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Multi-focused cut elimination

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We investigate cut elimination in multi-focused sequent calculi and the impact on the cut elimination proof of design choices in such calculi. The particular design we advocate is illustrated by a multi-focused calculus for full linear logic using an explicitly polarised syntax and incremental focus handling, for which we provide a syntactic cut elimination procedure. We discuss the effect of cut elimination on the structure of proofs, leading to a conceptually simple proof exploiting the strong structure of multi-focused proofs.

1. From focalisation to multi-focalisation

There is a fundamental tension in proof theory between the concept of a proof in normal form and the meta-theoretic properties of a formal system describing such proof objects for a given logic. This is a crucial aspect of the original proof-theoretic investigations into logical formalisms (Gentzen 1934) leading to the definition of natural deduction and the sequent calculus. Indeed, the former offers canonical proofs supposed to match the usual view of deduction in mathematical proofs at the cost of a complex notion of normal form, whereas the second has better meta-theoretic properties and in particular provides a simple framework for proving consistency of a given logic, through cut elimination.

The most important result in the proof theory of any natural deduction system is that normal forms are sufficient to obtain a complete proof system, but this is better observed in the sequent calculus as the admissibility of the cut rule. This yields a procedure that transforms any given proof into a cut-free proof of the same sequent, which is considered as equivalent. However, cut-free proofs in the sequent calculus are not canonical objects, due to the many possible permutations of rule instances yielding other valid proofs of the same sequent. In general, these proofs should be considered equivalent, as they reflect an irrelevant ordering of deduction steps, and they are, for example, identified in categorical semantics. This problem can be seen from two angles: It prevents the definition of a clear concept of *identity of proofs* for most logics, which are properly expressed in the sequent calculus but not in natural deduction, and it makes proof search more complicated even though the simple analytic nature of cut-free proofs in the sequent calculus should lend itself to efficient automated deduction.

These two problems, consequences of the lax structure of sequent calculus proofs, have been intensively studied in the setting of linear logic (Girard 1987). A particularly novel aspect of the proof theory of linear logic was the introduction of proof-nets as a graphical formalism meant to provide a proof representation akin to natural deduction in this logic

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enjoying an involutive negation and therefore difficult to handle in a canonical way. But the result is not yet satisfactory for anything other than the core fragment, multiplicative linear logic, and its purpose is not to be as versatile as the sequent calculus, or to offer the same proof search infrastructure. Another proposal emerged from the study of linear logic, with the definition of a complete subset of proofs called *focused* (Andreoli 1992), based on practical observations concerned with proof search and logic programming but also using the fundamental notion of *polarity*. The key idea is straightforward: Some inference rules are invertible and can therefore be applied eagerly, while the choices involved in applying non-invertible rules can be chained in large steps potentially leading to backtracking. The result is a sequent calculus enjoying a stronger structure in which phases called *inversion* and *chaining* alternate. In the one-sided calculus for linear logic, these rules decompose negative and positive connectives, respectively. Compare for example the proofs:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,\mathtt{ax}}{\vdash \bot, \overline{a}, a}\,\bot}{\vdash \bot \parr \overline{a}, a}\,\parr \qquad \dfrac{}{\vdash \overline{c}, c}\,\mathtt{ax}}{\vdash \bot \parr \overline{a}, \overline{c}, a \otimes c}\,\otimes \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,\mathtt{ax}}{\vdash \bot, \overline{a}, a}\,\bot \qquad \dfrac{}{\vdash \overline{c}, c}\,\mathtt{ax}}{\vdash \bot, \overline{a}, \overline{c}, a \otimes c}\,\otimes}{\vdash \bot \parr \overline{a}, \overline{c}, a \otimes c}\,\parr \qquad \dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,\mathtt{ax} \qquad \dfrac{}{\vdash \overline{c}, c}\,\mathtt{ax}}{\vdash \overline{a}, \overline{c}, a \otimes c}\,\otimes}{\vdash \bot, \overline{a}, a, \overline{c}, a \otimes c}\,\bot}{\vdash \bot \parr \overline{a}, \overline{c}, a \otimes c}\,\parr \quad , \tag{1}$$

which are syntactically distinct but equivalent up to permutations of rule instances. From the viewpoint of focusing, not all of them should be valid, and in fact only the one on the right is following the requirements to be a focused normal form – these requirements are discussed in more details below. Indeed, the $\otimes$ rule and the axioms should be chained together, as they apply on positive formulas. The focalisation result consists in granting a particular status to such proofs, proving that they form a complete subset. In other words, the uniform exploitation of permutability properties of inference rules leads to the restricted but complete sequent calculus that we call focused, for the particular logic considered. Here, we will follow this methodology to investigate normal forms of sequent calculus proofs, which are versatile enough to apply to a wide range of logics. The standard focusing technique has been studied for various logics, from its original presentation in linear logic to intuitionistic and classical logics (Liang and Miller 2009).

Permutability properties are intimately related to the notion of polarity of connectives and inference rules, which is best observed in linear logic. A complete understanding of the notions involved is most easily achieved by making polarities explicit in the language of formulas, following Laurent (2002). If we consider for instance the multiplicative core of linear logic, we obtain the grammar:

$$\begin{array}{rcl} P, Q & ::= & a \mid 1 \mid P \otimes Q \mid \downarrow N \\ N, M & ::= & \overline{a} \mid \bot \mid N \parr M \mid \uparrow P \end{array} . \tag{2}$$

The role of the polarity *shifts* introduced in this syntax is to mark the border between layers of positive and negative connectives. For the sake of simplicity, we always consider an atom a to be positive, whereas its negated counterpart $\overline{a}$ is negative, even though it was proven in Miller and Saurin (2007) that assigning a polarity to the atoms used in a proof is a flexible process. In a sequent calculus based on such a syntax, a sequent should only contain negative formulas if we interpret the meta-level comma as a $\parr$. This leads to a focused proof system where positive formulas being decomposed are enclosed in brackets and the two phases are distinguished by using sequents of the shape $\vdash \Gamma$ during inversion

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and $\vDash \Gamma, [P]$ during chaining, as follows:

$$\frac{\vDash \Gamma, [P]}{\vdash \Gamma, \uparrow P}\,\texttt{foc} \qquad \frac{\vdash \Gamma, N}{\vDash \Gamma, [\downarrow N]}\,\texttt{blur} \qquad \frac{}{\vDash \overline{a}, [a]}\,\texttt{ax} \qquad \frac{}{\vDash [1]}\,\mathbf{1}$$
$$\frac{\vdash \Gamma}{\vdash \Gamma, \bot}\,\bot \qquad \frac{\vdash \Gamma, N, M}{\vdash \Gamma, N \parr M}\,\parr \qquad \frac{\vDash \Gamma, [P] \quad \vDash \Delta, [Q]}{\vDash \Gamma, \Delta, [P \otimes Q]}\,\otimes \tag{3}$$

The distinction of a single-positive formula during chaining is meant to ensure that the formulas decomposed above a positive rule instance are the sub-formulas of the originally *focused* positive formula. However, this system enforces maximal inversion of the negative formulas only with a side-condition, that in the `foc` rule Γ contains only undecomposable negative formulas, of the shape $\overline{a}$ or $\uparrow P$. To summarise, the usual focused normal form can be described by its two key requirements:

1. negative formulas are eagerly decomposed, which implies that all of them are already decomposed when a positive formula is decomposed, and
2. positive formulas are hereditarily decomposed, meaning that positive sub-formulas of positive formulas are eagerly decomposed,

but here, we want to emphasise the primary importance of the second point. There is a dependency of positive formulas over negative formulas, leading to the non-permutability, in general, of positive rules below negative rules. The first requirement relies on this observation and it was originally introduced with proof search in mind, but in terms of normal forms, there is no good reason why a canonical proof should be one where *all* negative formulas are decomposed eagerly: The important point is that *some* must have been decomposed before a positive rule can be applied, when there is a dependency. For example, there is no dependency of $\otimes$ over $\parr$ in the proofs shown in Equation (1). These proofs also illustrate the importance of chaining: Moving the $\parr$ and $\bot$ instances downwards can be seen as applying the corresponding inversion principles, but one should notice that requiring decompositions of positives to be chained implies that the axiom rule `ax` must be applied to the atoms a and c as soon as they are made available. As a consequence, $a \otimes c$ cannot be focused on before the $\parr$ and $\bot$ connectives have been treated, leading to the focused proof on the right. These inversions are prerequisite to the decomposition of this positive formula and thus the essence of focalisation is expressed by chaining.

The issue of maximality of the inversion phase is related to subtleties of the design of focused systems, which will be addressed in more details in Section 4. The departure from the usual presentation using maximal inversion that we advocate here is important to the simplicity of the cut elimination procedure. We consider this change to be rather mild, given that the resulting system is a simple superset of the usual system, and because inversion lemmas yield relatively trivial proof transformations. The proof of completeness of chaining is also much more involved than the proof of these lemmas. Rather than being bent on the usual system stemming from a proof search perspective, we choose a flexible approach considering normal forms as the purpose of this study.

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Obviously, in a focused system as shown in Equation (3), progress has been made towards the definition of normal proofs within the framework of the sequent calculus, while preserving the good properties of the original system – analyticity by cut elimination. However, not all inessential distinctions have been ruled out because phases themselves can permute, leading to irrelevant orderings even if we consider each phase as a primitive, as done with *synthetic connectives* (Chaudhuri 2008). For example, consider the phases:

$$\dfrac{\dfrac{\dfrac{}{\vDash \overline{a}, [a]}\,\mathtt{ax} \quad \dfrac{\vdash \bot, \Gamma}{\vDash [\downarrow\bot], \Gamma}\,\mathtt{blur}}{\vDash \overline{a}, [a \otimes \downarrow\bot], \Gamma}\,\otimes}{\vdash \overline{a}, \uparrow(a \otimes \downarrow\bot), \Gamma}\,\mathtt{foc} \qquad \dfrac{\dfrac{\dfrac{}{\vDash \overline{c}, [c]}\,\mathtt{ax} \quad \dfrac{\vdash \overline{a}, \Delta}{\vDash [\downarrow\overline{a}], \Delta}\,\mathtt{blur}}{\vDash \overline{c}, [c \otimes \downarrow\overline{a}], \Delta}\,\otimes}{\vdash \overline{c}, \uparrow(c \otimes \downarrow\overline{a}), \Delta}\,\mathtt{foc}\ ,$$

which are parametric in Γ and Δ, respectively. These could be seen as basic components from which larger derivations are built, and in particular we can construct the following two proofs of the same sequent by combining the phases above in different orders:

$$\dfrac{\dfrac{\dfrac{}{\vDash \overline{a}, [a]}\,\mathtt{ax} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\vDash \overline{c}, [c]}\,\mathtt{ax} \quad \dfrac{\dfrac{\dfrac{}{\vDash \overline{a}, [a]}\,\mathtt{ax}}{\vdash \overline{a}, \uparrow a}\,\mathtt{foc}}{\vDash [\downarrow\overline{a}], \uparrow a}\,\mathtt{blur}}{\vDash \overline{c}, [c \otimes \downarrow\overline{a}], \uparrow a}\,\otimes}{\vdash \overline{c}, \uparrow(c \otimes \downarrow\overline{a}), \uparrow a}\,\mathtt{foc}}{\vdash \overline{c}, \bot, \uparrow(c \otimes \downarrow\overline{a}), \uparrow a}\,\bot}{\vDash \overline{c}, [\downarrow\bot], \uparrow(c \otimes \downarrow\overline{a}), \uparrow a}\,\mathtt{blur}}{\vDash \overline{a}, \overline{c}, [a \otimes \downarrow\bot], \uparrow(c \otimes \downarrow\overline{a}), \uparrow a}\,\otimes}{\vdash \overline{a}, \overline{c}, \uparrow(a \otimes \downarrow\bot), \uparrow(c \otimes \downarrow\overline{a}), \uparrow a}\,\mathtt{foc}$$

$$\dfrac{\dfrac{\dfrac{}{\vDash \overline{c}, [c]}\,\mathtt{ax} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\vDash \overline{a}, [a]}\,\mathtt{ax}}{\vdash \overline{a}, \uparrow a}\,\mathtt{foc}}{\vdash \bot, \overline{a}, \uparrow a}\,\bot}{\vDash [\downarrow\bot], \overline{a}, \uparrow a}\,\mathtt{blur}}{\vDash \overline{a}, [a \otimes \downarrow\bot], \overline{a}, \uparrow a}\,\otimes}{\vdash \overline{a}, \uparrow(a \otimes \downarrow\bot), \overline{a}, \uparrow a}\,\mathtt{foc}}{\vDash \overline{a}, \uparrow(a \otimes \downarrow\bot), [\downarrow\overline{a}], \uparrow a}\,\mathtt{blur}}{\vdash \overline{a}, \overline{c}, \uparrow(a \otimes \downarrow\bot), [c \otimes \downarrow\overline{a}], \uparrow a}\,\otimes}{\vdash \overline{a}, \overline{c}, \uparrow(a \otimes \downarrow\bot), \uparrow(c \otimes \downarrow\overline{a}), \uparrow a}\,\mathtt{foc}\ , \quad (4)$$

thus, exhibiting the permutability of phases in a focused sequent calculus. This leads to the notion of *multi-focalisation* (Chaudhuri et al. 2008a), where several positive formulas are selected and decomposed simultaneously. This is a second iteration of the focusing idea: To forget about permutations, it is sufficient to chain steps together into larger steps, from basic inference rules to phases now grouped into multi-chaining phases.

Although focalisation amounts to relatively strict requirements on the order in which rules are applied, multi-focalisation states that phases can be grouped together. It is therefore obviously complete for linear logic, since a usual focused proof is a form of multi-focused proof, albeit trivial. Soundness of focusing as well as multi-focusing is immediate since all rules of such a system are restrictions of the standard rules of linear logic. Our interest lies in the cut elimination argument and the dynamics of the associated procedure. This result, although crucial in structural proof theory, has often been left aside in the investigation of focused systems – maybe due to the historical relation between focusing and proof search. For standard linear logic, there is a compact argument (Chaudhuri 2008), and the intuitionistic and classical cases have also been studied (Liang and Miller 2009). Note that a more detailed analysis of cut elimination in the intuitionistic system LJF leads to a highly principled proof, which is not only useful to prove the focalisation result in a structural fashion (Simmons 2014) but also yields a

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natural computational interpretation which involves reduction strategies in the λ-calculus (Brock-Nannestad et al. 2015). Similarly, the proof can be made principled enough in a focused sequent calculus for standard linear logic to enjoy a simple computational interpretation (Brock-Nannestad and Guenot 2015b). Such results are desirable, as they show that focused proofs form a sensible subset of normal forms, and here we will show that multi-focused proofs are also well behaved with respect to cut elimination, even though they yield a system containing proofs in a stronger normal form than simple focusing – that we will call *mono-focusing* in the following.

Remark 1.1. Devising normal forms might lead to a debate on the strength of various proposals. The case of standard focalisation is obvious, as a single representative is picked among an equivalence class of proofs – this is nothing more than picking a subset, while staying in the original language. When moving to multi-focalisation, it is revealed that this scheme leads to arbitrary choices and thus endangers the notion of canonicity. Indeed, none of the two proofs shown above in Equation (4) is more canonical than the other, as they are in a way symmetric. The reason why we describe multi-focused proofs as a stronger normal form is that they can contain *more information* than their mono-focused counterparts. This qualitative difference will be illustrated by the proof shown in Equation (5) in the next section, which not only represents both proofs in Equation (4) but also contains the information that these two positive phases can permute with one another and are thus parallel.

From the perspective of canonicity, multi-focalisation offers the advantage of its wide applicability while providing a spectrum of normal forms, from mono-focused proofs to *maximally* multi-focused proofs, which were proved equivalent to proof-nets at least in the case of MLL (Chaudhuri et al. 2008a). Note that maximality of multi-chaining is a difficult question: Obtaining such a proof requires finding out which chaining phases can be permuted. In practice, increasing the degree of parallelism is already useful, and the question of the strength of multi-focusing is in our opinion better addressed by considering the full spectrum where chaining phases may or may not be grouped maximally. Indeed, there is to date no sequent calculus with a syntax expressive enough to statically enforce maximality, and it is quite unlikely that there can be one, since it is a global property that cannot be locally checked. Our system, where not all proofs are maximally multi-focused, is flexible and still contains these maximally normal forms.

In Section 2, we introduce a multi-focused sequent calculus for linear logic. It is slightly different from the original presentation (Chaudhuri et al. 2008a; Saurin 2008) as it only enforces chaining of positives and performs inversion non-maximally, and it involves the notion of *suspensions* (Simmons 2014). Then, Section 3 is devoted to a detailed proof of admissibility of the cut principle in this system, based on lemmas heavily using the notion of parametricity – which is more flexible than plain permutations, and would be, for example, much simpler to formalise in a proof assistant. Finally, Section 4 contains a discussion of the design choices available when defining multi-focused systems, with a particular emphasis on the question of maximal inversion, in linear logic or other logics.

The system discussed in the following section is best compared to the mono-focused sequent calculi and to the multi-focused system of Saurin (2008). Indeed, it is essentially

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a variant of the latter, in which foci are picked and released in a more flexible way, and in which inversion is not forcefully made maximal. A good understanding of focalisation and some basic knowledge about multi-focalisation is assumed, and we refer the reader to Saurin (2008) for a proper introduction to these topics. In comparison with the notion of multi-focused proof found there, we emphasise flexibility, the simplicity of the cut elimination argument and the generality of the methodology.

2. A multi-focused sequent calculus for linear logic

The step from mono-focalisation to multi-focalisation can be taken by considering a rather straightforward generalisation of the standard focused sequent calculus (Andreoli 1992) leading to the original multi-focalisation presentation (Chaudhuri et al. 2008a). But this system is not particularly well adapted to the task of cut elimination. While preserving important properties of a multi-focused system, we adopt here a more relaxed approach and consider a system with non-maximal inversion where the choice of the formulas to focus on is made incrementally. The structure of this expressive syntax for proofs is just relaxed enough to present a simple cut elimination argument using weak invertibility of positive rules and a decomposition of proofs isolating parametric derivations. The proof, presented in details in the next section, is thus hopefully amenable to mechanisation.

We adopt an explicitly polarised syntax for full linear logic, obtained by extending the grammar of formulas shown in Equation (2). We shall assume given some countable set $\mathcal{A}$ of *atoms*, partitioned among positive atoms written a, b or c, and their negative counterparts $\overline{a}$, $\overline{b}$ and $\overline{c}$. The grammar of formulas is then

$$\begin{array}{rcl} P,Q & ::= & a \mid \mathsf{1} \mid P \otimes Q \mid \mathsf{0} \mid P \oplus Q \mid\ !N \mid \downarrow N \\ N,M & ::= & \overline{a} \mid \bot \mid N \parr M \mid \top \mid N \,\&\, M \mid ?P \mid \uparrow P \end{array},$$

where P denotes a positive formula and N a negative one. We use $(-)^{\perp}$ to express the usual duality between formulas in linear logic. Note that because the syntax is polarised, the dual of a positive formula is always negative, and vice versa.

The particular notations we use require distinguishing between three forms of contexts:

$$\begin{array}{rcl} \Gamma,\Delta & ::= & \cdot \mid \Gamma, N \mid \Gamma, \langle \overline{a} \rangle \\ \Psi,\Phi & ::= & \Gamma \mid \Psi, [P] \\ \Omega & ::= & \cdot \mid \Omega, P \end{array}.$$

These will be identified by the letter used to denote them. The first kind is the basic case where no formula appears under focus – which implies negative formulas only, because positive ones can only appear under focus. Apart from plain negatives, such a Γ context can contain *suspended* negated atoms, written $\langle \overline{a} \rangle$, the only atoms available for use in an identity axiom. The general form Ψ of context enriches the basic context Γ with positives under focus, written $[P]$. Finally, contexts denoted by Ω are called *persistents*: They hold positive formulas that appeared under the replication modality ? and are therefore subject to structural rules. These forms of contexts are used in two kinds of judgements that have in general the following shapes:

$$\vdash \Omega : \Psi \qquad \vDash \Omega : \Psi,$$

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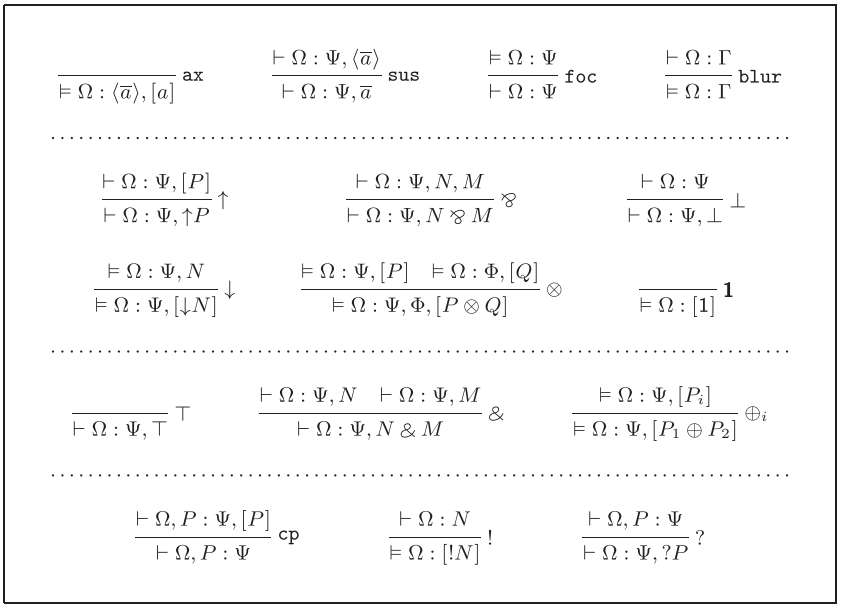


Fig. 1. Multi-focused sequent calculus LLM for linear logic. In the blur rule, we have the side condition that Γ must not contain any foci.

where the first one encapsulates *inversion,* whereas the second represents a *chaining* phase. The notation for separating the persistent context from the rest of the sequent is borrowed from Saurin (2008). Note that in the rules of the system shown above, and in the proof of cut elimination, such sequents sometimes appear in a particular configuration, where Ψ is of the particular shape Γ. This amounts to a side condition stating that no focused formula $[P]$ appears in that context. For this reason, the names used for contexts in the following is important to the understanding of the system.

The inference rules for our multi-focused sequent calculus, which we call LLM, are shown above in Figure 1. The key feature of this calculus is that selecting positive formulas that will be decomposed in the next chaining phase is done incrementally, during the inversion phase *before* the start of this chaining phase. This leads to the foc rule used here: For the change of phase to be meaningful, we require[†] that the context contains at least one focused formula. Similarly, the blur rule is applied only when the context contains only negative formulas or suspended atoms.

The notion of suspension has been introduced in a focused proof system to simplify the proof of cut elimination (Simmons 2014), but is more intimately related to the question of identity expansion. One could see this artifact as a means to make all negative formulas, including atomic ones, invertible when decomposed during the inversion phase. Another approach would consist in forcing all atoms to appear under a polarity shifting operator (Brock-Nannestad and Guenot 2015b), so that an atom can be viewed as

[†] This requirement is not strict, however. At worst, relaxing it would lead to the possibility of having empty chaining phases, consisting of a foc rule with a blur immediately above it.

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a placeholder for a formula of the same polarity, while preventing layers of negative or positive connectives from being merged, thus preventing phases from being merged during cut elimination. Note that we use suspensions only on atoms in our system, while they were originally used on arbitrary formulas. Suspensions are discussed further in Section 4.

The multiplicative fragment of the LLM system is standard, although one should notice that in a multi-focused system, a positive connective such as $\otimes$ does not induce an exact chaining. Rule instances decomposing several positive formulas can be interleaved if more than one positive formula appears under focus in the context. Similarly, additive connectives are treated by standard rules. With only these connectives, the border of phases are marked by polarity shifts $\downarrow$ and $\uparrow$ which are also controlling the selection of *foci*, and their release into the negative context. This can be illustrated by a multi-focused representation of the equivalent focused proofs that was shown in the previous section:

$$\dfrac{\dfrac{}{\vDash \overline{c}, [c]}\,\mathtt{ax} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vDash \overline{a}, [a]}\,\mathtt{ax}}{\vDash \overline{a}, [a \otimes \downarrow\bot], \overline{a}, \uparrow a}\,\otimes}{\vdash \overline{a}, \uparrow(a \otimes \downarrow\bot), \overline{a}, \uparrow a}\,\mathtt{foc} \qquad \dfrac{\dfrac{}{\vDash \overline{a}, [\downarrow\bot], \overline{a}, \uparrow a}\,\mathtt{blur}}{\vdash \overline{a}, \uparrow(a \otimes \downarrow\bot), \overline{a}, \uparrow a}\,\bot}{\vDash \overline{a}, \uparrow(a \otimes \downarrow\bot), [\downarrow\overline{a}], \uparrow a}\,\mathtt{blur}}{\vDash \overline{a}, \overline{c}, \uparrow(a \otimes \downarrow\bot), [c \otimes \downarrow\overline{a}], \uparrow a}\,\otimes}{\vdash \overline{a}, \overline{c}, \uparrow(a \otimes \downarrow\bot), \uparrow(c \otimes \downarrow\overline{a}), \uparrow a}\,\mathtt{foc}$$

$$\searrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vDash \cdot : \langle\overline{c}\rangle, [c]}\,\mathtt{ax} \qquad \dfrac{\dfrac{}{\vDash \cdot : \langle\overline{a}\rangle, [a]}\,\mathtt{ax}}{\vDash \cdot : \langle\overline{a}\rangle, [a \otimes \downarrow\bot], [\downarrow\overline{a}], \uparrow a}\,\otimes}{\vDash \cdot : \langle\overline{a}\rangle, \langle\overline{c}\rangle, [a \otimes \downarrow\bot], [c \otimes \downarrow\overline{a}], \uparrow a}\,\otimes}{\vdash \cdot : \langle\overline{a}\rangle, \langle\overline{c}\rangle, [a \otimes \downarrow\bot], [c \otimes \downarrow\overline{a}], \uparrow a}\,\mathtt{foc}}{\vdash \cdot : \langle\overline{a}\rangle, \overline{c}, [a \otimes \downarrow\bot], [c \otimes \downarrow\overline{a}], \uparrow a}\,\mathtt{sus}}{\vdash \cdot : \overline{a}, \overline{c}, [a \otimes \downarrow\bot], [c \otimes \downarrow\overline{a}], \uparrow a}\,\mathtt{sus}}{\vdash \cdot : \overline{a}, \overline{c}, [a \otimes \downarrow\bot], \uparrow(c \otimes \downarrow\overline{a}), \uparrow a}\,\uparrow}{\vdash \cdot : \overline{a}, \overline{c}, \uparrow(a \otimes \downarrow\bot), \uparrow(c \otimes \downarrow\overline{a}), \uparrow a}\,\uparrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vDash \cdot : \langle\overline{a}\rangle, [a]}\,\mathtt{ax}}{\vdash \cdot : \langle\overline{a}\rangle, [a]}\,\mathtt{foc}}{\vdash \cdot : \langle\overline{a}\rangle, \uparrow a}\,\uparrow}{\vdash \cdot : \overline{a}, \uparrow a}\,\mathtt{sus}}{\vdash \cdot : \bot, \overline{a}, \uparrow a}\,\bot}{\vDash \cdot : \bot, \overline{a}, \uparrow a}\,\mathtt{blur} \qquad \dfrac{\dfrac{}{\vDash \cdot : \bot, [\downarrow\overline{a}], \uparrow a}\,\mathtt{blur}}{\vDash \cdot : \bot, [\downarrow\overline{a}], \uparrow a}\,\downarrow}{\vDash \cdot : [\downarrow\bot], [\downarrow\overline{a}], \uparrow a}\,\downarrow$$

Here, the increase in size is due to the overhead of managing an incremental focus selection process and using the suspension rule on negative atoms used in an identity axiom. From the perspective of canonicity, we can observe that two of the three original chaining phases have been merged into a single multi-chaining phase. Within this phase, chaining cannot be enforced at the level of a single positive formula. For instance, the rule blurring focus on $\downarrow$ is used after both $\otimes$ connectives have been decomposed. However, the possible re-orderings of rule instances within a phase are not to be taken into account, since from

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the focalisation result we can obtain a calculus of synthetic connectives (Chaudhuri 2008) where these irrelevant details disappear. The entire multi-focused proof is then viewed as the following object:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vDash \cdot : \langle \overline{a} \rangle, [a]}\,\texttt{pos}}{\vdash \cdot : \langle \overline{a} \rangle, [a]}\,\texttt{foc}}{\vdash \cdot : \bot, \overline{a}, \uparrow a}\,\texttt{neg}}{\vDash \cdot : \bot, \overline{a}, \uparrow a}\,\texttt{blur}}{\dfrac{\dfrac{\vDash \cdot : \langle \overline{a} \rangle, \langle \overline{c} \rangle, [a \otimes \downarrow \bot], [c \otimes \downarrow \overline{a}], \uparrow a}{\vdash \cdot : \langle \overline{a} \rangle, \langle \overline{c} \rangle, [a \otimes \downarrow \bot], [c \otimes \downarrow \overline{a}], \uparrow a}\,\texttt{foc}}{\vdash \cdot : \overline{a}, \overline{c}, \uparrow (a \otimes \downarrow \bot), \uparrow (c \otimes \downarrow \overline{a}), \uparrow a}\,\texttt{neg}}\,\texttt{pos} \tag{5}$$

in which there are four synthetic `pos` and `neg` rules, mediated by the `foc` and `blur` rules, and a single branch since the two others were entirely part of a chaining phase.

Finally, the exponential fragment of `LLM` employs the dyadic presentation of the original focused sequent calculus (Andreoli 1992). It also adds the ! and ? modalities as possible markers of the border of phases. The selection of exponential foci is indirect: First, a formula $?P$ is decomposed by moving the sub-formula P into the persistent context Ω, and at any point in a subproof this P may be selected for a chaining phase using the `cp` rule. Because the persistent context is duplicated in the branching rules, a formula may be selected several times for decomposition.

Finally, it is quite easy to observe that the `LLM` sequent calculus is adequately capturing linear logic, because of its close ties to the original calculus introducing multi-focalisation (Chaudhuri et al. 2008a), and because the changes we have made are not endangering the completeness result.

Theorem 2.1. The `LLM` system is sound and complete with respect to linear logic.

Proof. Soundness is a trivial result in any focused system. Indeed, all the rules can be re-written to drop the information about focalisation, and we obtain a standard unfocused sequent calculus for linear logic. Completeness is the important part, but we can rely here on the standard focalisation result, since a proof that does not focus more than one formula at a time is *also* a multi-focused proof – albeit of a somewhat degenerate form. □

It should be noted that although it is straightforward to establish this form of *logical* or *external* soundness and completeness, certain design choices may prevent the system from being *internally* sound, i.e. lacking a strong cut elimination property, as we will show in the next section. The next step in the study of this system is therefore its main purpose: The proof of admissibility of the cut principle.

3. Cut elimination in multi-focused linear logic

Before we present the proof of admissibility of cut, we first discuss some of the challenges in adapting such a proof to a setting with multiple foci. We should note that it is not the fact that cut is admissible that is interesting in this respect, but rather the *proof* of

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this fact. For instance, a simple proof of cut admissibility would be to map the multi-focused proof to a (cut-full) mono-focused proof (e.g. as described in Section 4), eliminate the cuts there, and map the cut-free proof back into a degenerate multi-focused proof. This suffices to establish the admissibility of cut, but for our purposes such a proof is undesirable because it actively *destroys* information that was present in the original proof. Instead, we will present a proof that proceeds by structural induction on the cut formula and given derivations (occasionally referred to as a *structural cut admissibility* argument).

To understand the dynamics of such a cut admissibility procedure, we start by considering what happens in an unfocused calculus. To further simplify the presentation, we will not concern ourselves with the exponentials. The unfocused cut has the following form (shown here as part of a derivation, with subderivations $\mathcal{D}$ and $\mathcal{E}$):

$$\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vdash \Gamma, A & \vdash \Delta, A^{\perp}\end{array}}{\vdash \Gamma, \Delta}\,\mathtt{cut}\,.$$

To eliminate such a cut, we push it upwards in the proof until the formulas A and $A^{\perp}$ become principal. Once that happens, these formulas can be decomposed simultaneously to create cuts on smaller formulas. Note that in this setting, the cut can 'permute' up both $\mathcal{D}$ and $\mathcal{E}$, as the following examples show:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D} \\ \vdash \Gamma, B, A\end{array}}{\vdash \Gamma, B \oplus C, A}\oplus_1 \qquad \begin{array}{c}\mathcal{E} \\ \vdash \Delta, A^{\perp}\end{array}}{\vdash \Gamma, \Delta, B \oplus C}\,\mathtt{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vdash \Gamma, B, A & \vdash \Delta, A^{\perp}\end{array}}{\vdash \Gamma, \Delta, B}\,\mathtt{cut}}{\vdash \Gamma, B \oplus C}\oplus_1$$

$$\dfrac{\begin{array}{cc}\mathcal{D} & \dfrac{\begin{array}{c}\mathcal{E} \\ \vdash \Delta, B, C, A^{\perp}\end{array}}{\vdash \Delta, B \parr C, A}\parr \\ \vdash \Gamma, A & \end{array}}{\vdash \Gamma, \Delta, B \parr C}\,\mathtt{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vdash \Gamma, A & \vdash \Delta, B, C, A^{\perp}\end{array}}{\vdash \Gamma, \Delta, B, C}\,\mathtt{cut}.}{\vdash \Gamma, \Delta, B \parr C}\parr$$

In a weakly focused, polarised setting, the first of these permutations cannot occur, because the cut has the following form:

$$\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vdash \Gamma, [P] & \vdash \Delta, P^{\perp}\end{array}}{\vdash \Gamma, \Delta}\,\mathtt{cut}\,.$$

Because of the chaining requirement, the formula P must be decomposed in the final rule of $\mathcal{D}$, and hence it is never necessary to permute the cut up this derivation. Thus, moving to a focused setting in fact *streamlines* the cut admissibility argument, as noted in Simmons (2014). In a strongly focused setting, it is possible to present the system in such a way that the cut formula must be principal in *both* premises, which restricts the cut elimination procedure even further. See Brock-Nannestad and Guenot (2015b) for an example of this for strongly focused MELL.

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In a multi-focused setting, the exact nature of the cut rule is a bit more difficult to determine. If we simply generalise the mono-focused cut rule to allow several foci to be present, then we end up with the following rule:

$$\dfrac{\vdash \Psi, [P] \qquad \vdash \Delta, P^{\perp}}{\vdash \Psi, \Delta}\,\mathtt{cut}\,.$$

Unlike the mono-focused rule, here Ψ may contain additional foci. This makes the above rule somewhat ill-defined: Usually, foci are only allowed to appear during the *chaining* phases of a proof, and not during an *inversion* phase.

One way of avoiding this problem is to change the conclusion of the cut rule from $\vdash \Psi, \Delta$ to $\vDash \Psi, \Delta$. This breaks the $\mathtt{cut}/\parr$ permutation seen above, however

$$\dfrac{\mathcal{D} \quad \vDash \Psi, [P] \qquad \dfrac{\mathcal{E} \quad \vdash \Delta, N, M, P^{\perp}}{\vdash \Delta, N \parr M, P}\parr}{\vDash \Gamma, \Delta, N \parr M}\,\mathtt{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathcal{D} \quad \vDash \Psi, [P] \qquad \mathcal{E} \quad \vdash \Delta, N, M, P^{\perp}}{\vDash \Psi, \Delta, N, M}\,\mathtt{cut}.}{\vDash \Psi, \Delta N \parr M}\parr?$$

Note that we cannot directly use the $\parr$ rule, since it only works during an inversion phase. Of course, one could hope that this rule is admissible during a chaining phase as well, but this is not the case! For instance, the sequent $\vDash \bar{p}, \bar{q}, [p \otimes q]$ is derivable, but the sequent $\vDash \bar{p} \parr \bar{q}, [p \otimes q]$ is not, and hence such a 'focused $\parr$' rule is not admissible.

Despite this problem, it is in fact possible to complete a proof of cut admissibility with the above cut rule, as we show in Brock-Nannestad and Guenot (2015a). However, as we remark in that paper, this results in a cut elimination procedure that may split a single multi-chaining phase into two, and this we would like to avoid.

Another view, then, is that the potential foci in Ψ in $\vdash \Psi, \Delta$ can be seen as harmless. They are essentially *inert* until the next chaining phase begins. With this change in perspective, we can keep the straightforward generalisation of the cut rule with the expected judgment in the conclusion.

Of course, once this view has been adopted, it is natural to consider what other consequences it brings. This we consider further in Section 4.

The addition of multiple foci brings another wrinkle to the tale – we can no longer be sure that the cut formula is the principal formula of the first premise of the cut:

$$\dfrac{\dfrac{\mathcal{D} \quad \vDash \Omega : \Psi, [P_1], [Q \otimes R]}{\vDash \Omega : \Psi, [P_1 \oplus P_2], [Q \otimes R]}\oplus_1 \qquad \dfrac{\mathcal{E} \quad \vdash \Omega : \Phi, Q^{\perp}, R^{\perp}}{\vdash \Omega : \Phi, Q^{\perp} \parr R^{\perp}}\parr}{\vdash \Omega : \Psi, [P_1 \oplus P_2], \Phi}\,\mathtt{cut}\,.$$

No local permutation of the cut rule is possible at this point. Permuting the cut up the left premise yields a derivation of $\vdash \Omega : \Psi, [P_1], \Phi$ which unfortunately has the wrong judgment for applying the $\oplus_1$ rule. Nor can we apply the `blur` rule, as we have a focused formula present.

Instead, what we need to do is to reconsider the structure of the multi-chaining phases. The key observation is that there are no interactions *between* the formulas that are focused simultaneously, and hence we can always permute the rule introducing the cut

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formula to the *bottom* of the derivation. In the above case, this would entail re-arranging the derivation in the first premise of the cut such that the final rule introduces $[Q \otimes R]$. In that case, the cut would no longer be blocked, and the cut elimination procedure would proceed. We will refer to this property as *weak inversion*, and say that a formula P is *weakly invertible* if any proof of $\vDash \Omega : \Phi, [P]$ can be transformed into a proof that proceeds by decomposing P. The reason this is only a *weak* property is that it only tells us that a formula may be decomposed now, but not *how* it should be decomposed. As we will show in a later lemma, all formulas that appear under focus and inside a chaining phase are weakly invertible. This is immediate in a mono-focused system, but requires a bit more care when multiple foci are present, as indicated by the above analysis.

In the following, let us define an *active* formula to be a formula that is not a suspended atom $\langle \bar{a} \rangle$. In other words, an active formula is either of the form N or of the form $[P]$.

In the following lemmas, we will use the notation

$$\vDash \Omega : \Xi, \Phi \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vDash \Omega : \Psi, \Phi$$

to represent an open derivation from the sequent $\vDash \Omega : \Xi, \Phi$ to the sequent $\vDash \Omega : \Psi, \Phi$ which is *parametric* in Φ. Thus, we can choose to instantiate Φ with any context, and we will still have a valid open derivation. Likewise, we may apply our open derivation to a (possibly open) derivation ending in the sequent $\vDash \Omega : \Xi, \Phi$ to get a new (possibly open) derivation.

We begin with an important lemma concerning the structure of chaining phases. An intuitive way of interpreting this lemma is that there can be no dependencies between foci appearing in the same phase.

Lemma 3.1 (Focal decomposition). If $\vDash \Omega : \Psi, A$ where A is active, then there exists a Γ such that

1. $\vDash \Omega : \Gamma, A$ is derivable, and
2. for every Φ, there is an open derivation from $\vDash \Omega : \Gamma, \Phi$ to $\vDash \Omega : \Psi, \Phi$.

Furthermore, Ψ contains an active formula if and only if Γ contains an active formula.

Proof. The proof proceeds by induction over the derivation of $\vDash \Omega : \Psi, A$:

Case `ax`**:**

$$\mathcal{D} :: \frac{}{\vDash \Omega : \langle \bar{a} \rangle, [a]}\,\mathtt{ax}\,.$$

As A is assumed to be active, we must have $A = [a]$. Thus, we may set $\Gamma = \langle \bar{a} \rangle$ and the desired derivation is $\mathcal{D}$. As we now have $\Psi = \Gamma$, the desired open derivation is the trivial derivation from $\vDash \Omega : \Gamma, \Phi$ to itself. In this case, neither Γ nor Ψ contain active formulas.

Case $\downarrow$**,** $A = [\downarrow N]$ **principal:**

$$\frac{\begin{array}{c}\mathcal{D} \\ \vDash \Omega : \Psi, N\end{array}}{\vDash \Omega : \Psi, [\downarrow N]}\downarrow .$$

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For the first part, we reason as follows:

$$\begin{array}{lll} \mathcal{D}' :: \vDash \Omega : \Gamma, N \quad \text{and} \quad \vDash \Omega : \Gamma, \Phi \xrightarrow{\mathcal{E}_\Phi} \vDash \Omega : \Psi, \Phi & & \text{by i.h. on } \mathcal{D}. \\ \quad \vDash \Omega : \Gamma, [\downarrow N] & & \text{by } \downarrow \text{ on } \mathcal{D}'. \end{array}$$

For the second part:

$$\begin{array}{ll} \vDash \Omega : \Gamma, \Phi & \text{by assumption.} \\ \vDash \Omega : \Psi, \Phi & \text{by } \mathcal{E}_\Phi. \end{array}$$

Finally, by the induction hypothesis it follows that Ψ contains an active formula if and only if Γ does as well.

Case $\downarrow$, A non-principal:

$$\dfrac{\begin{array}{c}\mathcal{D} \\ \vDash \Omega : \Psi, N, A\end{array}}{\vDash \Omega : \Psi, [\downarrow N], A}\downarrow .$$

For the first part, we reason as follows:

$$\vDash \Omega : \Gamma, A \quad \text{and} \quad \vDash \Omega : \Gamma, \Phi \xrightarrow{\mathcal{E}_\Phi} \vDash \Omega : \Psi, N, \Phi \qquad \text{by i.h. on } \mathcal{D}.$$

For the second part:

$$\begin{array}{ll} \vDash \Omega : \Gamma, \Phi & \text{by assumption.} \\ \vDash \Omega : \Psi, N, \Phi & \text{by } \mathcal{E}_\Phi. \\ \vDash \Omega : \Psi, [\downarrow N], \Phi & \text{by } \downarrow. \end{array}$$

Finally, as Ψ, N contains an active formula, it follows from the induction hypothesis that Γ does as well. Thus, $\Psi, [\downarrow N]$ contains an active formula if and only if Γ contains an active formula.

Case 1:

$$\mathcal{D} :: \dfrac{}{\vDash \Omega : [1]}\mathbf{1} .$$

As A is assumed to be active, we must have $A = [1]$. Thus. we may set $\Gamma = \cdot$ and the desired derivation is then $\mathcal{D}$. As we now have $\Psi = \Gamma$, the desired open derivation is the trivial derivation from $\vDash \Omega : \Gamma, \Phi$ to itself. In this case, neither Γ nor Ψ contain active formulas.

Case $\otimes$, $A = [P \otimes Q]$ principal:

$$\dfrac{\begin{array}{c}\mathcal{D}_1 \\ \vDash \Omega : \Psi_1, [P]\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ \vDash \Omega : \Psi_2, [Q]\end{array}}{\vDash \Omega : \Psi_1, \Psi_2, [P \otimes Q]}\otimes .$$

For the first part, we reason as follows:

$$\begin{array}{lll} \mathcal{D}'_1 :: \vDash \Omega : \Gamma_1, [P] \quad \text{and} \quad \vDash \Omega : \Gamma_1, \Phi \xrightarrow{\mathcal{E}_\Phi} \vDash \Omega : \Psi_1, \Phi & & \text{by i.h. on } \mathcal{D}_1. \\ \mathcal{D}'_2 :: \vDash \Omega : \Gamma_2, [Q] \quad \text{and} \quad \vDash \Omega : \Gamma_2, \Phi \xrightarrow{\mathcal{F}_\Phi} \vDash \Omega : \Psi_2, \Phi & & \text{by i.h. on } \mathcal{D}_2. \\ \quad \vDash \Omega : \Gamma_1, \Gamma_2, [P \otimes Q] & & \text{by } \otimes \text{ on } \mathcal{D}'_1 \text{ and } \mathcal{D}'_2. \end{array}$$

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For the second part:

$$\begin{array}{ll} \vDash \Omega : \Gamma_1, \Gamma_2, \Phi & \text{by assumption.} \\ \vDash \Omega : \Psi_1, \Gamma_2, \Phi & \text{by } \mathcal{E}_{\Gamma_2,\Phi}. \\ \vDash \Omega : \Psi_1, \Psi_2, \Phi & \text{by } \mathcal{F}_{\Psi_1,\Phi}. \end{array}$$

Finally, if Ψ_1, Ψ_2 contains an active formula, then so does either Ψ_1 or Ψ_2. By the induction hypothesis, it now follows that either Γ_1 or Γ_2 contains an active formula, and thus so does Γ_1, Γ_2. Conversely, if neither Ψ_1 nor Ψ_2 contain active formulas, then neither does Γ_1 or Γ_2.

Case $\otimes$, A in linear context of first premise:

$$\dfrac{\begin{array}{cc} \mathcal{D}_1 & \mathcal{D}_2 \\ \vDash \Omega : \Psi_1, [P], A & \vDash \Omega : \Psi_2, [Q] \end{array}}{\vDash \Omega : \Psi_1, \Psi_2, [P \otimes Q], A}\otimes.$$

For the first part, we reason as follows:

$$\vDash \Omega : \Gamma, A \quad \text{and} \quad \vDash \Omega : \Gamma, \Phi \xrightarrow{\mathcal{E}_\Phi} \vDash \Omega : \Psi_1, [P], \Phi \qquad \text{by i.h. on } \mathcal{D}_1.$$

For the second part:

$$\begin{array}{ll} \vDash \Omega : \Gamma, \Phi & \text{by assumption.} \\ \mathcal{D}_1' :: \vDash \Omega : \Psi_1, [P], \Phi & \text{by } \mathcal{E}_\Phi. \\ \vDash \Omega : \Psi_1, \Psi_2, [P \otimes Q], \Phi & \text{by } \otimes \text{ on } \mathcal{D}_1' \text{ and } \mathcal{D}_2. \end{array}$$

In this case, both $\Psi_1, [P]$ and $\Psi_1, \Psi_2, [P \otimes Q]$ contain an active formula, and by the induction hypothesis on the former, it follows that Γ does as well. The case where A appears in the linear context of the second premise is analogous.

Case $\oplus_i$, $i \in \{1, 2\}$, $A = [P_1 \oplus P_2]$ principal:

$$\dfrac{\begin{array}{c} \mathcal{D} \\ \vDash \Omega : \Psi, [P_i] \end{array}}{\vDash \Omega : \Psi, [P_1 \oplus P_2]}\oplus_i.$$

For the first part, we reason as follows:

$$\begin{array}{lll} \mathcal{D}' :: \vDash \Omega : \Gamma, [P_i] & \text{and} \quad \vDash \Omega : \Gamma, \Phi \xrightarrow{\mathcal{E}_\Phi} \vDash \Omega : \Psi, \Phi & \text{by i.h. on } \mathcal{D}. \\ \vDash \Omega : \Gamma, [P_1 \oplus P_2] & & \text{by } \oplus_i \text{ on } \mathcal{D}'. \end{array}$$

For the second part:

$$\begin{array}{ll} \vDash \Omega : \Gamma, \Phi & \text{by assumption.} \\ \vDash \Omega : \Psi, \Phi & \text{by } \mathcal{E}_\Phi. \end{array}$$

Finally, by the induction hypothesis, it follows that Ψ contains an active formulas if and only if Γ does as well.

Case $\oplus_i$, $i \in \{1, 2\}$, A non-principal:

$$\dfrac{\begin{array}{c} \mathcal{D} \\ \vDash \Omega : \Psi, [P_i], A \end{array}}{\vDash \Omega : \Psi, [P_1 \oplus P_2], A}\oplus_i.$$

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For the first part, we reason as follows:

$$\vDash \Omega : \Gamma, A \quad \text{and} \quad \vDash \Omega : \Gamma, \Phi \xrightarrow{\mathcal{E}_\Phi} \vDash \Omega : \Psi, [P_i], \Phi \qquad \text{by i.h. on } \mathcal{D}.$$

For the second part:

$$\begin{array}{ll} \vDash \Omega : \Gamma, \Phi & \text{by assumption.} \\ \vDash \Omega : \Psi, [P_i], \Phi & \text{by } \mathcal{E}_\Phi. \\ \vDash \Omega : \Gamma, [P_1 \oplus P_2], \Phi & \text{by } \oplus_i. \end{array}$$

In this case, both $\Psi, [P_i]$ and $\Psi, [P_1 \oplus P_2]$ contain an active formula, and by the induction hypothesis on the former it follows that Γ does as well.

Case $!$**:**

$$\dfrac{\begin{array}{c}\mathcal{D} \\ \vdash \Omega : N\end{array}}{\vDash \Omega : [!N]}\,!\,.$$

In this case, $A = [!N]$, and the desired derivation is simply the given derivation. The open derivation from $\vDash \Omega : \Phi$ to itself is trivial. In this case, $\Gamma = \Psi = \cdot$, and neither of them contain active formulas.

Case `blur`**:**

$$\dfrac{\begin{array}{c}\mathcal{D} \\ \vdash \Omega : \Gamma, A\end{array}}{\vDash \Omega : \Gamma, A}\,\texttt{blur}\,.$$

Again, the desired derivation is simply the given derivation. The open derivation from $\vDash \Omega : \Gamma, \Phi$ to itself is trivial. In this case, $\Gamma = \Psi$, and they clearly contain the same active formulas, if any.

This completes the proof. □

A further lemma allows us to decompose open derivations inside the chaining phase:

Lemma 3.2. If for all Φ we have an open derivation

$$\vDash \Omega : \Psi'_1, \Psi'_2, \Phi \xrightarrow{\mathcal{D}_\Phi} \vDash \Omega : \Psi, \Phi,$$

then there exist Ψ_1 and Ψ_2 such that $\Psi = \Psi_1, \Psi_2$, and for all Φ there exist open derivations

$$\vDash \Omega : \Psi'_1, \Phi \xrightarrow{\mathcal{E}_\Phi} \vDash \Omega : \Psi_1, \Phi$$

$$\vDash \Omega : \Psi'_2, \Phi \xrightarrow{\mathcal{F}_\Phi} \vDash \Omega : \Psi_2, \Phi.$$

Proof. The proof proceeds by induction on the structure of the given open derivation, decomposing it from the top down. We first observe that since Φ is essentially parametric, in all cases except the base case, the topmost rule must involve a formula in either Ψ'_1 or Ψ'_2. Note that there is no distinction between these two contexts, hence we can without loss of generality assume it is always Ψ'_1 that contains the aforementioned formula.

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Base case: In this case, $\mathcal{D}_\Phi$ is the trivial open derivation from $\vDash \Omega : \Psi, \Phi$ to itself, and $\Psi = \Psi'_1, \Psi'_2$. Putting $\Psi_1 = \Psi'_1$ and $\Psi_2 = \Psi'_2$, with corresponding trivial open derivations $\mathcal{E}_\Phi$ and $\mathcal{F}_\Phi$ now yields the desired result.

Case $\otimes$, first premise:

$$\begin{array}{c} \vDash \Omega : (\Psi'_1, [P]), \Psi'_2, \Phi \\ \vdots\ \mathcal{D}_\Phi \\ \vDash \Omega : \Psi, \Phi \end{array} \quad = \quad \dfrac{\begin{array}{cc} \vDash \Omega : (\Psi'_1, [P]), \Psi'_2, \Phi & \begin{array}{c}\mathcal{D}\\ \vDash \Omega : \Psi'_3, [Q]\end{array} \end{array} \otimes}{\begin{array}{c} \vDash \Omega : (\Psi'_1, [P \otimes Q], \Psi'_3), \Psi'_2, \Phi \\ \vdots\ \mathcal{D}'_\Phi \\ \vDash \Omega : \Psi, \Phi \end{array}}.$$

Applying the induction hypothesis to the open derivation $\mathcal{D}'_\Phi$ yields two parametric open derivations:

$$\vDash \Omega : \Psi'_1, [P \otimes Q], \Psi'_3, \Phi \quad \xrightarrow{\mathcal{E}'_\Phi} \quad \vDash \Omega : \Psi_1, \Phi$$

$$\vDash \Omega : \Psi'_2, \Phi \quad \xrightarrow{\mathcal{F}_\Phi} \quad \vDash \Omega : \Psi_2, \Phi$$

such that $\Psi = \Psi_1, \Psi_2$. The second of these, $\mathcal{F}_\Phi$ is one of the desired open derivations. Pre-composing $\mathcal{E}'_\Phi$ with the following open derivation

$$\dfrac{\begin{array}{cc} \begin{array}{c}\vdots\\ \vDash \Omega : (\Psi'_1, [P]), \Phi\end{array} & \begin{array}{c}\mathcal{D}\\ \vdots\\ \vDash \Omega : \Psi'_3, [Q]\end{array} \end{array} \otimes}{\begin{array}{c} \vDash \Omega : \Psi'_1, \Psi'_3, [P \otimes Q], \Phi \\ \vdots \end{array}}$$

yields the desired open derivation $\mathcal{E}_\Phi$.

Case $\otimes$, second premise: This case is symmetrical to the preceding case, and has therefore been elided.

Case $\oplus_i$, $i \in \{1, 2\}$:

$$\begin{array}{c} \vDash \Omega : (\Psi'_1, [P_i]), \Psi'_2, \Phi \\ \vdots\ \mathcal{D}_\Phi \\ \vDash \Omega : \Psi, \Phi \end{array} \quad = \quad \dfrac{\vDash \Omega : (\Psi'_1, [P_i]), \Psi'_2, \Phi}{\begin{array}{c} \vDash \Omega : (\Psi'_1, [P_1 \oplus P_2]), \Psi'_2, \Phi \\ \vdots\ \mathcal{D}'_\Phi \\ \vDash \Omega : \Psi, \Phi \end{array}} \oplus_i.$$

Applying the induction hypothesis to the open derivation $\mathcal{D}'_\Phi$ yields two parametric open derivations:

$$\vDash \Omega : \Psi'_1, [P_1 \oplus P_2], \Phi \quad \xrightarrow{\mathcal{E}'_\Phi} \quad \vDash \Omega : \Psi_1, \Phi$$

$$\vDash \Omega : \Psi'_2, \Phi \quad \xrightarrow{\mathcal{F}_\Phi} \quad \vDash \Omega : \Psi_2, \Phi$$

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such that $\Psi = \Psi_1, \Psi_2$. Again, $\mathcal{F}_\Phi$ is one of the desired derivations. Pre-composing $\mathcal{E}'_\Phi$ with the following open derivation

$$\dfrac{\begin{array}{c}\vdots\\ \vdash \Omega : (\Psi'_1, [P_i]), \Phi\end{array}}{\vdash \Omega : \Psi'_1, [P_1 \oplus P_2], \Phi}\oplus_i$$

yields the desired derivation $\mathcal{E}_\Phi$.

Case $\downarrow$: This case is similar to the preceding case, and has been elided.

Case `foc`:

$$\begin{array}{c}\vdash \Omega : \Psi'_1, \Psi'_2, \Phi\\ \vdots\ \mathcal{D}_\Phi\\ \vdash \Omega : \Psi, \Phi\end{array} \quad = \quad \dfrac{\vdash \Omega : \Psi'_1, \Psi'_2, \Phi}{\begin{array}{c}\vdash \Omega : \Psi'_1, \Psi'_2, \Phi\\ \vdots\ \mathcal{D}'_\Phi\\ \vdash \Omega : \Psi, \Phi\end{array}}\texttt{foc}\,.$$

This case is in fact impossible, as the following argument shows. First, note that the presence of the `foc` rule results in a switch from the chaining judgment to the inversion judgment. As $\mathcal{D}'_\Phi$ ends in the chaining judgment, there must be a rule somewhere further down in this derivation that switches from the inversion judgment to the chaining judgment. The only rules that do this are the ! and `blur` rules:

$$\dfrac{\vdash \Omega : N}{\vDash \Omega : [!N]}\,! \qquad \dfrac{\vdash \Omega : \Gamma}{\vDash \Omega : \Gamma}\,\texttt{blur}.$$

Recall that we assumed the open derivation to exist for all Φ. This means that we should be able to instantiate Φ with a focused formula such as $[1]$, and still have a valid open derivation. The above two inference rules do not allow this, however, as they require the context in the premise to consist of only negative formulas. This is a contradiction, and hence our original assumption that the topmost rule in the open derivation was the `foc` rule must be impossible.

This completes the proof. □

Next, we show the weak inversion properties that were alluded to in the introduction of this section.

Lemma 3.3 (Weak inversion properties). All rules that end in the $\vDash$ judgment are weakly invertible. More specifically

1. If $\vDash \Omega : \Psi, [P_1 \oplus P_2]$ then $\vDash \Omega : \Psi, [P_i]$ for some $i \in \{1, 2\}$.
2. If $\vDash \Omega : \Psi, [\downarrow N]$ then $\vDash \Omega : \Psi, N$.
3. If $\vDash \Omega : \Psi, [!N]$ then $\Psi = \cdot$ and $\vdash \Omega : N$.
4. If $\vDash \Omega : \Psi, [1]$ then $\Psi = \cdot$.
5. If $\vDash \Omega : \Psi, [a]$ then $\Psi = \langle \bar{a} \rangle$.
6. $\vDash \Omega : \Psi, [0]$ is impossible.
7. If $\vDash \Omega : \Gamma$ then $\vdash \Omega : \Gamma$.

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8. If $\vDash \Omega : \Psi, [P \otimes Q]$ then there exist Ψ_1, Ψ_2 such that $\Psi_1, \Psi_2 = \Psi$ and $\vDash \Omega : \Psi_1, [P]$ and $\vDash \Omega : \Psi_2, [Q]$ are derivable.

Proof. Most of the cases are consequences of the first multi-focal decomposition lemma we proved. We will consider these cases first:

Case $[P_1 \oplus P_2]$**:** Let $\mathcal{D} :: \vDash \Omega : \Psi, [P_1 \oplus P_2]$ be given. By the decomposition lemma, we now get derivations

$$\mathcal{D}' :: \vDash \Omega : \Gamma, [P_1 \oplus P_2] \qquad \text{and} \qquad \vDash \Omega : \Gamma, \Phi \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vDash \Omega : \Psi, \Phi.$$

Now, $\mathcal{D}'$ is a focused derivation with only a single focus, $[P_1 \oplus P_2]$, and by inversion it therefore follows that $\vDash \Omega : \Gamma, [P_i]$ is provable for some $i \in \{1, 2\}$. Using $\mathcal{E}_{[P_i]}$, we now get the desired derivation of $\vDash \Omega : \Psi, [P_i]$.

Case $[\downarrow N]$**:** Let $\mathcal{D} :: \vDash \Omega : \Psi, [\downarrow N]$ be given. By the decomposition lemma, we now get derivations

$$\mathcal{D}' :: \vDash \Omega : \Gamma, [\downarrow N] \qquad \text{and} \qquad \vDash \Omega : \Gamma, \Phi \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vDash \Omega : \Psi, \Phi.$$

Now, $\mathcal{D}'$ is a focused derivation with only a single focus, $[\downarrow N]$, and by inversion it therefore follows that $\vDash \Omega : \Gamma, N$ is provable. Using $\mathcal{E}_N$, we now get the desired derivation of $\vDash \Omega : \Psi, N$.

Case $[!N]$**:** Let $\mathcal{D} :: \vDash \Omega : \Psi, [!N]$ be given. By the decomposition lemma, we now get derivations

$$\mathcal{D}' :: \vDash \Omega : \Gamma, [!N] \qquad \text{and} \qquad \vDash \Omega : \Gamma, \Phi \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vDash \Omega : \Psi, \Phi.$$

Now, $\mathcal{D}'$ is a focused derivation with only a single focus, $[!N]$, and by inversion it therefore follows that

1. $\vdash \Omega : N$ is provable, and
2. $\Gamma = \cdot$.

Because Γ is empty, it contains no active formulas and so by the decomposition lemma, neither does Ψ. Because Ψ has no active formulas, $[!N]$ is the only focus in the sequent $\vDash \Omega : \Psi, [!N]$, and by the above analysis it again follows that $\Psi = \cdot$, as desired.

Case $[\mathbf{1}]$**:** Let $\mathcal{D} :: \vDash \Omega : \Psi, [1]$ be given. By the decomposition lemma, we now get derivations

$$\mathcal{D}' :: \vDash \Omega : \Gamma, [1] \qquad \text{and} \qquad \vDash \Omega : \Gamma, \Phi \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vDash \Omega : \Psi, \Phi.$$

Now, $\mathcal{D}'$ is a focused derivation with only a single focus, $[1]$, and by inversion it therefore follows that $\Gamma = \cdot$. Because Γ is empty, it contains no active formulas and so by the decomposition lemma, neither does Ψ. Because Ψ has no active formulas, $[1]$ is the only focus in the sequent $\vDash \Omega : \Psi, [1]$, and by the above analysis it again follows that $\Psi = \cdot$, as desired.

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Case $[a]$**:** Let $\mathcal{D} :: \vDash \Omega : \Psi, [a]$ be given. By the decomposition lemma, we now get derivations

$$\mathcal{D}' :: \vDash \Omega : \Gamma, [a] \qquad \text{and} \qquad \vDash \Omega : \Gamma, \Phi \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vDash \Omega : \Psi, \Phi.$$

Now, $\mathcal{D}'$ is a focused derivation with only a single focus, $[a]$, and by inversion it therefore follows that $\Gamma = \langle \bar{a} \rangle$. As Γ contains no active formulas it follows from the decomposition lemma, that neither does Ψ. Because Ψ has no active formulas, $[a]$ is the only focus in the sequent $\vDash \Omega : \Psi, [a]$, and by the above analysis it again follows that $\Psi = \langle \bar{a} \rangle$, as desired.

Case $[\mathtt{0}]$**:** Let $\mathcal{D} :: \vDash \Omega : \Psi, [\mathtt{0}]$ be given. By the decomposition lemma, we now get derivations

$$\mathcal{D}' :: \vDash \Omega : \Gamma, [\mathtt{0}] \qquad \text{and} \qquad \vDash \Omega : \Gamma, \Phi \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vDash \Omega : \Psi, \Phi.$$

Now, $\mathcal{D}'$ is a focused derivation with only a single focus, $[\mathtt{0}]$, hence this derivation must end in a rule that decomposes this formula. This is absurd, however, as there is no way to introduce a formula of the form $[\mathtt{0}]$, and from this we get the desired contradiction.

Case linear context has no foci: By inspection of the inference rules, it follows that there is only one way to derive a sequent of the form $\vDash \Omega : \Gamma$, and that is by using the `blur` rule applied to a derivation of $\vdash \Omega : \Gamma$.

Case $[P \otimes Q]$**:** Let $\mathcal{D} :: \vDash \Omega : \Psi, [P \otimes Q]$ be given. By the decomposition lemma, we get derivations

$$\mathcal{D}' :: \vDash \Omega : \Gamma, [P \otimes Q] \qquad \text{and} \qquad \vDash \Omega : \Gamma, \Phi \quad \xrightarrow{\mathcal{E}'_\Phi} \quad \vDash \Omega : \Psi, \Phi.$$

Now, $\mathcal{D}'$ is a focused derivation with only a single focus, hence by inversion it follows that there exist Γ_1 and Γ_2 such that $\Gamma_1, \Gamma_2 = \Gamma$ and the following derivations:

$$\mathcal{D}'_1 :: \vDash \Omega : \Gamma_1, [P] \qquad \text{and} \qquad \mathcal{D}'_2 :: \vDash \Omega : \Gamma_2, [Q].$$

Applying Lemma 3.2 to $\mathcal{E}'_\Phi$ and the splitting of Γ into Γ_1 and Γ_2, we get the following open derivations:

$$\vDash \Omega : \Gamma_1, \Phi \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vDash \Omega : \Psi_1, \Phi \qquad\qquad \vDash \Omega : \Gamma_2, \Phi \quad \xrightarrow{\mathcal{F}_\Phi} \quad \vDash \Omega : \Psi_2, \Phi.$$

Applying $\mathcal{E}_{[P]}$ to $\mathcal{D}'_1$ and $\mathcal{F}_{[Q]}$ to $\mathcal{D}'_2$ now yields the desired derivations.

□

Next, we prove a decomposition lemma that shows foci are not decomposed during an inversion phase.‡

Like the first decomposition lemma, this will decompose a derivation of $\vdash \Omega : \Psi, [P]$ into two parts – an open derivation that witnesses the fact that $[P]$ is kept unchanged

‡ Strictly speaking, the lemma shows that any such decomposition is never necessary, not that it cannot take place.

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throughout the inversion phase, and a set of derivations decomposing $[P]$ inside the chaining phase. Because of the additive connectives, it is necessary to work with open derivations that have multiple leaves. We will write these in an extension of the earlier syntax. Thus,

$$(\vDash \Omega_1 : \Psi_1, \Phi), \ldots, (\vDash \Omega_n : \Psi_n, \Phi) \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vdash \Omega : \Psi, \Phi.$$

is an open derivation parametric in Φ, and with the leaves listed on the left.

We may now state the lemma:

Lemma 3.4 (Inversion decomposition). If $\vdash \Omega : \Psi, [P]$, then there exists an n such that for all $i \in \{1, \ldots, n\}$:

1. there exist contexts Ω_i and Ψ_i such that $\vDash \Omega, \Omega_i : \Psi_i, [P]$ is derivable, and
2. for all Φ there exists an open derivation

$$(\vDash \Omega, \Omega_1 : \Psi_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Psi_n, \Phi) \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vdash \Omega : \Psi, \Phi.$$

Proof. The proof proceeds by induction on the derivation of $\vdash \Omega : \Psi, [P]$:

Case `sus`**:**

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash \Omega : \Psi, \langle \bar{a} \rangle, [P]\end{array}}{\vdash \Omega : \Psi, \bar{a}, [P]}\,\mathtt{sus}\,.$$

We reason as follows:

$\mathcal{D}'_i :: \vDash \Omega, \Omega_i : \Psi_i, [P]$ for $i \in \{1, \ldots, n\}$, and

$(\vDash \Omega, \Omega_1 : \Psi_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Psi_n, \Phi) \quad \xrightarrow{\mathcal{E}'_\Phi} \quad \vdash \Omega : \Psi, \langle \bar{a} \rangle, \Phi$ by i.h. on $\mathcal{D}$.

$(\vDash \Omega, \Omega_1 : \Psi_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Psi_n, \Phi) \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vdash \Omega : \Psi, \bar{a}, \Phi$

by post-composing $\mathcal{E}'_\Phi$ with `sus`.

The cases for $\uparrow$, $\bot$ and $\parr$ are similar, and have been elided.

Case $\top$**:**

$$\mathcal{D} :: \dfrac{}{\vdash \Omega : \Psi, \top, [P]}\,\top\,.$$

In this case, we let $n = 0$, and reason as follows:

$\cdot \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vdash \Omega : \Psi, \top, \Phi$ by $\top$.

Case ?**:**

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash \Omega, Q : \Psi, [P]\end{array}}{\vdash \Omega : \Psi, ?Q, [P]}\,?\,.$$

We reason as follows:

$\mathcal{D}'_i :: \vDash \Omega, \Omega_i : \Psi_i, [P]$ for $i \in \{1, \ldots, n\}$, and

$(\vDash \Omega, \Omega_1 : \Psi_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Psi_n, \Phi) \quad \xrightarrow{\mathcal{E}'_\Phi} \quad \vdash \Omega, Q : \Psi, \Phi$ by i.h. on $\mathcal{D}$.

$(\vDash \Omega, \Omega_1 : \Psi_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Psi_n, \Phi) \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vdash \Omega : \Psi, ?Q, \Phi$

by post-composing $\mathcal{E}'_\Phi$ with ?.

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Case $\&$**:**

$$\dfrac{\begin{array}{cc}\mathcal{D}^1 & \mathcal{D}^2\\ \vdash \Omega : \Psi, N, [P] & \vdash \Omega : \Psi, M, [P]\end{array}}{\vdash \Omega : \Psi, N \,\&\, M, [P]}\&.$$

We reason as follows:

$$\begin{array}{ll}\mathcal{D}_i^1 :: \vDash \Omega, \Omega_i^1 : \Psi_i^1, [P] & \text{for } i \in \{1, \ldots, n\}, \text{ and}\\ \quad (\vDash \Omega, \Omega_1^1 : \Psi_1^1, \Phi), \ldots, (\vDash \Omega, \Omega_n^1 : \Psi_n^1, \Phi) \xrightarrow{\mathcal{E}_\Phi^1} \vdash \Omega : \Psi, N, \Phi & \text{by i.h. on } \mathcal{D}^1.\\ \mathcal{D}_i^2 :: \vDash \Omega, \Omega_i^2 : \Psi_i^2, [P] & \text{for } i \in \{1, \ldots, m\}, \text{ and}\\ \quad (\vDash \Omega, \Omega_1^2 : \Psi_1^2, \Phi), \ldots, (\vDash \Omega, \Omega_n^2 : \Psi_n^2, \Phi) \xrightarrow{\mathcal{E}_\Phi^2} \vdash \Omega : \Psi, M, \Phi & \text{by i.h. on } \mathcal{D}^2.\end{array}$$

Now, $\mathcal{D}_1^1, \ldots, \mathcal{D}_n^1, \mathcal{D}_1^2, \ldots, \mathcal{D}_m^2$ are the desired derivations, and we construct the desired open derivation

$$\begin{array}{l}(\vDash \Omega, \Omega_1^1 : \Psi_1^1, \Phi), \ldots, (\vDash \Omega, \Omega_n^1 : \Psi_n^1, \Phi),\\ (\vDash \Omega, \Omega_1^2 : \Psi_1^2, \Phi), \ldots, (\vDash \Omega, \Omega_n^2 : \Psi_n^2, \Phi) \xrightarrow{\mathcal{E}_\Phi} \vdash \Omega : \Psi, N \,\&\, M, \Phi\end{array}$$

by post-composing $\mathcal{E}_\Phi^1$ and $\mathcal{E}_\Phi^2$ with the $\&$ rule.

Case `cp`**:**

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash \Omega, Q : \Psi, [Q], [P]\end{array}}{\vdash \Omega, Q : \Psi, [P]}\texttt{cp}.$$

We reason as follows:

$$\begin{array}{ll}\mathcal{D}_i' :: \vDash \Omega, \Omega_i : \Psi_i, [P] & \text{for } i \in \{1, \ldots, n\}, \text{ and}\\ \quad (\vDash \Omega, \Omega_1 : \Psi_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Psi_n, \Phi) \xrightarrow{\mathcal{E}_\Phi'} \vdash \Omega, Q : \Psi, [Q], \Phi & \text{by i.h. on } \mathcal{D}.\\ \quad (\vDash \Omega, \Omega_1 : \Psi_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Psi_n, \Phi) \xrightarrow{\mathcal{E}_\Phi} \vdash \Omega, Q : \Psi, \Phi & \text{by post-composing } \mathcal{E}_\Phi' \text{ with } \texttt{cp}.\end{array}$$

Case `foc`**:**

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vDash \Omega : \Psi, [P]\end{array}}{\vdash \Omega : \Psi, [P]}\texttt{foc}.$$

In this case, $\mathcal{D}$ is the desired derivation, and the open derivation from $\vDash \Omega : \Psi, \Phi$ to $\vdash \Omega : \Psi, \Phi$ is simply the `foc` rule.

□

By composing the preceding lemma with the first decomposition lemma, we get the following corollary:

Corollary 3.5. If $\vdash \Omega : \Psi, [P]$, then there exists an n such that for all $i \in \{1, \ldots, n\}$:

1. there exist contexts Ω_i and Γ_i such that $\vDash \Omega, \Omega_i : \Gamma_i, [P]$ is derivable, and

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2. for all Φ there exists an open derivation

$$(\vDash \Omega, \Omega_1 : \Gamma_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Gamma_n, \Phi) \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vdash \Omega : \Psi, \Phi.$$

Proof. Let $\mathcal{D}$ be the given derivation of $\vdash \Omega : \Psi, [P]$. By applying the previous lemma, there is an n such that for all $i \in \{1, \ldots, n\}$ we have derivations $\mathcal{D}_i :: \vDash \Omega, \Omega_i : \Psi_i, [P]$ and an open derivation

$$(\vDash \Omega, \Omega_1 : \Psi_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Psi_n, \Phi) \quad \xrightarrow{\mathcal{E}'_\Phi} \quad \vdash \Omega : \Psi, \Phi.$$

Applying the first decomposition lemma to each $\mathcal{D}_i$, we get $\mathcal{D}'_i :: \vDash \Omega, \Omega_i : \Gamma_i, [P]$ (these are the desired derivations), and an open derivation

$$\vDash \Omega, \Omega_i : \Gamma_i, \Phi \quad \xrightarrow{\mathcal{E}^i_\Phi} \quad \vDash \Omega, \Omega_i : \Psi_i, \Phi.$$

By pre-composing $\mathcal{E}'_\Phi$ with $\mathcal{E}^i_\Phi$ for all $i \in \{1, \ldots, n\}$, we get the desired open derivation

$$(\vDash \Omega, \Omega_1 : \Gamma_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Gamma_n, \Phi) \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vdash \Omega : \Psi, \Phi.$$

□

Note that in none of the above lemmas do we prove that the decompositions and inversion properties are well behaved with regard to the heights of the derivations involved, although this is indeed the case. One might suspect that such properties might be needed for the cut admissibility argument to go through, but this is in fact not the case. The reason we can avoid this is that the cut admissibility argument proceeds by a lexicographic induction where the size of the cut formula is the most important part. This gives us *carte blanche* to freely apply the above lemmas in cases where the size of the cut formula decreases. Of special note in this case is the principal cut for $\uparrow/\downarrow$, where the above lemmas are applied to great effect.

Theorem 3.6 (Admissibility of cut). The following rules are admissible:

$$\frac{\vDash \Omega : \Psi, [P] \qquad \vdash \Omega : \Phi, P^\perp}{\vdash \Omega : \Psi, \Phi}\texttt{cut} \qquad \frac{\vDash \Omega : \Psi, [P] \qquad \vDash \Omega : \Phi, P^\perp}{\vDash \Omega : \Psi, \Phi}\texttt{fcut}$$

$$\frac{\vdash \Omega : P^\perp \qquad \vdash \Omega, P : \Phi}{\vdash \Omega : \Phi}\texttt{cut!} \qquad \frac{\vdash \Omega : P^\perp \qquad \vDash \Omega, P : \Phi}{\vDash \Omega : \Phi}\texttt{fcut!}.$$

Proof. By lexicographic induction over

1. the structure of the formula, followed by
2. the size of the cut, defining `cut!` and `fcut!` as 'big,' and `cut` and `fcut` as 'small,' followed by
3. the structure of the second derivation.

Thus, we allow appeals to the induction hypothesis if:

1. the cut formula gets strictly smaller (in which case the size of cut and second derivation may increase), or

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2.the cut formula stays the same, but the size of cut strictly decreases (and the second derivation may increase), or
3.the cut formula and size of cut stay the same, and the second derivation strictly decreases.

When we appeal to one of the four induction hypotheses, we will do so by the name of the admissible rule above. The 'input' will be a triple $(P, \mathcal{D}, \mathcal{E})_i$ consisting of the cut formula P, first derivation $\mathcal{D}$ and second derivation $\mathcal{E}$. The subscript i will be either 1, 2 or 3 to indicate the justification for applying the induction hypothesis, corresponding to the three cases above.

We begin by considering cases based on the second premise of the `cut` rule, where the cut formula $P^\perp$ is *not* principal. As many of these cases are handled in a similar way, we have elided some of them:

Case `sus`**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \dfrac{\mathcal{E}}{\vdash \Omega : \Phi, P^\perp, \langle \bar{a} \rangle}\,\mathtt{sus} \\ \vDash \Omega : \Psi, [P] & \vdash \Omega : \Phi, P^\perp, \bar{a}\end{array}}{\vdash \Omega : \Psi, \Phi, \bar{a}}\,\mathtt{cut}\ \text{(to show)}.$$

We reason as follows:

$\vdash \Omega : \Psi, \Phi, \langle \bar{a} \rangle$ by `cut` on $(P, \mathcal{D}, \mathcal{E})_3$.
$\vdash \Omega : \Psi, \Phi, \bar{a}$ by `sus`.

Case $\&$**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \dfrac{\mathcal{E}_1 \qquad \mathcal{E}_2}{\vdash \Omega : \Phi, P^\perp, N \qquad \vdash \Omega : \Phi, P^\perp, M}\,\& \\ \vDash \Omega : \Psi, [P] & \vdash \Omega : \Phi, P^\perp, N \,\&\, M\end{array}}{\vdash \Omega : \Psi, \Phi, N \,\&\, M}\,\mathtt{cut}\ \text{(to show)}.$$

We reason as follows:

$\mathcal{E}_1' :: \vdash \Omega : \Psi, \Phi, N$ by `cut` on $(P, \mathcal{D}, \mathcal{E}_1)_3$.
$\mathcal{E}_2' :: \vdash \Omega : \Psi, \Phi, M$ by `cut` on $(P, \mathcal{D}, \mathcal{E}_2)_3$.
$\vdash \Omega : \Psi, \Phi, N \,\&\, M$ by $\&$ on $\mathcal{E}_1'$ and $\mathcal{E}_2'$.

Case ?**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \dfrac{\mathcal{E}}{\vdash \Omega, Q : \Phi, P^\perp}\,? \\ \vDash \Omega : \Psi, [P] & \vdash \Omega : \Phi, P^\perp, ?Q\end{array}}{\vdash \Omega : \Psi, \Phi, ?Q}\,\mathtt{cut}\ \text{(to show)}.$$

We reason as follows:

$\mathcal{D}' :: \vDash \Omega, Q : \Psi, [P]$ by weakening§on $\mathcal{D}$.
$\vdash \Omega, Q : \Psi, \Phi$ by `cut` on $(P, \mathcal{D}', \mathcal{E})_3$.
$\vdash \Omega : \Psi, \Phi, ?Q$ by ?.

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Case `cp`**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vDash \Omega, Q : \Psi, [P] & \dfrac{\vdash \Omega, Q : \Phi, P^{\perp}, [Q]}{\vdash \Omega, Q : \Phi, P^{\perp}}\texttt{cp}\end{array}}{\vdash \Omega, Q : \Psi, \Phi}\texttt{cut (to show)}.$$

We reason as follows:

$\vdash \Omega, Q : \Psi, \Phi, [Q]$ by `cut` on $(P, \mathcal{D}, \mathcal{E})_3$.
$\vdash \Omega, Q : \Psi, \Phi$ by `cp`.

Case `foc`**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vDash \Omega : \Psi, [P] & \dfrac{\vDash \Omega : \Phi, P^{\perp}}{\vdash \Omega : \Phi, P^{\perp}}\texttt{foc}\end{array}}{\vdash \Omega : \Psi, \Phi}\texttt{cut (to show)}.$$

We reason as follows:

$\vDash \Omega : \Psi, \Phi$ by `fcut` on $(P, \mathcal{D}, \mathcal{E})_3$.
$\vdash \Omega : \Psi, \Phi$ by `foc`.

Next, we consider the cases where $P^{\perp}$ is the principal formula of the final rule of the second premise. These are the so-called principal cuts:

Case `sus`**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vDash \Omega : \Psi, [a] & \dfrac{\vdash \Omega : \Phi, \langle \bar{a} \rangle}{\vdash \Omega : \Phi, \bar{a}}\texttt{sus}\end{array}}{\vdash \Omega : \Phi, \Psi}\texttt{cut (to show)}.$$

By the weak inversion lemma, we get $\Psi = \langle \bar{a} \rangle$, and the desired derivation is thus $\mathcal{E}$.

Case $\uparrow$**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vDash \Omega : \Psi, [\downarrow P^{\perp}] & \dfrac{\vdash \Omega : \Xi, [P]}{\vdash \Omega : \Xi, \uparrow P}\uparrow\end{array}}{\vdash \Omega : \Xi, \Psi}\texttt{cut (to show)}.$$

By the weak inversion lemma, we get a derivation $\mathcal{D}' :: \vDash \Omega : \Psi, P^{\perp}$. We now reason as follows:

$\mathcal{D}'' :: \vdash \Omega : \Gamma, P^{\perp}$ and
$\vDash \Omega : \Gamma, \Phi \xrightarrow{\mathcal{F}'_{\Phi}} \vDash \Omega : \Psi, \Phi$ by decomposition lemma on $\mathcal{D}'$.
$\mathcal{E}_i :: \vDash \Omega, \Omega_i : \Gamma_i, [P]$ for $i \in \{1, \ldots, n\}$ and
$(\vDash \Omega, \Omega_1 : \Gamma_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Gamma_n, \Phi) \xrightarrow{\mathcal{F}''_{\Phi}} \vdash \Omega : \Xi, \Phi$

[§] That weakening is admissible and preserves the height of a derivation follows immediately from the fact that all inference rules are parametric in Ω. Alternatively, it can be proven by a straightforward induction.

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by previous corollary on $\mathcal{E}$.

Now, for every $i \in \{1, \ldots, n\}$, we reason as follows

$$\begin{array}{lr} \mathcal{D}_i'' :: \vdash \Omega, \Omega_i : \Gamma, P^\perp & \text{by weakening on } \mathcal{D}. \\ \quad \vdash \Omega, \Omega_i : \Gamma, \Gamma_i & \text{by } \texttt{cut} \text{ on } (P, \mathcal{E}_i, \mathcal{D}_i'')_1. \\ \quad \vDash \Omega, \Omega_i : \Gamma, \Gamma_i & \text{by } \texttt{blur}. \\ \mathcal{G}_i \ :: \vDash \Omega, \Omega_i : \Psi, \Gamma_i & \text{by } \mathcal{F}'_{\Gamma_i}. \end{array}$$

We may now construct the desired derivation as follows:

$$\vdash \Omega : \Xi, \Psi \qquad \text{by } \mathcal{F}''_\Psi \text{ applied to } \mathcal{G}_1, \ldots, \mathcal{G}_n.$$

Case $\perp$:

$$\dfrac{\begin{array}{cc} \mathcal{D} & \dfrac{\mathcal{E}}{\vdash \Omega : \Phi}\,\perp \\ \vDash \Omega : \Psi, [1] & \vdash \Omega : \Phi, \perp \end{array}}{\vdash \Omega : \Phi, \Psi}\,\texttt{cut} \text{ (to show)}.$$

By the weak inversion lemma, we get $\Psi = \cdot$, and the desired derivation is thus $\mathcal{E}$.

Case $\parr$:

$$\dfrac{\begin{array}{cc} \mathcal{D} & \dfrac{\mathcal{E}}{\vdash \Omega : \Phi, P^\perp, Q^\perp}\,\parr \\ \vDash \Omega : \Psi, [P \otimes Q] & \vdash \Omega : \Phi, P^\perp \parr Q^\perp \end{array}}{\vdash \Omega : \Phi, \Psi}\,\texttt{cut} \text{ (to show)}.$$

By the weak inversion lemma, there exist Ψ_1 and Ψ_2 such that $\Psi_1, \Psi_2 = \Psi$ and derivations $\mathcal{D}_1 :: \vDash \Omega : \Psi_1, [P]$ and $\mathcal{D}_2 :: \vDash \Omega : \Psi_2, [Q]$. We now reason as follows:

$$\begin{array}{lr} \mathcal{F} \ :: \vdash \Omega : \Phi, \Psi_1, Q^\perp & \text{by } \texttt{cut} \text{ on } (P, \mathcal{D}_1, \mathcal{E})_1. \\ \quad \vdash \Omega : \Phi, \Psi & \text{by } \texttt{cut} \text{ on } (Q, \mathcal{D}_2, \mathcal{F})_1. \end{array}$$

Case $\top$:

$$\dfrac{\begin{array}{cc} \mathcal{D} & \dfrac{}{\vdash \Omega : \Phi, \top}\,\top \\ \vDash \Omega : \Psi, [0] & \end{array}}{\vdash \Omega : \Phi, \Psi}\,\texttt{cut} \text{ (to show)}.$$

By the weak inversion lemma on $\mathcal{D}$, this case is impossible.

Case $\&$:

$$\dfrac{\begin{array}{cc} \mathcal{D} & \dfrac{\begin{array}{cc} \mathcal{E}_1 & \mathcal{E}_2 \\ \vdash \Omega : \Phi, P_1^\perp & \vdash \Omega : \Phi, P_2^\perp \end{array}}{\vdash \Omega : \Phi, P_1^\perp \,\&\, P_2^\perp}\,\& \\ \vDash \Omega : \Psi, [P_1 \oplus P_2] & \end{array}}{\vdash \Omega : \Phi, \Psi}\,\texttt{cut} \text{ (to show)}.$$

By the weak inversion lemma, there exists a derivation $\mathcal{D}' :: \vDash \Omega : \Psi, [P_i]$ for some $i \in \{1, 2\}$. We now reason as follows:

$$\vdash \Omega : \Phi, \Psi \qquad \text{by } \texttt{cut} \text{ on } (P_i, \mathcal{D}', \mathcal{E}_i)_1.$$

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Case ?**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vDash \Omega : \Psi, [!P^{\perp}] & \dfrac{\vdash \Omega, P \ : \Phi}{\vdash \Omega : \Phi, ?P}?\end{array}}{\vdash \Omega : \Phi, \Psi}\texttt{cut (to show)}.$$

By the weak inversion lemma, $\Psi = \cdot$, and there exists a derivation $\mathcal{D}' :: \vdash \Omega : P^{\perp}$. We now reason as follows:

$\vdash \Omega : \Phi$ by `cut!` on $(P, \mathcal{D}', \mathcal{E})_1$.

This completes the cases for the `cut` rule. Next, we consider cases on the second premise of the `fcut` rule:

Case `ax`**:** This case is impossible, as the context $\langle \bar{a} \rangle, [a]$ contains no negative formulas.

Case $\downarrow$**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vDash \Omega : \Psi, [P] & \dfrac{\vDash \Omega : \Phi, N, P^{\perp}}{\vDash \Omega : \Phi, [\downarrow N], P^{\perp}}\downarrow\end{array}}{\vDash \Omega : \Psi, \Phi, [\downarrow N]}\texttt{fcut (to show)}..$$

We reason as follows:

$\vDash \Omega : \Psi, \Phi, N$ by `fcut` on $(P, \mathcal{D}, \mathcal{E})_3$.
$\vDash \Omega : \Psi, \Phi, [\downarrow N]$ by $\downarrow$.

Case 1: Impossible

Case $\otimes$**,** $P^{\perp}$ **in linear context of first premise:**

$$\dfrac{\begin{array}{ccc}\mathcal{D} & \begin{array}{c}\mathcal{E}_1 \\ \vDash \Omega : \Phi_1, [P_1], P^{\perp}\end{array} & \begin{array}{c}\mathcal{E}_2 \\ \vDash \Omega : \Phi_2, [P_2]\end{array} \\ \vDash \Omega : \Psi, [P] & \multicolumn{2}{c}{\dfrac{}{\vDash \Omega : \Phi_1, \Phi_2, [P_1 \otimes P_2], P^{\perp}}\otimes}\end{array}}{\vDash \Omega : \Psi, \Phi_1, \Phi_2, [P_1 \otimes P_2]}\texttt{fcut (to show).}$$

We reason as follows:

$\mathcal{F}_1 :: \vDash \Omega : \Psi, \Phi_1, [P_1]$ by `fcut` on $(P, \mathcal{D}, \mathcal{E}_1)_3$.
$\vDash \Omega : \Psi, \Phi_1, \Phi_2, [P_1 \otimes P_2]$ by $\otimes$ on $\mathcal{F}_1$ and $\mathcal{E}_2$.

The case where $P^{\perp}$ is in the linear context of the second premise is analogous.

Case $\oplus_i$**,** $i \in \{1, 2\}$**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \begin{array}{c}\mathcal{E} \\ \vDash \Omega : \Phi, [P_i], P^{\perp}\end{array} \\ \vDash \Omega : \Psi, [P] & \dfrac{}{\vDash \Omega : \Phi, [P_1 \oplus P_2], P^{\perp}}\oplus_i\end{array}}{\vDash \Omega : \Psi, \Phi, [P_1 \oplus P_2]}\texttt{fcut (to show).}$$

We reason as follows:

$\vDash \Omega : \Psi, \Phi, [P_i]$ by `fcut` on $(P, \mathcal{D}, \mathcal{E})_3$.
$\vDash \Omega : \Psi, \Phi, [P_1 \oplus P_2]$ by $\oplus_i$.

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Case !**:** Impossible.

Case `blur`**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vDash \Omega : \Psi, [P] & \dfrac{\vdash \Omega : \Gamma, P^{\perp}}{\vDash \Omega : \Gamma, P^{\perp}}\texttt{blur}\end{array}}{\vDash \Omega : \Psi, \Gamma}\texttt{fcut (to show).}$$

We reason as follows:

$$\begin{array}{lll}\mathcal{D}' :: \vDash \Omega : \Gamma', [P] & \text{and} & \\ \vDash \Omega : \Gamma', \Phi \quad \xrightarrow{\mathcal{F}_{\Phi}} & \vDash \Omega : \Psi, \Phi & \text{by decomposition on } \mathcal{D}. \\ \vdash \Omega : \Gamma', \Gamma & & \text{by cut on } (P, \mathcal{D}', \mathcal{E})_3. \\ \vDash \Omega : \Gamma', \Gamma & & \text{by blur.} \\ \vDash \Omega : \Psi, \Gamma & & \text{by } \mathcal{F}_{\Gamma}.\end{array}$$

This takes care of the cases for the `fcut` rule. Next, we look at the cases for the `cut!` rule. The cases where some formula in the linear context of the second premise gets decomposed are all handled in the same way, and hence we only show a few cases here:

Case `sus`**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vdash \Omega : P^{\perp} & \dfrac{\vdash \Omega, P : \Phi, \langle \bar{a} \rangle}{\vdash \Omega, P : \Phi, \bar{a}}\texttt{sus}\end{array}}{\vdash \Omega : \Phi, \bar{a}}\texttt{cut!}\ \text{(to show).}$$

We reason as follows:

$$\begin{array}{ll}\vdash \Omega : \Phi, \langle \bar{a} \rangle & \text{by cut! on } (P, \mathcal{D}, \mathcal{E})_3. \\ \vdash \Omega : \Phi, \bar{a} & \text{by sus.}\end{array}$$

Case $\&$**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \begin{array}{cc}\mathcal{E}_1 & \mathcal{E}_2 \\ \vdash \Omega, P : \Phi, N & \vdash \Omega, P : \Phi, M\end{array} \\ \vdash \Omega : P^{\perp} & \dfrac{\qquad\qquad}{\vdash \Omega, P : \Phi, N \,\&\, M}\,\&\end{array}}{\vdash \Omega : \Phi, N \,\&\, M}\texttt{cut!}\ \text{(to show).}$$

We reason as follows:

$$\begin{array}{lll}\mathcal{E}_1' :: \vdash \Omega : \Phi, N & & \text{by cut! on } (P, \mathcal{D}, \mathcal{E}_1)_3. \\ \mathcal{E}_2' :: \vdash \Omega : \Phi, M & & \text{by cut! on } (P, \mathcal{D}, \mathcal{E}_2)_3. \\ \vdash \Omega : \Phi, N \,\&\, M & & \text{by } \& \text{ on } \mathcal{E}_1' \text{ and } \mathcal{E}_2'.\end{array}$$

Case ?**:**

$$\dfrac{\begin{array}{cc}\mathcal{D} & \mathcal{E} \\ \vdash \Omega : P^{\perp} & \dfrac{\vdash \Omega, Q, P : \Phi}{\vdash \Omega, P : \Phi, ?Q}?\end{array}}{\vdash \Omega : \Phi, ?Q}\texttt{cut!}\ \text{(to show).}$$

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We reason as follows:

$\mathcal{D}' :: \vdash \Omega, Q : P^\perp$ by weakening on $\mathcal{D}$.
$\vdash \Omega, Q : \Phi$ by `cut!` on $(P, \mathcal{D}', \mathcal{E})_3$.
$\vdash \Omega : \Phi, ?Q$ by ?.

Case `cp`**,** P **not principal:**

$$\dfrac{\mathcal{D} \quad \vdash \Omega, Q : P^\perp \qquad \dfrac{\mathcal{E} \quad \vdash \Omega, Q, P : \Phi, [Q]}{\vdash \Omega, Q, P : \Phi}\texttt{cp}}{\vdash \Omega, Q : \Phi}\texttt{cut!}\ \text{(to show).}$$

We reason as follows:

$\vdash \Omega, Q : \Phi, [Q]$ by `cut!` on $(P, \mathcal{D}, \mathcal{E})_3$.
$\vdash \Omega, Q : \Phi$ by `cp`.

Case `cp`**,** P **principal:**

$$\dfrac{\mathcal{D} \quad \vdash \Omega : P^\perp \qquad \dfrac{\mathcal{E} \quad \vdash \Omega, P : \Psi, [P]}{\vdash \Omega, P : \Psi}\texttt{cp}}{\vdash \Omega : \Psi}\texttt{cut!}\ \text{(to show).}$$

We reason as follows:

$\mathcal{E}' :: \vdash \Omega : \Psi, [P]$ by `cut!` on $(P, \mathcal{D}, \mathcal{E})_3$.
$\mathcal{E}'_i :: \vDash \Omega, \Omega_i : \Psi_i, [P]$ for $i \in \{1, \ldots, n\}$ and
$(\vDash \Omega, \Omega_1 : \Psi_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Psi_n, \Phi) \xrightarrow{\mathcal{F}_\Phi} \vdash \Omega : \Psi, \Phi$
by decomposition on $\mathcal{E}'$.
$\mathcal{D}_i :: \vdash \Omega, \Omega_i : P^\perp$ by weakening on $\mathcal{D}$.
$\mathcal{G}_i :: \vdash \Omega, \Omega_i : \Psi_i$ by `cut` on $(P, \mathcal{E}'_i, \mathcal{D}_i)_2$.
$\vdash \Omega : \Psi$ by $\mathcal{F}_\cdot$ applied to $\mathcal{G}_1, \ldots, \mathcal{G}_n$.

Recall that each secondary cut is justified because we consider `cut!` to be a 'bigger' cut than `cut`.

Case `foc`**:**

$$\dfrac{\mathcal{D} \quad \vdash \Omega : P^\perp \qquad \dfrac{\mathcal{E} \quad \vDash \Omega, P : \Phi}{\vdash \Omega, P : \Phi}\texttt{foc}}{\vdash \Omega : \Phi}\texttt{cut!}\ \text{(to show).}$$

We reason as follows:

$\vDash \Omega : \Phi$ by `fcut!` on $(P, \mathcal{D}, \mathcal{E})_3$.
$\vdash \Omega : \Phi$ by `foc`.

Finally, we come to the cases for `fcut!`. Most of these cases are completely straightforward, and have therefore been elided.

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Case `ax`**:**

$$\dfrac{\dfrac{\mathcal{D}}{\vdash \Omega : P^{\perp}} \qquad \dfrac{}{\vDash \Omega, P : \langle \bar{a} \rangle, [a]}\,\mathtt{ax}}{\vDash \Omega : \langle \bar{a} \rangle, [a]}\,\mathtt{fcut!}\ \text{(to show).}$$

We reason as follows:

$\vDash \Omega : \langle \bar{a} \rangle, [a]$ by `ax`.

Case $\downarrow$**:**

$$\dfrac{\dfrac{\mathcal{D}}{\vdash \Omega : P^{\perp}} \qquad \dfrac{\dfrac{\mathcal{E}}{\vDash \Omega, P : \Phi, N}}{\vDash \Omega, P : \Phi, [\downarrow N]}\downarrow}{\vDash \Omega : \Phi, [\downarrow N]}\,\mathtt{fcut!}\ \text{(to show).}$$

We reason as follows:

$\vDash \Omega : \Phi, N$ by `fcut!` on $(P, \mathcal{D}, \mathcal{E})_3$.
$\vDash \Omega : \Phi, [\downarrow N]$ by $\downarrow$.

Case $\otimes$**:**

$$\dfrac{\dfrac{\mathcal{D}}{\vdash \Omega : P^{\perp}} \qquad \dfrac{\dfrac{\mathcal{E}_1}{\vDash \Omega, P : \Phi_1, [P_1]} \qquad \dfrac{\mathcal{E}_2}{\vDash \Omega, P : \Phi_2, [P_2]}}{\vDash \Omega, P : \Phi_1, \Phi_2, [P_1 \otimes P_2]}\otimes}{\vDash \Omega : \Phi_1, \Phi_2, [P_1 \otimes P_2]}\,\mathtt{fcut!}\ \text{(to show).}$$

We reason as follows:

$\mathcal{F}_1 :: \vDash \Omega : \Phi_1, [P_1]$ by `fcut!` on $(P, \mathcal{D}, \mathcal{E}_1)_3$.
$\mathcal{F}_2 :: \vDash \Omega : \Phi_2, [P_2]$ by `fcut!` on $(P, \mathcal{D}, \mathcal{E}_2)_3$.
$\vDash \Omega : \Phi_1, \Phi_2, [P_1 \otimes P_2]$ by $\otimes$ on $\mathcal{F}_1$ and $\mathcal{F}_2$.

Case `blur`**:**

$$\dfrac{\dfrac{\mathcal{D}}{\vdash \Omega : P^{\perp}} \qquad \dfrac{\dfrac{\mathcal{E}}{\vdash \Omega, P : \Gamma}}{\vDash \Omega, P : \Gamma}\,\mathtt{blur}}{\vDash \Omega : \Gamma}\,\mathtt{fcut!}\ \text{(to show).}$$

We reason as follows:

$\vdash \Omega : \Gamma$ by `cut!` on $(P, \mathcal{D}, \mathcal{E})_3$.
$\vDash \Omega : \Gamma$ by `blur`.

This completes the proof. □

With this presentation of the cut elimination argument, we have attempted to strike a balance between elegance and intuition. The use of the `fcut` and `fcut!` rules – which simply propagate the cut to the next phase – is somewhat inelegant, and one might consider simply always moving instances of `cut` to the next principal case through the use of appropriate decomposition lemmas. This would result in a principal case for $\uparrow/\downarrow$

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where we would reduce

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdash \Omega, \Omega'_1 : \Phi_1, P^\perp\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}_n\\ \vdash \Omega, \Omega'_n : \Phi_n, P^\perp\end{array}}{\begin{array}{c}\ddots\vdots\iddots\mathcal{D}_{P^\perp}\\ \vdash \Omega : \Phi, P^\perp\end{array}}\downarrow \qquad \dfrac{\begin{array}{c}\mathcal{E}_1\\ \vdash \Omega, \Omega_1 : \Gamma_1, [P]\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{E}_m\\ \vdash \Omega, \Omega_m : \Gamma_m, [P]\end{array}}{\begin{array}{c}\ddots\vdots\iddots\mathcal{E}_{[P]}\\ \vdash \Omega : \Psi, [P]\end{array}}\uparrow}{\vdash \Omega : \Phi, \Psi}\,\mathtt{cut}$$

to

$$\left(\left(\dfrac{\begin{array}{c}\mathcal{E}_j\\ \vdash \Omega, \Omega_j : \Gamma_j, [P]\end{array} \qquad \begin{array}{c}\mathcal{D}_i\\ \vdash \Omega, \Omega'_i : \Phi_i, P^\perp\end{array}}{\vdash \Omega, \Omega_j, \Omega'_i : \Gamma_j, \Phi_i}\,\mathtt{cut}\right)_{i=1,\ldots,n} \begin{array}{c}\ddots\vdots\iddots\mathcal{D}_{\Gamma_j}\\ \vdash \Omega, \Omega_j : \Phi, \Gamma_j\end{array}\right)_{j=1,\ldots,m} \begin{array}{c}\ddots\vdots\iddots\mathcal{E}_\Phi\\ \vdash \Omega : \Phi, \Psi\end{array}.$$

This is already somewhat excessive, however, although it does shed some light on how foci are re-distributed after the reduction of the cut. Moreover, the corresponding attempt at avoiding the `fcut!` rule would be even more complex.

4. Design choices in multi-focused proof systems

The particular design of the inference rules of the LLM system leads to a relatively simple proof of cut elimination, as described in the previous section. But different choices could have been made, with various impacts on the cut elimination result. We discuss here the most important design choices, and show how they are tied to the kind of syntactic procedure we presented in the previous section.

4.1. *Selection and release of foci*

Perhaps the most important novelty of our system is the incremental treatment of the focus selection mechanism. It is obvious that instances of the selection rules $\uparrow$ and cp can be grouped in a single rule, which can then also proceed with the phase change marked by the switch from $\vdash$ to $\models$ in LLM. In the non-exponential fragment, the resulting rule can be written as follows:

$$\dfrac{\models \Gamma, [\Omega]}{\vdash \Gamma, \uparrow\Omega},$$

where $\uparrow\Omega$ denotes some context containing only positive formulas under a shift, therefore having the shape $\uparrow P_1, \ldots, \uparrow P_n$, whereas $[\Omega]$ contains only focused positive formulas, which in that case will correspond to $[P_1], \ldots, [P_n]$. If this approach is chosen, there should also be a grouped rule for releasing negative formulas obtained by decomposing a positive

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and wrapped under a shift. The resulting rule can also change phase, as follows:

$$\frac{\vdash \Gamma, \Sigma}{\vDash \Gamma, [\downarrow \Sigma]},$$

where Σ denotes a context containing only negative formulas. Although these rules seem simpler than the scheme used in LLM, and are obviously closer to the original presentation of multi-focalisation (Chaudhuri et al. 2008a), they are problematic for the cut elimination argument because of their rigidity.

The problem stems from the handling of the persistent assumptions. When selecting the foci that will be decomposed during the next multi-chaining phase, it is necessary to allow foci to be selected from the persistent context. Furthermore, if one is interested in *maximally* multi-focused proofs, then such a selection must allow for any number of copies of formulas in the persistent context to be focused at the same time. The resulting rule thus needs to have the following form:

$$\frac{\vDash \Omega : \Gamma, [\Omega^{\vec{n}}], [\Omega']}{\vdash \Omega : \Gamma, \uparrow \Omega'},$$

where $\vec{n}$ is a vector of natural numbers indicating how many copies of each formula in Ω are needed. For a cut elimination argument based on such a calculus, the proof is not straightforward. We direct the reader to Brock-Nannestad and Guenot (2015a) for an example of this. The main innovation of the present paper is the decoupling of the rule for initiating focus from the rule for selecting the multiple foci. With this slight change, the proof of cut elimination – and of the lemmas on which this result rests – becomes immediate, while still maintaining the firm distinction between positive and negative phases that is the hallmark characteristic of a focused calculus.

4.2. *Suspended atomic formulas*

The LLM system depends on having a notion of suspended atoms, which are not part of most focused systems found in the literature, but in our case they play an important role. A very straightforward argument why this device should be part of the system is that the admissibility of cut fails in its absence. Indeed, in the statement of Lemma 3.1, it is crucial that A must be an active formula, as it fails to be true if A is a suspended atom. To see this, simply consider the sequent $\vDash \cdot ; [a], \langle \overline{a} \rangle$, with $A = \langle \overline{a} \rangle$.

If instead we removed the sus rule, and had the usual rule for ax,

$$\frac{}{\vDash \Omega : \overline{a}, [a]}\,\mathtt{ax}$$

the system would fail to have the cut elimination property, as we could then form the following cut:

$$\dfrac{\dfrac{\dfrac{}{\vDash \cdot : [a], \overline{a}}\,\mathtt{ax}}{\vDash \cdot : [a], [\downarrow \overline{a}]}\downarrow \qquad \begin{array}{c}\vdots\\ \vdash \cdot : \overline{a}, [\downarrow \bot], \uparrow a\end{array}}{\vdash \cdot : [a], \overline{a}, [\downarrow \bot]}\,\mathtt{cut}.$$

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It is easy to show that the second premise of the above cut is derivable in this modified system, but that the conclusion is not. Indeed, an atomic focus such as $[a]$ cannot be released: The initial rule must be applied within the same multi-chaining phase, which contradicts the need to release the other focus $[\downarrow\bot]$. Note that the system would still be sound and complete with respect to unfocused linear logic, but would fail to be *internally* sound since cuts could be used to build improperly focused proofs.

Finally, the use of atomic suspensions makes the `LLM` calculus more symmetrical, as in most presentations of cut elimination for linear logic, the principal cases match up connectives with their duals. This view, however, is lost when it comes to atoms, because there is no inference rule which introduces the dual of an atomic formula, and therefore in the cut elimination argument, atomic cuts can be reduced at any time. Notice that this is not the case in our system: The `ax` rule is decomposed – in the principal case – exactly when it meets up with the `sus` rule.

4.3. *Maximal inversion and strongly focused proofs*

Beyond the grouping of focus selection and release, designing a calculus closer to the usual systems from the literature on focalisation raises the question of maximal inversion. Indeed, most focused calculi present so-called *strong* focusing, where negative connectives are decomposed fully before any chaining phase may be initiated. We have instead used a *weakly* focused calculus, for the two reasons that we will outline now. The first reason is technical. In principle it is easy to make our calculus strongly focused by adding a side condition on the rule:

$$\frac{\vDash \Omega : \Psi}{\vdash \Omega : \Psi}\,\mathtt{foc}\,,$$

stating that the context Ψ may only contain formulas of the shape $\langle \overline{a} \rangle$, $\uparrow P$ or $[P]$. The trouble with this solution is that it breaks our cut elimination argument. Consider the case where a cut permutes across a `foc` instance:

$$\frac{\mathcal{D} \quad \vDash \Omega : \Psi, [P] \qquad \dfrac{\mathcal{E} \quad \vDash \Omega : \Phi, P^{\perp}}{\vdash \Omega : \Phi, P^{\perp}}\,\mathtt{foc}}{\vdash \Omega : \Psi, \Phi}\,\mathtt{cut}\,,$$

where in a strongly focused setting, Φ must not contain any decomposable negative formulas. We would like to reduce this to

$$\frac{\dfrac{\mathcal{D} \quad \vDash \Omega : \Psi, [P] \qquad \mathcal{E} \quad \vDash \Omega : \Phi, P^{\perp}}{\vDash \Omega : \Psi, \Phi}\,\mathtt{fcut}}{\vdash \Omega : \Psi, \Phi}\,\mathtt{foc?}\,,$$

but we have no guarantee that Ψ does not contain invertible formulas, and hence it is not clear that we can actually apply the `foc` rule. This mirrors the corresponding cut where we permute above the `blur` rule, through a judicious use of the decomposition lemmas. Therefore, there is a problem in ensuring that inversion phases are still maximal after cut elimination. For intuitionistic calculi, this has been studied in Chaudhuri et al. (2008b)

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and Simmons (2014), and more recently for the multiplicative-exponential fragment of linear logic in Brock-Nannestad and Guenot (2015b), however, none of these systems deal with the question of having multiple foci. Beyond this technical difficulty, there is a more fundamental reason to choose a weakly focused calculus. Indeed, strongly focused proofs are in general somewhat lopsided. Inside such a proof, there is an elegant back-and-forth where a focused formula $[\downarrow N]$ gives rise to a negative formula N that is decomposed fully before the next positive formula is focused upon. But at the very bottom of the proof, there is a great flurry of activity where all the negatives present in the end sequent are decomposed fully before the first chaining phase begins. Thus, a negative formula may be decomposed long before its sub-formulas actually are chosen as foci.

This should lead one to reconsider the standard treatment of negatives in focused calculi, and possibly even suggests that negatives should be decomposed as late as possible rather than the opposite. Finally, one could also consider a middle ground in the discussion on maximal inversion of negative connectives, by requiring that not all available negatives are decomposed in an inversion phase, but rather that a selected negative should be decomposed entirely. This amounts to a transfer of the chaining requirement on positives to the realm of negatives, making the system more symmetric while staying obviously complete – since it is less restrictive than the usual strongly focused approach. However, as in the case of the strongly focused systems, it seems this significantly complicates the proof of cut elimination.

In the absence of a clear answer to the question of how to handle the negative connectives, we have decided to remain agnostic, and work with a weakly focused system instead.

4.4. *Soundness*

Soundness comes in many shapes and sizes. At a very basic level, there is the notion of 'external' soundness, i.e. whether the judgments in one system can be faithfully interpreted as belonging to some other, hopefully well-known system. As we noted previously, showing that LLM is sound in this way with regard to unfocused LL is a straightforward matter.

At the same time, there is the notion of 'internal' soundness. This is usually seen as the question of whether the system has the kind of metatheoretical properties that one usually expects. Chief among these properties is that of *cut admissibility*, which was the subject of the majority of this paper. However, there is a third variation on the theme of soundness that is worth mentioning in our case. We argue that a truly multi-focused system is more expressive than a mono-focused system because it allows the proof objects to *intrinsically* encode the fact that certain chaining phases may permute past each other. In small examples, this is easy to see, but to show this soundness in general, we need to show that any multi-chaining phase can be decomposed – or rather, serialised – into a sequence of mono-focused phases, and that these many phases can be ordered in any way one wishes.

We will now sketch how this may be proved by yet another use of the decomposition lemmas. Translating a multi-focused proof to a mono-focused proof can be done in two simple steps:

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1. First, we note that any LLM proof can be transformed into a proof of the same end sequent in which no sequent contains more than a single focused formula. Intuitively, this 'degenerate' multi-focused proof should be seen as essentially mono-focused.
2. Secondly, we show that the degenerate proofs can indeed be translated into proofs in focused LL.

Of course, not all LLM proofs have mono-focused counterparts. A sequent of the form $\vDash \Omega : \Psi, [P], [Q]$ cannot even be represented in a mono-focused setting, and hence it is only a subset of the proofs that are translatable. In practice, however, this is not a problem. As long as we only consider translating proofs of judgments that contain *no* foci, this is sufficient.

There are two rules that allow us to pick multiple foci, the $\uparrow$ rule and the cp rule. What we will now show is that these rules can be made admissible in a way that is compatible with having at most one focus at any given point in the proof.

Consider a derivation $\mathcal{D}$ of a sequent of the form $\vdash \Omega : \Psi, [P]$ where Ψ contains at least one focus. By the decomposition corollary, we know that there exists a number of derivations $\mathcal{D}_i$, where $i = 1, \ldots, n$ such that

$$\mathcal{D}_i :: \vDash \Omega, \Omega_i : \Gamma_i, [P],$$

and an open derivation $\mathcal{E}_\Phi$

$$(\vDash \Omega, \Omega_1 : \Gamma_1, \Phi), \ldots, (\vDash \Omega, \Omega_n : \Gamma_n, \Phi) \quad \xrightarrow{\mathcal{E}_\Phi} \quad \vdash \Omega : \Psi, \Phi.$$

For each i, we now construct the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_i}{\vDash \Omega, \Omega_i : \Gamma_i, [P]}\mathtt{foc}}{\vdash \Omega, \Omega_i : \Gamma_i, [P]}\uparrow}{\vdash \Omega, \Omega_i : \Gamma_i, \uparrow P}\mathtt{blur}},$$

and composing these with $\mathcal{E}_{\uparrow P}$ yields a derivation of $\vdash \Omega : \Psi, \uparrow P$.

Note that the bottommost phase of $\mathcal{D}_i$ contains only the focus $[P]$, since Γ_i contains only negative formulas.

Essentially, this (and a similar argument for cp) shows that the following inference rules (which are identical to the usual $\uparrow$ and cp rules) are admissible *without* needing multiple foci present at the same time:

$$\dfrac{\vdash \Omega : \Psi, [P]}{\vdash \Omega : \Psi, \uparrow P}\uparrow \qquad \dfrac{\vdash \Omega, P : \Psi, [P]}{\vdash \Omega, P : \Psi}\mathtt{cp}.$$

Using these admissibility results, we can turn any multi-focused proof into one that never has more than one focus present at a time. We now observe that all uses of cp and $\uparrow$ in such a proof can be permuted up to the corresponding foc rule of the given phase. Similarly, whenever we have an instance of the $\downarrow$ rule, the rule above it must be the blur rule (since the context contains at most one focus). Thus, we can now translate these to the mono-focused system as follows:

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$$\dfrac{\dfrac{\vdash \Omega : \Gamma, N}{\models \Omega : \Gamma, N}\mathtt{blur}}{\models \Omega : \Gamma, [\downarrow N]}\downarrow \quad \rightsquigarrow \quad \dfrac{\vdash \Omega : \Gamma, N}{\models \Omega : \Gamma, [\downarrow N]}\downarrow$$

$$\dfrac{\dfrac{\models \Omega : \Gamma, [P]}{\vdash \Omega : \Gamma, [P]}\mathtt{foc}}{\vdash \Omega : \Gamma, \uparrow P}\uparrow \quad \rightsquigarrow \quad \dfrac{\models \Omega : \Gamma, [P]}{\vdash \Omega : \Gamma, \uparrow P}\uparrow$$

$$\dfrac{\dfrac{\models \Omega, P\ : \Gamma, [P]}{\vdash \Omega, P\ : \Gamma, [P]}\mathtt{foc}}{\vdash \Omega, P\ : \Gamma}\mathtt{cp} \quad \rightsquigarrow \quad \dfrac{\models \Omega, P\ : \Gamma, [P]}{\vdash \Omega, P\ : \Gamma}\mathtt{cp}.$$

5. Further developments of focalisation

The sequent calculus presented here, along with its proof of cut elimination, illustrates the benefits of the focusing methodology, and of its generalisation to multi-focusing, beyond the usual scope of proof search. Indeed, the associated restrictions on the structure of proofs guide the reductions induced by cut elimination: The proof and its computational contents are more intelligible when a large part of the commutative cases are avoided. To a certain extent, multi-focalisation is an alternative route towards canonicity, less obvious but more robust so far than proof-nets (Girard 1996). There are essentially two necessary lines of investigation based on our current understanding of multi-focalisation.

First, there is the question of the nature of multi-focalisation. Although it appeared as a mere artifact of the linear sequent calculus, the original notion of focused proof turned out to be strongly rooted in the pervasive notion of polarity, and applies to many different logics. Moreover, it is not dependent on the structure of the sequent calculus and it can be expressed in some other proof formalisms, such as natural deduction in the intuitionistic variant of linear logic (Brock-Nannestad and Schürmann 2010). Formalisms based upon the deep inference methodology, despite their differences with sequent calculi, exhibit the focalisation result in a naturally incremental way, as studied in the calculus of structures (Chaudhuri et al. 2011). When phases can be isolated, the question of permuting phases themselves yields a notion of multi-focalisation that could be studied and compared to the notion obtained in the linear sequent calculus: Understanding the mechanisms at work in multi-focused proofs thus requires adapting it to more proof representations. Interestingly enough, focalisation is itself a unifying mechanism since it provides the expressivity that we need to encode proof systems in the sequent calculus (Miller and Nigam 2010).

Then, multi-focalisation introduces some notion of degree of canonicity where the mono-focalisation notion was only drawing a border between focused and unfocused. The whole spectrum of permutability and chaining properties should therefore be explored within a common framework: We have left aside here the maximal inversion of negative connectives usually performed in focused systems, and we believe that the *strongly focused*

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approach is detrimental to the proof of cut elimination. But then, inversion is either left uncontrolled and essentially lazy, or it has to be harnessed in a more subtle way – for example, we can chain the decomposition of negatives, as mentioned in Section 4, but this yields questions concerning cut elimination. Further away in the spectrum of canonicity, the problem of a maximisation of multi-focusing is a challenge from two viewpoints: Characterising maximal groupings of chaining phases is difficult, and showing that there exists a cut elimination procedure preserving this maximality seems also to require a complex, global approach to proof objects. However, this is necessary if we want to use focalisation as the foundation for canonicity in the same way as one can use proof-nets. This objective has been partially reached (Chaudhuri et al. 2008a), but only in a very restricted fragment of linear logic. An extension to full linear logic, or to other logics such as intuitionistic and classical logics, is the next natural goal – although this still leaves open the question of maintaining maximality through cut elimination.

Beyond the purely proof-theoretic notion of canonicity, a more practical aspect of the study of multi-focused proofs and of their cut elimination procedure is of course the computational interpretation we would assign to proofs, with the purpose of representing programs or designing type systems based on logical foundations. Just as proof-nets have proved useful to understand computation (Guerrini et al. 2001; Laurent 2003), focused proofs provide us with insights on computation even when considering logics not as well understood as intuitionistic logic (Brock-Nannestad and Guenot 2015b). It might even be a methodology of choice to understand complex logics such as those involving fixpoints for induction and co-induction (Baelde 2012). The generality of the two phenomena of polarisation and permutation is such that multi-focalisation offers a unique opportunity to establish a general, elegant theory of proof objects that would form solid foundations for a large class of correspondences in the tradition of the Curry–Howard methodology.

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