Bull. Aust. Math. Soc. **114** (2026), 323–332
doi:10.1017/S0004972726101166

SLENDER SEMIGROUPS

ATTILA NAGY

(Received 20 December 2025; accepted 4 February 2026; first published online 21 April 2026)

Abstract

We define the concept of slenderness for arbitrary semigroups containing at least one idempotent element, and give necessary and sufficient conditions for semigroups to be slender. We describe the semigroup of all homomorphisms of the direct product of a nonempty family of groups into a slender rectangular abelian group.

2020 *Mathematics subject classification*: primary 20M10; secondary 20K30.

Keywords and phrases: group, semigroup.

1. Introduction

The concept of slender group first appeared in the book [3]; it is due to J. Łoś, who defined it originally for abelian groups which were written additively. In this paper, we write a binary operation as multiplication. Let $C = \langle c \rangle$ be an infinite cyclic group generated by the element c and let $\prod_C$ denote the direct product of countably many copies of C, that is, the group of all sequences $x = (x_1, x_2, \ldots)$ of elements of C with term-wise multiplication. For each positive integer n, let δ_n denote the sequence of elements of C whose nth element is c and the other elements are e, where e denotes the identity element of C. An abelian group A is said to be slender if every homomorphism h of $\prod_C$ into A maps all but finitely many of the δ_n to the identity element of A; in other words, h sends δ_n into the identity element of A for almost all indices n. From this definition, it follows (see [4, page 489]) that a slender abelian group is torsion-free. The notion of slenderness was also defined for arbitrary (not necessarily commutative) groups. In [6], an arbitrary group G is said to be slender if it is torsion-free, and every homomorphism of $\prod_C$ into G sends δ_n into the identity element of G for almost all indices n. The slenderness of groups has been examined by many authors (see, for example, the papers [5, 6, 8, 11, 12] and the book [4]).

In our present paper, we extend the notion of slenderness to semigroups. We say that a semigroup S containing at least one idempotent element is slender if, for every homomorphism h of $\prod_C$ into S, $(\delta_n)h$ is an idempotent element of S for almost all indices n. We prove that a semigroup S is slender if and only if every maximal subgroup

© The Author(s), 2026. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.

https://doi.org/10.1017/S0004972726101166 Published online by Cambridge University Press

(or equivalently, every subgroup) of S is slender. We show that if a semigroup S is a semilattice Y of subsemigroups S_α ($\alpha \in Y$) such that every S_α contains at least one idempotent element, then S is a slender semigroup if and only if the subsemigroups S_α are slender for all $\alpha \in Y$. As a consequence of this result, we prove that a Clifford semigroup is slender if and only if it is a semilattice of slender groups. We also prove that a Rees matrix semigroup over a group G is slender if and only if G is a slender group. Applying this result, we show that a completely simple semigroup is slender if and only if it is isomorphic to a Rees matrix semigroup over a slender group.

Starting from the infinite cyclic group C, we define two operations $*$ and $\diamond$ on two copies of the set C such that $(C;*)$ is a left zero semigroup and $(C;\diamond)$ is a right zero semigroup. We consider the direct product $\mathcal{R}G_C^\infty$ of countably many copies of the rectangular abelian group $(C;*)\times(C;\diamond)\times C$, that is, the semigroup of all sequences $(x_1,x_2,x_3,\ldots)$ of the elements of C with term-wise multiplication: $(x_1,x_2,x_3,\ldots)(y_1,y_2,y_3,\ldots)=(x_1,y_2,x_3y_3,\ldots)$. We show that a semigroup S containing at least one idempotent element is slender if and only if every homomorphism of $\mathcal{R}G_C^\infty$ into S sends δ_n into an idempotent element of S for almost all indices n. We prove that if H is a group and $I\times\Lambda\times G$ is a rectangular abelian group, then the set $\mathcal{H}om(H,I\times\Lambda\times G)$ of all homomorphisms of H into $I\times\Lambda\times G$ is a semigroup. We also prove that if G_α ($\alpha\in Y$) is an arbitrary nonempty set of groups and $I\times\Lambda\times G$ is a slender rectangular abelian group, then $\mathcal{H}om(\prod_{\alpha\in Y}G_\alpha, I\times\Lambda\times G)\cong I\times\Lambda\times\nabla\mathcal{H}om(\prod_{\alpha\in Y}G_\alpha,G)$, where (by [6]) $\nabla\mathcal{H}om(\prod_{\alpha\in Y}G_\alpha,G)$ denotes the set of all $\sigma\in\mathcal{H}om(\prod_{\alpha\in Y}G_\alpha,G)$ with the following property: there is a subset $Y^*\subseteq Y$ such that the restriction of σ to $(\prod_{\alpha\in Y^*}G_\alpha)^\iota\subseteq\prod_{\alpha\in Y}G_\alpha$ (where ι denotes the canonical embedding of $\prod_{\alpha\in Y^*}G_\alpha$ into $\prod_{\alpha\in Y}G_\alpha$) sends every element of $(\prod_{\alpha\in Y^*}G_\alpha)^\iota$ into the identity element of G, and $Y\setminus Y^*$ is finite.

2. Preliminaries

A nonempty set together with an associative multiplication is called a *semigroup*. Let a be an arbitrary element of a semigroup S. The subsemigroup $\langle a\rangle$ of S generated by a consists of all the positive integer powers of a: $\langle a\rangle=\{a,a^2,a^3,\ldots\}$. The order of $\langle a\rangle$ is called the *order* of a. Assume that $a^r=a^s$ is satisfied for some positive integers $r<s$. Let s be chosen as small as possible, that is, a^s is the first power of a which equals some lower power of a. Since $a,a^2,\ldots,a^{s-1}$ are pairwise distinct elements, r is the only positive integer such that $r<s$ and $a^r=a^s$. Let $m=s-r$. We call r the *index* and m the *period* of a. By [2, Theorem 1.9], if the order of a is infinite, then all the powers of a are distinct; if the order of a is finite, then there exist two positive integers, the index r and the period m of a such that $a^{m+r}=a^r$ and $\langle a\rangle=\{a,a^2,\ldots,a^{m+r-1}\}$, in which case, $K_a=\{a^r,a^{r+1},\ldots,a^{m+r-1}\}$ is a cyclic subgroup of S of order m.

An element e of a semigroup is called an *idempotent element* if $e^2=e$. If e is an idempotent element of a semigroup S, then $eSe=eS\cap Se$ is a subsemigroup of S in which e is the identity element. The group of units of eSe is denoted by H_e. By [2, Theorem 1.11.], if e is an idempotent element of a semigroup S, then H_e contains

https://doi.org/10.1017/S0004972726101166 Published online by Cambridge University Press

every subgroup of S that meets H_e. Thus, H_e is a *maximal subgroup* of S. The set of all idempotent elements of a semigroup is a partially ordered set with respect to the partial order: $e \leq f$ if $ef = fe = e$. An idempotent element e of a semigroup S is said to be a *primitive idempotent* element if the only idempotents of S under e are e itself and 0 (if S has a zero element 0), and $e \neq 0$.

A semigroup is said to be a *band* if every element in it is an idempotent element. A semigroup S is called a *left zero semigroup* (respectively, a *right zero semigroup*) if $ab = a$ (respectively, $ab = b$) is satisfied for all $a, b \in S$. It is clear that every left zero semigroup and every right zero semigroup is a band. The direct product of a left zero semigroup and a right zero semigroup is called a *rectangular band*. The direct product $B \times G$ of a rectangular band B and a group G is called a *rectangular group*; if G is commutative, then $B \times G$ is said to be a *rectangular abelian group*. A commutative band is called a *semilattice*. A congruence η on a semigroup S is said to be a *semilattice congruence* if the factor semigroup S/η is a semilattice. Denoting the semilattice S/η by Y, the η-classes S_α ($\alpha \in Y$) of S are subsemigroups of S. In this case, we say that S is a semilattice Y of subsemigroups S_α ($\alpha \in Y$). The universal relation on a semigroup is a semilattice congruence. A semigroup S is said to be *semilattice indecomposable* if the only semilattice congruence on S is the universal relation on S.

Let S_α ($\alpha \in Y$) be a nonempty family of pairwise disjoint semigroups satisfying the following conditions:

(i) for every $\alpha, \beta \in Y$ with $\alpha \geq \beta$, there is a homomorphism $f_{\alpha,\beta}$ of S_α into S_β;
(ii) if $\alpha > \beta > \gamma$, then $f_{\alpha,\beta} f_{\beta,\gamma} = f_{\alpha,\gamma}$;
(iii) $f_{\alpha,\alpha}$ is the identity mapping on S_α.

Then, $S = \bigcup_{\alpha \in Y} S_\alpha$ is a semigroup with respect to the multiplication: for $a \in S_\alpha$ and $b \in S_\beta$, $ab = (af_{\alpha,\alpha\beta})(bf_{\beta,\alpha\beta})$, where the multiplication on the right-hand side is in $S_{\alpha\beta}$. This semigroup (which is a semilattice Y of semigroups S_α) is called a *strong semilattice Y of semigroups S_α*.

A semigroup S is said to be a *regular semigroup* if, for every element a of S, there is an element $x \in S$ such that $axa = a$. In this case, ax and xa are idempotent elements of S. A semigroup S is called a *Clifford semigroup* if it is regular and the idempotent elements of S are central, that is, $es = se$ for every $s \in S$ and every idempotent element e of S. If G is a group, I, Λ are nonempty sets and $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix with entries in G, then $S = I \times G \times \Lambda$ is a semigroup with the multiplication $(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j} b, \mu)$. This semigroup is called the *Rees $I \times \Lambda$ matrix semigroup over the group G with the sandwich matrix P*. By [7, Theorem 3.3.1], every Rees matrix semigroup over a group is completely simple, that is, it contains no proper ideal and contains a primitive idempotent element. Conversely, every completely simple semigroup is isomorphic to a Rees matrix semigroup over a group. By [9], a semigroup S is called a generalised group if, for every element $s \in S$, there is a unique element $e(s) \in S$ such that $se(s) = e(s)s = s$ and, for every element $s \in S$, there is an element s^{-1} such that $ss^{-1} = s^{-1}s = e(s)$. In [1], it is proved that a semigroup S is a generalised group if and only if it is completely simple.

https://doi.org/10.1017/S0004972726101166 Published online by Cambridge University Press

For other notions, results and notation used in this paper, we refer to the books [2, 4, 10].

3. Results

DEFINITION 3.1. A semigroup S containing at least one idempotent element is said to be slender if, for every homomorphism h of $\prod_C$ into S, $(\delta_n)h$ is an idempotent element of S for almost all positive integers n.

THEOREM 3.2. *For a semigroup S containing at least one idempotent element, the following assertions are equivalent:*

(1) *S is slender;*
(2) *every subgroup of S is slender;*
(3) *every maximal subgroup of S is slender.*

PROOF. (1) $\Longrightarrow$ (2). Let G be a subgroup of a slender semigroup S and let h be a homomorphism of $\prod_C$ into G. Then, h is a homomorphism of $\prod_C$ into S. Since S is slender, $(\delta_n)h$ is the identity element of G for almost all positive integers n. Thus, G is a slender group. Thus, assertion (2) is satisfied. Since (2) implies (3), it remains to prove that (3) implies (1).

(3) $\Longrightarrow$ (1). Let S be a semigroup containing at least one idempotent element such that every maximal subgroup of S is slender. Let h be a homomorphism of $\prod_C$ into S. Since $(\prod_C)h$ is a subgroup of S, [2, Theorem 1.11] implies that there is an idempotent element e of S such that $(\prod_C)h$ is contained by the maximal subgroup H_e of S. Thus, h is a homomorphism of $\prod_C$ into the group H_e. By the assumption, H_e is a slender group and, hence, $(\delta_n)h$ equals the identity element e of H_e for almost all positive integers n. Thus, S is a slender semigroup. □

COROLLARY 3.3. *Every band is slender.*

PROOF. This is evident by Theorem 3.2, because the one-element group is slender and the elements of a band B are maximal subgroups of B. □

THEOREM 3.4. *Let S be a semigroup which is a semilattice Y of subsemigroups S_α ($\alpha \in Y$) such that every S_α contains at least one idempotent element. Then, S is a slender semigroup if and only if the subsemigroups S_α are slender for all $\alpha \in Y$.*

PROOF. If S is a slender semigroup, then it follows, by the definition of slenderness, that the subsemigroups S_α are slender for all $\alpha \in Y$.

Conversely, assume that the subsemigroups S_α are slender for all $\alpha \in Y$. Let η denote the semilattice congruence on S, whose classes are the subsemigroups S_α ($\alpha \in Y$). Let h be a homomorphism of $\prod_C$ into S. Then, $(\prod_C)h$ is a subgroup of S. It is clear that η' is a semilattice congruence on $(\prod_C)h$, where η' is defined in the following way: for elements $a, b \in (\prod_C)h$, $(a, b) \in \eta'$ if and only if $(a, b) \in \eta$. Since every group is semilattice indecomposable, η' is the universal relation on $(\prod_C)h$ and

https://doi.org/10.1017/S0004972726101166 Published online by Cambridge University Press

so $(\prod_C)h \subseteq S_{\alpha_0}$ for some $\alpha_0 \in Y$. Since S_{α_0} is a slender subsemigroup of S, $(\delta_n)h$ equals an idempotent element of S_{α_0} (and so of S) for almost all indices n. Thus, S is a slender semigroup. □

REMARK 3.5. By Theorem 3.4, if S_α $(\alpha \in Y)$ is an arbitrary nonempty family of slender semigroups S_α, then the strong semilattice Y of semigroups S_α is slender.

THEOREM 3.6. *A Clifford semigroup is slender if and only if it is a semilattice of slender groups.*

PROOF. By [7, Theorem 4.2.1], a semigroup is a Clifford semigroup if and only if it is a semilattice Y of groups G_α. Then, by Theorem 3.4, a Clifford semigroup is slender if and only if the subgroups G_α are slender for every $\alpha \in Y$. □

THEOREM 3.7. *A Rees matrix semigroup $\mathcal{M}(G; I, \Lambda; P)$ over a group G is slender if and only if G is a slender group.*

PROOF. Let $\mathcal{M}(G; I, \Lambda; P)$ be a Rees matrix semigroup over a group G. By [2, Lemma 3.2 (iv)], the set $H_{i\lambda} = \{i\} \times G \times \{\lambda\}$ $(i \in I, \lambda \in \Lambda)$ is a maximal subgroup of $\mathcal{M}(G; I, \Lambda; P)$. Then, by Theorem 3.2, $\mathcal{M}(G; I, \Lambda; P)$ is slender if and only if all the subgroups $H_{i\lambda}$ $(i \in I, \lambda \in \Lambda)$ are slender. Using also [2, Lemma 3.2 (iv)], the mapping $a \mapsto (i, ap_{\lambda i}^{-1}, \lambda)$ is an isomorphism of G upon $H_{i\lambda}$ for all $i \in I$ and $\lambda \in \Lambda$. Thus, the Rees matrix semigroup $\mathcal{M}(G; I, \Lambda; P)$ is slender if and only if G is a slender group. □

THEOREM 3.8. *A completely simple semigroup is slender if and only if it is isomorphic to a Rees matrix semigroup over a slender group.*

PROOF. By [7, Theorem 3.3.1], a semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup over a group. Thus, the assertion follows from Theorem 3.7. □

THEOREM 3.9. *Let G be a group, let I be a left zero semigroup and let Λ be a right zero semigroup. Then, the rectangular group $I \times \Lambda \times G$ is slender if and only if G is a slender group.*

PROOF. It is clear that a rectangular group $I \times \Lambda \times G$ is a Rees matrix semigroup $\mathcal{M}(G; I, \Lambda; P)$ over the group G such that every element of the sandwich matrix P equals the identity element of G. Thus, the assertion of the theorem follows from Theorem 3.7. □

Starting from the infinite cyclic group C, we define two operations $*$ and $\diamond$ on two copies of the set C as follows: for arbitrary $a, b \in C$, let $a * b = a$ and $a \diamond b = b$. Then, $(C; *)$ is a left zero semigroup and $(C; \diamond)$ is a right zero semigroup. Thus, $\mathcal{R}G_C = (C; *) \times (C; \diamond) \times C$ is a rectangular abelian group. It is easy to see that

$$\xi : x \mapsto (e, e, x) \tag{3.1}$$

is a homomorphism of the group C into the rectangular abelian group $\mathcal{R}G_C$. Moreover,

$$\sigma : (x_1, x_2, x_3) \mapsto x_3 \tag{3.2}$$

https://doi.org/10.1017/S0004972726101166 Published online by Cambridge University Press

is a homomorphism of the rectangular abelian group $\mathcal{R}G_C$ into the group C. Let $\mathcal{R}G_C^\infty$ denote the direct product of countably many copies of the rectangular abelian group $(C;*)\times(C;\diamond)\times C$, that is, the semigroup of all sequences $(x_1,x_2,x_3,\ldots)$ of the elements of C with term-wise multiplication: $(x_1,x_2,x_3,\ldots)(y_1,y_2,y_3,\ldots) = (x_1,y_2,x_3y_3,\ldots)$. We note that the elements of $\prod_C$ and $\mathcal{R}G_C^\infty$ are sequences of elements of C, and so δ_n can be considered as an element of both $\prod_C$ and $\mathcal{R}G_C^\infty$. It is clear that

$$(x_1,x_2,x_3,x_4,x_5,x_6,\ldots)\mapsto((x_1,x_4,\ldots),(x_2,x_5,\ldots),(x_3,x_6,\ldots))$$

is an isomorphism of the direct product $\mathcal{R}G_C^\infty$ upon the rectangular abelian group $\prod_{n=1}^\infty(C;*)\times\prod_{n=1}^\infty(C;\diamond)\times\prod_{n=1}^\infty C$. Thus, $\mathcal{R}G_C^\infty$ is a rectangular abelian group. In [13, Example 1], the authors define an operation $*$ on the Cartesian product $\prod_{n=1}^\infty\mathbb{Z}$ of the additive group $\mathbb{Z}$ of integers as follows: $(x_1,x_2,x_3,\ldots)*(y_1,y_2,y_3,\ldots) = (x_1,y_2,x_3+y_3,\ldots)$. The Cartesian product $\prod_{n=1}^\infty\mathbb{Z}$ is a generalised group with respect to this operation which is denoted by $\prod_N^g\mathbb{Z}$. The sequence of elements of $\mathbb{Z}$ whose nth element is 1 and the other elements are 0 is denoted by i_n. With the help of $\prod_N^g\mathbb{Z}$, the authors define the notion of a slender generalised group. A generalised group G is said to be slender if every homomorphism of $\prod_N^g\mathbb{Z}$ into G sends i_n into an idempotent element of G for almost all indices n. The authors show that an abelian group A is a slender generalised group if and only if A is a slender group. In the next theorem, we generalise this result. First of all, we note that $\prod_N^g\mathbb{Z}$ is isomorphic to the rectangular abelian group $\mathcal{R}G_C^\infty$; the sequences i_n and δ_n correspond to each other.

THEOREM 3.10. *A semigroup S containing at least one idempotent element is slender if and only if every homomorphism of the rectangular abelian group $\mathcal{R}G_C^\infty$ into S sends δ_n into an idempotent element of S for almost all indices n.*

PROOF. Assume that S is a slender semigroup. Let h be a homomorphism of the rectangular abelian group $\mathcal{R}G_C^\infty$ into S. We show that $(\delta_n)h$ is an idempotent element of S for almost all indices n. If $n\neq 3k$, then δ_n is an idempotent element of $\mathcal{R}G_C^\infty$ and, hence, $(\delta_n)h$ is an idempotent element of S, because a homomorphic image of an idempotent element is idempotent. By the above, it is sufficient to show that $(\delta_{3k})h$ is an idempotent element of S for almost all positive integers k. Let ξ' be the following mapping of $\prod_C$ into $\mathcal{R}G_C^\infty$:

$$\xi' : (x_1,x_2,x_3,x_4,x_5,x_6,\ldots)\mapsto(e,e,x_3,e,e,x_6,\ldots).$$

Since ξ' is an extension of ξ defined by (3.1), ξ' is a homomorphism of $\prod_C$ into $\mathcal{R}G_C^\infty$. Thus, $\xi'h$ is a homomorphism of $\prod_C$ into S. Since S is a slender semigroup, $(\delta_{3k})\xi'h$ is an idempotent element of S for almost all positive integers k. Since $(\delta_{3k})\xi' = \delta_{3k}$,

$$(\delta_{3k})\xi'h = ((\delta_{3k})\xi')h = (\delta_{3k})h$$

and so $(\delta_{3k})h$ is an idempotent element of S for almost all positive integers k.

To prove the converse, assume that every homomorphism of the rectangular abelian group $\mathcal{R}G_C^\infty$ into a semigroup S sends δ_n into an idempotent of S for almost all indices n. We show that S is a slender semigroup. Let h be a homomorphism of the group $\prod_C$

https://doi.org/10.1017/S0004972726101166 Published online by Cambridge University Press

into S. We show that $(\delta_n)h$ is an idempotent element of S for almost all indices n. Let σ' be the following mapping of $\mathcal{R}G_C^\infty$ into $\prod_C$:

$$\sigma' : (x_1, x_2, x_3, x_4, x_5, x_6, \ldots) \mapsto (x_3, x_6, \ldots).$$

Since σ' is an extension of σ defined by (3.2), σ' is a homomorphism of $\mathcal{R}G_C^\infty$ into $\prod_C$, and so $\sigma' h$ is a homomorphism of $\mathcal{R}G_C^\infty$ into S. By the assumption, $(\delta_n)\sigma' h$ is an idempotent element of S for almost all indices n. Since $(\delta_{3n})\sigma' = \delta_n$ for all positive integers n,

$$(\delta_{3n})\sigma' h = (\delta_n)h,$$

and so $(\delta_n)h$ is an idempotent element of S for almost all indices n. Hence, S is a slender semigroup. □

THEOREM 3.11. *Let H be a group and let $I \times \Lambda \times G$ be a rectangular abelian group. Then, the set $\mathcal{H}om(H, I \times \Lambda \times G)$ of all homomorphisms of the group H into $I \times \Lambda \times G$ is a semigroup with respect to the following multiplication: for arbitrary $f_1, f_2 \in \mathcal{H}om(H, I \times \Lambda \times G)$ and arbitrary $h \in H$, $(h)(f_1 f_2) = (h)f_1(h)f_2$, where the multiplication on the right-hand side is in $I \times \Lambda \times G$.*

PROOF. Assume $f_1, f_2 \in \mathcal{H}om(H, I \times \Lambda \times G)$ and $x, y \in H$. Let $(x)f_1 = (i, \lambda, a)$ and $(x)f_2 = (j, \mu, b)$, $(y)f_1 = (i', \lambda', a')$, $(y)f_2 = (j', \mu', b')$. Then,

$$\begin{aligned}(xy)(f_1 f_2) &= (xy)f_1(xy)f_2 = (x)f_1(y)f_1(x)f_2(y)f_2 \\ &= (i, \lambda, a)(i', \lambda', a')(j, \mu, b)(j', \mu', b') = (i, \mu', aa'bb') \\ &= (i, \mu', aba'b') = (i, \lambda, a)(j, \mu, b)(i', \lambda', a')(j', \mu', b') \\ &= (x)f_1(x)f_2(y)f_1(y)f_2 = (x)(f_1 f_2)(y)(f_1 f_2).\end{aligned}$$

Thus, $f_1 f_2 \in \mathcal{H}om(H, I \times \Lambda \times G)$. Since $(f_1 f_2)f_3 = f_1(f_2 f_3)$ for arbitrary homomorphisms f_1, f_2, f_3 of H into $I \times \Lambda \times G$, it follows that $\mathcal{H}om(H, I \times \Lambda \times G)$ is a semigroup. □

THEOREM 3.12. *Let H be an arbitrary group and let $I \times \Lambda \times G$ be a rectangular abelian group. Then, $\mathcal{H}om(H, I \times \Lambda \times G) \cong I \times \Lambda \times \mathcal{H}om(H, G)$.*

PROOF. By Theorem 3.11, $\mathcal{H}om(H, I \times \Lambda \times G)$ is a semigroup. Since $\mathcal{H}om(H, G)$ is an abelian group, $I \times \Lambda \times \mathcal{H}om(H, G)$ is a rectangular abelian group. Let f be an arbitrary element of $I \times \Lambda \times \mathcal{H}(H, G)$. Then, there are elements $i \in I$, $\lambda \in \Lambda$ and $\alpha_f \in \mathcal{H}om(H, G)$ such that

$$f = (i, \lambda, \alpha_f).$$

Let Φ_f denote the mapping of H into $I \times \Lambda \times G$ defined as follows: for arbitrary $h \in H$, let

$$(h)\Phi_f = (i, \lambda, (h)\alpha_f).$$

https://doi.org/10.1017/S0004972726101166 Published online by Cambridge University Press

Then, for every $h_1, h_2 \in H$,

$$\begin{aligned}(h_1h_2)\Phi_f &= (i,\lambda,(h_1h_2)\alpha_f) = (i,\lambda,(h_1)\alpha_f(h_2)\alpha_f)\\ &= (i,\lambda,(h_1)\alpha_f)(i,\lambda,(h_2)\alpha_f) = (h_1)\Phi_f(h_2)\Phi_f.\end{aligned}$$

Thus,

$$\Phi_f \in \mathcal{H}om(H, I\times\Lambda\times G).$$

We show that $\Phi : f \mapsto \Phi_f$ is an isomorphism which maps $I\times\Lambda\times\mathcal{H}om(H,G)$ onto $\mathcal{H}om(H, I\times\Lambda\times G)$.

First, we show that Φ is a homomorphism. Let

$$f_1 = (i,\lambda,\alpha_{f_1}),\ f_2 = (j,\mu,\alpha_{f_2}) \in I\times\Lambda\times\mathcal{H}om(H,G)$$

be arbitrary elements. Then, $f_1f_2 = (i,\mu,\alpha_{f_1}\alpha_{f_2})$. For arbitrary $h \in H$,

$$\begin{aligned}(h)\Phi_{f_1f_2} &= (h)\Phi_{(i,\mu,\alpha_{f_1}\alpha_{f_2})} = (i,\mu,(h)\alpha_{f_1}\alpha_{f_2}) = (i,\mu,(h)\alpha_{f_1}(h)\alpha_{f_2})\\ &= (i,\lambda,(h)\alpha_{f_1})(j,\mu,(h)\alpha_{f_2}) = (h)\Phi_{f_1}(h)\Phi_{f_2} = (h)\Phi_{f_1}\Phi_{f_2},\end{aligned}$$

and so

$$(f_1f_2)\Phi = (f_1)\Phi(f_2)\Phi.$$

To show that Φ is injective, assume

$$(f_1)\Phi = (f_2)\Phi$$

for elements

$$f_1 = (i,\lambda,\alpha_{f_1}),\ f_2 = (j,\mu,\alpha_{f_2}) \in I\times\Lambda\times\mathcal{H}om(H,G).$$

Then, for all $h \in H$,

$$(i,\lambda,(h)\alpha_{f_1}) = (h)\Phi_{f_1} = (h)\Phi_{f_2} = (j,\mu,(h)\alpha_{f_2})$$

from which it follows that

$$i = j,\quad \lambda = \mu,\quad (h)\alpha_{f_1} = (h)\alpha_{f_2},$$

that is,

$$f_1 = f_2.$$

To show that Φ is surjective, consider an arbitrary homomorphism φ of H into $I\times\Lambda\times G$. Since $(H)\varphi$ is a subgroup of $I\times\Lambda\times G$, [2, Theorem 1.11] implies that there are elements $i \in I$ and $\lambda \in \Lambda$ such that $(H)\varphi$ is contained by the maximal subgroup $H_{i\lambda} = \{i\}\times\{\lambda\}\times G$ of $I\times\Lambda\times G$. Let

$$f = (i,\lambda,\varphi\varphi_{i\lambda}),$$

where $\varphi_{i\lambda}$ is an isomorphism of $\{i\}\times\{\lambda\}\times G$ onto G defined as follows: for arbitrary (i,λ,g),

$$(i,\lambda,g)\varphi_{i\lambda} = g.$$

https://doi.org/10.1017/S0004972726101166 Published online by Cambridge University Press

Since $\varphi\varphi_{i\lambda} \in \mathcal{H}om(H, G)$, we have

$$f \in I \times \Lambda \times \mathcal{H}om(H, G).$$

Let $h \in H$ be an arbitrary element. Then, $(h)\varphi = (i, \lambda, h^*)$ for some $h^* \in G$. Since

$$(h)\Phi_f = (i, \lambda, (h)\varphi\varphi_{i\lambda}) = (i, \lambda, (i, \lambda, h^*)\varphi_{i\lambda}) = (i, \lambda, h^*) = (h)\varphi,$$

we see that

$$(f)\Phi = \varphi.$$

Thus, Φ is an isomorphism of the rectangular group $I \times \Lambda \times \mathcal{H}om(H, G)$ onto $\mathcal{H}om(H, I \times \Lambda \times G)$. □

Let G_α $(\alpha \in Y)$ be an arbitrary nonempty set of groups and let G be an arbitrary group. As in [6], let $\triangledown\mathcal{H}om(\prod_{\alpha\in Y} G_\alpha, G)$ denote the set of all $\sigma \in \mathcal{H}om(\prod_{\alpha\in Y} G_\alpha, G)$ having the following property: there is a subset $Y^* \subseteq Y$ such that the restriction of σ to $(\prod_{\alpha\in Y^*} G_\alpha)^\iota \subseteq \prod_{\alpha\in Y} G_\alpha$ (where ι denotes the canonical embedding of $\prod_{\alpha\in Y^*} G_\alpha$ into $\prod_{\alpha\in Y} G_\alpha$) sends every element of $(\prod_{\alpha\in Y^*} G_\alpha)^\iota$ into the identity element of G, and $Y \setminus Y^*$ is finite. If G is an abelian group, then $\triangledown\mathcal{H}om(\prod_{\alpha\in Y} G_\alpha, G)$ is a subgroup of the group $\mathcal{H}om(\prod_{\alpha\in Y} G_\alpha, G)$.

THEOREM 3.13. *If $I \times \Lambda \times G$ is a slender rectangular abelian group and G_α $(\alpha \in Y)$ is an arbitrary nonempty set of groups, then*

$$\mathcal{H}om\Big(\prod_{\alpha\in Y} G_\alpha, I \times \Lambda \times G\Big) \cong I \times \Lambda \times \triangledown\mathcal{H}om\Big(\prod_{\alpha\in Y} G_\alpha, G\Big).$$

PROOF. Let $I \times \Lambda \times G$ be a rectangular abelian group and let G_α $(\alpha \in Y)$ be an arbitrary nonempty set of groups. By Theorem 3.12,

$$\mathcal{H}om\Big(\prod_{\alpha\in Y} G_\alpha, I \times \Lambda \times G\Big) \cong I \times \Lambda \times \mathcal{H}om\Big(\prod_{\alpha\in Y} G_\alpha, G\Big).$$

If the rectangular abelian group $I \times \Lambda \times G$ is slender, then G is a slender group by Theorem 3.9. It follows from [6, Theorem 4.3] that

$$\mathcal{H}om\Big(\prod_{\alpha\in Y} G_\alpha, G\Big) = \triangledown\mathcal{H}om\Big(\prod_{\alpha\in Y} G_\alpha, G\Big).$$

Thus,

$$\mathcal{H}om\Big(\prod_{\alpha\in Y} G_\alpha, I \times \Lambda \times G\Big) \cong I \times \Lambda \times \triangledown\mathcal{H}om\Big(\prod_{\alpha\in Y} G_\alpha, G\Big).$$ □

References

[1] J. Araújo and J. Konieczny, 'Molaie's generalized groups are completely simple semigroups', *Bull. Polytechnic Inst. Iaşi* **48**(1-2) (2002), 1–5.

[2] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups I* (American Mathematical Society, Providence, RI, 1961).

https://doi.org/10.1017/S0004972726101166 Published online by Cambridge University Press

[3] L. Fuchs, *Abelian Groups* (Publishing House of the Hungarian Academy of Sciences, Budapest, 1958).
[4] L. Fuchs, *Abelian Groups*, Springer Monographs in Mathematics (Springer, Cham, 2015).
[5] L. Fuchs and R. Göbel, 'Union of slender groups', *Arch. Math. (Basel)* **87** (2006), 6–14.
[6] R. Göbel, 'On stout and slender groups', *J. Algebra* **35** (1975), 39–55.
[7] J. M. Howie, *Fundamentals of Semigroup Theory* (The Clarendon Press, Oxford, 1995).
[8] J. Łoś, 'On the complete direct sum of countable abelian groups', *Publ. Math. Debrecen* **3** (1954), 269–272.
[9] M. R. Molaei, 'Generalized groups', *Bull. Polytechnic Inst. Iaşi* **45**(3-4) (1999), 21–24.
[10] A. Nagy, *Special Classes of Semigroups* (Kluwer Academic Publishers, Dordrecht–Boston–London, 2001).
[11] R. J. Nunke, 'Slender groups', *Bull. Amer. Math. Soc.* **67** (1961), 274–275.
[12] B. Wald, 'A note on slender groups', *Arch. Math. (Basel)* **31** (1978), 432–434.
[13] M. R. A. Zand and H. T. Ardikani, 'On slender generalized groups', Preprint, 2025, arXiv:2511.14498.

ATTILA NAGY, Department of Algebra and Geometry,
Budapest University of Technology and Economics,
Műegyetem rkp. 3, 1111, Budapest, Hungary
e-mail: nagyat@math.bme.hu

https://doi.org/10.1017/S0004972726101166 Published online by Cambridge University Press