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A problem of Ulam about magnetic fields generated by knotted wires

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Abstract. In the context of magnetic fields generated by wires, we study the connection between the topology of the wire and the topology of the magnetic lines. We show that a generic knotted wire has a magnetic line of the same knot type, but that given any pair of knots there is a wire isotopic to the first knot having a magnetic line isotopic to the second. These questions can be traced back to Ulam in 1935.

1. *Introduction*

It is classical that the magnetic field generated by a closed wire is given by the Biot–Savart law. That is, if $\gamma(s)$ is a closed curve in $\mathbb{R}^3$ of length ℓ and parametrized, say, by the arc-length parameter, its associated magnetic field is given by the Biot–Savart integral

$$B_\gamma(x) := \int_0^\ell \frac{\dot\gamma(s) \times (x - \gamma(s))}{4\pi |x - \gamma(s)|^3}\, ds.$$

It is apparent that the vector field B_γ, which is divergence-free, is in fact independent of the parametrization of the curve, although it does depend on its orientation. As is well known, the magnetic field $B_\gamma(x)$ diverges as the inverse of the distance to γ as x gets closer to the wire.

While the magnetic fields created by simple curves, such as a circular wire, are well understood and have been utilized to describe a wide range of physical phenomena, our understanding of the magnetic fields created by more complicated wires remains strikingly limited in spite of their manifold applications in engineering [**8**]. This led Ulam to consider the relationship between the degree of knottedness of the wire and that of the associated periodic magnetic lines. This should not be regarded as a peripheral problem, but rather as a concrete attempt to develop new ideas to analyze the structure of the magnetic lines created by wires.

Specifically, Ulam posed the question of whether the magnetic field created by a knotted wire must have a periodic magnetic line of non-trivial topology. This is among the first problems of *The Scottish Book* [**11**, Problem 18], the original manuscript of which goes back to 1935. This question of Ulam was set in a broader context in his later problem

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collection [**15**, §VII.7], where he asks whether the magnetic lines topologically reflect the knottedness of the wire.

Unlike many of the problems collected in *The Scottish Book*, the progress made on these questions has been scarce. This is particularly remarkable, on the one hand, because the existence of knotted fields plays a significant role in a number of areas of physics (see, for example, [**3**, **10**, **14**]) and has been addressed in recent research on the Euler equations [**4**, **5**] that goes back to conjectures of Arnold [**1**] and Lord Kelvin [**13**], and on the other hand, because it has been recently proposed to employ knotted wires to construct a new magnetic confinement device, called the knotatron [**9**]. The reason for the lack of rigorous contributions to this line of research is probably that, by the very nature of the problems, the proof of these results must combine ideas from the theory of dynamical systems, which are key to studying the knot types of the magnetic lines, with fine analytic estimates for the Biot–Savart integral associated with a curve of complicated geometry.

It is worth emphasizing that these techniques are completely different from those employed in [**4**, **5**]. Indeed, in these references we construct Beltrami fields on $\mathbb{R}^3$, i.e., analytic vector fields satisfying $\operatorname{curl} w = \lambda w$ for some non-zero constant λ, which have integral curves and invariant tori of arbitrary topology. This relies on fine partial differential equation (PDE) estimates, KAM theory and a Runge-type theorem. These techniques are of little use in the present context, which is not surprising given that here one does not need to derive estimates for a PDE (because the magnetic field is explicitly given by the Biot–Savart law) and invariant tori do not play a role in our proof. Notice, furthermore, that the magnetic field B_γ is not everywhere smooth, as it becomes singular on the curve γ, and that Runge-type theorems cannot be applied to irrotational fields such as B_γ.

Our objective in this paper is to provide two complementary results concerning the relationship between the knottedness of the wire and the topology of the magnetic lines. The first result asserts that there are wires and magnetic lines whose knot types can be prescribed at will (and independently). In particular, there are wires knotted in an arbitrarily complicated way with an unknotted periodic magnetic line and there are wires isotopic to the unknot with a magnetic line isotopic to any given knot. More precisely, our first result can be stated as follows.

THEOREM 1.1. *Let γ and $\widetilde{\gamma}$ be any two closed curves in $\mathbb{R}^3$. Then there is a wire isotopic to γ whose associated magnetic field has a periodic magnetic line isotopic to $\widetilde{\gamma}$.*

Our second result asserts, however, that Ulam's original question can be answered in the affirmative at least for C^0-generic curves. That is, given any curve in the space, there is a smooth C^0-small deformation of it which is isotopic to the original curve and has a magnetic line of the same knot type. To state this result, we will say that two isotopic curves are *arbitrarily C^k-close* if the isotopy can be taken arbitrarily close to the identity in the C^k norm.

THEOREM 1.2. *Let γ be a closed curve in $\mathbb{R}^3$. Given any integer k, there is a wire isotopic to γ and arbitrarily C^0-close to it whose associated magnetic field has a periodic magnetic line isotopic to γ and arbitrarily C^k-close to it.*

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It is worth emphasizing that the magnetic lines constructed in the above theorems are not just a mathematical curiosity, but should be observable in actual experiments. This is because they are structurally stable, so any C^1 small perturbation of the magnetic field (e.g. the magnetic field associated with a C^1 small deformation of the wire) still has a magnetic line of the same topology. In this direction, a feature of this magnetic line that we find quite surprising is that it is in fact hyperbolic, not elliptic, so in particular there are no invariant tori in a neighborhood of it.

The paper is organized as follows. In §2 we will consider magnetic fields generated by a current density field defined on a toroidal surface and compute the asymptotic behavior of the corresponding Biot–Savart integral as the width of the surface tends to zero. In §3 we will connect magnetic fields of this kind with fields created by wires. This will hinge on a measure-theoretic argument that exploits the structure of divergence-free vector fields on a torus whose integral curves are all periodic. These results are put to use in §§4 and 5, where we respectively prove Theorems 1.1 and 1.2.

We conclude the Introduction with a word about notation. Throughout the paper, all the curves are assumed to be smooth and without self-intersections unless stated otherwise, and the isotopies and diffeomorphisms are always of class C^∞. When it does not give rise to confusion, we will use the same notation γ for a parametrized curve in space, which is a map $\gamma(t)$ from the real line (or the circle) to $\mathbb{R}^3$, and for its image $\gamma \equiv \gamma(\mathbb{R}) \subset \mathbb{R}^3$.

2. *Asymptotics for magnetic fields generated by surface currents*

A key ingredient in the proof of Theorems 1.1 and 1.2 will be the use of magnetic fields that are not generated by wires, but by current densities supported on toroidal surfaces. Our goal in this section is to analyze the behavior of magnetic fields of this kind. We will be particularly interested in the case where the toroidal surface is very thin. One word of warning: observe that the toroidal surface is used as a support of a current measure so, in general, it will not be an invariant torus of the field. In particular, using current measures supported on tori does not pave the way to extracting any dynamical information via KAM theory.

To make things precise, let us consider the toroidal surface

$$\mathcal{S}_{\gamma,\epsilon} := \{x \in \mathbb{R}^3 : \operatorname{dist}(x, \gamma) = \epsilon\},$$

which is a smooth torus of the same knot type as the curve γ provided that the width ϵ is small enough. We will define coordinates (s, y) on the domain bounded by $\mathcal{S}_{\gamma,\epsilon}$ as follows. Let s be an arc-length parametrization of the curve γ, whose length we will denote by ℓ. This amounts to saying that the tangent field $\dot\gamma$ has unit norm and $\gamma(s)$ is ℓ-periodic, so in particular we can assume that $s \in \mathbb{R}/\ell\mathbb{Z}$. Without loss of generality, we will make the assumption that the curve γ does not have any inflection points, which allows us to define the normal and binormal vectors $\mathbf{n}(s)$, $\mathbf{b}(s)$ at any point of the curve $\gamma(s)$. It is well known that this assumption is satisfied for generic curves [**2**, p. 184] (roughly speaking, 'generic' here refers to an open and dense set, with respect to a reasonable C^k topology, in the space of smooth curves in $\mathbb{R}^3$).

Using the normal and binormal vector fields and denoting by $\mathbb{D}^2$ the two-dimensional unit disk, we can introduce coordinates $(s, y) \in (\mathbb{R}/\ell\mathbb{Z}) \times \mathbb{D}^2$ in the solid torus bounded

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by $\mathcal{S}_{\gamma,\epsilon}$ via the diffeomorphism

$$(s,\, y) \mapsto \gamma(s) + \epsilon y_1 \mathbf{n}(s) + \epsilon y_2 \mathbf{b}(s). \tag{2.1}$$

Recall that the unit tangent vector is $\mathbf{t}(s) := \dot{\gamma}(s)$.

In the coordinates $(s,\, y)$, a short computation using the Frenet formulas yields the formula for the Jacobian of this change of coordinates, which shows that the volume measure is written in these coordinates as

$$dx := \epsilon^2 [1 - \epsilon\kappa(s) y_1]\, ds\, dy. \tag{2.2}$$

Here and in what follows, $\kappa(s)$ and $\tau(s)$ respectively denote the curvature and torsion of the curve γ. We will sometimes take polar coordinates $(r,\, \theta)$ in the disk $\mathbb{D}^2$, which are defined as

$$y_1 = r\cos\theta, \quad y_2 = r\sin\theta.$$

In terms of these coordinates, the volume is given by

$$dx = \epsilon^2 r[1 - \epsilon\kappa(s) r\cos\theta]\, ds\, dr\, d\theta.$$

We shall next consider magnetic fields created by current densities supported on $\mathcal{S}_{\gamma,\epsilon}$. The current distribution will be given by $J\, dS$, where dS is the surface measure on the surface $\mathcal{S}_{\gamma,\epsilon}$ and the density J is a smooth tangent vector field defined on $\mathcal{S}_{\gamma,\epsilon}$. The associated magnetic field is then

$$\mathcal{B}_J(x) := \int_{\mathcal{S}_{\gamma,\epsilon}} \frac{J(x') \times (x - x')}{4\pi |x - x'|^3}\, dS(x').$$

We will always assume that the divergence of J on the surface is zero (which is equivalent to demanding that the divergence of the vector-valued measure $J\, dS$ is zero, in the sense of distributions), so that $\mathcal{B}_J$ indeed has the physical interpretation of a magnetic field. In particular, $\mathcal{B}_J$ satisfies Maxwell's equations

$$\operatorname{curl} \mathcal{B}_J = J\, dS, \quad \operatorname{div} \mathcal{B}_J = 0.$$

The main result of this section is the following lemma, which provides the asymptotic behavior as $\epsilon \to 0$ of the magnetic field $\mathcal{B}_J$ for a wide family of current densities J. This family has two important features: firstly, the fields are divergence-free; and secondly, they have a special dependence on ϵ that is crucial to proving the existence of hyperbolic periodic magnetic lines isotopic to γ. Since we will only be interested in the behavior of the field near γ, let us restrict our attention to a fixed small neighborhood of the core curve γ (r.g. the region $|y| < \frac{1}{10}$). In this region, if $h(\epsilon)$ is a function that tends to zero as $\epsilon \to 0$ (typically ϵ^m or $\epsilon \log \epsilon$), let us agree to say that a scalar quantity $q(s,\, y,\, \epsilon)$ is of order $O(h(\epsilon))$ if

$$|\partial_s^j D_y^k q(s,\, y,\, \epsilon)| \le C_{j,k} h(\epsilon) \tag{2.3a}$$

for all $j,\, k$. Likewise, given a non-negative integer m, we will say that $q(s,\, y,\, \epsilon)$ is of order $O(|y|^m)$ if

$$|\partial_s^j D_y^k q(s,\, y,\, \epsilon)| \le C_{j,k} |y|^{\max\{m-k,0\}} \tag{2.3b}$$

in the above region for all $j,\, k$ and uniformly in ϵ. We will use the notation

$$O(1) \equiv O(\epsilon^0) \quad \text{and} \quad O(h(\epsilon) + |y|^k) := O(h(\epsilon)) + O(|y|^k). \tag{2.3c}$$

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LEMMA 2.1. *Let us take the divergence-free tangent vector field*

$$J := \frac{1}{1-\epsilon\kappa(s)\cos\theta}\left(F(\theta)\partial_s + \frac{G(s)}{\epsilon}\partial_\theta\right), \tag{2.4}$$

defined on $\mathcal{S}_{\gamma,\epsilon}$ in terms of two smooth functions F and G of period 2π and ℓ, respectively. We will also assume that G does not vanish. Consider the current distribution on $\mathcal{S}_{\gamma,\epsilon}$ given by $J\,dS$. Then, using the order notation (2.3),

$$\mathcal{B}_J = [G(s) + O(\epsilon\log\epsilon + |y|)]\partial_s + \frac{b_1 + b_2 y_1 - a_2 y_2 + O(\epsilon\log\epsilon + |y|^2)}{2\epsilon}\partial_{y_1} - \frac{a_1 + a_2 y_1 + b_2 y_2 + O(\epsilon\log\epsilon + |y|^2)}{2\epsilon}\partial_{y_2},$$

where a_n, b_n are the Fourier coefficients of F:

$$F(\theta) =: \sum_{n=0}^{\infty}(a_n\cos n\theta + b_n\sin n\theta).$$

Proof. Let us start by recalling that equation (2.1) and the Frenet formulas imply that the vector fields ∂_s, ∂_r and ∂_θ corresponding to the coordinates (s, r, θ) can be written in terms of the curvature κ and torsion τ of the curve γ as

$$\partial_s \equiv \frac{\partial x}{\partial s} = (1-\epsilon\kappa y_1)\mathbf{t} + \epsilon\tau y_2\mathbf{n} - \epsilon\tau y_1\mathbf{b}, \tag{2.5}$$

$$\partial_r \equiv \frac{\partial x}{\partial r} = \frac{\epsilon y_1\mathbf{n} + \epsilon y_2\mathbf{b}}{|y|}, \tag{2.6}$$

$$\partial_\theta \; equiv \frac{\partial x}{\partial\theta} = -\epsilon y_2\mathbf{n} + \epsilon y_1\mathbf{b}. \tag{2.7}$$

Here and in what follows, the curvature, torsion and elements of the Frenet basis are evaluated at s unless stated otherwise.

As the vector field ∂_r has norm ϵ, it follows from the expression (2.2) for the volume in the coordinates (s, r, θ) that the surface measure of $\mathcal{S}_{\gamma,\epsilon}$ can be written as

$$dS = \epsilon A\, ds\, d\theta,$$

with $A(s,\theta) := 1 - \epsilon\kappa(s)\cos\theta$. Hence the field J is indeed divergence-free on $\mathcal{S}_{\gamma,\epsilon}$, since the usual formula for the divergence yields

$$\operatorname{div}_{\mathcal{S}_{\gamma,\epsilon}} J = \frac{1}{A(s,\theta)}\frac{\partial}{\partial s}\left(\frac{A(s,\theta)F(\theta)}{1-\epsilon\kappa(s)\cos\theta}\right) + \frac{1}{A(s,\theta)}\frac{\partial}{\partial\theta}\left(\frac{A(s,\theta)G(s)}{\epsilon(1-\epsilon\kappa(s)\cos\theta)}\right) = 0.$$

Let us fix a point x in a small neighborhood of the curve γ that we will describe by its coordinates $(s, y) \in (\mathbb{R}/\ell\mathbb{Z}) \times \mathbb{D}^2$, which will remain fixed in the argument. Let us fix a small number δ and write the equation for $\mathcal{B}_J$ as

$$\mathcal{B}_J(x) = \int_{\mathcal{S}_{\gamma,\epsilon}} \frac{J(x')\times(x-x')}{4\pi|x-x'|^3}\,dx' = I + O(\epsilon), \tag{2.8}$$

where

$$I = \epsilon\int_0^{2\pi}\int_{s-\delta}^{s+\delta}\frac{J(x')\times(x-x')}{4\pi|x-x'|^3}A(s',\theta')\,ds'\,d\theta'$$

and (s', θ') are the coordinates used to parametrize the point $x' \in \mathcal{S}_{\gamma,\epsilon}$.

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To see why this is true, it suffices to observe that J is uniformly bounded in ϵ (here one has to use the fact that the norm of $(1/\epsilon)\partial_\theta$ is 1 on the surface) and that the distance $|x - x'|$ is at least δ when $|s - s'| > \delta$. This readily yields

$$|\mathcal{B}_J(x) - I| = \left| \epsilon \int_0^{2\pi} \int_{|s'-s|>\delta} \frac{J(x') \times (x - x')}{4\pi |x - x'|^3} A(s', \theta')\, ds'\, d\theta' \right| \leq C_\delta |\mathcal{S}_{\gamma,\epsilon}| = O(\epsilon).$$

An analogous reasoning can obviously be applied to the derivatives of $\mathcal{B}_J(x) - I$, so equation (2.8) follows.

We are interested in the asymptotic behavior of I for small ϵ. In order to compute it, we will need to expand the quantity $x - x'$ in the variables ϵ and $s' - s$, which can be achieved using equation (2.1) and the Frenet formulas:

$$x - x' = -(s' - s)\mathbf{t} + \epsilon(y_1 - \cos\theta')\mathbf{n} + \epsilon(y_2 - \sin\theta')\mathbf{b} + \mathcal{O}(2).$$

Hereafter we will use the notation $\mathcal{O}(k)$ to denote terms in the Taylor expansion that are at least a kth power of $(\epsilon, s' - s)$. For example,

$$(s' - s)^{2j}, \quad \epsilon(s' - s)^j, \quad \epsilon^{1+j}$$

are all $\mathcal{O}(2)$ if $j \geq 1$. Likewise, using formulas (2.5)–(2.7), the vector field $J(x')$ can be written as

$$J(x') = F(\theta')\mathbf{t} + G(s)(\cos\theta'\mathbf{b} - \sin\theta'\mathbf{n}) + \mathcal{O}(1),$$

where we recall that $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are evaluated at s.

Since the orthonormal basis $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is positively oriented, it then follows that

$$\begin{aligned} J(x') \times (x - x') = {} & \epsilon G(s)(1 - y_1 \cos\theta' - y_2 \sin\theta')\mathbf{t} \\ & - [(s' - s)G(s)\cos\theta' + \epsilon(y_2 - \sin\theta')F(\theta')]\mathbf{n} \\ & - [(s' - s)G(s)\sin\theta' - \epsilon(y_1 - \cos\theta')F(\theta')]\mathbf{b} + \mathcal{O}(2) \end{aligned} \tag{2.9}$$

and

$$|x - x'|^2 = (s' - s)^2 + \epsilon^2(1 + \beta) + \mathcal{O}(3), \tag{2.10}$$

with

$$\beta := -2y_1 \cos\theta' - 2y_2 \sin\theta' + |y|^2.$$

As the surface measure is

$$dS(x') = \epsilon(1 + \mathcal{O}(1))\, ds'\, d\theta',$$

one can now use formulas (2.9) and (2.10) to write I as

$$I = \frac{\epsilon}{4\pi} \int_0^{2\pi} \int_{s-\delta}^{s+\delta} \frac{h_1\mathbf{t} + h_2\mathbf{n} + h_3\mathbf{b}}{[(s' - s)^2 + \epsilon^2(1 + \beta) + \mathcal{O}(3)]^{3/2}}\, ds'\, d\theta'. \tag{2.11}$$

Here we have set

$$\begin{aligned} h_1 &:= \epsilon\, G(s)(1 - y_1 \cos\theta' - y_2 \sin\theta') + \mathcal{O}(2), \\ h_2 &:= -[(s' - s)G(s)\cos\theta' + \epsilon(y_2 - \sin\theta')F(\theta')] + \mathcal{O}(2), \\ h_3 &:= -[(s' - s)G(s)\sin\theta' - \epsilon(y_1 - \cos\theta')F(\theta')] + \mathcal{O}(2). \end{aligned}$$

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Let us begin with the tangent component of I,

$$I_1 := \frac{\epsilon}{4\pi} \int_0^{2\pi} \int_{s-\delta}^{s+\delta} \frac{h_1}{[(s'-s)^2 + \epsilon^2(1+\beta) + \mathcal{O}(3)]^{3/2}}\, ds'\, d\theta'.$$

Using again the fact that $1+\beta$ is positive for small enough y, clearly

$$(s'-s)^2 + \epsilon^2(1+\beta) + \mathcal{O}(3) = ((s'-s)^2 + \epsilon^2(1+\beta))(1+\mathcal{O}(1)),$$

so one can decompose I_1 as

$$\begin{aligned} I_1 &= \frac{\epsilon}{4\pi} \int_0^{2\pi} \int_{s-\delta}^{s+\delta} \frac{\epsilon G(s)(1 - y_1 \cos\theta' - y_2 \sin\theta')}{[(s'-s)^2 + \epsilon^2(1+\beta)]^{3/2}}\, ds'\, d\theta' \\ &\quad + \epsilon \int_0^{2\pi} \int_{s-\delta}^{s+\delta} \frac{\mathcal{O}(2)}{[(s'-s)^2 + \epsilon^2(1+\beta)]^{3/2}}\, ds'\, d\theta' \\ &= I_{11} + I_{12}. \end{aligned} \tag{2.12}$$

Since $1+\beta > 0$ for small y, the second integral can be easily bounded as

$$\begin{aligned} |I_{12}| &\leq C\epsilon \int_0^{2\pi} \int_{s-\delta}^{s+\delta} \frac{(s'-s)^2 + \epsilon|s-s'| + \epsilon^2}{[(s'-s)^2 + \epsilon^2(1+\beta)]^{3/2}}\, ds'\, d\theta' \\ &\leq C\epsilon \int_{s-\delta}^{s+\delta} \frac{1}{[(s'-s)^2 + \epsilon^2]^{1/2}}\, ds' \\ &\leq C\epsilon \log \frac{1}{\epsilon}. \end{aligned}$$

To study the integral I_{11} in (2.12), let us introduce the variable

$$t := \frac{s'-s}{\epsilon(1+\beta)^{1/2}},$$

in terms of which the integral is expressed as

$$\begin{aligned} I_{11} &= \frac{1}{4\pi} \int_0^{2\pi} \int_{-\delta/[\epsilon(1+\beta)^{1/2}]}^{\delta/[\epsilon(1+\beta)^{1/2}]} \frac{G(s)(1 - y_1 \cos\theta' - y_2 \sin\theta')}{(1+\beta)(t^2+1)^{3/2}}\, dt\, d\theta' \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{G(s)(1 - y_1 \cos\theta' - y_2 \sin\theta')}{(1+\beta)(t^2+1)^{3/2}}\, dt\, d\theta' + O(\epsilon^2) \\ &= \frac{G(s)}{2\pi} \int_0^{2\pi} (1 - y_1 \cos\theta' - y_2 \sin\theta')(1 + 2y_1 \cos\theta' + 2y_2 \sin\theta')\, d\theta' \\ &\quad + O(\epsilon^2 + |y|^2) \\ &= G(s) + O(\epsilon^2 + |y|). \end{aligned}$$

To pass to the third line we have expanded in y and used the fact that

$$\int_{-\infty}^{\infty} \frac{dt}{(t^2+1)^{3/2}} = 2.$$

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Now that we are done with the computation of I_1, let us consider next the normal component of I and decompose it as before:

$$\begin{aligned} I_2 &:= \frac{\epsilon}{4\pi}\int_0^{2\pi}\int_{s-\delta}^{s+\delta}\frac{h_2}{[(s'-s)^2+\epsilon^2(1+\beta)]^{3/2}}\,ds'\,d\theta' \\ &= -\frac{\epsilon}{4\pi}\int_0^{2\pi}\int_{s-\delta}^{s+\delta}\frac{(s'-s)G(s)\cos\theta'}{[(s'-s)^2+\epsilon^2(1+\beta)]^{3/2}}\,ds'\,d\theta' \\ &\quad -\frac{\epsilon^2}{4\pi}\int_0^{2\pi}\int_{s-\delta}^{s+\delta}\frac{(y_2-\sin\theta')F(\theta')}{[(s'-s)^2+\epsilon^2(1+\beta)]^{3/2}}\,ds'\,d\theta' \\ &\quad +\epsilon\int_0^{2\pi}\int_{s-\delta}^{s+\delta}\frac{\mathcal{O}(2)}{[(s'-s)^2+\epsilon^2(1+\beta)]^{3/2}}\,ds'\,d\theta' \\ &=: I_{21}+I_{22}+I_{23}. \end{aligned}$$

Arguing as in the case of I_1, one immediately gets that

$$|I_{23}| \le C\epsilon\log\frac{1}{\epsilon},$$

and the fact that the integrand is an odd function of $s'-s$ immediately implies that

$$I_{21}=0.$$

Moreover, the integral I_{22} can be analyzed just as in the case of I_{11}, yielding

$$I_{22}=\frac{b_1+b_2y_1-a_2y_2}{2}+O(\epsilon+|y|^2).$$

The binormal component of I,

$$I_3 := \frac{\epsilon}{4\pi}\int_0^{2\pi}\int_{s-\delta}^{s+\delta}\frac{h_3}{[(s'-s)^2+\epsilon^2(1+\beta)+\mathcal{O}(3)]^{3/2}}\,ds'\,d\theta',$$

can be computed as in the case of I_2, obtaining

$$I_3=-\frac{a_1+a_2y_1+b_2y_2}{2}+O(\epsilon+|y|^2).$$

As equations (2.5)–(2.7) obviously imply that

$$\mathbf{t}=(1+O(\epsilon))\partial_s+O(|y|)\partial_{y_1}+O(|y|)\partial_{y_2},\quad \mathbf{n}=\frac{1}{\epsilon}\partial_{y_1},\quad \mathbf{b}=\frac{1}{\epsilon}\partial_{y_2},$$

one obtains the desired asymptotic formula for $\mathcal{B}_J$.

It is clear that the same method yields formulas for the derivatives of the components of $\mathcal{B}_J$, which correspond to the derivatives of the terms that we have already computed (e.g. in the case of first-order derivatives, $\partial_s I_k$ and $\partial_{y_j} I_k$). To illustrate the reasoning, let us consider $\partial_s I_1$. Since the point of coordinates $(s,\,y)$ is in the interior of the solid torus bounded by $\mathcal{S}_{\gamma,\epsilon}$, one can safely differentiate under the integral sign to find

$$\begin{aligned} \partial_s I_1 &= \frac{\epsilon}{4\pi}\int_0^{2\pi}\int_{s-\delta}^{s+\delta}\frac{\partial}{\partial s}\left(\frac{\epsilon G(s)(1-y_1\cos\theta'-y_2\sin\theta')}{[(s'-s)^2+\epsilon^2(1+\beta)]^{3/2}}\right)ds'\,d\theta' \\ &\quad +\epsilon\int_0^{2\pi}\int_{s-\delta}^{s+\delta}\frac{\partial}{\partial s}\left(\frac{\mathcal{O}(2)}{[(s'-s)^2+\epsilon^2(1+\beta)]^{3/2}}\right)ds'\,d\theta'. \end{aligned}$$

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Here we have used the fact that the boundary terms cancel out by parity. Using the fact that for all q of order $\mathcal{O}(2)$ one can write

$$\partial_s q = \partial_{s'} q_1 + q_2$$

with q_j also of order $\mathcal{O}(2)$, we can further simplify these integrals as

$$\begin{aligned}\partial_s I_1 &= \frac{\epsilon}{4\pi}\int_0^{2\pi}\int_{s-\delta}^{s+\delta} \frac{\epsilon G'(s)(1-y_1\cos\theta'-y_2\sin\theta')+\mathcal{O}(2)}{[(s'-s)^2+\epsilon^2(1+\beta)]^{3/2}}\,ds'\,d\theta' \\ &\quad -\epsilon\int_0^{2\pi}\int_{s-\delta}^{s+\delta}\frac{\partial}{\partial s'}\left(\frac{\epsilon G(s)(1-y_1\cos\theta'-y_2\sin\theta')+\mathcal{O}(2)}{[(s'-s)^2+\epsilon^2(1+\beta)]^{3/2}}\right)ds'\,d\theta' \\ &= \frac{\epsilon}{4\pi}\int_0^{2\pi}\int_{s-\delta}^{s+\delta} \frac{\epsilon G'(s)(1-y_1\cos\theta'-y_2\sin\theta')+\mathcal{O}(2)}{[(s'-s)^2+\epsilon^2(1+\beta)]^{3/2}}\,ds'\,d\theta'.\end{aligned}$$

Here we are using a parity argument both to get rid of the boundary terms that appear when one integrates by parts and to neglect the terms of $\mathcal{O}(2)$ that are odd functions of $s'-s$, as they do not contribute to the integral. As the above integral is of the same form as I_1, the previous reasoning immediately yields

$$\partial_s I_1 = G'(s) + O(\epsilon\log\epsilon + |y|).$$

The derivatives with respect to y, which are in fact easier, can be handled with a completely analogous argument. □

Remark 2.2. The norm of the field $\mathcal{B}_J$ is of order 1, and the reason why the components of $\mathcal{B}_J$ along the fields ∂_{y_j} are of order $1/\epsilon$ is simply that the norm $|\partial_{y_j}|$ is ϵ (recall that they are simply the normal or binormal vector divided by ϵ).

3. *From surface currents to closed wires*

In this section we will derive tools that permit us to show that there are configurations of wires that create magnetic fields which approximate, in a certain sense, magnetic fields generated by current densities of the form studied in §2. For simplicity, throughout this section we will denote by $\mathcal{S}$ a surface of $\mathbb{R}^3$ diffeomorphic to a torus.

An important ingredient in the proof will be the idea of convergence of measures. We recall that a sequence of vector-valued measures $d\Lambda_n$ supported on $\mathcal{S}$ converges weakly to $d\Lambda$ if, given any continuous function $u:\mathcal{S}\to\mathbb{R}^3$, one has

$$\lim_{n\to\infty}\int_{\mathcal{S}} u\cdot d\Lambda_n = \int_{\mathcal{S}} u\cdot d\Lambda.$$

In this direction, an easy but very useful result is the following lemma.

LEMMA 3.1. *Let K be a compact subset of $\mathbb{R}^3$. Consider a sequence of vector-valued measures $d\Lambda_n$ whose supports are contained in a compact set K', and assume that this sequence converges weakly to $d\Lambda$. If K does not intersect K', then*

$$\lim_{n\to\infty}\left\|\int_{K'}\frac{d\Lambda_n(x')\times(x-x')}{4\pi|x-x'|^3}-\int_{K'}\frac{d\Lambda(x')\times(x-x')}{4\pi|x-x'|^3}\right\|_{C^m(K)}=0 \tag{3.1}$$

for any integer m.

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Proof. Observe that the kernel

$$(x, x') \mapsto \frac{x - x'}{4\pi |x - x'|^3} \tag{3.2}$$

is continuous in the set $(x, x') \in K \times K'$. The convergence of the measures $d\Lambda_n$ to $d\Lambda$ and the fact that these measures are supported on K' imply that, uniformly for all $x \in K$,

$$\lim_{n\to\infty} \int_{K'} \frac{d\Lambda_n(x') \times (x - x')}{4\pi |x - x'|^3} = \int_{K'} \frac{d\Lambda(x') \times (x - x')}{4\pi |x - x'|^3}.$$

Since the derivatives of the kernel (3.2) with respect to x are also continuous on $K \times K'$, the same argument yields the C^k convergence (3.1) on the set K. □

The next lemma shows how to approximate the magnetic field created by a surface distribution $J\,dS$ through the Biot–Savart integral (cf. equation (2.8)) by that of a collection of magnetic wires. Concerning the statement of the lemma, it is worth noting that we require the tangent vector field J to be divergence-free. This is not an additional condition that we impose because we want to interpret this field as a current density, but an actual necessary technical condition for the statement of the lemma to hold true. This is because the divergence of any measure of the form $\dot\gamma_k(t)\,dt$ can be easily seen to be zero, in the sense of distributions, so the fact that there is a collection of measures $d\Lambda_n$ of the form (3.3) that converge to $J\,dS$ automatically implies that the tangent field J is divergence-free on $\mathcal{S}$ (or, equivalently, that the divergence of $J\,dS$ is zero).

LEMMA 3.2. *Let J be a tangent vector field on the toroidal surface $\mathcal{S}$ whose divergence on the surface is zero. Let us assume that J does not vanish and that all its integral curves are periodic. Then there exist a positive constant c_0 and a sequence of finite collections of (distinct) periodic integral curves of J, $\gamma_k : (\mathbb{R}/T_k\mathbb{Z}) \to \mathcal{S}$, such that the vector-valued measure*

$$d\Lambda_n := \frac{c_0}{n} \sum_{k=1}^{n} \dot\gamma_k(t)\,dt \tag{3.3}$$

converges weakly to $J\,dS$ as $n \to \infty$. Here T_k is the minimal period of the integral curve γ_k.

Proof. It is known (cf. [**6**, 4.1.14]) that if J is a non-vanishing divergence-free field on the torus whose integral curves are all periodic, then:

(i) there is a closed transverse curve $\mathcal{C}$ on $\mathcal{S}$ which intersects all the integral curves of J at exactly one point;

(ii) the period function $T : \mathcal{S} \to (0, \infty)$, which maps each point on the surface to the minimal period of the integral curve of J passing through it, is smooth.

Let us now define the isochronous field associated to J,

$$\widetilde{J} := T J, \tag{3.4}$$

whose integral curves are all closed and of period 1. Consider a periodic coordinate on $\mathcal{C}$ of period 1,

$$\Theta : \mathcal{C} \to \mathbb{T}^1, \tag{3.5}$$

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with $\mathbb{T}^1 := \mathbb{R}/\mathbb{Z}$. Let us now construct a map $\Psi : \mathcal{S} \to \mathbb{T}^2$ by setting

$$\Psi^{-1}(\alpha, \sigma) := \phi_\sigma[\Theta^{-1}(\alpha)], \tag{3.6}$$

where ϕ_σ is the flow at time σ of the field $\widetilde{J}$. Since $\mathcal{C}$ intersects each integral curve exactly once and all integral curves of $\widetilde{J}$ have period 1, it is obvious that Ψ is a diffeomorphism. Moreover, it is apparent that one can write the push-forward of $\widetilde{J}$ under the diffeomorphism Ψ as

$$\Psi_* \widetilde{J} = \partial_\sigma.$$

As the period function takes the same value on each integral curve of J, there is a smooth function $\widetilde{T} : \mathbb{T}^1 \to (0, \infty)$ such that

$$T[\Psi^{-1}(\alpha, \sigma)] = \widetilde{T}(\alpha). \tag{3.7}$$

In particular, T is a first integral of J, so it is trivial that the divergence of $\widetilde{J}$ on the surface is also zero. Hence, it follows that the push-forward $\widetilde{\sigma}$ of the area 2-form on $\mathcal{S}$ to $\mathbb{T}^2$ can be written as

$$\widetilde{\sigma} = B(\alpha)\, d\alpha \wedge d\sigma, \tag{3.8}$$

where $B : \mathbb{T}^1 \to \mathbb{R}$ is a positive smooth function. In order to see this, it suffices to write $\widetilde{\sigma} = B\, d\alpha \wedge d\sigma$ and notice that the field $\Psi_* \widetilde{J} = \partial_\sigma$ is divergence-free with respect to $\widetilde{\sigma}$, which implies that

$$0 = (\mathrm{div}_{\widetilde{\sigma}}\, \partial_\sigma)\, \widetilde{\sigma} = d(i_{\partial_\sigma} \widetilde{\sigma}) = -d(B\, d\alpha),$$

which shows that B is independent of σ.

Let us consider a sequence of points $\alpha_k \in \mathbb{T}^1$ that is uniformly distributed with respect to the probability measure on $\mathbb{T}^1$ given by

$$d\rho := \frac{B(\alpha)}{c_0 \widetilde{T}(\alpha)}\, d\alpha, \tag{3.9}$$

where $B(\alpha)$ is the positive function defined in (3.8) and

$$c_0 := \int_{\mathbb{T}^1} \frac{B(\alpha)}{\widetilde{T}(\alpha)}\, d\alpha.$$

Specifically, this means that, for any interval I of $\mathbb{T}^1$,

$$\lim_{n\to\infty} \frac{\#\{k \le n : \alpha_k \in I\}}{n} = \int_I d\rho. \tag{3.10}$$

Let $\widetilde{\gamma}_k : \mathbb{T}^1 \to \mathcal{S}$ be the integral curve of $\widetilde{J}$ with initial condition $\widetilde{\gamma}_k(0) = \Theta^{-1}(\alpha_k)$. We shall next check that

$$d\Lambda_n := \frac{c_0}{n} \sum_{k=1}^{n} \dot{\widetilde{\gamma}}_k(\sigma)\, d\sigma$$

converges weakly to $J\, dS$. Before doing so, let us observe that this implies the lemma, because if $\gamma_k(t)$ denotes the integral curve of J with initial condition $\gamma_k(0) = \Theta^{-1}(\alpha_k)$, it is obvious from the invariance of the measure under reparametrization (namely, the fact that $\dot{\widetilde{\gamma}}_k(\sigma)\, d\sigma = \dot{\gamma}_k(t)\, dt$) that $d\Lambda_n$ is also given by the formula (3.3) provided in the statement.

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To show that $d\Lambda_n$ converges weakly to $J\,dS$, recall that the fact that the points $\{\alpha_k\}$ are uniformly distributed with respect to the probability measure $d\rho$ is equivalent to saying that the measure

$$d\rho_n := \frac{1}{n}\sum_{k=1}^{n}\delta_{\alpha_k}$$

on $\mathbb{T}^1$ converges weakly to $d\rho$ as $n\to\infty$. Hence,

$$\begin{aligned}\int u\cdot d\Lambda_n &= \frac{c_0}{n}\sum_{k=1}^{n}\int_{\mathbb{T}^1}\langle\Psi_* u,\,\partial_\sigma\rangle|_{(\alpha_k,\sigma)}\,d\sigma\\ &= c_0\int_{\mathbb{T}^2}\langle\Psi_* u,\,\partial_\sigma\rangle\,d\rho_n\,d\sigma\end{aligned}$$

satisfies

$$\begin{aligned}\lim_{n\to\infty}\int u\cdot d\Lambda_n &= c_0\int_{\mathbb{T}^2}\langle\Psi_* u,\,\partial_\sigma\rangle\,d\rho\,d\sigma\\ &= \int_{\mathbb{T}^2}\langle\Psi_* u,\,\partial_\sigma\rangle\frac{B(\alpha)}{\widetilde{T}(\alpha)}\,d\alpha\,d\sigma\\ &= \int_{\mathcal{S}} u\cdot J\,dS,\end{aligned}$$

where we have used the fact that $\widetilde{J} = T\,J$. □

In the proof of Theorem 1.2 we will need the following lemma, which is basically a refinement of Lemma 3.2 that provides a versatile sufficient condition for a sequence of vector-valued measures supported on curves (not necessarily integral curves of the field J) to converge to the current distribution $J\,dS$.

LEMMA 3.3. *Let J be as in Lemma 3.2 and consider the associated map $\Theta:\mathcal{C}\to\mathbb{T}^1$ defined in (3.5). Suppose that $\Gamma_n:\mathbb{R}/T_n\mathbb{Z}\to\mathcal{S}$ is a sequence of periodic curves without self-intersections that satisfy the following properties, with T_n being the minimal period.*

(i) *The curves Γ_n intersect $\mathcal{C}$ transversally and their points of intersection are uniformly distributed with respect to the measure $d\rho$ defined in (3.9), that is, for any open subset $I\subset\mathcal{C}$ one has*

$$\lim_{n\to\infty}\frac{\#(\Gamma_n\cap I)}{\#(\Gamma_n\cap\mathcal{C})} = \int_{\Theta(I)} d\rho.$$

(ii) *The tangent vectors $\dot\Gamma_n(\sigma)$ converge uniformly to the isochronized field $\widetilde{J}$, defined in (3.4):*

$$\lim_{n\to\infty}\sup_{\sigma\in\mathbb{R}}|\dot\Gamma_n(\sigma)-\widetilde{J}(\Gamma_n(\sigma))| = 0.$$

Then there is a positive constant c_0 such that the vector-valued measures

$$d\lambda_n := \frac{c_0}{\#(\Gamma_n\cap\mathcal{C})}\dot\Gamma_n(\sigma)\,d\sigma,$$

which are supported on Γ_n, converge weakly to $J\,dS$ as $n\to\infty$.

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Proof. We will use the notation introduced in the proof of Lemma 3.2 without further notice. Setting

$$N_n := \#(\Gamma_n \cap \mathcal{C}),$$

let us denote the intersection points of Γ_n with the transverse curve $\mathcal{C}$ by

$$\Gamma_n \cap \mathcal{C} = \{p_{n,k}\}_{k=1}^{N_n},$$

where we are labeling the points $p_{n,k}$ so that they correspond to consecutive intersection points. We will denote the intersection times of the curve Γ_n by $\sigma_{n,k} \in [0, T_n)$, with $\sigma_{n,1} = 0$ and

$$p_{n,k} = \Gamma_n(\sigma_{n,k}).$$

The point in the unit circle associated to $p_{n,k}$ under the map Θ will be denoted by $\alpha_{n,k} := \Theta(p_{n,k})$.

The uniform convergence of $\dot{\Gamma}_n(\sigma)$ to $\widetilde{J}(\Gamma_n(\sigma))$ and the fact that all integral curves of $\widetilde{J}$ are closed with period 1 obviously imply that the time between consecutive intersections with $\mathcal{C}$ tends to 1:

$$\lim_{n\to\infty} \max_{1\leq k\leq N_n} |\sigma_{n,k+1} - \sigma_{n,k} - 1| = 0. \tag{3.11}$$

Throughout we are identifying $\sigma_{n,N_n+1} := T_n$. In particular, this implies that T_n/N_n tends to 1. Letting $\widetilde{\gamma}_{n,k} : \mathbb{T}^1 \to \mathcal{S}$ be the integral curve of the isochronized field $\widetilde{J}$ with initial condition $\widetilde{\gamma}_{n,k}(0) = p_{n,k}$, we then infer that the integral curve $\widetilde{\gamma}_{n,k}$ is close to Γ_n in the sense that

$$\lim_{n\to\infty} \max_{1\leq k\leq N_n} \|\Gamma_n(\cdot) - \widetilde{\gamma}_{n,k}(\cdot - \sigma_{n,k})\|_{C^1((\sigma_{n,k},\sigma_{n,k+1}))} = 0. \tag{3.12}$$

Let us define c_0 as in the proof of Lemma 3.2. To prove that

$$d\lambda_n := \frac{c_0}{N_n} \dot{\Gamma}_n(\sigma)\, d\sigma$$

converges weakly to $J\, dS$, let us take an arbitrary smooth function $u : \mathcal{S} \to \mathbb{R}^3$, which without loss of generality can be thought of as a tangent vector field on $\mathcal{S}$. Notice that

$$\begin{aligned}\mathcal{E}_n := \max_{1\leq k\leq N_n} \left| \int_{\sigma_{n,k+1}-\sigma_{n,k}}^{1} u(\widetilde{\gamma}_{n,k}(\sigma)) \cdot \dot{\widetilde{\gamma}}_{n,k}(\sigma)\, d\sigma \right| \\ + \max_{1\leq k\leq N_n} \int_{\sigma_{n,k}}^{\sigma_{n,k+1}} |u(\Gamma_n(\sigma)) \cdot \dot{\Gamma}_n(\sigma) - u(\widetilde{\gamma}_{n,k}(\sigma - \sigma_{n,k})) \cdot \dot{\widetilde{\gamma}}_{n,k}(\sigma - \sigma_{n,k})|\, d\sigma\end{aligned}$$

tends to zero as $n \to \infty$ by (3.11) and (3.12). Setting

$$d\Lambda_n := \frac{c_0}{N_n} \sum_{k=1}^{N_n} \dot{\widetilde{\gamma}}_{n,k}(\sigma)\, d\sigma,$$

one then has that

$$\begin{aligned}\left| \int_{\mathcal{S}} u \cdot d\lambda_n - \int_{\mathcal{S}} u \cdot d\Lambda_n \right| &\leq \frac{c_0}{N_n} \sum_{k=1}^{N_n} \left| \int_{\sigma_{n,k+1}-\sigma_{n,k}}^{1} u(\widetilde{\gamma}_{n,k}(\sigma)) \cdot \dot{\widetilde{\gamma}}_{n,k}(\sigma)\, d\sigma \right| \\ &\quad + \frac{c_0}{N_n} \sum_{k=1}^{N_n} \int_{\sigma_{n,k}}^{\sigma_{n,k+1}} |u(\Gamma_n(\sigma)) \cdot \dot{\Gamma}_n(\sigma) - u(\widetilde{\gamma}_{n,k}(\sigma - \sigma_{n,k})) \cdot \dot{\widetilde{\gamma}}_{n,k}(\sigma - \sigma_{n,k})|\, d\sigma \\ &\leq c_0\, \mathcal{E}_n\end{aligned}$$

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also tends to zero as $n \to \infty$. Here we have used the fact that

$$\int_{\sigma_{n,k}}^{\sigma_{n,k}+1} u(\widetilde{\gamma}_{n,k}(\sigma)) \cdot \dot{\widetilde{\gamma}}_{n,k}(\sigma)\, d\sigma = \int_{\sigma_{n,k}}^{\sigma_{n,k+1}} u(\widetilde{\gamma}_{n,k}(\sigma)) \cdot \dot{\widetilde{\gamma}}_{n,k}(\sigma)\, d\sigma + \int_{\sigma_{n,k+1}-\sigma_{n,k}}^{1} u(\widetilde{\gamma}_{n,k}(\sigma)) \cdot \dot{\widetilde{\gamma}}_{n,k}(\sigma)\, d\sigma.$$

As the points $\alpha_{n,k}$ are equidistributed with respect to the probability measure $d\rho$, it follows directly from the proof of Lemma 3.2 that the measures $d\Lambda_n$ converge to $J\, dS$ as $n \to \infty$. The lemma is then proved. □

Remark 3.4. Using the coordinates (α, σ) introduced in the proof of Lemma 3.2, one can visually understand these results as follows. In these coordinates on $\mathcal{S}$ (which do not generally arise from a diffeomorphism of the ambient space $\mathbb{R}^3$, as they change the knot type of the integral curves of J), the integral curves of J are the vertical circles $\{\alpha = \text{constant}\}$. The transverse curve $\mathcal{C}$ corresponds to the equatorial circle $\{\sigma = 0\}$ and the collection of integral curves γ_k constructed in Lemma 3.2 is simply a collection of vertical circles $\{\alpha = \alpha_k\}$ with α_k distributed according to certain probability measure. The curves Γ_n satisfying the assumptions of Lemma 3.3 correspond to nearly vertical periodic curves which close after winding once in the horizontal direction and N_n times in the vertical direction.

4. *Magnetic lines and wires of arbitrary topology*

In this section we will prove Theorem 1.1. For the sake of clarity, let us divide the argument into several steps.

Step 1: Construction of a surface current distribution with a hyperbolic magnetic line isotopic to $\widetilde{\gamma}$. Let us consider the toroidal surface $\mathcal{S}_{\widetilde{\gamma},\epsilon}$ of core curve $\widetilde{\gamma}$ and small width ϵ, and the divergence-free tangent vector field J on $\mathcal{S}_{\widetilde{\gamma},\epsilon}$ given by

$$J := \frac{1}{1 - \epsilon\kappa(s)\cos\theta} J_0, \quad J_0 := 2\cos 2\theta\, \partial_s + \frac{1}{\epsilon}\partial_\theta. \tag{4.1}$$

Notice that the field J is of the form (2.4), so Lemma 2.1 ensures that the magnetic field $\mathcal{B}_J$ generated by the surface current distribution $J\, dS$ is given by

$$\mathcal{B}_J = [1 + q_0(\epsilon, y)]\partial_s - \frac{y_2 + q_1(\epsilon, y)}{\epsilon}\partial_{y_1} - \frac{y_1 + q_2(\epsilon, y)}{\epsilon}\partial_{y_2},$$

where

$$q_0(\epsilon, y) = O(\epsilon \log \epsilon + |y|), \quad q_1(\epsilon, y) = O(\epsilon \log \epsilon + |y|^2),$$
$$q_2(\epsilon, y) = O(\epsilon \log \epsilon + |y|^2).$$

Let us consider the vector field on the domain $\mathcal{T}$ bounded by $\mathcal{S}_{\widetilde{\gamma},\epsilon}$ given by

$$X := -(y_2 + q_1(0, y))\partial_{y_1} - (y_1 + q_2(0, y))\partial_{y_2},$$

which vanishes identically on the curve $\widetilde{\gamma} \equiv \{y = 0\}$ (of course, for $\epsilon = 0$ we are taking $\epsilon \log \epsilon$ as zero). In terms of the coordinates

$$\tilde{y}_1 := y_1 + y_2, \quad \tilde{y}_2 := y_1 - y_2,$$

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the integral curves of the linearization

$$\widetilde{X} := -y_2\partial_{y_1} - y_1\partial_{y_2}$$

of X are given by

$$s(t) = s_0, \quad \tilde{y}_1(t) = \tilde{y}_{10}e^{-t}, \quad \tilde{y}_2(t) = \tilde{y}_{20}e^{t}.$$

Therefore the invariant set $\widetilde{\gamma} \equiv \{y = 0\}$ of X is normally hyperbolic because at each point of the circle there is a one-dimensional stable component (corresponding to the variable $\tilde{y}_1$ with Lyapunov exponent -1) where the flow is exponentially contracting and a one-dimensional unstable component (corresponding to $\tilde{y}_2$ with Lyapunov exponent 1) where the flow is exponentially expanding.

It is therefore well known that the invariant circle $\{y = 0\}$ is preserved under small perturbations of the field X. More precisely, let us take any integer $k \geq 1$, a compact set $K \subset \mathcal{T}$ defined as

$$K = \text{a compact neighborhood of the closed curve } \{y = 0\} \equiv \widetilde{\gamma},$$

and define the C^k norm of a vector field,

$$\|Y\|_{C^k(K)},$$

as the sum of the $C^k(K)$ norms of the components of Y in the basis $\{\partial_s, \partial_{y_1}, \partial_{y_2}\}$ (this is important to avoid having to deal with inessential factors of ϵ). One then has [**7**, Theorem 4.1] that there exists some positive constant δ_1 such that any field Y with

$$\|X - Y\|_{C^k(K)} < \delta_1 \tag{4.2}$$

has a one-dimensional invariant set isotopic to the curve $\{y = 0\}$, and the distance between the corresponding isotopy Θ and the identity in the C^k norm is of order δ_1:

$$\|\Theta - \mathrm{id}\|_{C^k(K)} < C\delta_1.$$

Since

$$\|X - \epsilon\mathcal{B}_J\|_{C^k(K)} < C\epsilon \log \frac{1}{\epsilon},$$

it follows from (4.2) that, for small enough ϵ, $\epsilon\mathcal{B}_J$ (and therefore $\mathcal{B}_J$) has a one-dimensional invariant set $\widetilde{\gamma}_2$ isotopic to the curve $\widetilde{\gamma} \equiv \{y = 0\}$ and contained in a small neighborhood $\{|y| < C\epsilon \log(1/\epsilon)\}$. As the field $\mathcal{B}_J$ does not vanish in the set $\{|y| < C\epsilon \log(1/\epsilon)\}$ for small enough ϵ, $\widetilde{\gamma}_2$ must be a periodic integral curve of $\mathcal{B}_J$. The normal hyperbolicity of the invariant set $\{y = 0\}$ of X implies that $\widetilde{\gamma}_2$ is a hyperbolic periodic integral curve of $\mathcal{B}_J$, so in particular it is robust in the sense that there is some $\delta_2 > 0$ such that any field Z with

$$\|\mathcal{B}_J - Z\|_{C^k(K)} < \delta_2 \tag{4.3}$$

must have a periodic integral curve isotopic to $\widetilde{\gamma}$, and the isotopy can be chosen close to the identity in C^k.

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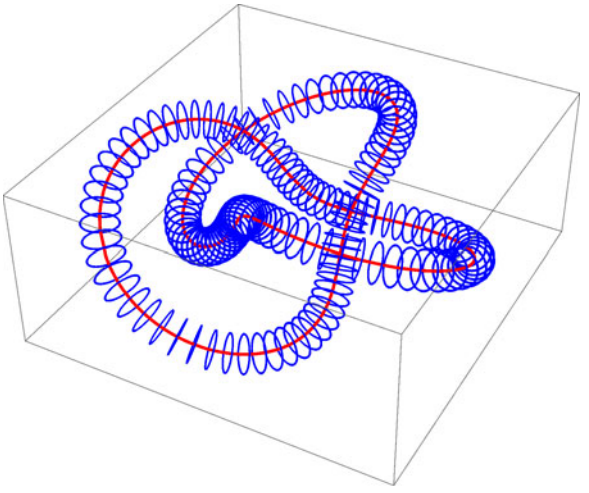


FIGURE 1. A collection of closed wires (represented as thin unknotted lines) the sum of whose magnetic fields has a magnetic line diffeomorphic to the core knot (in this case, a trefoil).

Step 2: Approximation of the magnetic field created by the surface current distribution by the sum of the fields of a finite collection of unknotted wires. Let us next analyze the integral curves of J_0, which coincide with those of J up to a reparametrization of the curve. These are the solutions to the system of ordinary differential equations (ODEs)

$$\dot{s} = 2\cos 2\theta, \quad \dot{\theta} = \frac{1}{\epsilon},$$

with an arbitrary initial condition (s_0, θ_0), that is,

$$s(t) = s_0 + \epsilon \sin\left(2\theta_0 + \frac{2t}{\epsilon}\right), \quad \theta(t) = \theta_0 + \frac{t}{\epsilon}. \tag{4.4}$$

These curves are all periodic with period $2\pi\epsilon$. Geometrically, for small ϵ this integral curve is a small deformation of, and isotopic to, the circle $\{s = s_0\}$ contained in the torus $\{r = 1\}$. This curve is obviously isotopic to the unknot.

Lemmas 3.1 and 3.2 then ensure that there is a finite collection of periodic integral curves $\{\gamma_k\}_{k=1}^N$ of J such that the sum of the magnetic fields that they create,

$$Y := \sum_{k=1}^{N} B_{\gamma_k},$$

is close to $\mathcal{B}_J$ in the set K modulo multiplication by a positive constant $c := c_0/N$:

$$\|\mathcal{B}_J - cY\|_{C^k(K)} < \frac{\delta_2}{3}. \tag{4.5}$$

This collection of curves is depicted in Figure 3.

Step 3: Replacing the finite collection of unknotted wires by a single unknot Γ_δ. Let us denote by

$$d\Lambda := \sum_{k=1}^{N} \dot{\gamma}_k(t)\, dt$$

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the vector-valued measure associated with the above periodic integral curves, so that the field Y can be written as

$$Y(x) = \int \frac{d\Lambda(x') \times (x - x')}{4\pi |x - x'|^3}.$$

For concreteness, we will henceforth assume that the curves γ_k are parametrized as in (4.4), so t ranges over $(0, 2\pi\epsilon)$ in each curve. Notice that the measure $d\Lambda$ is independent of the way the curves are parametrized, provided that the orientation is preserved.

We will need the following observation. Let $I_1, \ldots, I_N$ be intervals of length at most δ and let us denote by $d\Lambda_\delta$ the measure obtained from $d\Lambda$ after removing the intervals I_k from the curves. That is, for any continuous vector-valued function F we set

$$\int F \cdot d\Lambda_\delta := \sum_{k=1}^{N} \int_{0<t<2\pi\epsilon, t\notin I_k} F(\gamma_k(t)) \cdot \dot\gamma_k(t)\, dt.$$

Then $d\Lambda_\delta$ obviously converges weakly to $d\Lambda$ as $\delta \to 0$, for any choice of the intervals I_k.

Another useful observation is the following. Let p, q be points in $\mathbb{R}^3 \backslash K$. Then, given any other pair of points $\tilde p, \tilde q$ with

$$|p - \tilde p| + |q - \tilde q| < \delta,$$

it is clear that one can choose curves $\Gamma_0, \widetilde\Gamma_0 : [0, 1] \to \mathbb{R}^3 \backslash K$ such that

$$\Gamma_0(0) = p, \quad \Gamma_0(1) = q, \quad \widetilde\Gamma_0(0) = \tilde p, \quad \widetilde\Gamma_0(1) = \tilde q,$$

and the distance between the curves is of order δ in the sense that

$$\|\Gamma_0 - \widetilde\Gamma_0\|_{C^k([0,1])} < C\delta. \tag{4.6}$$

An immediate consequence of (4.6) is that, if we reverse the orientation of $\widetilde\Gamma_0$ by setting $\widehat\Gamma_0(t) := \widetilde\Gamma_0(1 - t)$, then the vector-valued measure

$$\dot\Gamma_0(t)\, dt + \dot{\widehat\Gamma}_0(t)\, dt$$

converges weakly to zero as $\delta \to 0$.

With these two observations, for each $\delta > 0$ we will construct a closed oriented curve Γ (piecewise smooth, although eventually we will smooth things out) such that the associated measure

$$\dot\Gamma(t)\, dt$$

converges weakly to $d\Lambda$ as $\delta \to 0$. To this end, on each curve γ_k we will fix two distinct points p_k, q_k. For each $\delta > 0$, let us take two points $\tilde p_k, \tilde q_k$ satisfying

$$|p_k - \tilde p_k| + |q_k - \tilde q_k| < \delta.$$

The second observation above ensures that one can connect the points $(p_k, \tilde p_k)$ with $(q_{k+1}, \tilde q_{k+1})$ through curves $\Gamma_k, \widehat\Gamma_k : [0, 1] \to \mathbb{R}^3$ that do not intersect one another and such that the measure

$$\sum_{k=1}^{N} (\dot\Gamma_k(t)\, dt + \dot{\widehat\Gamma}_k(t)\, dt)$$

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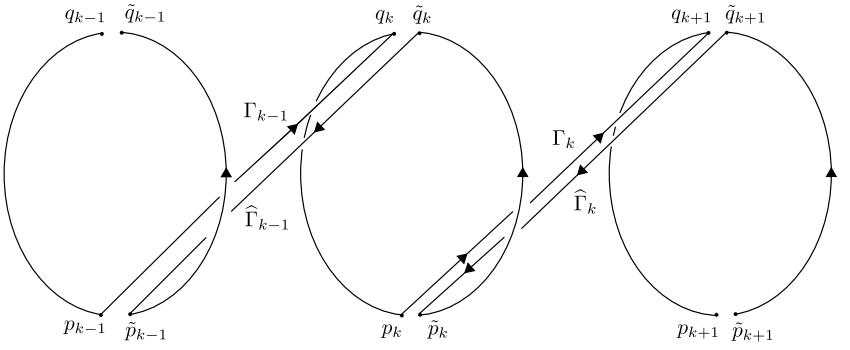


FIGURE 2. Passing from a collection of unknots to a single unknot.

converges weakly to zero as $\delta \to 0$. Here k ranges from 1 to N and we identify $N+1$ with 1. Let us define the piecewise smooth curve Γ as the union of the curves Γ_k, $\widehat{\Gamma}_k$ and the integral curves γ_k, from which we remove the $2N$ arcs of the curves of length of order δ connecting the points p_k with $\tilde{p}_k$ and q_k with $\tilde{q}_k$. It can easily be seen that one can choose the orientation of the curves Γ_k, $\widehat{\Gamma}_k$ so that the curve Γ has a well-defined orientation, which coincides with the orientation of each γ_k (cf. Figure 2).

The two observations that we made above then imply that the measure $\dot{\Gamma}(t)\,dt$ converges weakly to $d\Lambda$ as $\delta \to 0$, so we infer from Lemma 3.1 that the magnetic field created by Γ is close to Y in the sense that

$$\|B_\Gamma - Y\|_{C^k(K)} < \frac{\delta_2}{3c} \tag{4.7}$$

whenever the constant δ is small enough. Furthermore, since Γ has been constructed as the connected sums of the unknots $\gamma_1, \ldots, \gamma_N$, it is standard that it is also an unknot.

Step 4: From the unknot Γ to γ through a connected sum taking place far from the magnetic line. Let us take a large number R that will be fixed later. Translating the curve γ if necessary, we can assume that the distance between γ and the set K is at least R, and that γ does not intersect Γ.

Let us fix points $P \in \Gamma$, $Q \in \gamma$. For small δ, let us take another couple of points $\tilde{P} \in \Gamma$, $\tilde{Q} \in \gamma$ with

$$|P - \tilde{P}| + |Q - \tilde{Q}| < \delta_3$$

for a small enough constant δ_3. The second observation in Step 3 ensures that there are oriented curves Γ_0, $\widehat{\Gamma}_0$ connecting the points P, $\tilde{P}$ with Q, $\tilde{Q}$, respectively, and such that the measure

$$\dot{\Gamma}_0(t)\,dt + \hat{\dot{\Gamma}}_0(t)\,dt$$

converges weakly to 0 as $\delta_3 \to 0$. One can obviously assume that the distance from these curves to the set K is uniformly bounded away from zero. We can now define a piecewise smooth curve Γ' as the union of the curves Γ_0, $\widehat{\Gamma}_0$, Γ and γ, without the two arcs of length of order δ_3 that connect the points P and Q with $\tilde{P}$ and $\tilde{Q}$, in each case. We choose the orientation of Γ' so that it coincides with that of γ and Γ.

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It follows from the construction that Γ' is isotopic to γ (because it is the connected sum of γ with an unknot) and that

$$\dot{\Gamma}'(t)\,dt \to \dot{\Gamma}(t)\,dt + \dot{\gamma}(t)\,dt$$

as $\delta_3 \to 0$. By Lemma 3.1, one then has

$$\lim_{\delta_3 \to 0} \|B_{\Gamma'} - B_\Gamma - B_\gamma\|_{C^k(K)} = 0.$$

Since the distance between K and γ is at least R, it is clear that

$$\|B_\gamma\|_{C^k(K)} \leq \frac{C}{R^2},$$

so $B_{\Gamma'}$ converges to B_Γ on K as $R \to \infty$ and $\delta_3 \to 0$. For convenience, let us denote by Γ'' the curve that one obtains by slightly rounding off the corners of Γ', which can then be chosen (for large R and small δ_3) to satisfy

$$\|B_{\Gamma''} - B_\Gamma\|_{C^k(K)} < \frac{\delta_2}{3c}.$$

By equations (4.5) and (4.7), it then follows that

$$\begin{aligned}\|\mathcal{B}_J - cB_{\Gamma''}\|_{C^k(K)} &\leq \|\mathcal{B}_J - cY\|_{C^k(K)} + c\|B_\Gamma - Y\|_{C^k(K)} + c\|B_\Gamma - B_{\Gamma''}\|_{C^k(K)} \\ &< \delta_2,\end{aligned}$$

so condition (4.3) ensures that the magnetic field $B_{\Gamma''}$ generated by the wire Γ'' (which is isotopic to γ) has a periodic magnetic line that is isotopic to (and actually a C^k small deformation of) $\widetilde{\gamma}$. Theorem 1.1 then follows.

Remark 4.1. It is worth noticing that, although we have chosen a very concrete current J in the proof of Theorem 1.1, the same argument works in much greater generality. In particular, the argument goes through for any current of the form considered in Lemma 2.1 provided that the function $G(s)$ does not vanish and the Fourier coefficients of $F(\theta)$ satisfy

$$a_1 = b_1 = 0, \quad a_2^2 + b_2^2 \neq 0.$$

5. *Existence of knotted magnetic lines for generic knotted wires*

In this section we will prove Theorem 1.2. For concreteness, let us take again the tangent vector field J on the toroidal surface $\mathcal{S}_{\gamma,\epsilon}$ given by (4.1), where ϵ is a small constant.

Our objective is to show that there are periodic curves $\Gamma_n : \mathbb{R}/n\mathbb{Z} \to \mathcal{S}_{\gamma,\epsilon}$ satisfying the hypotheses of Lemma 3.3 for the field J that are isotopic to γ. Notice that, as the curve lies on $\mathcal{S}_{\gamma,\epsilon}$, the C^0 norm of the difference between the isotopy and the identity is of order ϵ, and can therefore be made as small as one wishes. Furthermore, we will choose the curve so that the number of intersection points $\#(\Gamma_n \cap \mathcal{C})$, as defined in Lemma 3.3, is precisely n. Lemma 3.3 then implies that there is a constant c_0 such that the measure

$$\frac{c_0}{n}\dot{\Gamma}_n(t)\,dt$$

converges weakly to $J\,dS$, so Lemma 2.1 ensures that for any compact set K that does not cut $\mathcal{S}_{\gamma,\epsilon}$ one has

$$\lim_{n\to\infty} \left\| \frac{c_0}{n} B_{\Gamma_n} - \mathcal{B}_J \right\|_{C^k(K)} = 0. \tag{5.1}$$

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Since we proved in Step 1 of §4 that the field $\mathcal{B}_J$ has a hyperbolic periodic magnetic line isotopic to γ and C^k close to it, it stems from (5.1) that for large enough n the magnetic field B_{Γ_n} also has a periodic magnetic line isotopic and C^k close to γ. Hence Theorem 1.2 will follow once we construct the curve Γ_n.

The construction of the curve Γ_n is simpler in the coordinates (α, σ) introduced in (3.6). We will henceforth use the notation developed in the proof of Lemma 3.2 without further mention. Let us consider a sequence of points

$$\{\alpha_k\}_{k=1}^{\infty} \subset \mathbb{T}^1 \tag{5.2}$$

that are uniformly distributed with respect to the probability measure (3.9). We can safely assume that $\alpha_k \neq \alpha_{k'}$ for all $k \neq k'$. For each n, let us write

$$\{\alpha_k : 1 \leq k \leq n\} = \{\alpha_{n,k} : 1 \leq k \leq n\},$$

where $\alpha_{n,k}$ is a relabeling of the n first points α_k chosen so that, identifying the points in $\mathbb{T}^1$ with numbers in $[0, 1)$, one has

$$0 \leq \alpha_{n,1} < \alpha_{n,2} < \cdots < \alpha_{n,n}.$$

Since the probability measure (3.9) is absolutely continuous, the sequence (5.2) is dense on $\mathbb{T}^1$, so the difference

$$[0, 1) \ni \Delta_{n,k} := \alpha_{n,k+1} - \alpha_{n,k} \mod 1,$$

understood as a number in $[0, 1)$ where we are using the convention $\alpha_{n,n+1} \equiv \alpha_{n,1}$, must tend uniformly to zero in the sense that

$$\lim_{n\to\infty} \max_{1\leq k\leq n} \Delta_{n,k} = 0. \tag{5.3}$$

Take a smooth function $\chi : \mathbb{R} \to \mathbb{R}$ such that

$$\chi(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ 1 & \text{for } t \geq 1. \end{cases}$$

Let us define smooth curves $\widetilde{\Gamma}_n : \mathbb{R}/n\mathbb{Z} \to \mathbb{T}^2$ in terms of the coordinates (α, σ) as $\widetilde{\Gamma}_n(t) := (\alpha_n(t), \sigma_n(t))$, where we set

$$\alpha_n(t) := \alpha_{n,1} + \textstyle\sum_{k=1}^{n} \Delta_{n,k}\chi(t + 1 - k) \mod 1,$$
$$\sigma_n(t) := t \mod 1.$$

It is clear that $\widetilde{\Gamma}_n$ has period n and that in each period the curve winds once along the coordinate α and n times along the coordinate σ. Moreover,

$$\dot{\widetilde{\Gamma}}_n(t) = \left(\sum_{k=1}^{n} \Delta_{n,k}\, \chi'(t + 1 - k) \right) \partial_\alpha + \partial_\sigma,$$

where χ' denotes the derivative of χ. Since at most one of the functions $\chi'(t + 1 - k)$ can be non-zero at any time $t \in \mathbb{R}$, it is apparent from (5.3) that

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} |\dot{\widetilde{\Gamma}}_n(t) - \partial_\sigma| = 0. \tag{5.4}$$

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Finally, let us now define the curve $\Gamma_n : \mathbb{R}/n\mathbb{Z} \to \mathcal{S}_{\gamma,\epsilon}$ as

$$\Gamma_n(t) := \Psi^{-1}\widetilde{\Gamma}_n(t).$$

We claim that, as $\widetilde{\Gamma}_n$ winds once in the α-direction and n times in the σ-direction, the curve Γ_n is a $(1 : n')$ cable over the core curve γ, where

$$n' := n - N_0$$

with N_0 a fixed number. In particular, for any large enough n, Γ_n is isotopic to γ. For this we will need to compute the expression for the curves in the Frenet coordinates and to take into account the curves' own twist. The reason is that, as changes of coordinates in the torus do not necessarily come from an ambient diffeomorphism, one cannot use arbitrary coordinates on the torus to check the isotopy type of a curve.

To check this, we start by taking the set $\mathcal{C}$ as

$$\mathcal{C} := \{r = \epsilon, \theta = 0\},$$

so the coordinate $\alpha : \mathcal{C} \to \mathbb{T}^1$ can be chosen as

$$\alpha := \frac{s}{\ell},$$

where we recall that ℓ denotes the length of the curve γ. The equation for the integral curves of J,

$$\dot{s} = \frac{2\cos 2s}{1 - \epsilon\kappa(s)\cos\theta},$$
$$\dot{\theta} = \frac{1}{\epsilon(1 - \epsilon\kappa(s)\cos\theta)},$$

implies that in terms of the time variable t' defined by the ODE

$$\frac{dt'}{dt} = \frac{1}{2\pi\epsilon[1 - \epsilon\kappa(s(t))\cos\theta(t)]}, \quad t'|_{t=0} = 0,$$

the integral curves are given by

$$s = s_0 + \epsilon\sin(2\theta_0 + 4\pi t'), \quad \theta = \theta_0 + 2\pi t'.$$

Since

$$t' = \frac{t(1 + O(\epsilon))}{2\pi\epsilon},$$

it turns out that the period is

$$T = 2\pi\epsilon + O(\epsilon^2),$$

so

$$\widetilde{J} = TJ = 2\pi\partial_\theta + O(\epsilon).$$

Hence the variable σ can be written in terms of (s, θ) as

$$\sigma = \frac{\theta}{2\pi} + O(\epsilon).$$

In view of the expression for (α, σ) in terms of (s, θ), it is apparent that the curve Γ_n winds once along the coordinate s and n times along the coordinate θ. The coordinates

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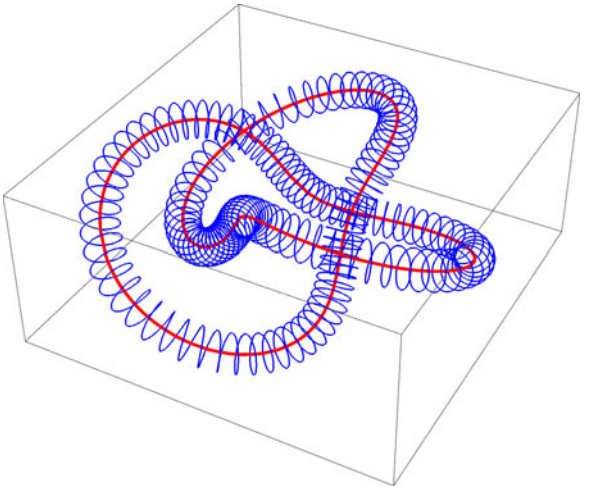


FIGURE 3. A wire (thin line) isotopic to the core knot (in this case, a trefoil) and C^0 close to it that has a magnetic line isotopic to the core knot. The wire and the core curve can be chosen arbitrarily close.

(s, θ) correspond to the Frenet frame, which is well known [**12**] to twist N_0 times along the curve γ, where

$$\pm N_0 = \frac{1}{2\pi} \int_0^\ell \tau(s)\, ds + \mathrm{Writhe}(\gamma)$$

is the total torsion of the curve plus its writhe (the sign here depends of the orientation of the frame). Hence we infer that Γ_n is a $(1 : n')$ cable over γ, so it is isotopic to γ (see Figure 3).

By construction, the intersection of Γ_n with the set $\mathcal{C}$ is the image under the diffeomorphism Θ^{-1} of the points $\{\alpha_k : 1 \le k \le n\}$, so as $n \to \infty$ they are distributed with respect to the measure (3.9). Moreover, since the push-forward of the field $\widetilde{J}$ under Ψ is precisely ∂_σ, it follows from (5.4) that

$$\lim_{n\to\infty} \sup_{\sigma\in\mathbb{R}} |\dot{\Gamma}_n(\sigma) - \widetilde{J}(\Gamma_n(\sigma))| = 0.$$

Hence Γ_n is a sequence of curves that has the properties that we required above, so Theorem 1.2 follows.

Remark 5.1. Although we have taken a concrete example of current field J for which all the computations can be made in a very explicit way, the argument carries over verbatim to a much more general class of fields J. In particular, there are two sufficient conditions for the argument to remain valid.

(i) The integral curves of J are all small deformations of (and isotopic to) the circles $\{s = \text{constant}\}$, in the coordinates that we defined on the surface. (In fact, while the fact that the integral curves are all periodic is key, the condition that they are small deformations of the circles $\{s = \text{constant}\}$ can be relaxed significantly, as it is only used to control the isotopy type of the curve.)

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(ii) The magnetic field $\mathcal{B}_J$ has a hyperbolic periodic magnetic line isotopic to and C^k close to γ. As we saw in §4, a sufficient condition for this is that the functions F and G that appear in the field J satisfy the conditions of Remark 4.1.

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