HYPERCYCLIC AND CHAOTIC CONVOLUTION OPERATORS

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ABSTRACT

Every convolution operator on a space of ultradifferentiable functions of Beurling or Roumieu type and on the corresponding space of ultradistributions is hypercyclic and chaotic when it is not a multiple of the identity. The operator of differentiation is hypercyclic on the space $A^{-\infty}$, but it need not be hypercyclic on radial weighted algebras of entire functions.

1. *Introduction*

The purpose of this paper is to study the hypercyclic and chaotic behaviour of certain differential operators. We prove in Theorem 4 that convolution operators on spaces of ultradifferentiable functions of Beurling or Roumieu type and also on the spaces of ultradistributions of these two types are hypercyclic and chaotic when they are not scalar multiples of the identity. In particular the result holds for Gevrey classes and for linear partial differential operators of infinite order, but it is also new for the classical distributions. We observe that many of these spaces of functions or distributions are not Fréchet spaces since they need not be metrizable. The behaviour of the operator of differentiation on certain weighted inductive limits of spaces of holomorphic functions is discussed in Section 4. We consider the space $A^{-\infty}$ of Korenblum in Corollary 10 and certain weighted algebras of entire functions [**3**, Chapter 2] in Theorem 11. Our results show that concrete operators may have an interesting hypercyclic behaviour in non-metrizable locally convex spaces which appear in complex analysis or convolution equations. There has recently been great interest in hypercyclic and chaotic linear operators on infinite dimensional spaces and in their applications. We mention [**1**, **4**, **5**, **11**, **19**, **22**, **31**]. Paper [**21**] is an excellent survey of the state of the art concerning hypercyclic operators on Banach and more general locally convex spaces.

Godefroy and Shapiro [**19**, 5.5, 6.2] proved that every partial differential operator on $\mathbb{R}^N$ which is not a scalar multiple of the identity is chaotic on $\mathscr{C}^\infty(\mathbb{R}^N)$, as a consequence of a theorem on continuous linear operators on $H(\mathbb{C}^N)$ which commute with translations. This theorem extended classical work of Birkhoff and MacLane showing that the operators of translation and differentiation on the space of entire functions of one complex variable are hypercyclic. In [**5**] Bernal-Gonzalez has studied the hypercyclicity of sequences of differential operators on spaces of holomorphic functions defined on open subsets of $\mathbb{C}$. Our proofs combine Godefroy and Shapiro's result [**19**, 5.1, 6.2] with an extension of the hypercyclic comparison principle

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[**34**, p. 111] presented in Lemma 3. They also utilize the special structure of the spaces and of the operators which appear in the statements. In Section 4 the results of [**20**] are also essential.

The surjectivity of convolution operators between spaces of $\mathscr{C}^\infty$-functions or distributions on open sets of $\mathbb{R}^N$ was characterized by Ehrenpreis [**18**] and Hörmander [**23**]; also see [**24**]. The surjectivity of convolution operators on spaces of ultradifferentiable functions on arbitrary open sets was studied by several authors. We refer to [**13**] and the references given there. Theorem 4 below shows that hypercyclic convolution operators need not be surjective.

2. *Notation*

Our notation for locally convex spaces and functional analysis is standard; see [**29**, **32**]. A continuous and linear operator T from a locally convex space E into itself is called *hypercyclic* if there is a vector x (which is called a hypercyclic vector) in E such that its orbit $(x, Tx, T^2x, \ldots)$ is dense in E. Every hypercyclic operator T on E is topologically transitive in the sense of dynamical systems, that is, for every pair of non-empty open subsets U and V of E there is n such that $T^n(U)$ meets V. Devaney [**17**] proposed that a continuous map on a metric space be called *chaotic* if it is topologically transitive, has a dense set of periodic points and possesses a certain 'sensitivity to initial conditions'. Every linear operator on a Fréchet space which is hypercyclic satisfies the sensitive dependence (see [**19**, 6.1] or [**2**]). Every continuous linear operator T on a locally convex space E which is topologically transitive satisfies the following sensitive dependence with respect to initial conditions: for every $x \in E$ and for absolutely convex 0-neighbourhoods U and V in E, $U \neq E$, there are $y \in x + V$ and n such that $T^n(x) - T^n(y) \notin U$. As an extension of Devaney's definition to our present context of non-metrizable locally convex spaces E, we propose to call a continuous linear operator on E *chaotic* if it is topologically transitive and has a dense set of periodic points. It is important to observe the following.

EXAMPLE 1. A chaotic continuous linear operator on an arbitrary locally convex space need not be hypercyclic.

Let $T = \lambda B$, $\lambda > 1$, be the weighted backward shift on l_2. The operator T is hypercyclic and has a dense set P of periodic points; for example see [**19**]. Let e_i denote the ith canonical unit vector of l_2. We denote by E the dense topological subspace of l_2 which is the linear span of $\{e_i; i \in \mathbb{N}\} \cup P$. Clearly T acts continuously from E into itself, it has a dense set of periodic points on E, and it is topologically transitive by an easy density argument. Clearly T cannot be hypercyclic since every element of E has a finite orbit.

We briefly recall the definition of the standard spaces of ultradifferentiable functions of Beurling type, convolution operators on these spaces and most of the notation which will be used in the sequel. The definitions are taken from Braun, Meise and Taylor [**15**]; also compare with Beurling [**6**] and Björck [**10**]. The reader familiar with the classical theory of distributions [**32**] may proceed directly to Theorem 4 and read the proof for the classical case.

Let $\omega: \mathbb{R} \longrightarrow [0, \infty[$ be a continuous even function which is increasing on $[0, \infty[$ and satisfies $\omega(0) = 0$ and $\omega(1) > 0$. It is called a *weight function* if it satisfies the following conditions:

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(α) $\omega(2t) \leqslant K(1+\omega(t))$ for all $t \in \mathbb{R}$;
(β) $\int_{-\infty}^{\infty} \omega(t)/(1+t^2)\,dt < \infty$;
(γ) $\log(1+t^2) = O(\omega(t))$ as t tends to ∞;
(δ) $\varphi: t \longrightarrow \omega(e^t)$ is convex on $\mathbb{R}$.

The Young conjugate of φ is defined by $\varphi^*(x) = \sup_{y>0}\{xy - \varphi(y)\}$.

The most important example of weight functions is given by $\omega(t) = t^\beta$, $0 < \beta < 1$. These are the weights which permit us to define the classical Gevrey classes of exponent $d := 1/\beta$. For $\omega(t) = \log(1+t)$ the spaces of functions and ultradistributions of Beurling type defined below are the usual C^∞ functions and Schwartz distributions; see [**24**]. When we deal with the Roumieu case below it is necessary to assume (and we do it without explicit formulation) that the weight ω satisfies $\log(1+t) = o(\omega(t))$ instead of condition (γ); see [**15**]. We refer also to this article for more examples and details.

DEFINITION 2. Let ω be a weight function.

(a) For a compact set $K \subset \mathbb{R}^N$ and $\ell > 0$ we set

$$\mathscr{E}_\omega^\ell(K) = \{f \in C^\infty(K) : \|f\|_{K,\ell} < \infty\},$$

where

$$\|f\|_{K,\ell} := \sup_{x\in K} \sup_\alpha |f^{(\alpha)}(x)| \exp\left(-\ell \varphi^*\left(\frac{|\alpha|}{\ell}\right)\right).$$

(b) For an open set $G \subset \mathbb{R}^N$ we define

$$\mathscr{E}_{(\omega)}(G) := \{f \in C^\infty(G) : \|f\|_{K,\ell} < \infty,\ \forall\ K \subset\subset G,\ \forall\ \ell \in \mathbb{N}\}$$

and

$$\mathscr{E}_{\{\omega\}}(G) := \{f \in C^\infty(G) : \forall\ K \subset\subset G,\ \exists\ \ell \in \mathbb{N} : \|f\|_{K,\ell} < \infty\},$$

and we endow these spaces with the natural topologies. The elements of $\mathscr{E}_{(\omega)}(G)$ (respectively $\mathscr{E}_{\{\omega\}}(G)$) are called *$\omega$-ultradifferentiable functions of Beurling* (*respectively Roumieu*) *type on G*. We write $\mathscr{E}_*(G)$, where $*$ can be either (ω) or $\{\omega\}$ at all occurring places.

(c) For a compact set K in $\mathbb{R}^N$ we set

$$\mathscr{D}_*(K) = \{f \in \mathscr{E}_*(\mathbb{R}^N) : \operatorname{supp}(f) \subset K\}$$

endowed with the induced topology. For an open set $G \subset \mathbb{R}^N$ and a fundamental sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets of G we define $\mathscr{D}_*(G) := \operatorname{ind}_{j\to} \mathscr{D}_*(K_j)$. The dual $\mathscr{D}'_*$ of $\mathscr{D}_*$ is endowed with its strong topology. The elements of $\mathscr{D}'_{(\omega)}(G)$ (respectively $\mathscr{D}'_{\{\omega\}}(G)$) are called *$\omega$-ultradistributions of Beurling* (*respectively Roumieu*) *type on G*. Given $\mu \in \mathscr{E}'_*(\mathbb{R}^N)$ and two open sets G and G' in $\mathbb{R}^N$ with $G' - \operatorname{supp}\mu \subset G$ we define the *convolution operator*

$$T_\mu : \mathscr{E}_*(G) \longrightarrow \mathscr{E}_*(G') \quad \text{by} \quad T_\mu(f) = \mu * f,\ \mu * f(x) = \langle \mu_y, f(x-y)\rangle.$$

T_μ is a continuous linear operator on $\mathscr{E}_*(G)$, and μ induces also a continuous linear operator $S_\mu : \mathscr{D}'_*(G) \longrightarrow \mathscr{D}'_*(G')$ where

$$S_\mu(v)(\varphi) = \langle \mu * v, \varphi\rangle = \langle v, \check{\mu} * \varphi\rangle \quad \text{and} \quad \langle \check{\mu}, \varphi\rangle = \langle \mu, \check{\varphi}\rangle,\ \check{\varphi}(x) = \varphi(-x),$$

for $v \in \mathscr{D}'_*(G)$ and $\varphi \in \mathscr{D}_*(G')$. Note that for $\mu = \sum_{|\alpha|\leqslant m} a_\alpha i^{-|\alpha|} \delta_0^\alpha$ the corresponding convolution operator is just the linear partial differential operator with constant coefficients $P(D) = \sum_{|\alpha|\leqslant m} a_\alpha i^{-|\alpha|} (\partial/\partial x)^\alpha$, while for $\mu = \varphi \in \mathscr{D}_*(\mathbb{R}^N)$ we get $T_\mu(f) =$

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$\int f(y)\,\varphi(x-y)\,dy$. For $\mu \in \mathscr{E}'_*$ its *Fourier–Laplace transform* is the entire function $\hat{\mu}$ on $\mathbb{C}^N$ defined by

$$\hat{\mu}(z) = \mu(\exp(-i\langle ., z\rangle)).$$

The Fourier–Laplace transform $\mathscr{F} : \mu \longrightarrow \hat{\mu}$ defines a linear topological isomorphism between $\mathscr{E}'_*(\mathbb{R}^N)$ and a weighted space of entire functions on $\mathbb{C}^N$; see the details in [**15**, 3.5, 7.4].

3. *Convolution operators on spaces of ultradifferentiable functions*

We need the following extension of the hypercyclic comparison principle of [**34**, p. 111].

LEMMA 3. *Let T be a continuous linear operator on a locally convex space E. Let F be a locally convex space which is continuously and densely contained in E. If $T|F : F \longrightarrow F$ is a well-defined, continuous and hypercyclic (respectively chaotic) operator, then T is also hypercyclic (respectively chaotic) on E. In particular T has a hypercyclic vector which belongs to F.*

In fact, in [**34**, p. 111] the result is mentioned for hypercyclic operators, but if the restriction of the operator has a dense set of periodic points in the subspace F, the operator itself must also have it by density. Moreover, if $T|F$ is topologically transitive, again a density argument yields that T is also topologically transitive.

THEOREM 4. *Let μ be an element of $\mathscr{E}'_*(\mathbb{R}^N)$ which is not a scalar multiple of the Dirac measure. Let G be a non-empty open subset of $\mathbb{R}^N$ such that $G \supset G + \operatorname{supp} \check{\mu} = G - \operatorname{supp} \mu$. Then the convolution operators*

(a) $T_\mu : \mathscr{E}_*(G) \longrightarrow \mathscr{E}_*(G)$;

(b) $S_\mu ; \mathscr{D}'_*(G) \longrightarrow \mathscr{D}'_*(G)$;

are hypercyclic and chaotic.

Proof. In order to apply Lemma 3 we have to explain in which sense $H(\mathbb{C}^N)$ is a dense subspace of $\mathscr{E}_*(G)$ such that the convolution operator T_μ acts continuously on $H(\mathbb{C}^N)$. The case of $\mathscr{D}'_*(G)$ follows from the density of $\mathscr{E}_*(G)$ in the space of ultradistributions, since the restriction of S_μ coincides with T_μ. (See [**15**, 5.4].)

We define $j : H(\mathbb{C}^N) \longrightarrow \mathscr{E}_*(G)$ by $j(f) := f|G$. It is well known that j is a well-defined, continuous injection. To see that j has dense range, for each $z \in \mathbb{C}^N$, we define the exponential function $e_z(x) := \exp(i\langle x, z\rangle)$, $x \in G$. Clearly each function e_z belongs to the image of j in $\mathscr{E}_*(\mathbb{R}^N)$. Assume that $v \in \mathscr{E}'_*(G)$ vanishes on the image of j; then it satisfies $\langle v, e_z\rangle = 0$ for every $z \in \mathbb{C}^N$. The ultradistribution v can be considered as an element of $\mathscr{E}'_*(\mathbb{R}^N)$ and $\langle v, e_z\rangle = \hat{v}(-z)$ for each z. Consequently the entire function $\hat{v}$ vanishes at all the points of $\mathbb{C}^N$. This implies that $\hat{v} = 0$, and hence $v = 0$ as an element of $\mathscr{E}'_*(\mathbb{R}^N)$. In particular $v(g) = 0$ for all $g \in \mathscr{D}_*(G)$. Since $\mathscr{D}_*(G)$ is dense in $\mathscr{E}_*(G)$, it follows that $v = 0$ in $\mathscr{E}'_*(G)$. The density of the range of j follows by the Hahn–Banach theorem.

It remains to show that the restriction of T_μ acts continuously on $H(\mathbb{C}^N)$ in the precise sense established above. We define the continuous linear functional $\eta := \mu \circ j$ on $H(\mathbb{C}^N)$ and the convolution operator T_η on this space associated with η by

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$T_\eta(f)(z) = (\eta * f)(z) := \langle \eta_t, f(z-t)\rangle$ for all $z \in \mathbb{C}^N$ and all $f \in H(\mathbb{C}^N)$. The operator T_η acts continuously on $H(\mathbb{C}^N)$; if $F(z) = \hat{\eta}(-z) = \sum_\alpha b_\alpha z^\alpha$, then $T_\eta(f)(z) = (F(D)f)(z) = \sum_\alpha b_\alpha f^{(\alpha)}(z)$ for all f and z (see [**36**, Section 9; **3**, 1.5]). Therefore T_η is an operator on $H(\mathbb{C}^N)$ which commutes with translations, and it is not a scalar multiple of the identity by our assumption on μ. Finally, it follows directly from the definitions that $j((T_\eta)(g)) = T_\mu(j(g))$ for all $g \in H(\mathbb{C}^N)$, and the proof is completed by applying [**19**, 5.1 and 6.2]. □

The classes of quasianalytic functions are defined by replacing the non-quasianalytic condition (β) for the weight ω by the condition $\omega(t) = O(t)$ as t goes to ∞. The most important example, which is obtained for $\omega(t) = t$, is the space $A(G)$ of real analytic functions on an open set G of $\mathbb{R}^N$. We refer to [**14**] for details. The injectivity of the Fourier–Laplace transform on the dual of the space of quasianalytic ultradifferentiable functions on a convex open set permits us to adapt the arguments of Theorem 4 to show that convolution operators on $\mathscr{E}_*(G)$ which are not scalar multiples of the identity are also chaotic in the quasianalytic case. In particular we have the following.

PROPOSITION 5. *Every convolution operator on the space of real analytic functions $A(G)$ on an open convex set G on $\mathbb{R}^N$ is hypercyclic and chaotic when it is not a scalar multiple of the identity.*

4. *The operator of differentiation on weighted inductive limits of spaces of holomorphic functions*

In this section we study the hypercyclic behaviour of the operator of differentiation in certain weighted inductive limits of spaces of holomorphic functions. In [**11**] it was shown that many (but not all) countable inductive limits of separable Banach spaces admit hypercyclic operators. All the operators constructed there act continuously from a certain Banach step into itself. The operator of differentiation defined below does not act in general continuously from any step of the weighted inductive limit into itself.

Let G be a balanced open subset of $\mathbb{C}^N$. A *weight* v on G is a bounded radial continuous strictly positive function on G. (The function v is called radial if $v(\alpha z) = v(z)$ for all $z \in G$ and all $|\alpha| = 1$.) We define the weighted Banach spaces

$$Hv(G) := \{f \in H(G) \mid \sup_{z\in G} v(z)|f(z)| < \infty\},$$
$$Hv_0(G) := \{f \in H(G) \mid v|f| \text{ vanishes at } \infty \text{ on } G\}.$$

(A function $g: G \longrightarrow \mathbb{R}$ is said to vanish at infinity on G if for every $\varepsilon > 0$ there is a compact subset K of G such that $|g(z)| < \varepsilon$ for every $z \in G\backslash K$.) Given a decreasing sequence $V = (v_n)_n$ of weights on G, the *weighted inductive limit* $VH(G)$ *of spaces of holomorphic functions* is then defined by $VH(G) := \operatorname{ind}_n Hv_n(G)$. We refer the reader to [**7–9**]. In what follows we assume that every polynomial on $\mathbb{C}^N$ is contained in the space $H(v_1)_0(G)$ and that the following condition (S) on the sequence V holds: for every n there is $m > n$ such that v_m/v_n vanishes at infinity on G. This important condition is satisfied in all the examples below; we refer to [**8**, **9**] for the relevance of this condition.

We study conditions to ensure that the weighted inductive limit $VH(G)$ is closed under differentiation. We need the following notation. For $z \in G$, we set $r(z) := (1/2)\operatorname{dist}(z, \mathbb{C}^N\backslash G)$ if $G \neq \mathbb{C}^N$ and $r(z) = 1$ otherwise. Given a sequence of weights

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$V = (v_n)_n$ on G, we write $w_n := 1/v_n$ for each n. The ith canonical unit vector in $\mathbb{C}^N$ is denoted by e_i, $i = 1, \dots, N$. The proof of the following result follows by a standard application of the Cauchy formula; see [**37**, 7.2] or [**36**, Section 1].

PROPOSITION 6. *The space $VH(G)$ is closed under differentiation if the following condition is satisfied: for all n there are $m > n$ and $C > 0$ such that for all $z \in G$*

$$r(z)^{-1} \max_{1 \leqslant i \leqslant N} \max_{|z_i - \zeta_i| \leqslant r(z)} w_n(z + \zeta_i e_i) \leqslant C w_m(z).$$

If G is an open subset of $\mathbb{C}$, the operator of differentiation on $H(G)$ is denoted by $\partial(f) = f'$. It is possible to give complete characterizations of the weighted inductive limits $VH(G)$ which are stable under differentiation in terms of the sequence of weights in special cases. To do this we need the concept of *associated weight* $\tilde{v}$ for a weight v (cf. [**8**]). It is defined as follows: $\tilde{v}(z) := 1/(\sup\{|f(z)| : f \in H(G), |f| \leqslant 1/v \text{ on } G\})$. A weight is called *essential* if there is $C > 0$ such that $v \leqslant \tilde{v} \leqslant Cv$ on G. We refer to [**8**] for properties and examples of associated weights. The methods of [**8**] (cf. [**12**, 2.1]) permit us to show that the weighted Banach space $Hv(G)$ is contained in $Hu(G)$ if and only if there is $C > 0$ such that $C\tilde{v} \geqslant \tilde{u}$.

We first consider the case $G = D$, the unit disc of $\mathbb{C}$. We refer to [**35**] for the definition of *normal weights* on D; a weight v is normal if and only if it satisfies the conditions $(*)$ and $(**)$ of Lusky [**27**, 2.1] (see also [**26**]). Every normal weight is essential [**8**, 3.4]. If v is a normal weight on D, then $u(z) := (1 - |z|)\, v(z)$, $z \in D$, is also normal, and, by [**27**, 3.1], the operator $\partial : Hv(D) \longrightarrow Hu(D)$ is surjective.

PROPOSITION 7. *Let $V = (v_n)_n$ be a decreasing sequence of normal weights on D satisfying the general assumptions of this section. The space $VH(D)$ is closed under differentiation if and only if for every n there are $m > n$ and δ such that $(1 - r)\, v_n(r) \geqslant \delta v_m(r)$ for each $r \in [0, 1[$.*

Proof. By the closed graph theorem $VH(D)$ is closed under differentiation if and only if the operator $\partial : VH(D) \longrightarrow VH(D)$ is continuous. By Grothendieck's factorization theorem [**29**, 24.33], this holds if and only if for each n there is $m > n$ such that $\partial(Hv_n(D)) \subset Hv_m(D)$. With the notation $u_n(z) := (1 - |z|)\, v_n(z)$, $z \in D$, by [**27**, 3.1] the latter inclusion holds if and only if $Hu_n(D) \subset Hv_m(D)$. The remarks before the proposition complete the proof. □

Observe that the necessary and sufficient condition in Proposition 7 already implies condition (S) for the sequence V. The situation might be different for $G = \mathbb{C}$. In fact the Banach space $Hv(\mathbb{C})$, $v(z) = \exp(-|z|)$, $z \in \mathbb{C}$, is closed under differentiation, as follows easily from Proposition 6.

Let u, v be weights on $\mathbb{C}$ which are non-increasing on $[0, \infty[$. If there is $c > 0$ such that $v(1 + r) \geqslant cu(r)$ for all $r \geqslant 1$, then $\partial : Hv(\mathbb{C}) \longrightarrow Hu(\mathbb{C})$ is well-defined and continuous. This follows, for example, by [**37**, 7.2]. On the other hand, if there is $c > 0$ such that $cru(r) \leqslant v(r)$ for all $r \geqslant 1$, then $Hv(\mathbb{C}) \subset \partial(Hu(\mathbb{C}))$ by the proof of [**36**, 2.3]. Accordingly we have the following.

PROPOSITION 8. *Let $V = (v_n)$ be a decreasing sequence of weights on $\mathbb{C}$ which are non-increasing in $[0, \infty[$ and satisfy the general assumptions of this section and the following two conditions:*

(i) *for each n there are m and $c > 0$ such that $v_n(1 + r) \geqslant cv_m(r)$ if $r \geqslant 1$;*

(ii) *for each k there are l and $c > 0$ with $v_k(r) \geqslant crv_l(r)$ if $r \geqslant 1$.*

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Then the operator of differentiation $\partial: VH(\mathbb{C}) \longrightarrow VH(\mathbb{C})$ *is well-defined, continuous and surjective.*

Conditions (i) and (ii) in Proposition 8 are satisfied in the case of the spaces $A_p(\mathbb{C})$ defined below.

THEOREM 9. *Let* G *be a balanced bounded open subset of* $\mathbb{C}^N$. *Let* $V = (v_n)_n$ *be a decreasing sequence of radial weights on* G *such that* $H(v_1)_0(G)$ *contains the polynomials and such that* V *satisfies condition* (S). *If the space* $VH(G)$ *is stable under differentiation, then every linear partial differential operator with constant coefficients* $P(\partial): VH(G) \longrightarrow VH(G)$ *which is not a scalar multiple of the identity is hypercyclic and chaotic.*

Proof. Since the sequence V satisfies condition (S), the weighted inductive limit $VH(G)$ coincides with $\operatorname{ind}_n H(v_n)_0(G)$. We can apply [**7**, 1.6] to conclude that the polynomials are dense in $VH(G)$. Since G is bounded, the restriction of every entire function on $\mathbb{C}^N$ belongs to $VH(G)$. We can apply [**19**, 5.1, 6.2] and Lemma 3 to conclude. □

We denote by B_N the unit ball of $\mathbb{C}^N$. The Korenblum space $A^{-\infty}$ in several complex variables is defined as the smallest algebra of holomorphic functions that contains the class of bounded holomorphic functions on B_N and is closed by differentiation. This space is a countable inductive limit of weighted Banach spaces of holomorphic functions on B_N. In fact $A^{-\infty} = \operatorname{ind}_n Hv_n(B_N)$, where v_n is the weight $v_n(z) = (1-|z|)^n$, $z \in B_N$. We refer to [**28**] or to [**16**, **25**] for the one-dimensional case.

COROLLARY 10. *Every linear partial differential operator with constant coefficients on the space* $A^{-\infty}$ *is hypercyclic and chaotic if it is not a scalar multiple of the identity.*

Similar results can be obtained for the algebras of analytic functions on the unit disc in $\mathbb{C}$ considered by Momm in [**30**].

The operator of differentiation need not be hypercyclic on weighted inductive limits of spaces of entire functions. Let $p: \mathbb{C} \longrightarrow [0, \infty[$ be a continuous subharmonic radial function such that $\log(1+|z|^2) = o(p(z))$, and $p(2z) = O(p(z))$ as $|z|$ goes to ∞. The function p is a radial weight function in the sense of [**3**, Chapter 2]. The radial Hörmander algebra $A_p(\mathbb{C})$ is the weighted inductive limit $VH(\mathbb{C})$ of Banach spaces defined for the sequence of weights $v_n(z) = \exp(-np(z))$, $z \in \mathbb{C}$. The space $A_p(\mathbb{C})$ contains the polynomials and is stable under differentiation; see [**3**, 2.1].

For the convenience of the reader we recall Kitai's hypercyclicity criterion [**19**, 1.5] which is used in the proof of part (ii) of Theorem 11. Suppose that T is a continuous linear operator on a separable Fréchet space E for which the sequence of iterations $(T^n)_n$ tends pointwise to zero on a dense subset. If there is a (possibly different) dense subset X of E, and a map $S: X \longrightarrow X$ such that TS coincides with the identity on X, and $(S^n)_n$ tends pointwise to zero on X, then the operator T is hypercyclic.

THEOREM 11. (i) *The operator of differentiation* ∂ *is not hypercyclic on* $A_p(\mathbb{C})$ *if* $p(r) = o(r - (1/2)\log(r))$ *as* r *goes to* ∞.

(ii) *The operator of differentiation* ∂ *is hypercyclic on* $A_p(\mathbb{C})$ *if* $r = O(p(r))$ *as* r *goes to* ∞.

(iii) *The operator of differentiation* ∂ *has a dense set of periodic points on* $A_p(\mathbb{C})$ *if* $r = O(p(r))$ *as* r *goes to* ∞.

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Proof. (i) Assume that the operator ∂, which is a continuous linear operator on $A_p(\mathbb{C})$, is hypercyclic. Since $A_p(\mathbb{C})$ is dense in $H(\mathbb{C})$, we can apply Lemma 3 to conclude that ∂, as an operator on $H(\mathbb{C})$, has a hypercyclic vector $f \in A_p(\mathbb{C})$. This is a contradiction, since part (B) of the main theorem of [**20**] asserts there is no hypercyclic vector f for the operator of differentiation on $H(\mathbb{C})$ such that $|f(z)| = O(|z|^{-1/2}\exp(|z|))$ as $|z|$ goes to ∞.

(ii) We denote by v the weight $v(z) = \exp(-|z|)$, $z \in \mathbb{C}$. The Banach space $Hv_0(\mathbb{C})$ is continuously contained in $A_p(\mathbb{C})$. The polynomials are dense in $Hv_0(\mathbb{C})$; hence the Banach space is dense in $A_p(\mathbb{C})$. Moreover the differential operator ∂ acts continuously on $Hv_0(\mathbb{C})$. Indeed, if $f \in H(\mathbb{C})$ satisfies $|f(z)| \leqslant \varepsilon/v(z)$ for $|z| \geqslant r$, we can apply the Cauchy formula for $|z| > r+1$ to get

$$|f'(z)| = \left|\frac{1}{2\pi i}\int_{|z-\zeta|=1} \frac{f(\zeta)}{(z-\zeta)^2}\,d\zeta\right| \leqslant (3\varepsilon)/v(z).$$

By Lemma 3, it is now enough to show that ∂ is hypercyclic on $Hv_0(\mathbb{C})$. We apply the hypercyclicity criterion [**19**, 1.5] as follows. The sequence of iterations $(\partial^n)_n$ tends pointwise to 0 on the set X of polynomials which is dense in $Hv_0(\mathbb{C})$. We denote by S the operator of integration on the set of polynomials X. Clearly $\partial \circ S$ coincides with the identity on X. To conclude it remains to prove that the sequence $(S^n)_n$ converges pointwise to 0 on X. Since $S^n(z^k) = k!z^{k+n}/(k+n)!$, it is enough to show that the sequence $(z^n/n!)_n$ tends to 0 in the space $Hv_0(\mathbb{C})$. An easy calculation shows that the norm of the polynomial $z^n/n!$ in $Hv_0(\mathbb{C})$ coincides with $n^n e^{-n}/n!$, which tends to 0 by Stirling's formula. This completes the proof.

(iii) For $\alpha \in \mathbb{C}$, we define $e_\alpha(z) = \exp(\alpha z)$, $z \in \mathbb{C}$. By our assumption, the function e_α belongs to $A_p(\mathbb{C})$. Every element of the linear span P of the set of all e_α such that α is a root of unity is a periodic point of the operator ∂. We show that P is dense in $A_p(\mathbb{C})$. As in [**9**] we denote by $\bar{V}$ the set of all the weights $\bar{v}$ on $\mathbb{C}$ such that $\bar{v}/v_n$ is bounded for each n. By [**9**, 1.6], the space $VH(\mathbb{C}) = A_p(\mathbb{C})$ coincides algebraically and topologically with the locally convex space $H\bar{V}(\mathbb{C}) := \operatorname{proj}_{\bar{v}\in\bar{V}} H\bar{v}(\mathbb{C})$. Moreover this is a space of the type considered in [**36**] (and denoted there as $E(K)$). By [**36**, 3.2], if L is a continuous linear form on $A_p(\mathbb{C})$, the function $F(\alpha) := L(\exp(\alpha.))$, $\alpha \in \mathbb{C}$, defines an analytic function for $\alpha \in \mathbb{C}$ and the correspondence between L and F is one-to-one. Assume that L vanishes on P. The analytic function F vanishes on all the roots of unity. This implies that $F = 0$; hence $L = 0$. The density of P follows from the Hahn–Banach theorem. □

COROLLARY 12. *Let $p_s(z) = |z|^s$, $s > 0$. The operator of differentiation is not hypercyclic on $A_{p_s}(\mathbb{C})$ if $0 < s < 1$ and it is hypercyclic and chaotic if $s \geqslant 1$. In particular it is hypercyclic and chaotic in the space $A_{p_1}(\mathbb{C}) = \mathrm{Exp}(\mathbb{C})$ of entire functions of exponential type.*

OPEN QUESTIONS 13. (1) Characterize the weighted inductive limits $VH(G)$ on open connected subsets of $\mathbb{C}^N$ such that the operator of differentiation acts continuously on $VH(G)$, and is hypercyclic or chaotic.

(2) Is every chaotic continuous linear operator on a complete separable locally convex space hypercyclic?

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Note added in proof, *April* 2000. Open Question 13(2) has been solved by K. G. Grosse-Erdmann (personal communication). Let T be a chaotic weighted backward shift on l_2, and let E be a countable direct sum of copies of l_2, which is complete. The operator $S : E \longrightarrow E$, $S((x_n)_n) := (Tx_n)_n$, is chaotic but not hypercyclic. The details will appear elsewhere.

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