta \in \mathbb{R}$ and $r \in [a, \infty[$. Then f has infinitely many complete orbits $(\theta_n^j, r_n^j)_{n\in\mathbb{Z}}$ such that $R_j \le r_n^j \le R_{j+1}$ for $n \in \mathbb{N}$, where $R_j < R_{j+1} \to \infty$ as $j \to \infty$. Moreover,

$$R_j \le \liminf_{n\to\pm\infty} \frac{\theta_n^j}{n} \le \limsup_{n\to\pm\infty} \frac{\theta_n^j}{n} \le R_{j+1}.$$

Proof. Define

$$M = \sup_{\theta_0\in\mathbb{R}} \sup_{r\in[a,\infty[} \min_{\theta\in[\theta_0,\theta_0+T]} \left|\hat{h}(\theta, r) - \frac{1}{2}r^2\right| < \infty \tag{4.2}$$

and

$$m = \max\{\|F\|_\infty, \|G\|_\infty\} < \infty.$$

Next, fix $b > a$ such that $b \ge 1$ and

$$b \ge 205(a + m), \quad b \ge 235300800 \cdot (M + m + Tm) \tag{4.3}$$

both hold. For $f : \mathbb{R} \times [a, b] \to \mathbb{R}^2$ we are going to use similar arguments as in the proof of Theorem 3.3; however, since f does not preserve the boundaries, the technicalities are more involved. As before, we consider the strictly increasing maps $[a, b] \ni r \mapsto \theta_1(\theta, r) = \theta + r + F(\theta, r) \in \mathbb{R}$ and their inverses

$$[\theta_1(\theta, a), \theta_1(\theta, b)] \ni \theta_1 \mapsto r(\theta, \theta_1) \in [a, b].$$

Taking $\hat{h}$ from Definition 4.1, let

$$h(\theta, \theta_1) = \hat{h}(\theta, r(\theta, \theta_1)) \quad \text{and} \quad V(\theta, \theta_1) = h(\theta, \theta_1) - \tfrac{1}{2}(\theta_1 - \theta)^2, \quad (\theta, \theta_1) \in \Omega_0,$$

where

$$\Omega_0 = \{(\theta, \theta_1) \in \mathbb{R}^2 : \theta \in \mathbb{R},\ \theta_1(\theta, a) \le \theta_1 \le \theta_1(\theta, b)\}.$$

Then

$$\partial_\theta h = -r, \quad \partial_{\theta_1} h = r_1, \quad \partial_\theta V = F, \quad \partial_{\theta_1} V = G - F. \tag{4.4}$$

Also,

$$|\partial^2_{\theta\theta_1} V(\theta, \theta_1)| = |\partial^2_{\theta\theta_1} h(\theta, \theta_1) + 1| \le \frac{1/43}{1 - 1/43} = \frac{1}{42}, \quad (\theta, \theta_1) \in \Omega_0 \tag{4.5}$$

from (3.8) and (4.1). Put

$$\Omega_1 = \{(\theta, \theta_1) \in \mathbb{R}^2 : \theta \in \mathbb{R},\ \theta + a_1 \le \theta_1 \le \theta + b_1\}$$

for $a_1 = a + m$ and $b_1 = b - m$. Then $\theta_1(\theta, a) = \theta + a + F(\theta, a) \le \theta + a_1$ and $\theta_1(\theta, b) = \theta + b + F(\theta, b) \ge \theta + b_1$ imply that $\Omega_1 \subset \Omega_0$. The qualitative assumption (4.2) on $\hat{h}$ results in the following estimate, by Lemma 4.4 below:

$$|V(\theta, \theta_1)| \le M + \tfrac{22}{43}(a_1 + b_1 + 4m)m + \tfrac{3}{2}(b_1 - a_1 + 2T)m \quad (\theta, \theta_1) \in \Omega_1. \tag{4.6}$$

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Now put

$$a_2 = a_1 + \eta \quad \text{and} \quad b_2 = b_1 - \eta \quad \text{where } \eta = \tfrac{1}{200}b.$$

Then $a_2 < b_2$. Next, fix a cut-off function $\chi \in C^\infty(\mathbb{R})$ such that $\chi(s) = 1$ for $s \in [a_2, b_2]$, $\chi(s) = 0$ for $s \le a_1$, $\chi(s) = 0$ for $s \ge b_1$, and

$$|\chi(s)| \le 1, \quad |\chi'(s)| \le \frac{4}{\eta}, \quad |\chi''(s)| \le \frac{20}{\eta^2} \quad \text{for } s \in \mathbb{R}.$$

Let $\tilde V(\theta, \theta_1) = \chi(\theta_1 - \theta)V(\theta, \theta_1)$ for $(\theta, \theta_1) \in \Omega_1$. Then $\tilde V(\theta, \theta + a_1) = 0$ for $\theta \in \mathbb{R}$. Moreover,

$$\begin{aligned}
\partial_{\theta_1}\tilde V(\theta, \theta_1) &= \chi'(\theta_1 - \theta)V(\theta, \theta_1) + \chi(\theta_1 - \theta)\partial_{\theta_1}V(\theta, \theta_1),\\
\partial^2_{\theta\theta_1}\tilde V(\theta, \theta_1) &= -\chi''(\theta_1 - \theta)V(\theta, \theta_1) + \chi'(\theta_1 - \theta)[\partial_\theta V(\theta, \theta_1) - \partial_{\theta_1}V(\theta, \theta_1)]\\
&\quad + \chi(\theta_1 - \theta)\partial^2_{\theta\theta_1}V(\theta, \theta_1).
\end{aligned}$$

In particular, $\partial_{\theta_1}\tilde V(\theta, \theta + a_1) = 0$ and

$$\begin{aligned}
|\partial^2_{\theta\theta_1}\tilde V(\theta, \theta_1)| &\le 20\eta^{-2}(M + \tfrac{22}{43}(a + b + 4m)m + \tfrac{3}{2}(b - a - 2m + 2T)m)\\
&\quad + 12\eta^{-1}m + \tfrac{1}{42} =: \varepsilon_1 \quad \text{for } (\theta, \theta_1) \in \Omega_1,
\end{aligned}$$

by (4.6), (4.4), and (4.5). Hence Lemma 3.5 applies, giving

$$|\tilde V(\theta, \theta_1)| \le \frac{\varepsilon_1}{2}(\theta_1 - \theta)^2, \quad (\theta, \theta_1) \in \Omega_1.$$

Thus, upon defining $\tilde h(\theta, \theta_1) = (\theta_1 - \theta)^2/2 + \tilde V(\theta, \theta_1)$ for $(\theta, \theta_1) \in \Omega_1$, it follows that

$$\underline{\alpha}(\theta_1 - \theta)^2 \le \tilde h(\theta, \theta_1) \le \overline{\alpha}(\theta_1 - \theta)^2, \quad (\theta, \theta_1) \in \Omega_1, \tag{4.7}$$

where $\underline{\alpha} = (1 - \varepsilon_1)/2$ and $\overline{\alpha} = (1 + \varepsilon_1)/2$. Because of (4.3), $\varepsilon_1 \le 1/21$ is satisfied. In fact, the inequalities $a + b + 4m \le 4(a + m) + b \le (209/205)b$ and $b \ge 1$ yield

$$\begin{aligned}
\varepsilon_1 &\le 20\,\eta^{-2}(M + \tfrac{3}{5}bm + 3(2b + T)m) + 12\eta^{-1}m + \tfrac{1}{42}\\
&\le 140\eta^{-2}(M + (b + T)m) + 12\eta^{-1}m + \tfrac{1}{42}\\
&\le 28\,000\eta^{-1}(M + Tm) + 28\,000\eta^{-1}m + 12\eta^{-1}m + \tfrac{1}{42}\\
&\le 28\,012\eta^{-1}(M + m + Tm) + \tfrac{1}{42} \le \tfrac{1}{21}
\end{aligned}$$

by (4.3). Since $\varepsilon_1 \le 1/21$, the bound $\overline{\alpha}/\underline{\alpha} = (1 + \varepsilon_1)/(1 - \varepsilon_1) \le 11/10$ is obtained. In particular, on $\Omega_2 = \{(\theta, \theta_1) \in \mathbb{R}^2 : a_2 \le \theta_1 - \theta \le b_2\} \subset \Omega_1$ the estimate (4.7) holds for h, as $\tilde h = h$ on Ω_2 by definition of χ. Next, observe that

$$100a_2 = 100(a + m) + \tfrac{1}{2}b < \tfrac{199}{200}b - m = b - m - \eta = b_2$$

by (4.3). Consequently we can use Corollary 2.8 and Theorem 2.1 for h on Ω_2 to get a sequence $(\theta_n)_{n\in\mathbb{Z}}$ such that $|\theta_0| \le b_2$, $a_2 \le \theta_{n+1} - \theta_n \le b_2$ for $n \in \mathbb{Z}$, and

$$\partial_2 h(\theta_{n-1}, \theta_n) + \partial_1 h(\theta_n, \theta_{n+1}) = 0, \quad n \in \mathbb{Z}.$$

In addition, $a_2 \le \liminf_{n\to\pm\infty}(\theta_n/n) \le \limsup_{n\to\pm\infty}(\theta_n/n) \le b_2$. If we put $r_n = -\partial_\theta h(\theta_n, \theta_{n+1})$, then the argument from the proof of Theorem 3.3 shows that $(\theta_n, r_n)_{n\in\mathbb{Z}} \subset \mathbb{R} \times [a, b]$ is a complete orbit for the map f.

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We have thus obtained a first complete orbit for f. The next step is to replace $R_1 = a$ by $R_2 = b + 1$ in the preceding argument and select a new R_3 which satisfies $R_3 \geq 205(R_2 + m)$. As the conditions (4.3) hold for R_3, a further complete orbit for $f : \mathbb{R} \times [R_2, R_3] \to \mathbb{R}^2$ is obtained. Continuing in this way, we get infinitely many complete orbits for f which have the desired properties. □

We present one more technical result that played a role in the proof of Theorem 4.3.

LEMMA 4.4. *Using the notation from the proof of Theorem 4.3,*

$$|V(\theta, \theta_1)| \leq M + \tfrac{22}{43}(a_1 + b_1 + 4m)m + \tfrac{3}{2}(b_1 - a_1 + 2T)m \quad \text{for } (\theta, \theta_1) \in \Omega_1.$$

Proof. Consider a fixed $(\tilde\theta, \tilde\theta_1) \in \Omega_1$, so that $a_1 \leq \tilde\theta_1 - \tilde\theta \leq b_1$ holds. As a consequence of (4.2), for $\tilde\theta \in \mathbb{R}$ and $c_1 = (a_1 + b_1)/2 \geq a$ there is $\theta \in [\tilde\theta, \tilde\theta + T]$ such that $|\hat h(\theta, c_1) - c_1^2/2| \leq M$. Now, $\theta + c_1 = \theta + r + F(\theta, r)$ for $r = r(\theta, \theta + c_1)$ gives $|r(\theta, \theta + c_1) - c_1| \leq \|F\|_\infty \leq m$. By (3.5), $|\partial_r \hat h(\theta, r)| = |r_1\, \partial_r \theta_1| = (r + G)\,(1 + \partial_r F) \leq (44/43)(c_1 + 2m)$ for $|r - c_1| \leq m$, and it follows that

$$\begin{aligned} |V(\theta, \theta + c_1)| &= |\hat h(\theta, r(\theta, \theta + c_1)) - c_1^2/2| \\ &\leq |\hat h(\theta, r(\theta, \theta + c_1)) - \hat h(\theta, c_1)| + |\hat h(\theta, c_1) - c_1^2/2| \\ &\leq \left| \int_{c_1}^{r(\theta, \theta + c_1)} \partial_r \hat h(\theta, r)\, dr \right| + M \leq \frac{44}{43}(c_1 + 2m)m + M. \end{aligned}$$

According to (4.4),

$$V(\tilde\theta, \tilde\theta_1) = V(\theta, \theta + c_1) + \int_\gamma dV = V(\theta, \theta + c_1) + \int_\gamma F\, d\theta + (G - F)\, d\theta_1$$

for the path γ connecting first $(\tilde\theta, \tilde\theta_1)$ to $(\tilde\theta, \tilde\theta + c_1)$ along the vertical line $\{\theta = \tilde\theta\}$, and then $(\tilde\theta, \tilde\theta + c_1)$ to $(\theta, \theta + c_1)$ along the straight line $\{\theta_1 - \theta = c_1\}$. In particular, γ has length $|\gamma| \leq |\tilde\theta_1 - \tilde\theta - c_1| + T \leq (b_1 - a_1)/2 + T$. Thus the preceding estimates imply

$$|V(\tilde\theta, \tilde\theta_1)| \leq \tfrac{44}{43}(c_1 + 2m)m + M + 3(\tfrac{1}{2}(b_1 - a_1) + T)m,$$

as claimed. □

Finally, we consider an application of Theorem 4.3.

THEOREM 4.5. *Let $\phi \in C^2(\mathbb{R})$ be such that ϕ and ϕ' are bounded. Then the map*

$$\theta_1 = \theta + r, \quad r_1 = r + \phi'(\theta + r)$$

has infinitely many complete orbits enjoying the properties described in Theorem 4.3.

In particular, this applies to the special case of the maps

$$\theta_1 = \theta + r, \quad r_1 = r + \lambda(\sin \omega_1(\theta + r) + \sin \omega_2(\theta + r)),$$

for all $\lambda, \omega_1, \omega_2 > 0$.

Proof. Put $\hat h(\theta, r) = r^2/2 + \phi(\theta + r)$, $F = 0$ and $G(\theta, r) = \phi'(\theta + r)$. Then $r_1 \partial_\theta \theta_1 - r = (r + \phi'(\theta, r)) - r = \phi'(\theta + r) = \partial_\theta \hat h$ and $r_1 \partial_r \theta_1 = r + \phi'(\theta + r) = \partial_r \hat h$ show that $r_1\, d\theta_1 - r\, d\theta = d\hat h$ holds. Since $|\hat h(\theta, r) - r^2/2| = |\phi(\theta, r)| \leq \|\phi\|_\infty$, f is c-exact

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symplectic with $c(r) = r^2/2$. Concerning (4.1), we only have to note that $\|G\|_\infty = \|\phi'\|_\infty < \infty$ by assumption; hence the first claim follows from Theorem 4.3. For the special case, $\phi(\theta) = -\lambda(\omega_1^{-1} \cos \omega_1\theta + \omega_2^{-1} \cos \omega_2\theta)$ satisfies $\|\phi\|_\infty < \infty$ and $\|\phi'\|_\infty < \infty$. □

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