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Smooth, mixing transformations with loosely Bernoulli Cartesian square

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Abstract. A zero-entropy system is said to be loosely Bernoulli if it can be induced from an irrational rotation of the circle. We provide a criterion for zero-entropy systems to be loosely Bernoulli that is compatible with mixing. Using this criterion, we show the existence of smooth mixing zero-entropy loosely Bernoulli transformations whose Cartesian square is loosely Bernoulli.

Key words: classical ergodic theory, classification in ergodic theory, smooth dynamics
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1. *Introduction*

One of the main goals of ergodic theory is to describe invariants and models which classify broad classes of systems with respect to certain equivalence relations. Some of these relations, such as metric isomorphism, may prove too restrictive, whereas some others, such as orbit equivalence, may prove too flexible (recall that, by Dye's theorem, [2] any two ergodic measure-preserving transformations on non-atomic spaces are orbit-equivalent). Kakutani equivalence, initially introduced by Kakutani in [15], can be thought of as an intermediate notion of classification. Two ergodic measure-preserving transformations (X, T) and (Y, S) are said to be *Kakutani equivalent* if there exist positive measure sets $A \subset X$, $B \subset Y$ such that the induced maps T_A, S_B are metrically isomorphic.

Particularly important in the theory of Kakutani is the equivalence class of irrational circle rotations. This class was initially considered by Katok [20] under the name of *standard* or *loosely Kronecker* transformations, and independently introduced by Feldman [7], who coined the term *loosely Bernoulli (LB)* to denote the ergodic automorphisms that are Kakutani equivalent to an irrational circle rotation in the zero-entropy case, or to a Bernoulli shift in the positive entropy case.

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Most of the existing criteria for loose-Bernoullicity rely on the presence of particular periodic approximations for the underlying transformation (a remarkable exception to this is the horocycle flow [25]). Unfortunately, these approximations do not easily coexist with mixing and some of these criteria directly forbid it (see, for example, [18, Proposition 3.5]). Furthermore, the coexistence of mixing with certain periodic approximations, such as rank one transformations, becomes even more difficult in the smooth setting (see [3]).

In this work, we aim to provide criteria that are compatible with mixing which guarantee the loose-Bernoullicity of a zero-entropy system and of its Cartesian square. Our criteria will rely on localized periodic approximations in the same spirit as those introduced by Fayad in [4]. We then show that these criteria are verified by a class of infinitely differentiable zero-entropy mixing transformations in $\mathbb{T}^3$. These systems are obtained by modifying the class of mixing transformations introduced in [6].

1.1. *Periodic approximations.* Approximation of measure-preserving systems by periodic transformations was first proposed in the forties by Halmos [12] who provided a general framework for this theory and used it to prove the genericity of several ergodic properties. Halmos [13] successfully applied his method to prove that the set of weak mixing transformations is a G_δ dense set in the space of measure-preserving automorphisms endowed with the weak topology. Shortly after and by similar methods, Rokhlin [27] proved that the set of mixing transformations is of the first category under the same weak topology.

Two decades later, the theory of periodic approximations regained strength after the works of Katok and Stepin, [19, 21, 22, 28]. These works laid out the basis of the so-called *method of approximations* and introduced a different point of view on periodic approximations which, in contrast to Halmos' approach, allowed the study of individual automorphisms by approximation. Katok [20] provided periodic approximation criteria for standardicity. He proved that a system with *good cyclic* approximations must be standard and that the existence of *excellent linked approximations of type* $(h, h+1)$ for a given transformation guarantee the standardicity of its Cartesian square. For definitions and proofs of these statements, we refer the interested reader to [18, 20].

1.2. *Cartesian products and Cartesian squares.* Several notions related to periodic approximations, such as rank one, finite rank, local rank one and very weakly Bernoulli, are known to imply loose-Bernoullicity (see [9, 24]). All of these properties, which attempt to capture precise aspects of the long-time behaviour of measure-preserving transformations, provide important tools for classification and illustrate the richness of existing phenomena in abstract ergodic theory.

However, given two measure-preserving automorphisms satisfying any of the ergodic properties mentioned above, it is not necessarily true that their Cartesian product and, in particular, the Cartesian square of any of these transformations, verifies the same property. The loose-Bernoullicity of Cartesian products of LB transformations was first considered by Swanson [29] who associated to every totally ergodic LB transformation a class of zero-entropy LB automorphisms whose Cartesian product with the initial transformation is LB. On the other hand, Ornstein, Rudolph and Weiss [24] constructed

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rank one transformations whose Cartesian square is not LB. Shortly after this, M. Ratner showed that any non-trivial transformation T in the horocycle flow is LB [25] but its Cartesian square $T \times T$ is not [26]. In a recent work, Kanigowski and Wei [17] studied the loose-Bernoullicity of Cartesian products of Kochergin flows and showed that, in general, the Cartesian product of Kochergin flows with different power of singularities is not LB. Cartesian products of rank one staircase transformations that are not LB were constructed by Kanigowski and de la Rue in [16].

The question of the existence of zero-entropy LB transformations with LB Cartesian square was proposed by Swanson in [29] and was positively answered by A. Katok who constructed examples of weak mixing transformations verifying this property (a detailed account of his construction can be found in [10]). Katok's example was later modified by Gerber [10] who showed the existence of a zero-entropy LB transformation that is mixing of all orders and whose Cartesian square is LB. More recently, Gerber and Kunde [11] provided examples of smooth LB transformations whose Cartesian square is LB. The loose-Bernoullicity of their examples relies on a slight modification of the aforementioned loose-Bernoullicity criterion proved by Katok [20], namely, the existence of excellent linked approximations of type $(h, h+1)$. As in Katok's case, their criterion also guarantees that the transformation is weak mixing but not mixing (see [11, Proposition 4.1] for the exact statement of the criterion).

1.3. *Plan of the work.* This work is organized as follows. In §2, we introduce the notation and basic objects we use throughout the paper. Proposition 2.2 is an adaptation of the loose-Bernoullicity criterion (Proposition 2.1) proved by E. Janvresse and T. de la Rue in [14].

In §3, we combine Proposition 2.2 and the existence of an appropriate sequence of Rokhlin towers to prove a loose-Bernoullicity criterion for ergodic zero-entropy processes (item 3 in Proposition 3.4). We stress the fact that although the measures of the towers in the criterion verify certain non-summability conditions, their measures are not necessarily bounded from below. As a direct consequence of Proposition 3.4, we deduce, in Theorem 3.5, a similar loose-Bernoullicity criterion for ergodic zero-entropy transformations. We then adapt these criteria to obtain analogous results for the Cartesian square of weak mixing zero-entropy transformations (Proposition 3.7 and Theorem 3.8).

Finally, in §4, we prove the existence of a family of smooth, mixing, zero-entropy, LB automorphisms of $\mathbb{T}^3$ whose Cartesian square is LB (Theorem 4.3). The Appendix contains technical tools used in the construction of these transformations.

2. *Preliminaries*

Throughout this work, $(X, \mu, \mathcal{E})$ will denote a Lebesgue measure space and T will denote a zero-entropy measure-preserving automorphism on X. If there is no risk of confusion, we will denote $(X, \mu, \mathcal{E})$ simply by X.

2.1. *Partitions and $\mathcal{P}$-names.* We denote by $\mathcal{P}_f(X)$ the set of finite measurable partitions of X. We consider a *partition* $\mathcal{P} \in \mathcal{P}_f(X)$ to be a collection of disjoint sets $\mathcal{P} = \{P_1, \ldots, P_m\}$ with a fixed ordering of its elements and whose union is X. Given

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that $A \in \mathcal{E}$, the induced partition on A is defined as

$$\mathcal{P}|_A = \{P_i \cap A \mid 1 \le i \le m\}.$$

Given two partitions $\mathcal{P} = \{P_1, \dots, P_m\}$, $\mathcal{Q} = \{Q_1, \dots, Q_l\}$ of X, we define its Cartesian product $\mathcal{P} \times \mathcal{Q}$ as the finite partition of $X \times X$ given by

$$\mathcal{P} \times \mathcal{Q} = \{P_i \times Q_j \mid 1 \le i \le m, 1 \le j \le l\}$$

and ordered lexicographically. We say that $\mathcal{P}$ *refines* $\mathcal{Q}$ if, for every $P \in \mathcal{P}$, there exists $Q \in \mathcal{Q}$ such that $P \subset Q$. We denote this relation by $\mathcal{P} \le \mathcal{Q}$. Whenever $l = m$, we denote

$$D(\mathcal{P}, \mathcal{Q}) = \sum_{k=1}^{m} \mu(P_k \Delta Q_k).$$

Given that $\mathcal{P} \in \mathcal{P}_f(X)$, the pair $(T, \mathcal{P})$ is called a *process*. For $x \in X$, we define its $\mathcal{P}$-*name* as the unique vector $x^{\mathcal{P}} \in \mathbb{Z}^m$ satisfying, for all $i \in \mathbb{Z}$,

$$x_i^{\mathcal{P}} = j \qquad \text{if and only if} \qquad T^i(x) \in P_j,$$

where $x_i^{\mathcal{P}}$ denotes the ith coordinate of $x^{\mathcal{P}}$. Recall that whenever $\mathcal{P}$ is a *generating partition* of $\mathcal{E}$, that is, $\mathcal{E}$ is the smallest T invariant σ-algebra containing $\mathcal{P}$, the system (X, T) is equivalent (in a measure theoretic sense) to the shift over the space of $\mathcal{P}$-names of X. Given that $n \in \mathbb{N}$, we define the $\mathcal{P}$-n-*name* associated to x as the n-tuple

$$x^{\mathcal{P},n} = (x_i^{\mathcal{P}})_{0 \le i < n}.$$

For a partition with exactly m elements, the $\mathcal{P}$-n-names are naturally a subset of the space of *words of length n* over the finite alphabet $\{1, 2, \dots, m\}$, which we denote by W_m^n. The *length* of a word w is denoted by $|w|$. The *concatenation* of two words v, w is denoted by $v \cdot w$. To simplify the notation, we denote the concatenation of k words $v_1, v_2, \dots, v_k$ by $\prod_{i=1}^{k} v_i$. We denote the *Hamming distance* in the space of n-words by

$$d_n(v, w) = \frac{1}{n} \#\{0 \le i < n \mid v_i \ne w_i\}$$

and the *modified Hamming distance* or $\overline{f}$ *distance* by

$$\overline{f}_n(v, w) = \frac{n - l}{n},$$

where l is the maximum natural number for which there exist sequences

$$0 \le i_1 < i_2 < \cdots < i_l < n, \quad 0 \le j_1 < j_2 < \cdots < j_l < n,$$

obeying $v_{i_s} = w_{j_s}$ for $s = 1, \dots, l$. Given that $x, y \in X$, we denote

$$d_n^{\mathcal{P}}(x, y) = d_n(x^{\mathcal{P},n}, y^{\mathcal{P},n}), \quad \overline{f}_n^{\mathcal{P}}(x, y) = \overline{f}_n(x^{\mathcal{P},n}, y^{\mathcal{P},n}).$$

For any $W \in \mathcal{E}$, denote

$$\overline{f}_n^{\mathcal{P}}(W) = \sup_{x,y \in W} \overline{f}_n^{\mathcal{P}}(x, y).$$

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2.2. *Loose-Bernoullicity.* Since we will only consider zero-entropy transformations, we provide a simplified definition of loose-Bernoullicity for this particular case. For a more general definition, we refer the reader to [7, 24]. In the following, we will refer to a zero-entropy measure-preserving transformation $(X, \mu, \mathcal{E}, T)$ equipped with a finite measurable partition $\mathcal{P} \in \mathcal{P}_f(X)$ as a *zero-entropy process* which, by an abuse of notation, we denote simply by $(T, \mathcal{P})$.

A zero-entropy process $(T, \mathcal{P})$ is called LB if, for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, there exists $x \in X$ obeying

$$\mu(\{y \in X \mid \overline{f}_n^{\mathcal{P}}(x, y) < \epsilon\}) > 1 - \epsilon.$$

A zero-entropy automorphism T is said to be LB if the associated process $(T, \mathcal{P})$ is LB for some generating partition $\mathcal{P}$ or, equivalently, if there exists a refining sequence of finite measurable partitions $(\mathcal{P}_n)_{n\in\mathbb{N}} \subset \mathcal{P}_f(X)$ obeying $\mathcal{P}_n \to \mathcal{E}$ such that $(T, \mathcal{P}_n)$ is LB for all $n \in \mathbb{N}$. Recall that $\mathcal{P}_n \to \mathcal{E}$ means that, for any $A \in \mathcal{E}$, there exists a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{E}$ such that A_n is $\mathcal{P}_n$-measurable and $\mu(A_n \Delta A) \to 0$. Let us point out that a transformation is LB if and only if, for any finite measurable partition $\mathcal{P}$, the process $(T, \mathcal{P})$ is LB. For the proof of this fact and a detailed account of LB transformations, see [24].

The following criterion for loose-Bernoullicity was proved by E. Janvresse and T. de la Rue in [14].

PROPOSITION 2.1. *Let $(T, \mathcal{P})$ be a zero-entropy process. Suppose that, for all $\epsilon > 0$ and for $\mu \times \mu$-almost every $(x, y) \in X \times X$, there exists $l(x, y) \in \mathbb{N}$ such that*

$$\overline{f}_{l(x,y)}^{\mathcal{P}}(x, y) < \epsilon.$$

Then the process $(T, \mathcal{P})$ is LB.

2.2.1. *The $P(\mathcal{P}, \alpha, \delta, n)$ property.* Given that $\alpha, \delta \in [0, 1)$ and $n \in \mathbb{N}$, we say that the transformation T verifies property $P(\mathcal{P}, \alpha, \delta, n)$ or, equivalently, that the process $(T, \mathcal{P})$ verifies property $P(\alpha, \delta, n)$ if there exists $W \in \mathcal{E}$ obeying

$$\mu(W) > 1 - \delta, \quad \overline{f}_n^{\mathcal{P}}(W) < 1 - \alpha.$$

With this notation, a process $(T, \mathcal{P})$ is LB if and only if, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, the process verifies property $P(1 - \epsilon, \epsilon, n)$. Proposition 2.1 implies the following criterion for loose-Bernoullicity.

PROPOSITION 2.2. *Let $(T, \mathcal{P})$ be a zero-entropy process. Suppose that there exist sequences $\delta_n \searrow 0$, $\alpha_n \nearrow 1$, $(m_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $(T, \mathcal{P})$ verifies $P(\alpha_n, \delta_n, m_n)$ for all $n \in \mathbb{N}$. Then the process $(T, \mathcal{P})$ is LB.*

Proof. By hypothesis, for all $n \in \mathbb{N}$, there exists $W_n \in \mathcal{E}$ such that

$$\mu(W_n) > 1 - \delta_n, \quad \overline{f}_{m_n}^{\mathcal{P}}(W_n) < 1 - \alpha_n.$$

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Define

$$E = \limsup_{n\in\mathbb{N}}(W_n \times W_n) = \bigcap_{i=1}^{\infty}\bigcup_{n=i}^{\infty} W_n \times W_n.$$

Clearly, $(\mu\times\mu)(E) = 1$. Suppose $\epsilon > 0$ and let $n_0 \in \mathbb{N}$ satisfy $1 - \alpha_{n_0} < \epsilon$. Given that $(x, y) \in E$, there exists $n > n_0$ such that $(x, y) \in W_n \times W_n$. Therefore

$$\overline{f}^{\mathcal{P}}_{m_n}(x, y) < 1 - \alpha_n < \epsilon.$$

By Proposition 2.1, the process $(T, \mathcal{P})$ is LB. □

2.3. *Rokhlin towers.* Given that $F \in \mathcal{E}$ and $h \in \mathbb{N}$, we say that the collection

$$\mathcal{T} = \mathcal{T}(F, h) = \{F, TF, \cdots T^{h-1}\}$$

is a *Rokhlin tower* of *height* h if the sets $F, TF, \ldots, T^{h-1}F$ are pairwise disjoint. The sets T^iF are called *levels* of the tower. The first and last elements of the collection are known as the *base* and the *roof* of the tower. The union of all levels is called the *support* of the tower. By an abuse of notation, $\mathcal{T}$ will denote the support of the tower and also the tower itself. The measure of the support is called the *size* of the tower. We denote by $\mathcal{T}(X)$ the set of Rokhlin towers of X. Given that $\mathcal{T} \in \mathcal{T}(X)$, we denote its height by $h_{\mathcal{T}}$ and its size by $\mu_{\mathcal{T}}$.

As we shall see, Rokhlin towers can be used to easily *match* coordinates between two given $\mathcal{P}$-names. Given two words v, w, not necessarily having the same length, and a natural number l, we say that we can match at least l coordinates of v and w if there exist sequences

$$0 \le i_1 < i_2 < \cdots < i_l \le |v|, \quad 0 \le j_1 < j_2 < \cdots < j_l \le |w|$$

obeying

$$v_{i_s} = w_{j_s} \quad \text{for } s = 1, \ldots, l.$$

Whenever $|v| = n = |w|$, this is equivalent to saying that $\overline{f}_n(v, w) \le (n - l)/n$. To illustrate how to use Rokhlin towers to match two $\mathcal{P}$-n-names associated to different points, let $(T, \mathcal{P})$ be a process, let $\mathcal{T}(F, h)$ be a Rokhlin tower and consider the finite measurable partition $\mathcal{Q} \in \mathcal{P}_f(X)$ given by

$$\mathcal{Q} = \mathcal{T} \cup \mathcal{P}|_{\mathcal{T}^c}.$$

Notice that

$$x, y \in F \implies x^{\mathcal{Q},h} = y^{\mathcal{Q},h}.$$

Thus, for any $x, y \in X$, finding iterates

$$0 \le i_1 < i_2 < \cdots < i_L < n - h, \quad 0 \le j_1 < j_2 < \cdots < j_L < n - h$$

for which $T^{i_l}x, T^{j_l}y \in F$ means that we can match at least hL coordinates of $x^{\mathcal{Q},n}$ and $y^{\mathcal{Q},n}$. Such iterates can be easily found whenever the system possesses some recurrence property. For ergodic systems, the relation between $\mathcal{P}$-n-names and $\mathcal{Q}$-n-names, for sufficiently large values of n (see Lemma 2.3), is determined by how well the tower $\mathcal{T}$

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approximates the restricted partition $\mathcal{P}|_{\mathcal{T}}$. Given a partition $\mathcal{P} = \{P_1, \ldots, P_m\} \in \mathcal{P}_f(X)$ and a Rokhlin tower $\mathcal{T} \in \mathcal{T}(X)$, we say that $\mathcal{T}$ is *δ-monochromatic with respect to $\mathcal{P}$* if

$$\Delta(\mathcal{T}, \mathcal{P}) = \min_{\substack{\mathcal{T} \leq \xi, \\ |\xi| = m}} \sum_{k=1}^{m} \mu(\xi_k \setminus P_k) < \delta,$$

where the minimum is taken over all the partitions of the support of the tower with exactly m sets and whose elements are unions of levels of the tower. If

$$\Delta(\mathcal{T}, \mathcal{P}) = 0,$$

we say that the tower is *monochromatic with respect to $\mathcal{P}$*. Let us point out that this is equivalent to

$$\mathcal{T} \leq \mathcal{P}|_{\mathcal{T}}.$$

We can provide an explicit expression of a minimizing partition in the definition of $\Delta(\mathcal{T}, \mathcal{P})$, and thus of $\Delta(\mathcal{T}, \mathcal{P})$ itself, as follows. Given a tower $\mathcal{T} = \mathcal{T}(F, h)$ and a partition $\mathcal{P} = \{P_1, \ldots, P_m\}$, define, for all $0 < k < h$,

$$I_{\mathcal{P}}^{\mathcal{T}}(k) = \min \left\{ 0 < j \leq m \;\middle|\; \mu(T^k(F) \setminus P_j) = \min_{0 < i \leq m} \mu(T^k(F) \setminus P_i) \right\}.$$

If there is no risk of confusion, we denote $I_{\mathcal{P}}^{\mathcal{T}}(k)$ simply by $I(k)$. Let

$$P_j^{\mathcal{T}} = \bigcup_{I(k) = j} T^k F$$

for all $0 < j \leq m$ and define $\mathcal{P}^{\mathcal{T}} = \{P_1^{\mathcal{T}}, \ldots, P_m^{\mathcal{T}}\}$. Denote

$$D_{\mathcal{P}}^{\mathcal{T}} = \bigcup_{k=1}^{h} T^k(F) \setminus P_{I(k)}. \tag{1}$$

The set $D_{\mathcal{P}}^{\mathcal{T}}$ is called the *difference set* of $\mathcal{T}$ and $\mathcal{P}$. By construction,

$$\Delta(\mathcal{T}, \mathcal{P}) = \sum_{k=1}^{m} \mu(P_k^{\mathcal{T}} \setminus P_k) = \mu(D_{\mathcal{P}}^{\mathcal{T}}).$$

Notice that, for all $x, y \in X \setminus D_{\mathcal{P}}^{\mathcal{T}}$,

$$x_0^{\mathcal{Q}} = y_0^{\mathcal{Q}} \implies x_0^{\mathcal{P}} = y_0^{\mathcal{P}}.$$

By applying the pointwise ergodic theorem to $D_{\mathcal{P}}^{\mathcal{T}}$, we obtain the following lemma.

LEMMA 2.3. *Let $(T, \mathcal{P})$ be an ergodic process. Suppose that $\mathcal{T} \in \mathcal{T}(X)$ and denote*

$$\mathcal{Q} = \mathcal{T} \cup \mathcal{P}|_{\mathcal{T}^c}.$$

Given that $\delta > 0$, there exists $n_0 \in \mathbb{N}$ and $W \in \mathcal{E}$ with $\mu(W) > 1 - \delta$ such that

$$d_n^{\mathcal{P}}(x, y) \leq d_n^{\mathcal{Q}}(x, y) + 3\Delta(\mathcal{T}, \mathcal{P})$$

for all $n > n_0$ and for all $x, y \in W$.

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Proof. If $\Delta(\mathcal{T}, \mathcal{P}) = 0$, then $\mathcal{T} \leq \mathcal{P}|_{\mathcal{T}}$ which yields $d_n^{\mathcal{P}}(x, y) = d_n^{\mathcal{Q}}(x, y)$ for all $n \in \mathbb{N}$ and all $x, y \in X$. Otherwise, assume that $\Delta(\mathcal{T}, \mathcal{P}) > 0$. Denote by $D_{\mathcal{P}}^{\mathcal{T}}$ the difference set of $\mathcal{T}$ and $\mathcal{P}$ as defined in (1). By ergodicity, there exists $n_0 \in \mathbb{N}$ so that the measure of the set W of points in X satisfying

$$\#\{0 \leq i < n \mid T^i x \in D_{\mathcal{P}}^{\mathcal{T}}\} \leq \tfrac{3}{2}\mu(D_{\mathcal{P}}^{\mathcal{T}})n = \tfrac{3}{2}\Delta(\mathcal{T}, \mathcal{P})n$$

for all $n > n_0$ has measure at least $1 - \delta$. Hence, for all $n > n_0$ and all $x, y \in W$,

$$\begin{aligned} n(d_n^{\mathcal{P}}(x, y) - d_n^{\mathcal{Q}}(x, y)) \leq & \#\{0 \leq i < n \mid x_i^{\mathcal{Q}} = y_i^{\mathcal{Q}} \text{ and } x_i^{\mathcal{P}} \neq y_i^{\mathcal{P}}\} \\ \leq & \#\{0 \leq i < n \mid T^i x \in D_{\mathcal{P}}^{\mathcal{T}} \text{ or } T^i y \in D_{\mathcal{P}}^{\mathcal{T}}\} \\ \leq & 3\Delta(\mathcal{T}, \mathcal{P})n. \end{aligned}$$

□

3. *Loose-Bernoullicity criteria*

3.1. *Main proposition.* The following result is at the core of this work. It embodies the idea of using Rokhlin towers to improve existent bounds on the $\overline{f}$ distance associated to a given finite measurable partition. More precisely, given an ergodic process $(T, \mathcal{P})$, a natural number n and a positive measure set $W \in \mathcal{E}$, the presence of a sufficiently high, large and monochromatic Rokhlin tower implies the existence of a set W_+, whose measure can be taken to be as close to one as desired, such that

$$\overline{f}_m^{\mathcal{P}}(W_+) < \overline{f}_n^{\mathcal{P}}(W)$$

for sufficiently large m. Proposition 3.1 provides precise estimates on $\overline{f}_m(W_+)$. Recall that, for any $W \in \mathcal{E}$, we denote

$$\overline{f}_n^{\mathcal{P}}(W) = \sup_{x,y \in W} \overline{f}_n^{\mathcal{P}}(x, y)$$

and that a process $(T, \mathcal{P})$ is said to verify property $P(\alpha, \delta, n)$ if there exists $W \in \mathcal{E}$ such that

$$\mu(W) > 1 - \delta, \quad \overline{f}_n(W) < 1 - \alpha.$$

The following proof is inspired by S. Ferenczi's argument in [**8**, Proposition 1] which relies heavily on the ideas introduced in [**24**, §7] by Orstein, Rudolph and Weiss.

PROPOSITION 3.1. *Let $(T, \mathcal{P})$ be an ergodic process verifying $P(\alpha, \delta, N)$ for some $\alpha \geq 0$, $\delta > 0$ and $N \in \mathbb{N}$. Suppose that there exists a Rokhlin tower $\mathcal{T} \in \mathcal{T}(X)$ obeying*

$$h_{\mathcal{T}} > \frac{2N}{\delta}.$$

Then, for any $\delta > 4\delta_+ > 0$, there exists $N_+ \in \mathbb{N}$ such that the process $(T, \mathcal{P})$ verifies $P(\alpha_+, \delta_+, N_+)$, where

$$\alpha_+ = \alpha + \frac{\mu_{\mathcal{T}}^2}{10}(1 - \alpha)^2 - 6\delta - 3\Delta(\mathcal{T}, \mathcal{P}). \tag{2}$$

Proof. Denote $\mathcal{T} = \mathcal{T}(F, h)$ and fix $0 < c < 1$. Let

$$\mathcal{T}_c = \mathcal{T}(F, \lceil ch \rceil - 1)$$

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and denote the size of $\mathcal{T}_c$ by μ_c. Let N_0 be sufficiently large so that the set W_0 of points in X satisfying

$$\#\{0 \le i < n \mid T^i x \in \mathcal{T}_c\} \ge \tfrac{3}{4}\mu_c n \quad \text{for all } n \ge N_0$$

has measure at least $1 - (\delta_+)/2$. Denote by W_1 the set given by property $P(\alpha, \delta, N)$. We will suppose, without loss of generality, that $4\delta < (1/10)$. Let N_+ be sufficiently large so that

$$N_+ > \max\left\{N, \frac{N_0}{\delta_+}\right\}$$

and such that the set W of points in X satisfying:

(1) $\#\{0 \le i < n \mid T^i x \in W_0\} \ge (1 - \delta_+)n$;
(2) $\#\{0 \le i < n \mid T^i x \in W_1\} \ge (1 - 2\delta)n$; and
(3) $\#\{0 \le i < n \mid T^i x \in F\} \ge (3\mu_{\mathcal{T}}/4h)n$,

for all $n > (N_+/2)$, has measure at least $1 - \delta_+$. Notice that both N_0 and N_+ are well defined by the pointwise ergodic theorem. Fix $x, y \in W$, $n \ge N_+$ and denote $v = x^{\mathcal{P},n}$, $w = y^{\mathcal{P},n}$. To prove the result, it suffices to show that

$$\overline{f}_n(v, w) < 1 - \alpha_+.$$

For an appropriate choice of c depending only on α (see equation (3)), the proposition will follow from the following claims.

Using the recurrence of x and y to $\mathcal{T}_c$, one can prove the following claim.

CLAIM 3.2. *There exists a natural number L and splittings $v = v^1 \cdot \cdots \cdot v^{2L+1}$, $w = w^1 \cdot \cdots \cdot w^{2L+1}$ such that:*

(1) $(nc/2)\mu_{\mathcal{T}}^2 \le hL \le nc\mu_{\mathcal{T}}^2$;
(2) $|v^{2k}| = h = |w^{2k}|$ *for all* $k = 1, \ldots, L$;
(3) $v_0^{2k}, w_0^{2k} \in F$ *for all* $k = 1, \ldots, L$; *and*
(4) $\sum_{k=0}^{L} ||v^{2k+1}| - |w^{2k+1}|| \le chL + \delta n$. □

Using the recurrence of x and y to the set W_1 given by $P(\alpha, \delta, N)$ and the fact that most of the odd blocks will be sufficiently big (condition (1) in Claim 1) and have comparable size (condition (4) in Claim 1), one can prove the following claim.

CLAIM 3.3. *There exist partial couplings between the odd blocks of the splittings of v, w in Claim 1 allowing one to match at least*

$$n\alpha\left(1 - \frac{Lh}{n}\left(1 + \frac{c}{2}\right) - 6\delta\right)$$

coordinates of v and w.

Before proving Claims 1 and 2, let us show how they imply the result. Let

$$\mathcal{Q} = \mathcal{T} \cup \mathcal{P}|_{\mathcal{T}^c}.$$

By Lemma 2.3, we can suppose, without loss of generality, that

$$d_n^{\mathcal{P}}(x, y) < d_n^{\mathcal{Q}}(x, y) + 3\Delta(\mathcal{T}, \mathcal{P}).$$

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Since any two points in the floor set F of the tower define the same $\mathcal{Q}$-h-name and by conditions (2) and (3) in Claim 1,

$$\sum_{k=1}^{L} \#\{0 \le i < h \mid v_i^{2k} \ne w_i^{2k}\} \le 3\Delta(\mathcal{T}, \mathcal{P})n.$$

Thus we can match at least

$$hL - 3\Delta(\mathcal{T}, \mathcal{P})n$$

coordinates of the even blocks associated to the splittings of v, w. Therefore, by Claims 1 and 2, we can match at least

$$n\left(\alpha + \frac{hL}{n}\left(1 - \alpha\left(1 + \frac{c}{2}\right)\right) - 6\delta - 3\Delta(\mathcal{T}, \mathcal{P})\right)$$

coordinates of v and w. Hence

$$\overline{f}_n(v, w) \le 1 - \alpha - \frac{hL}{n}\left(1 - \alpha\left(1 + \frac{c}{2}\right)\right) + 6\delta + 3\Delta(\mathcal{T}, \mathcal{P}).$$

Therefore, for

$$c = \min\left\{\frac{9}{10}, \frac{1-\alpha}{\alpha}\right\}, \tag{3}$$

the result follows. In fact, with this definition of c and by (1) in Claim 1,

$$\frac{hL}{n}\left(1 - \alpha\left(1 + \frac{c}{2}\right)\right) \ge \frac{\mu_{\mathcal{T}}^2}{10}(1-\alpha)^2. \qquad \square$$

Proof of Claim 3.2. Let $m = (1-2\delta)n$ and denote $l = \lceil (3\mu_{\mathcal{T}}/4h)m \rceil$. By properties (1) and (3) in the definition of W, there exist sequences $0 < t_1 < \cdots < t_l \le m$ and $0 < t_1' < \cdots < t_l' \le n$ obeying:

- $T^{t_k}x \in F$, $T^{t_k'}y \in W_0$;
- $t_k' \ge t_k$;
- $t_{k+1}' - t_k' \ge t_{k+1} - t_k$; and
- $\sum_{k=1}^{l}(t_k' - t_k) < \delta_+ n$.

Thus, for any $1 \le k \le l$ and by the definition of W_0 and N_0,

$$\#|\{0 \le j \le N_0 \ \mid \ T^{t_k'+j}y \in \mathcal{T}_c\}| \ge \frac{3c\mu_{\mathcal{T}}}{4}N_0.$$

Hence there exists $0 \le j \le N_0$ such that

$$\#|\{0 \le k \le l \ \mid \ T^{t_k'+j}y \in \mathcal{T}_c\}| \ge \frac{3c\mu_{\mathcal{T}}}{4}l. \tag{4}$$

Let $L := \lceil (3c\mu_{\mathcal{T}}/4)l \rceil$. By the definition of t_k, t_k' and by (4), there exist natural numbers

$$\begin{aligned} 0 \le i_1 < i_1 + h < i_2 < i_2 + h < \cdots < i_L < i_L + h < n, \\ 0 \le j_1 < j_1 + h < j_2 < j_2 + h < \cdots < j_L < j_L + h < n \end{aligned} \tag{5}$$

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such that $T^{i_k}x, T^{j_k}y \in F$ for all $1 \le k \le L$ and

$$\sum_{k=1}^{L} |i_k - j_k - j| < Lch + 2\delta_+ n.$$

Let us assume, for simplicity, that $i_1, j_1 > 0$. Then we can use the index sequences in (5) to split v, w as a concatenation of non-empty words

$$v = v^1 \cdot v^2 \cdots v^{2L+1}, \quad w = w^1 \cdot w^2 \cdots w^{2L+1}$$

such that, for all $1 \le k \le L$,

$$v_0^{2k}, w_0^{2k} \in F, \quad |v^{2k}| = h = |w^{2k}|$$

and

$$\sum_{k=0}^{L} ||v^{2k+1}| - |w^{2k+1}|| < chL + 2\delta_+ n + 2N_0$$
$$< chL + \delta n.$$

□

Proof of Claim 3.3. Notice that any word u admits a unique decomposition of the form

$$u = \prod_{i=1}^{s} \sigma^i,$$

where $s \in \mathbb{N}$ and each σ^i satisfies one of the following:

(1) $\sigma_j^i \notin W_1$ for all $0 \le j < |\sigma^i|$;
(2) $\sigma_0^i \in W_1$ and $|\sigma^i| = N$;
(3) $i = s, \sigma_0^s \in W_1$ and $|\sigma^s| < N$.

Given a word u, let us denote

$$R(u) = N|\{0 < i \le s \mid \sigma_0^i \in W_1,\ |\sigma^i| = N\}|,$$

where σ^i denotes the aforementioned decomposition of u. Define

$$R_k = \min\{R(v^k), R(w^k)\}.$$

By the definition of W_1, for all $0 \le k \le L$ we can match at least αR_{2k+1} coordinates of the blocks (v^{2k+1}, w^{2k+1}). By the definition of W, at most $2\delta n$ of the first n iterates of x and y do not belong to W_1. Recall that the sum of the lengths of the odd blocks associated to each splitting is equal to $n - Lh$. Hence

$$\sum_{k=0}^{L} R(v^{2k+1}), \sum_{k=1}^{L} R(w^{2k+1}) \ge n - Lh - 2\delta n - LN.$$

Notice that

$$R_k = \tfrac{1}{2}(R(v^k) + R(w^k) - |R(v^k) - R(w^k)|)$$
$$\ge R(v^k) + R(w^k) - \tfrac{1}{2}(|v^k| + |w^k| + ||v^k| - |w^k||).$$

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Thus, by (4) in Claim 1 and since $h > 2N/\delta$, it follows that

$$\sum_{k=0}^{L} R_{2k+1} \geq n - Lh - 2LN - \frac{c}{2}hL - 5\delta n$$
$$\geq n\left(1 - \frac{Lh}{n}\left(1 + \frac{c}{2}\right) - 6\delta\right). \qquad \square$$

3.2. *LB criterion for ergodic zero-entropy transformations.* In this section, we combine Propositions 2.2 and 3.1 to provide a new loose-Bernoullicity criterion for ergodic zero-entropy processes (item 3 of Proposition 3.4) which relies only on the existence of an appropriate sequence of Rokhlin towers. As a direct consequence, we obtain a loose-Bernoullicity criterion for ergodic zero-entropy transformations (Theorem 3.5).

By iterating Proposition 3.1, we will show the following proposition.

PROPOSITION 3.4. *Let $(T, \mathcal{P})$ be an ergodic zero-entropy process and let $L \in \mathbb{N} \cup \{+\infty\}$ and $\mathfrak{T} = \{\mathcal{T}_n\}_{1 \leq n < L}$ be a collection of Rokhlin towers. Suppose $\mathcal{T}_n$ is δ_{n-1}-monochromatic with respect to $\mathcal{P}$ for all $1 \leq n < L$, where*

$$\alpha_0 = 0, \quad \alpha_n = \alpha_{n-1} + \delta_{n-1}, \quad \delta_{n-1} = \mu_{\mathcal{T}_n}^2\left(\frac{1 - \alpha_{n-1}}{10}\right)^2.$$

Then there exist natural numbers $N_n(T, \mathcal{P}, \mathcal{T}_1, \ldots, \mathcal{T}_n)$ for $0 \leq n < L$ such that, for any $1 \leq l \leq L$ satisfying

$$h_{\mathcal{T}_n} > N_{n-1} \quad \textit{for all } 1 \leq n < l,$$

the following hold.

(1) *There exist $(m_n)_{0 \leq n < l} \subset \mathbb{N}$ such that $(T, \mathcal{P})$ verifies $P(\alpha_n, \delta_n, m_n)$ for all $0 \leq n < l$.*
(2) *If $l < +\infty$ and there exists a Rokhlin tower $\mathcal{T}$, $\delta_{\mathcal{T}}$-monochromatic with respect to $\mathcal{P}$, where*

$$\delta_{\mathcal{T}} = \mu_{\mathcal{T}}^2\left(\frac{1 - \alpha_{l-1}}{10}\right)^2$$

obeying

$$h_{\mathcal{T}} > N_{l-1},$$

then the process $(T, \mathcal{P})$ verifies $P(\alpha_{l-1} + \delta_{\mathcal{T}}, \delta_{\mathcal{T}}, m_l)$ for some $m_l \in \mathbb{N}$.
(3) *If $l = +\infty$ and*

$$\sum_{n=1}^{\infty} \mu_{\mathcal{T}_n}^2 = +\infty, \tag{6}$$

then the process $(T, \mathcal{P})$ is LB.

Proof. By the pointwise ergodic theorem, there exists m_0 depending only on T and $\mathcal{P}$ so that property $P(\alpha_0, \delta_0, m_0)$ is verified. In fact, given that $P \in \mathcal{P}$ with positive measure, it suffices to take m_0 sufficiently large so that the set of points x in X for which at least one

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of the iterates $T^i x$ with $0 \le i < m_0$ belongs to P has measure bigger than $1 - \delta_0$. Let

$$N_0 = \left\lceil \frac{2m_0}{\delta_0} \right\rceil.$$

We will define recursively sequences of natural numbers m_n, $N_n = \lceil 2m_n/\delta_n \rceil$, as follows. Let $1 \le n < L$ and suppose that $m_0, \ldots, m_{n-1}$ (and thus $N_0, \ldots, N_{n-1}$) have been defined. If

$$h_{\mathcal{T}_n} < N_{n-1},$$

we define $N_k = N_{n-1}$, $m_k = m_{n-1}$ for all $n \le k < L$ and the recursive process ends. Otherwise, if

$$h_{\mathcal{T}_n} \ge N_{n-1},$$

we can apply Proposition 3.1 with $\alpha = \alpha_{n-1}$, $\delta = \delta_{n-1}$, $N = m_{n-1}$, $\mathcal{T} = \mathcal{T}_n$, $\delta_+ = \delta_n$ and define

$$m_n = N_+(\alpha_{n-1}, \delta_{n-1}, m_{n-1}, \mathcal{T}_n, \delta_n).$$

Notice that, in this case, for $\alpha = \alpha_{n-1}$, $\delta = \delta_{n-1}$ and $\mathcal{T} = \mathcal{T}_n$ in equation (2),

$$\alpha_+ \ge \alpha_{n-1} + \mu^2_{\mathcal{T}_n}(1 - \alpha_{n-1})^2\left(\frac{1}{10} - \frac{9}{100}\right) = \alpha_n$$

and thus property $P(\alpha_n, \delta_n, m_n)$ is satisfied. By induction, there exist sequences of natural numbers $(N_n)_{0 \le n < L}$ and $(m_n)_{0 \le n < L}$ for which the first assertion holds. The second assertion now follows from Proposition 3.1. Finally, if $l = +\infty$, it follows from (6) that $\alpha_n \nearrow 1$ and $\delta_n \searrow 0$. Therefore, the process $(T, \mathcal{P})$ is LB by Proposition 2.2. □

By applying item 3 in Proposition 3.4 to a sequence of refining measurable partitions, we immediately obtain the following loose-Bernoullicity criterion.

THEOREM 3.5. *Let (X, T) be an ergodic zero-entropy transformation and let $\{\mathcal{P}_n\}_{n \in \mathbb{N}} \subset \mathcal{P}_f(X)$ be a sequence of refining partitions satisfying $\mathcal{P}_n \to \mathcal{E}$. Suppose that there exists a collection of Rokhlin towers $\mathfrak{T} = \{\mathcal{T}_n\}_{n \ge 1} \subset \mathcal{T}(X)$ obeying:*

(i) $\sum_{n=1}^{\infty} \mu^2_{\mathcal{T}_n} = +\infty$*; and*

(ii) $\mathcal{T}_{n+1}$ *is $\delta_{i,n-i}$-monochromatic with respect to $\mathcal{P}_i$ for all $0 \le i \le n$, where*

$$\alpha_{i,0} = 0, \quad \alpha_{i,n+1} = \alpha_{i,n} + \delta_{i,n}, \quad \delta_{i,n} = \mu_{\mathcal{T}^2_{n+1}}\left(\frac{1 - \alpha_{i,n}}{10}\right)^2.$$

There exist natural numbers $M_n(T, \mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{T}_1, \ldots, \mathcal{T}_n)$ such that if

$$h_{\mathcal{T}_n} > M_{n-1}$$

holds for all $n \ge 1$, then the transformation T is LB.

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Remark. M_n in the previous theorem can be taken as

$$M_n = \max_{0\le i\le n} N_i(T, \mathcal{P}_i, \mathcal{T}_i, \dots, \mathcal{T}_n),$$

with N_i as defined in Proposition 3.4.

Let us point out that the hypotheses of Theorem 3.5 are always satisfied by local rank one systems and thus Theorem 3.5 can be seen as a generalization, in the zero-entropy case, of Ferenczi's result in [**8**, Proposition 1], which shows that ergodic local rank one transformations are LB.

Recall that a system (X, T) is said to have *local rank one* if there exists $a > 0$ such that, for every finite measurable partition $\mathcal{P} \in \mathcal{P}_f(X)$ and for every $\epsilon > 0$, there exists a Rokhlin tower $\mathcal{T} \in \mathcal{T}(X)$, with $\mu_{\mathcal{T}} > a$, and a measurable partition of the support of the tower $\mathcal{Q}$, obeying $\mathcal{T} \le \mathcal{Q}$, such that

$$D(\mathcal{Q}, \mathcal{P}|_{\mathcal{T}}) < \epsilon.$$

In short, the local rank one property guarantees that any finite measurable partition can be arbitrarily well approximated by towers whose size is bounded from below by a fixed positive constant.

We stress the fact that the size of the towers in Theorem 3.5 is not necessarily bounded from below.

3.3. *LB criterion for the Cartesian square.* In this section, we adapt Proposition 3.4 and Theorem 3.5 to obtain similar loose-Bernoullicity criteria, compatible with mixing, for the Cartesian square of ergodic zero-entropy processes (item 3 in Proposition 3.7) and transformations (Theorem 3.8).

Notice that any pair of Rokhlin towers $\mathcal{T}^{\pm} = \mathcal{T}(F^{\pm}, h^{\pm})$ with sizes $\mu^{\pm}$ induces a tower $\mathcal{T} = \mathcal{T}(F^+ \times F^-, \max\{h^+, h^-\})$ for the Cartesian square $T \times T$ whose size is given by

$$\mu_{\mathcal{T}} = \frac{\mu^+\mu^-}{\min\{h^+, h^-\}}.$$

The height of this product tower can be increased, at the cost of slightly shrinking the base, if some almost periodicity holds for the initial towers, that is, if most of their roof's image is contained in their base. To illustrate this, let us suppose for a moment that

$$T^{h^{\pm}}(F^{\pm}) = F^{\pm}. \tag{7}$$

Then, if h^+, h^- are relatively prime, $\mathcal{T}(F^+ \times F^-, h^+h^-)$ is a well-defined Rokhlin tower of size $\mu^+\mu^-$. Clearly, if the system is mixing, equation (7) cannot hold unless $F^{\pm} = X$. Nevertheless, this argument can be adapted provided the *precision* of the towers $\mathcal{T}^{\pm}$ is sufficiently small. Recall that the precision of a Rokhlin tower $\mathcal{T}(F, h)$, which we denote by $\rho_{\mathcal{T}}$, is given by

$$\rho_{\mathcal{T}} = \mu(F \Delta T^h(F)).$$

The following lemma gives explicit relations between the towers $\mathcal{T}^{\pm}$ and its Cartesian product.

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LEMMA 3.6. *Let $(X, \mu, \mathcal{E}, T)$ be a finite measure-preserving transformation, let $\mathcal{P}^{\pm} \in \mathcal{P}_f(X)$ and let $\mathcal{T}^{\pm} = \mathcal{T}(F^{\pm}, h^{\pm}) \in \mathcal{T}(X)$be Rokhlin towers of sizes $\mu^{\pm}$ with h_+, h_- relatively prime. If*

$$(h^{\mp} - 1)\rho_{\mathcal{T}^{\pm}} < (1 - c)\frac{\mu^{\pm}}{h^{\pm}}, \tag{8}$$

for some $0 < c < 1$, then there exists $F \subset F^+ \times F^-$ such that $\mathcal{T}(F, h^+h^-) \in \mathcal{T}(X \times X, T \times T)$ is a well-defined Rokhlin tower of size $\mu_{\mathcal{T}} > c^2\mu_+\mu_-$ satisfying

$$\Delta(\mathcal{T}, \mathcal{P}^+ \times \mathcal{P}^-) \le \Delta(\mathcal{T}^+, \mathcal{P}^+)\mu_- + \Delta(\mathcal{T}^-, \mathcal{P}^-)\mu_+.$$

Proof. Let

$$E^{\pm} = \bigcap_{k=1}^{h^{\mp}-1} T^{-kh^{\pm}}(F^{\pm}).$$

By (8),

$$\mu(E^{\pm}) > \frac{c\mu^{\pm}}{h^{\pm}}. \tag{9}$$

Let $F = E^+ \times E^-$. Notice that, for any $0 < i < h^+h^-$,

$$F \cap (T \times T)^i(F) \neq \emptyset \quad \text{implies} \quad h^{\pm} \mid i.$$

Since h^+, h^- are relatively prime, this shows that $\mathcal{T}(F, h^+h^-)$ is a well-defined Rokhlin tower. The estimate on the size of this tower follows trivially from (9). Denote $\mathcal{P}^{\pm} = \{P_1^{\pm}, \dots, P_N^{\pm}\}$ and let $\xi^{\pm} = \{\xi_1^{\pm}, \dots, \xi_N^{\pm}\}$ be partitions of $\mathcal{T}^{\pm}$ such that

$$\mathcal{T}^{\pm} \le \xi^{\pm}, \quad \Delta(\mathcal{T}^{\pm}, \mathcal{P}^{\pm}) = \Delta(\xi^{\pm}, \mathcal{P}^{\pm}).$$

Denote $\mathcal{P} = \mathcal{P}^+ \times \mathcal{P}^-$. Notice that $\mathcal{T} = \mathcal{T}(T \times T, F, h^+h^-) \subset \mathcal{T}^+ \times \mathcal{T}^-$, and since $\xi^+ \times \xi^-$ is a partition of $\mathcal{T}^+ \times \mathcal{T}^-$, then $\xi = (\xi^+ \times \xi^-)|_{\mathcal{T}}$ is a well-defined partition of $\mathcal{T}$. Furthermore, $\mathcal{T} \le \xi$ and

$$\begin{aligned}
\Delta(\mathcal{T}, \mathcal{P}) &\le \sum_{i,j=1}^{N} \mu((\xi_i^+ \times \xi_j^- \cap \mathcal{T}) \setminus (P_i^+ \times P_j^-)) \\
&\le \sum_{i,j=1}^{N} \mu(\xi_i^+ \setminus P_i^+ \times \xi_j^-) + \mu(\xi_i^+ \times \xi_j^- \setminus P_j^-) \\
&\le D(\xi^+, \mathcal{P}^+)\mu_- + D(\xi^-, \mathcal{P}^-)\mu_+.
\end{aligned}$$

□

Taking into account the previous lemma, the proof of the following result is similar to that of Proposition 3.4.

PROPOSITION 3.7. *Let $(T, \mathcal{P})$ be a weak mixing zero-entropy process, let $L \in \mathbb{N} \cup \{+\infty\}$ and let $\mathfrak{T} = \{\mathcal{T}_n^{\pm}\}_{1 \le n < L}$ be a collection of Rokhlin towers. Suppose $h_{\mathcal{T}_n^+}, h_{\mathcal{T}_n^-}$ are relatively prime and*

$$\rho_{\mathcal{T}_n^{\pm}} \le \frac{\mu_{\mathcal{T}_n}^{\pm}}{2h_{\mathcal{T}_n}^+ h_{\mathcal{T}_n}^-}$$

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for all $1 \le n < L$. *Further, assume that* $\mathcal{T}_{n+1}^{\pm}$ *are* δ_n*-monochromatic with respect to* $\mathcal{P}$ *for all* $1 \le n < L$, *where*

$$\alpha_0 = 0, \quad \alpha_{n+1} = \alpha_n + \delta_n, \quad \delta_n = \left(\frac{\mu_{\mathcal{T}_{n+1}^+}\mu_{\mathcal{T}_{n+1}^-}}{2}\right)^2\left(\frac{1-\alpha_n}{10}\right)^2.$$

Then there exist natural numbers $N_n(T, \mathcal{P}, \mathcal{T}_1^{\pm}, \dots, \mathcal{T}_n^{\pm})$ *for* $0 \le n < L$ *such that, for any* $1 \le l \le L$ *satisfying*

$$h_{\mathcal{T}_n^+}h_{\mathcal{T}_n^-} > N_{n-1} \quad \textit{for all } 1 \le n < l, \tag{10}$$

the following hold.

(1) *There exist* $(m_n)_{0\le n<l} \subset \mathbb{N}$ *such that the process* $(T \times T, \mathcal{P} \times \mathcal{P})$ *verifies* $P(\alpha_n, \delta_n, m_n)$ *for all* $0 \le n < l$.

(2) *If* $l < +\infty$ *and there exist Rokhlin towers* $\mathcal{T}^{\pm}$, $\delta_{\mathcal{T}}$*-monochromatic with respect to* $\mathcal{P}$, *where*

$$\delta_{\mathcal{T}} = \left(\frac{\mu_{\mathcal{T}^+}\mu_{\mathcal{T}^-}}{2}\right)^2\left(\frac{1-\alpha_{l-1}}{10}\right)^2,$$

obeying

$$h_{\mathcal{T}^+}h_{\mathcal{T}^-} > N_{l-1},$$

then the process $(T \times T, \mathcal{P} \times \mathcal{P})$ *verifies* $P(\alpha_{l-1} + \delta_{\mathcal{T}}, \delta_{\mathcal{T}}, m_l)$ *for some* $m_l \in \mathbb{N}$.

(3) *If* $l = +\infty$ *and*

$$\sum_{n=1}^{\infty}(\mu_{\mathcal{T}_{n+1}^+}\mu_{\mathcal{T}_{n+1}^-})^2 = +\infty, \tag{11}$$

then the process $(T \times T, \mathcal{P} \times \mathcal{P})$ *is LB.*

In the previous proposition, we assume that the system is weak mixing, instead of ergodic, just to guarantee that its Cartesian square is ergodic. Proposition 3.7 immediately implies the following theorem.

THEOREM 3.8. *Let* (X, T) *be a weak mixing zero-entropy transformation and let* $\{\mathcal{P}_n\}_{n\in\mathbb{N}} \subset \mathcal{P}_f(X)$ *be a sequence of refining partitions satisfying* $\mathcal{P}_n \to \mathcal{E}$. *Suppose that there exists a collection of Rokhlin towers* $\mathfrak{T} = \{\mathcal{T}_n^{\pm}\}_{n\ge1} \subset \mathcal{T}(X)$ *obeying:*

(i) $\sum_{n=1}^{\infty}\left(\mu_{\mathcal{T}_n^+}\mu_{\mathcal{T}_n^-}\right)^2 = +\infty$;

(ii) $\rho_{\mathcal{T}_n^{\pm}} \le \mu_{\mathcal{T}_n}^{\pm}/(2h_{\mathcal{T}_n}^{+}h_{\mathcal{T}_n}^{-})$ *for all* $n \ge 1$;

(iii) $h_{\mathcal{T}_n^+}, h_{\mathcal{T}_n^-}$ *are relatively prime for all* $n \ge 1$; *and*

(iv) $\mathcal{T}_{n+1}^{\pm}$ *is* $\delta_{i,n-i}^{\pm}$*-monochromatic with respect to* $\mathcal{P}_i$ *for all* $0 \le i \le n$, *where*

$$\alpha_{i,0} = 0, \quad \alpha_{i,n+1} = \alpha_{i,n} + \delta_{i,n},$$

$$\delta_{i,n} = \left(\frac{\mu_{\mathcal{T}_{n+1}^+}\mu_{\mathcal{T}_{n+1}^-}}{2}\right)^2\left(\frac{1-\alpha_{i,n}}{10}\right)^2.$$

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There exist natural numbers $M_n(T, \mathcal{P}_1, \dots, \mathcal{P}_n, \mathcal{T}_1^{\pm}, \dots, \mathcal{T}_n^{\pm})$ *such that if*

$$h_{\mathcal{T}_n^+} h_{\mathcal{T}_n^-} > M_{n-1}$$

holds for all $n \geq 1$*, then the transformation* $T \times T$ *is LB.*

Remark. M_n in the previous theorem can be taken as

$$M_n = \max_{0 \leq i \leq n} N_i(T, \mathcal{P}_i, \mathcal{T}_i^{\pm}, \dots, \mathcal{T}_n^{\pm}),$$

with N_i as defined in Proposition 3.7.

A similar criterion for loose-Bernoullicity of the Cartesian square of a measure-preserving transformation was proved by A. Katok and A. Stepin in [**23**] assuming the existence of *excellent approximations of type* $(n, n+1)$. Precise definitions and a proof of this criterion can be found in [**18**, §3.2]. We stress the fact that transformations within the hypotheses of Katok and Stepin's criterion cannot be mixing (see [**18**]). Also, for a given finite partition $\mathcal{P}$, systems with excellent approximations of type $(n, n+1)$ admit couples of arbitrarily high towers whose heights differ by one, whose size is bounded from below and which approximate $\mathcal{P}$ arbitrarily well. Therefore, we can apply Theorem 3.8 to recover Katok and Stepin's criterion.

4. *Smooth mixing transformations with LB Cartesian square*

In this section, we will apply the loose-Bernoullicity criterion in Theorem 3.8 to show the existence of zero-entropy smooth mixing LB transformations on $\mathbb{T}^3$ whose Cartesian square is LB. These systems will be obtained as the limits of the recursive procedure described in §4.3 and which we summarize in Lemma 4.7.

The transformations that we consider are time one maps of continuous special flows over minimal translations of $\mathbb{T}^2$. As we shall see in the following, these maps can be naturally identified with continuous homeomorphisms of $\mathbb{T}^3$ and they possess a natural invariant ergodic measure. Under some additional assumptions, these maps will be actually mixing with respect to this measure. Furthermore, by Abramov's [**1**] entropy formula (12), these maps will have zero entropy.

4.1. *Special flows.* Given a measure-preserving transformation $(X, \mu, \mathcal{E}, T)$ and an integrable function $\varphi : X \to (0, +\infty)$, denote by $\sim_{T,\varphi}$ the equivalence relation on $X \times \mathbb{R}$ given by

$$(x, s + \varphi(x)) \sim_{T,\varphi} (T(x), s).$$

The *special flow over* T *with roof function* φ which we denote $\Psi_{T,\varphi}^t : X_\varphi \to X_\varphi$, where $X_\varphi = X \times \mathbb{R}/_{\sim_{T,\varphi}}$ is defined as the identification by $\sim_{T,\varphi}$ of the action

$$(t, x, s) \mapsto (x, s + t).$$

Using the Birkhoff sums of φ with respect to T, which we denote by

$$S_m^T \varphi = \sum_{k=0}^{m-1} \varphi \circ T^k$$

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for any $m \in \mathbb{N}$, and noticing that

$$X_\varphi = \{(x, s) \in X \times \mathbb{R} \mid 0 \le s \le \varphi(x)\}/_{\sim_{T,\varphi}},$$

we can provide a more explicit expression of the special flow.

Given $(x, s) \in X \times \mathbb{R}$ obeying $0 \le s < \varphi(x)$ and for any $t \ge 0$,

$$\Psi^t_{T,\varphi}(x, s) = (T^{\pi(t,x,s)}(x), t + s - S^T_{\pi(t,x,s)}\varphi(x)),$$

where

$$\pi(t, x, s) = \min\{n \in \mathbb{N} \mid t + s < S^T_n\varphi(x)\}.$$

A similar expression can be obtained for $t < 0$. The special flow $\Psi^t_{T,\varphi}$ preserves the normalized product measure

$$\lambda_{T,\varphi} = \frac{(\mu \times \lambda)\,|_{X_\varphi}}{\int_X \varphi d\mu},$$

where λ denotes the Lebesgue measure on $\mathbb{R}$. We recall Abramov's [1] entropy formula

$$h_{\lambda_{T,\varphi}}(\Psi^t_{T,\varphi}) = t\frac{h_\mu(T)}{\int_X \varphi d\mu}. \tag{12}$$

4.1.1. *Time one maps of special flows over* $\mathbb{T}^2$. Let us recall that the toral translation $R_\omega : \mathbb{T}^2 \to \mathbb{T}^2$, where $\omega \in \mathbb{R}^2$ denotes the *translation vector*, is minimal if and only if $(1, \omega)$ is *non-resonant*. A vector $v \in \mathbb{R}^d$ is said to be non-resonant if its coordinates are *rationally independent*, that is, if the equation $\langle v, k\rangle = 0$ does not admit a solution $k \in \mathbb{Z}^d \setminus \{0\}$; otherwise, it is called *resonant*.

Given $r \in \mathbb{N} \cup \{+\infty\}$, let $C^r_+(\mathbb{T}^2)$ denote the space of strictly positive C^r functions on $\mathbb{T}^2$ endowed with the C^r topology. We denote $C^0_+(\mathbb{T}^2)$ simply by $C_+(\mathbb{T}^2)$. To simplify the notation, for $\omega \in \mathbb{R}^2$ fixed, we denote

$$T_{\omega,\varphi} = \Psi^1_{R_\omega,\varphi}, \quad \lambda_{\omega,\varphi} = \lambda_{R_\omega,\varphi},$$

for every $\varphi \in C_+(\mathbb{T}^2)$. By (12),

$$h_{\lambda_{\omega,\varphi}}(T_{\omega,\varphi}) = 0.$$

Notice that, for $\varphi, \psi \in C_+(\mathbb{T}^2)$, the transformations $T_{\omega,\varphi}$ and $T_{\omega,\psi}$ are not necessarily defined on the same space. To be able to compare these two mappings, we first normalize them as follows. Let $\omega = (\Omega, \Omega')$, where $\Omega, \Omega' \in \mathbb{R}^*$, and let us identify the torus $\mathbb{T}^3$ with $\mathbb{R}^3/(\Omega\mathbb{Z} \times \Omega'\mathbb{Z} \times \mathbb{Z})$. For every $\varphi \in C_+(\mathbb{T}^2)$, let

$$\begin{array}{rccc} \Phi_{\omega,\varphi} : & \mathbb{T}^3 & \longrightarrow & X_\varphi \\ & (x, y, z) & \mapsto & (x, y, z\varphi(x, y)). \end{array}$$

It is easy to show that $\Phi_{\omega,\varphi}$ is a well-defined homeomorphism. Define

$$G_{\omega,\varphi} = \Phi^{-1}_{\omega,\varphi} \circ T_{\omega,\varphi} \circ \Phi_{\omega,\varphi}, \quad \mu_{\omega,\varphi} = \Phi^*_{\omega,\varphi}(\lambda_{\omega,\varphi}). \tag{13}$$

Thus $(G_{\omega,\varphi}, \mu_{\omega,\varphi})$ is a well-defined measure-preserving homeomorphism of $\mathbb{T}^3$ for any $\varphi \in C_+(\mathbb{T}^2)$. Furthermore, the transformation $\Phi_{\omega,\varphi}$ (and therefore $G_{\omega,\varphi}$ and $\mu_{\omega,\varphi}$) is as regular as the roof function φ.

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4.2. *Mixing criterion.* Given $\omega = (\Omega, \Omega') \in \mathbb{R}^2$ such that $(1, \omega)$ is non-resonant, we denote by q_n, q_n' the first return times of Ω and Ω', respectively. Recall that given that $\alpha \in (0, 1)$ is irrational and by setting $q_0 = 1$, the *first return times* of α can be defined recursively for all $n \in \mathbb{N}$ by

$$q_{n+1} = \min\{k \in \mathbb{N}^* \mid |||k\alpha||| < |||q_n\alpha|||\},$$

where $|||\cdot|||$ denotes the distance to the closest integer. We extend the definition of $|||\cdot|||$ to vectors in $\mathbb{R}^d$ by

$$|||x||| = \min_{k\in\mathbb{Z}^d} |k - x|_1.$$

For $\alpha \in \mathbb{R}$, let us denote

$$\{\alpha\} = \alpha - \lfloor\alpha\rfloor.$$

The following mixing criterion is a consequence of the mixing criterion introduced by B. Fayad in [6, Proposition 3]. See also [4, Proposition 3.3] and [5, Proposition 2] for similar applications of this criterion.

PROPOSITION 4.1. *Suppose that*

$$q_n' \geq e^{3q_n}, \quad q_{n+1} \geq e^{3q_n'} \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

If, for every $n \in \mathbb{N}$ sufficiently large, there exist two sets, I_n, I_n', each of which are equal to the circle minus a finite number of intervals whose lengths converge to zero, and if:

- *for any $y \in \mathbb{T}$, any x such that $\{q_n x\} \in I_n$ and any $m \in [e^{2q_n}/2, 2e^{2q_n'}]$,*

$$\left|\partial_x S_m^{R_\omega}\varphi(x, y)\right| \geq \frac{m}{e^{q_n}}\frac{q_n}{n};$$

- *for any $x \in \mathbb{T}$, any y such that $\{q_n y\} \in I_n'$ and any $m \in [e^{2q_n}/2, 2e^{2q_{n+1}}]$,*

$$\left|\partial_y S_m^{R_\omega}\varphi(x, y)\right| \geq \frac{m}{e^{q_n'}}\frac{q_n'}{n},$$

then the special flow $\Psi_{R_\omega,\varphi}^t$ is mixing.

Let us point out that the set of vectors verifying (14) is uncountable and dense. For a proof of this, see [30, Appendix 1].

4.3. *Construction.* In the following, we fix a vector $\omega = (\Omega, \Omega') \in \mathbb{R}^2$ such that $(1, \omega)$ is non-resonant and denote by q_n, q_n' the first return times of Ω and Ω', respectively. We further assume that (14) holds. Let

$$\mathcal{C}_{\text{Mix}} = \{\varphi \in C_+^\infty(\mathbb{T}^2) \mid (G_{\omega,\varphi}, \mu_{\omega,\varphi}) \text{ is mixing}\}.$$

Denote by $\varphi_0 : \mathbb{T}^2 \to \mathbb{R}$ the real analytic function given by

$$\varphi_0(x, y) = 1 + \sum_{n=1}^{\infty} X_n(x) + Y_n(y), \tag{15}$$

where

$$X_n(x) = e^{-q_n}\cos(2\pi q_n x), \quad Y_n(y) = e^{-q_n'}\cos(2\pi q_n' y). \tag{16}$$

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Applying the mixing criterion in Proposition 4.1, one can prove the following proposition (see [6, Theorem 1]).

PROPOSITION 4.2. *For any trigonometric polynomial $P : \mathbb{T}^2 \to \mathbb{R}$ such that $\varphi_0 + P$ is strictly positive,*

$$\varphi_0 + P \in \mathcal{C}_{Mix}$$

In particular, $\mathcal{C}_{Mix}$ is C^∞-dense in $C^\infty_+(\mathbb{T}^2)$.

Our goal is to prove the following theorem.

THEOREM 4.3. *There exists a set $S \subset C^\infty_+(\mathbb{T}^2)$, C^∞-dense in*

$$\left\{\varphi \in C^\infty_+(\mathbb{T}^2) \;\middle|\; \int_{\mathbb{T}^2} \varphi(\theta)d\theta = 1\right\},$$

such that, for every $\varphi \in S$, the transformation $(G_{\omega,\varphi}, \mu_{\omega,\varphi})$ given by (13) is mixing and its Cartesian square $(G_{\omega,\varphi} \times G_{\omega,\varphi}, \mu_{\omega,\varphi} \times \mu_{\omega,\varphi})$ is LB.

Let us give an outline of the proof. We prove Theorem 4.3 by showing that, for any trigonometric polynomial P of zero average such that $\varphi_0 + P$ is strictly positive, the following holds. For all $r \in \mathbb{N}$ and all $\epsilon > 0$, there exist an increasing sequence of natural numbers m_n and a sequence of trigonometric polynomials $P_{m_n} : \mathbb{T} \to \mathbb{R}$ of zero average and degree at most q_{m_n} obeying

$$\sum_{n=1}^{\infty} \|P_{m_n}\|_{C^r} < \epsilon,$$

for which, redefining φ_0 as

$$\overline{\varphi}_0(x, y) = \sum_{k=1}^{\infty} \overline{X}_k(x) + Y_k(y),$$

where $\overline{X}_k$ is equal to P_{m_n} if $k = m_n$ for some $n \geq 1$ or to X_k otherwise,

$$\psi = \overline{\varphi}_0 + P$$

is a well-defined function in $C^\infty_+(\mathbb{T}^2)$ such that $(G_{\omega,\psi}, \mu_{\omega,\psi})$ is mixing and its Cartesian square $(G_{\omega,\psi} \times G_{\omega,\psi}, \mu_{\omega,\psi} \times \mu_{\omega,\psi})$ is LB. Notice that, up to taking m_1 sufficiently large, we can suppose, without loss of generality, that

$$\|\psi - (\varphi_0 + P)\|_{C^r} = \left\| \sum_{n=1}^{\infty} P_{m_n} - X_{m_n} \right\|_{C^r} < 2\epsilon.$$

The sequences $\{m_n\}_{n\geq 1}$ and $\{P_{m_n}\}_{n\geq 1}$ are defined recursively with respect to a sequence of refining partitions $\mathcal{P}_n \to \mathcal{E}$ set beforehand. We define m_n, P_{m_n} by guaranteeing (Proposition 4.6), at each step, the existence of Rokhlin towers $\mathcal{T}(G_{\omega,\psi_n}, F_n^{\pm}, h_n^{\pm})$ within

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the hypotheses of Lemma 3.6 for the processes $(G_{\omega,\psi_n}, \mathcal{P}_i)$ for $i = 0, \ldots, n$, where

$$\psi_n(x, y) = \varphi_0(x, y) + P(x, y) + \sum_{k=1}^{n} (P_{m_k}(x) - X_{m_k}(x)).$$

The towers $\mathcal{T}(G_{\omega,\psi_n}, F_n^{\pm}, h_n^{\pm})$ will be sufficiently monochromatic with respect to the partitions $\mathcal{P}_0, \ldots, \mathcal{P}_n$, and will be of size $\sim (n+1)^{-1/4}$. By ensuring that ψ_{n+1} is sufficiently close to ψ_n at each step, we can also guarantee that $\mathcal{T}(G_{\omega,\psi_m}, F_n^{\pm}, h_n^{\pm})$ are well-defined Rokhlin towers for all $m \geq n$.

By carefully choosing the sequence $\{h_n^{\pm}\}_{n\geq 1}$, we can guarantee that, for all $1 \leq n \leq m$, the process $(G_{\omega,\psi_m}, \mathcal{P}_n)$ and the towers $\{\mathcal{T}(G_{\omega,\psi_m}, F_i^{\pm}, h_i^{\pm})\}_{1\leq i\leq n}$ verify (10) with $l = n - 1$. Taking

$$\psi = \lim_{n\to\infty} \psi_n,$$

the loose-Bernoullicity of $(G_{\omega,\psi} \times G_{\omega,\psi}, \mu_{\omega,\psi} \times \mu_{\omega,\psi})$ will follow from Theorem 3.8. By Proposition 4.1, the transformation $(G_{\omega,\psi}, \mu_{\omega,\psi})$ will be mixing.

A precise statement of the recursive procedure described above is given in Lemma 4.7.

4.4. *Proof of Theorem 4.3.* Let us start by noticing that, under some particular conditions on $\mathcal{P}$, the property $P(\mathcal{P}, \alpha, \delta, n)$ is open in the space of transformations

$$\{(G_{\omega,\varphi}, \mu_{\omega,\varphi})\}_{\varphi\in C_+(\mathbb{T}^2)} \subset \mathrm{Hom}(\mathbb{T}^3)$$

endowed with the C^0 topology.

PROPOSITION 4.4. *Let $\alpha, \delta > 0$, let $n \in \mathbb{N}$ and suppose that $\mathcal{P}$ is a finite partition of $\mathbb{T}^3$ whose elements are either open or have zero Lebesgue measure. Then the set of $\varphi \in C_+(\mathbb{T}^2)$ for which $(G_{\omega,\varphi}, \mu_{\omega,\varphi})$ verifies $P(\mathcal{P}, \alpha, \delta, n)$ is open in $C_+(\mathbb{T}^2)$.*

Proof. Let us assume that $\delta < 1$; otherwise, the result is trivial. Let $\varphi \in C_+(\mathbb{T}^2)$ such that $(G_{\omega,\varphi}, \mu_{\omega,\varphi})$ verifies $P(\mathcal{P}, \alpha, \delta, n)$ and denote by W_0 the set associated to this property. Since we consider $\mathbb{T}^3$ endowed with the Borel σ-algebra, we can suppose, without loss of generality, up to slightly reducing the measure of W_0, that W_0 is a compact set. Furthermore, we can assume that all the iterates of points in W_0 belong to open sets of the partition since those who do not verify this condition form a set of zero Lebesgue measure (recall that $\mu_{\omega,\varphi}$ is equivalent to the Lebesgue measure on $\mathbb{T}^3$). By continuity, there exists $\epsilon > 0$ such that, for any $\psi \in C_+(\mathbb{T}^2)$ obeying

$$\|\psi - \varphi\|_{C^0} < \epsilon,$$

we have $\mu_{\omega,\psi}(W) > 1 - \delta$ and, for all $x \in W$, the $\mathcal{P}$-n-names associated to x with respect to $(G_{\omega,\varphi}, \mu_{\omega,\varphi})$ and $(G_{\omega,\psi}, \mu_{\omega,\psi})$ coincide. Therefore $(G_{\omega,\psi}, \mu_{\omega,\psi})$ verifies $P(\mathcal{P}, \alpha, \delta, n)$. □

Let us introduce the family of polynomials to be used in the recursive procedure described in §4.3. The class we define here is a slight modification of the one considered in [4, Proposition 3.5] by B. Fayad.

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PROPOSITION 4.5. *Let $r \in \mathbb{N}$ and $0 < \mu < \frac{1}{4}$. For all $n \in \mathbb{N}$ sufficiently large, there exists*

$$\frac{q_n e^{-q_n}}{2} \leq \eta_n \leq \frac{3q_n e^{-q_n}}{4}$$

and a trigonometric polynomial $P_{\mu,n}$ of zero average and degree at most q_{n+1} satisfying:

(1) $1/\eta_n \in 2q_n\mathbb{N}$;
(2) $\|P_{\mu,n}\|_{C^r} \leq e^{-3q_n/4}$;
(3) $|P_{\mu,n}(x) \mp \eta_n| \leq e^{3q_n/4}/(q_{n+1})$ *for* $|\{q_n x\} - \frac{1}{2} \pm \frac{1}{4}| < 3\mu$; *and*
(4) $\mp P'_{\mu,n}(x) \geq q_n^2/e^{q_n}$ *for* $|\{q_n x\} - \frac{1}{2} \pm \frac{1}{4}| > 4\mu$.

Furthermore, for any increasing sequence $\{m_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ and any decreasing sequence $\mu_n \searrow 0$ such that

$$\psi(x, y) = \varphi_0(x, y) + P(x, y) + \sum_{n=1}^{\infty} (P_{\mu_n,m_n}(x) - X_{m_n}(x))$$

is a well-defined function in $C^\infty_+(\mathbb{T}^2)$, the transformation $(G_{\omega,\psi}, \mu_{\omega,\psi})$ is mixing.

The proof of the previous proposition is completely analogous to that of Proposition 3.5 and Theorem 3.1 in [4] so we defer it to the Appendix. Using the family of polynomials just defined, we will prove the following proposition.

PROPOSITION 4.6. *Let P be a trigonometric polynomial with zero average such that $\varphi_0 + P$ is strictly positive. Suppose $\mathcal{P}$ is a finite partition of $\mathbb{T}^3$ whose elements are either open or have zero Lebesgue measure. For all $r, N \in \mathbb{N}$ and all $0 < \beta, \mu, \epsilon < 1/10$, there exists $m \in \mathbb{N}$ obeying*

$$|q_m^r e^{-q_m}| < \epsilon$$

such that, for $P_{\mu,m}$ as defined in Lemma 4.5,

$$\varphi(x, y) = \varphi_0(x, y) + P(x, y) + P_{\mu,m}(x) - X_m(x)$$

is a well-defined function in $C^\infty_+(\mathbb{T}^2)$ and the transformation $(G_{\omega,\varphi}, \mu_{\omega,\varphi})$ admits two Rokhlin towers $\mathcal{T}^\pm$ with closed sets as floor sets obeying

$$\Delta(\mathcal{T}^\pm, \mathcal{P}) \leq \beta, \quad \mu_{\mathcal{T}^\pm} > \mu, \quad h_{\mathcal{T}^-} = h_{\mathcal{T}^+} - 2 > N, \quad \rho_{\mathcal{T}^\pm} < \frac{\mu}{2(h_{\mathcal{T}^+})^2}.$$

Proof. Let $m \in \mathbb{N}$ and define

$$L_m = \sum_{n=1}^{m-1} X_n(x) + Y_n(y), \quad H_m = Y_m(y) + \sum_{n>m} X_n(x) + Y_n(y).$$

Thus we can express φ in the statement as

$$\varphi = 1 + P + L_m + P_{\mu,m} + H_m. \tag{17}$$

Take m sufficiently large so that

$$\deg(P) \leq q_{m-2}, \quad q_m^r e^{-q_m} < \epsilon, \quad \frac{e^{q_m}}{q_m^2} > N.$$

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Let

$$I^{\pm} = \left\{ x \in \mathbb{T} \;\middle|\; \left| x - \frac{1}{2q_m} \pm \frac{1}{4q_m} \right| \leq \frac{2\mu}{q_m} \right\}$$

and define

$$F^{\pm} = I^{\pm} \times \mathbb{T} \times \left[\frac{q_m \eta_m}{4}, \frac{3q_m \eta_m}{4} \right], \quad h^{\pm} = \eta_m^{-1} \pm 1,$$

with η_m as defined in Proposition 4.5. We claim that, for m sufficiently large,

$$\mathcal{T}^{\pm} = \mathcal{T}(F^{\pm}, h^{\pm})$$

are well-defined Rokhlin towers for the map $(G_{\omega,\varphi}, \mu_{\omega,\varphi})$ satisfying the desired conditions. We will prove this only for F^- as the proof for F^+ is analogous.

First, notice that $h^- > N$ by our initial hypothesis on m. Suppose, by contradiction, that there exists $0 < k < h^-$ for which $T^k(F^-) \cap F^- \neq \emptyset$. Then there exist $(x, y, z) \in F^-$ such that, for $n = \pi(k, x, y, z)$,

$$x + n\Omega \in I^-, \quad 0 \leq k - S_n^{R_\omega} \varphi(x, y) \leq \frac{q_m \eta_m}{2}. \tag{18}$$

Let $0 < c < 1$ such that $\varphi + P > c$. Notice that $n \leq kc^{-1}$ and take m sufficiently large so that $h^- c^{-1} < q_{m+1}$. As $x + n\Omega \in I^-$, then

$$|||n\Omega||| \leq \frac{4\mu}{q_m},$$

which implies that $n > q_{m-1}$. Since $n \leq kc^{-1} < q_{m+1}$, it follows that $n = lq_m$ for some $l \in \mathbb{N}$. By the definition of η_m, it follows that $l < e^{q_m}$ for m sufficiently large.

CLAIM. *For all $x \in I^-$, $y \in \mathbb{T}$ and $0 < l < e^{q_m}$,*

$$|S_{lq_m}^{R_\omega} \varphi(x, y) - lq_m + lq_m \eta_m| < \frac{1}{e^{2q_m}} \tag{19}$$

for m sufficiently large.

Proof of the Claim. We decompose φ as in (17) and calculate each Birkhoff sum separately. Since P and L_m are polynomials of degree less than q_{m-1} and q_m, respectively, by Corollary A.2 there exists a positive constant C_0 such that

$$\|S_{lq_m}^{R_\omega} P\|_{C^0} \leq \frac{lC_0 q_{m-1}}{q_{m+1}}, \quad \|S_{lq_m}^{R_\omega} L_m\|_{C^0} \leq \frac{lC_0 q_m}{q_{m+1}}.$$

Given that $x \in I^-$ and $0 < l < e^{q_m}$,

$$\left| \{q_m(x + l\Omega)\} - \frac{3}{4} \right| < 2\mu + \frac{l}{q_{m+1}} < 3\mu$$

for sufficiently large m. Thus, by condition (3) of Proposition 4.5,

$$|S_{lq_m}^{R_\omega} P_{\mu,m}(x) + lq_m \eta_m| \leq \frac{lq_m e^{3q_m/4}}{q_{m+1}}.$$

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Notice that $\|H_m\|_{C^0} \leq 2e^{-q'_m}$ for m sufficiently large. Let $x \in I^-$, $y \in \mathbb{T}$ and $0 < l < e^{q_m}$. Then

$$\begin{aligned}|S^{R_\omega}_{lq_m}\varphi(x, y) - lq_m + lq_m\eta_m| &\leq |S^{R_\omega}_{lq_m}P_{\mu,m}(x, y) + lq_m\eta_m| \\ &\quad + |S^{R_\omega}_{lq_m}P| + |S^{R_\omega}_{lq_m}L_m| + |S^{R_\omega}_{lq_m}H_m| \\ &\leq \frac{lq_m e^{3q_m/4}}{q_{m+1}} + \frac{lC_0}{q_{m+1}} + \frac{lC_0q_m}{q_{m+1}} + \frac{2lq_m}{e^{q'_m}} \\ &< \frac{1}{e^{2q_m}}\end{aligned}$$

for sufficiently large m. □

By (18) and the previous claim,

$$lq_m - lq_m\eta_m - e^{-2q_m} < k < lq_m - lq_m\eta_m + e^{-2q_m} + \frac{q_m\eta_m}{2}. \tag{20}$$

Since $k \leq \eta_m^{-1} - 2$, the left-hand side of the previous equation yields

$$lq_m\eta_m \leq 1 - \frac{\eta_m}{1-\eta_m}(1 - e^{-2q_m}).$$

Thus, by applying the previous inequality in the left-hand side of (20), it follows that

$$k > lq_m - 1$$

for m sufficiently large. But the right-hand side of (20) implies that

$$k < lq_m$$

for m sufficiently large, which is a contradiction since k is a natural number. Therefore $\mathcal{T}^- = \mathcal{T}(F^-, h^-)$ is a well-defined Rokhlin tower for $(G_{\omega,\varphi}, \mu_{\omega,\varphi})$. Furthermore,

$$\mu_{\mathcal{T}^-} = h^-\mu(F^-) = 2\mu(1 - \eta_m) > \mu.$$

Let $(x, y, z) \in F^-$. Taking $l = (\eta_m q_m)^{-1}$ in (19), we obtain

$$|h^- - S^{R_\omega}_{h^-+1}\varphi(x, y)| < \frac{1}{e^{2q_m}}.$$

Hence $\pi(h^-, x, y, z) = h^- + 1 = \eta_m^{-1}$ and

$$|G^{h^-}_{\omega,\varphi}(x, y, z) - (x, y, z)| = (|||\eta_m^{-1}\Omega|||, |||\eta_m^{-1}\Omega'|||, h^- - S^{R_\omega}_{h^-+1}\varphi(x, y)).$$

Since $|||\eta_m^{-1}\Omega||| \leq \eta_m^{-1}/(q_{m+1})$, it easily follows that

$$\rho_{\mathcal{T}^-} < \frac{\mu}{2(h^+)^2}$$

for sufficiently large m. It remains to show that $d(\mathcal{T}^-, \mathcal{P}) \leq \beta$. For this, consider an open cover of $\mathbb{T}^3$,

$$\mathcal{U} = \{P \mid P \in \mathcal{P}, P \neq \emptyset\} \cup \{U\},$$

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where U is an open set of μ_{φ_0+P}-measure less than $\beta/2$ and let a be the Lebesgue number of this cover. The existence of such cover follows directly from the hypotheses on $\mathcal{P}$. We claim that, for m sufficiently large, the diameter of every level of the towers $\mathcal{T}^-$ is smaller than a. By (19), for all $0 \le l < e^{q_m}$,

$$\begin{aligned}\operatorname{diam}(G_{\omega,\varphi}^{lq_m}(F^-)) &\le q_m\eta_m + \frac{1}{e^{2q_m}} + \frac{3\mu}{q_m}\|\varphi'\|_{C^0}\\ &\le \frac{2q_m^2}{e^{q_m}} + \frac{1}{e^{2q_m}} + \frac{6\mu}{q_m}\\ &\le \frac{10\mu}{q_m}\end{aligned}$$

for m sufficiently large. By definition of $\mathcal{T}^-$ and by Corollary A.2, there exists a positive constant C such that, for all $0 < r < q_m$,

$$\begin{aligned}&\operatorname{diam}(G_{\omega,\varphi}^{lq_m+r}(F^-)) \le \operatorname{diam}(G_{\omega,\varphi}^{lq_m}(F^-))\Big(1 + \max_{0\le i<q_m}\big\|S_i^{R_\omega}\varphi'\big\|_{C^0}\Big)\\ &\le \frac{10\mu}{q_m}\Big(1 + \max_{0\le i<q_m}\big\|S_i^{R_\omega}P'\big\|_{C^0} + \max_{0\le i<q_m}\big\|S_i^{R_\omega}L'_{m-1}\big\|_{C^0}\\ &\quad + q_m\|X_{m-1} + Y_{m-1} + P_{\mu,m} + H_m\|_{C^1}\Big)\\ &\le \frac{10\mu}{q_m}\left(1 + C + Cq_{m-1} + q_m\left(\frac{q_{m-1}}{e^{q_{m-1}}} + \frac{q'_{m-1}}{e^{q'_{m-1}}} + \frac{1}{e^{q_m/4}} + \frac{2q'_m}{e^{q'_m}}\right)\right)\\ &\le a\end{aligned}$$

for m sufficiently large. Therefore every level of the tower $\mathcal{T}^-$ is contained in at least one element of the open cover $\mathcal{U}$. By definition of $\mathcal{U}$, it easily follows that

$$\Delta(\mathcal{T}^-, \mathcal{P}) < \mu_{\omega,\varphi}(U).$$

Since $\mu_{\omega,P+\varphi_0}(U) < \beta/2$, we have $\mu_{\omega,\varphi}(U) < \beta$ for m sufficiently large. □

We now have all the tools to prove Theorem 4.3.

Proof of Theorem 4.3. Let $\varphi \in C_+^\infty(\mathbb{T}^2)$ with average equal to one, let $r \in \mathbb{N}$ and let $\epsilon > 0$. We will define $\psi \in C_+^\infty(\mathbb{T}^2)$ with average equal to one such that

$$\|\varphi - \psi\|_{C^r} < \epsilon,$$

$(G_{\omega,\psi}, \mu_{\omega,\psi})$ is mixing and $(G_{\omega,\psi} \times G_{\omega,\psi}, \mu_{\omega,\psi} \times \mu_{\omega,\psi})$ is LB. We will obtain ψ as the limit of a recursive procedure. By Proposition 4.2, there exists a trigonometric polynomial P_0 with zero average such that $\psi_0 = \varphi_0 + P_0 \in \mathcal{C}_{\text{Mix}}$ and

$$\|\varphi - \psi_0\|_{C^r} < \frac{\epsilon}{2}.$$

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We define a sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of refining partitions as follows. For all $n \in \mathbb{N}$, $\mathcal{P}_n$ is the partition of $\mathbb{T}^3$ into 2^{3n} open cubes of the form

$$\left(\frac{i}{2^n}, \frac{i+1}{2^n}\right) \times \left(\frac{j}{2^n}, \frac{j+1}{2^n}\right) \times \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right),$$

for some $i, j, k \in \mathbb{N}$, together with a zero Lebesgue measure set given by the union of the boundaries of the cubes. The order of the elements of each partition is not relevant. Clearly, $\mathcal{P}_n \to \mathcal{B}(\mathbb{T}^3)$. Let $\mu_n = (n+1)^{-1/4}$ and define

$$\alpha_0 = 0, \quad \alpha_{n+1} = \alpha_n + \delta_n, \quad \delta_n = \frac{\mu_n^4}{4}\left(\frac{1-\alpha_n}{10}\right)^2$$

for all $n \in \mathbb{N}$. Propositions 4.4, 4.5, and 4.6 imply the following lemma.

LEMMA 4.7. *There exist closed sets* $\{F_n^{\pm}\}_{n\geq 1} \subset \mathcal{E}$ *and sequences* $\{h_n^{\pm}\}_{n\geq 1} \subset \mathbb{N}$, $\{N_{n,k}\}_{n,k\in\mathbb{N}} \subset \mathbb{N}$, $\{m_n\}_{n\geq 1} \subset \mathbb{N}$, $\{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$, *with* m_n *increasing and* ϵ_n *decreasing, such that, for all* $0 \leq k < l \leq n$, *denoting*

$$P_l = P_{\mu_l, m_l} - X_{m_l},$$

with P_{μ_l,m_l} *as in Proposition 4.5, and*

$$\psi_n(x, y) = \varphi_0(x, y) + P_0(x, y) + P_1(x) + \cdots + P_n(x),$$

the following hold.

(1) $\psi_n \in \mathcal{C}_{Mix}$.
(2) $\|P_n\|_{C^{r+n}} \leq \epsilon_{n-1} \leq (1/2^n)$.
(3) $h_n^+ = h_n^- + 2$, *and* $h_n^+ h_n^- > N_{k,n-k}$.
(4) *For all* $\psi \in \mathcal{C}_{Mix}$ *obeying*

$$\|\psi - \psi_n\|_{C^0} \leq \epsilon_n,$$

$\mathcal{T}_l^{\pm} = \mathcal{T}(G_{\omega,\psi}, F_l^{\pm}, h_l^{\pm})$ *are well-defined Rokhlin towers,* δ_l*-monochromatic with respect to* $\mathcal{P}_k$, *obeying*

$$\mu_{\mathcal{T}_l}^{\pm} \geq \frac{1}{l^{1/4}}, \quad \rho_{\mathcal{T}_l^{\pm}} < \frac{\mu_{\mathcal{T}_l}}{2(h_l^+)^2}.$$

Moreover, $(G_{\omega,\psi}, \mathcal{P}_k)$ *and* $\mathfrak{T} = \{\mathcal{T}_i\}_{k\leq i<n}$ *verify the hypotheses of Proposition 3.7. For* $0 \leq i \leq n-k$, *the numbers* $N_i(G_{\omega,\psi}, \mathcal{P}_k, \mathcal{T}_k^{\pm}, \ldots, \mathcal{T}_{k+i-1}^{\pm})$ *given by Proposition 3.7 can be taken as* $N_{k,i}$.

Let us assume that the lemma has been proved and let

$$\psi = \lim_{n\to\infty} \psi_n.$$

By condition (2) in the lemma, ψ is a well-defined C^∞ function obeying

$$\|\varphi - \psi\|_{C^r} < \epsilon.$$

We can suppose, without loss of generality, that $\psi \in C_+^\infty(\mathbb{T}^2)$. By Proposition 4.5, $\psi \in \mathcal{C}_{\mathrm{Mix}}$ and thus $(G_{\omega,\psi}, \mu_{\omega,\psi})$ is mixing. By (4) in the lemma, the transformation $G_{\omega,\psi}$ and the collection $\{\mathcal{T}_n^{\pm}\}_{n\geq 1}$ verify the hypotheses of Theorem 3.8. Furthermore, by (3) and (4)

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in the lemma and by the remark in Theorem 3.8, it follows that $(G_{\omega,\psi} \times G_{\omega,\psi}, \mu_{\omega,\psi} \times \mu_{\omega,\psi})$ is LB. This proves Theorem 4.3. □

Proof Lemma 4.7. We prove the lemma by induction. Define $\epsilon_0 = \epsilon/4$, $N_{0,0} = 0$ and let $n \geq 1$. Suppose $N_{k,i}$ has been defined for all $0 \leq k < n$, $0 \leq i < n-k$ and that the other sequences in the statement have been defined up to their $(n-1)$th term. For all $0 \leq k \leq n$, define

$$N_{k,n-k} = N_{n-k}(G_{\omega,\psi_{n-1}}, \mathcal{P}_k, \mathcal{T}(G_{\omega,\psi_{n-1}}, F_k^{\pm}, h_k^{\pm}), \ldots, \mathcal{T}(G_{\omega,\psi_{n-1}}, F_{n-1}^{\pm}, h_{n-1}^{\pm})),$$

with N_{n-k} as in Proposition 3.7. Applying Proposition 4.6 with

$$P = P_0 + \cdots + P_{n-1}, \quad \mathcal{P} = \mathcal{P}_n, \quad \beta = \delta_n,$$

$$\mu = \mu_n, \quad \epsilon = \epsilon_{n-1}, \quad N = \frac{2}{\delta_n} \max_{0 \leq k \leq n} N_{k,n-k},$$

there exists $m_n \in \mathbb{N}$ such that, for $P_n = P_{\mu_n,m_n} - X_{m_n}$,

$$\psi_n = \psi_{n-1} + P_n \in C_+^{\infty}(\mathbb{T}^2), \quad \|P_n\|_{C^{r+n}} < \epsilon_{n-1},$$

and the system $(G_{\omega,\psi_n}, \mu_{\omega,\psi_n})$ admits Rokhlin towers $\mathcal{T}^{\pm} = \mathcal{T}(G_{\omega,\psi_n}, F_n^{\pm}, h_n^{\pm})$, δ_n-monochromatic with respect to $\mathcal{P}_n$, with closed floor sets such that

$$h_n^+ = h_n^- + 2 > N, \quad \mu_{\mathcal{T}^{\pm}} > \mu_n, \quad \rho_{\mathcal{T}^{\pm}} < \frac{\mu_n}{2(h_n^+)^2}.$$

By Proposition 4.2, ψ_n satisfies the first assertion. By construction, the second and third assertions also hold for P_n and $h_n^{\pm}$, respectively. Thus, it remains to define ϵ_n such that the fourth assertion holds. Let $0 \leq k < l \leq n$. Since

$$\|\psi_n - \psi_{n-1}\|_{C^0} < \epsilon_{n-1},$$

it follows from (4) in the lemma that

$$N_{k,n-k} = N_{n-k}(G_{\omega,\psi_n}, \mathcal{P}_k, \mathcal{T}(G_{\omega,\psi_n}, F_k^{\pm}, h_k^{\pm}), \ldots, \mathcal{T}(G_{\omega,\psi_n}, F_{n-1}^{\pm}, h_{n-1}^{\pm})).$$

By (4) in the lemma, if $k < n-1$, the transformation $(G_{\omega,\psi_n}, \mathcal{P}_k)$ and the collection $\mathcal{T} = \{\mathcal{T}(G_{\omega,\psi_n}, F_i^{\pm}, h_i^{\pm})\}_{k \leq i < n}$ verify the hypotheses of Proposition 3.7. By construction, if $k = n-1$, then $(G_{\omega,\psi_n}, \mathcal{P}_{n-1})$ and $\mathcal{T} = \{\mathcal{T}(G_{\omega,\psi_n}, F_n^{\pm}, h_n^{\pm})\}$ verify the hypotheses of Proposition 3.7. By Proposition 4.4 and by continuity, there exist

$$0 < \epsilon_n \leq \frac{\epsilon}{2^{n+2}}$$

such that the conclusions in (4) hold for any $\psi \in \mathcal{C}_{\text{Mix}}$ satisfying

$$\|\psi - \psi_n\|_{C^0} \leq \epsilon_n.$$

This completes the proof of Lemma 4.7. □

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A. *Appendix*

LEMMA A.1. *Let $\omega \in \mathbb{R}^d$ such that $(1, \omega)$ is non-resonant. Given a trigonometric polynomial $P : \mathbb{T}^d \to \mathbb{R}$ with zero average and of degree $n \geq 1$, there exists a trigonometric polynomial Q of degree n such that*

$$P = Q \circ R_\omega - Q, \tag{A.1}$$

where $R_\omega : \mathbb{T}^d \to \mathbb{T}^d$ denotes the toral translation of rotation vector ω. In particular, the Birkhoff sums of P with respect to R_ω obey

$$\|S_m^{R_\omega} P\|_{C^r} \leq \min\{2\|Q\|_{C^r}, |||m\omega|||\,\|Q\|_{C^{r+1}}\} \tag{A.2}$$

for all $m \geq 1$. Furthermore, there exists a constant $C(d, r)$ such that

$$\|Q\|_{C^r} \leq C\|P\|_{C^{d+r+1}} \max_{0<|k|_1\leq n} |||\langle k, \omega\rangle|||^{-1}.$$

Proof. Consider the Fourier expansion of P

$$P(\theta) = \sum_{|k|_1\leq n} p_k e^{2\pi i k\cdot\theta}$$

and define

$$Q(\theta) = \sum_{|k|_1\leq n} \frac{p_k}{e^{2\pi i k\cdot\omega} - 1} e^{2\pi i k\cdot\theta}. \tag{A.3}$$

Then Q satisfies (A.1). Equation (A.2) is a direct consequence of (A.1). Recall that, given that $r \in \mathbb{N}$, there exists a constant $C(d, r)$ such that, for any $f \in C^\infty(\mathbb{T}^d, \mathbb{R})$ with Fourier coefficients f_k,

$$C^{-1}|k|_1^r |f_k| \leq \|f\|_{C^r} \leq \sup_{k\in\mathbb{N}^d} C|k|_1^{r+d+1}|f_k|.$$

Since

$$|e^{2\pi i k\cdot\omega} - 1| \geq 2\sin(\pi\,|||k\cdot\omega|||) \geq |||k\cdot\omega|||,$$

the bound for the C^r-norm of Q now follows from (A.3). □

Remark. For $\omega \in \mathbb{R}\setminus\mathbb{Q}$ with return times $(q_n)_{n\in\mathbb{N}}$,

$$q_n \leq \max_{0<|k|_1<q_n} |||k\omega|||^{-1} = |||q_{n-1}\alpha|||^{-1} \leq 2q_n.$$

COROLLARY A.2. *Let $\omega \in \mathbb{R}^d$ such that $(1, \omega)$ is non-resonant. Given that $r \in \mathbb{N}$, there exists a constant $C(d, r)$ such that, for any trigonometric polynomial $P : \mathbb{T}^d \to \mathbb{R}$ with zero average and for all $m \geq 1$,*

$$\|S_m^{R_\omega} P\|_{C^r} \leq C\|P\|_{C^{r+d+2}} |||m\omega||| \max_{0<|k|_1\leq \deg(P)} |||\langle\omega, k\rangle|||^{-1}.$$

In particular, if $d = 1$, $\omega \in \mathbb{R}\setminus\mathbb{Q}$ with return times $(q_n)_{n\in\mathbb{N}}$ and $\deg(P) < q_{m_0}$, then

$$\|S_{q_{m-1}}^{R_\omega} P\|_{C^r} \leq 2C\frac{q_{m_0}}{q_m}\|P\|_{C^{r+2}}$$

for all $m \geq 1$.

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The construction in Proposition 4.5 is analogous to that of the polynomial families defined in [4, Proposition 3.5] and in [5, Proposition 4]. The proof of the last part of Proposition 4.5 is completely analogous to the proofs of [4, Theorem 3.1] and [5, Theorem 3], which rely on the mixing criterion in Proposition 4.1. We will briefly sketch the proofs here. For more details, we refer the interested reader to [4–6].

Proof of Proposition 4.5. Let $r, n \in \mathbb{N}$ and let $0 < \mu < 1/16$. Let $\widetilde{\xi}_n : \mathbb{R} \to \mathbb{R}$ be the $1/q_n$-periodic odd function defined by

$$\widetilde{\xi}_n(x) = \begin{cases} x & \text{if } x \in \left[0, \dfrac{1}{4q_n} - \dfrac{7\mu}{2q_n}\right], \\ \dfrac{1-14\mu}{4q_n} & \text{if } \left|x - \dfrac{1}{4q_n}\right| < \dfrac{7\mu}{2q_n}, \\ -x + \dfrac{1}{2q_n} & \text{if } x \in \left[\dfrac{1}{4q_n} + \dfrac{7\mu}{2q_n}, \dfrac{1}{2q_n}\right]. \end{cases}$$

Fix

$$\frac{q_n e^{-q_n}}{2} \leq \eta_n \leq \frac{3q_n e^{-q_n}}{4}$$

such that $1/\eta_n \in 2q_n\mathbb{N}$ (which is always possible for n sufficiently large) and define

$$\xi_n(x) = \frac{4q_n\eta_n}{1-14\mu}\widetilde{\xi}_n(x).$$

Let $K : \mathbb{R} \to \mathbb{R}$ be an even, positive, smooth bump function obeying

$$\mathrm{Supp}(K) \subset (-1, 1), \quad \int_{\mathbb{R}} K(x)dx = 1.$$

Define $K_n(x) = n^2 q_n K(n^2 q_n x)$. Clearly,

$$\mathrm{Supp}(K_n) \subset \left(\frac{-1}{n^2 q_n}, \frac{1}{n^2 q_n}\right), \quad \int_{\mathbb{R}} K_n(x)dx = 1.$$

Let

$$X = K_n * \xi_n$$

and notice that X is a well-defined $1/q_n$-periodic odd function. Define $P_{\mu,n}$ as the truncation of the Fourier series of $X = K_n * \xi_n$ to degree $q_{n+1} - 1$, namely,

$$P_{\mu,n}(x) = \sum_{k=-q_{n+1}+1}^{q_{n+1}-1} \widehat{X}(k)e^{2\pi ikx}.$$

Let us denote

$$I = \left\{x \in \mathbb{T} \,\middle|\, \left|\{q_n x\} + \tfrac{1}{4}\right| < 3\mu\right\}, \quad I' = \left\{x \in \mathbb{T} \,\middle|\, \left|\{q_n x\} + \tfrac{3}{4}\right| < 3\mu\right\},$$

$$J = \left\{x \in \mathbb{T} \,\middle|\, \left|\{q_n x\} + \tfrac{1}{4}\right| > 4\mu\right\}, \quad J' = \left\{x \in \mathbb{T} \,\middle|\, \left|\{q_n x\} + \tfrac{3}{4}\right| > 4\mu\right\}.$$

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Notice that the sets $I \cup I'$ and $J \cup J'$ correspond to the points satisfying the inequalities in assertions (3) and (4), respectively. We have

$$X|_I = \eta_n, \quad X|_{I'} = -\eta_n,$$

$$X'|_J = \frac{4q_n\eta_n}{1-14\mu}, \quad X'|_{J'} = -\frac{4q_n\eta_n}{1-14\mu}.$$

In particular,

$$\left|X'|_{J\cup J'}\right| \geq 2q_n^2 e^{-q_n}.$$

By the definition of X,

$$\begin{aligned}\|X\|_{C^{r+2}} &\leq \eta_n (n^2 q_n)^{r+3} \|K\|_{C^{r+2}} \\ &\leq n^{2r+6} q_n^{r+4} e^{-q_n} \|K\|_{C^{r+2}} \\ &\leq e^{-4q_n/5}\end{aligned}$$

for sufficiently large n. Furthermore,

$$\begin{aligned}\|X - P_{\mu,n}\|_{C^r} &\leq \sum_{|k|\geq q_{n+1}} (2\pi k)^r |\widehat{X}(k)| \\ &\leq \frac{1}{2\pi} \sum_{|k|\geq q_{n+1}} \frac{\|X\|_{C^{r+2}}}{k^2} \\ &\leq \frac{e^{3q_n/4}}{q_{n+1}}\end{aligned}$$

for n sufficiently large. Therefore conditions (2)–(4)in the proposition are verified by $P_{\mu,n}$ for n sufficiently large. It remains to show that the limit function belongs to $\mathcal{C}_{\text{Mix}}$. We do this by applying the mixing criterion in Proposition 4.1. We check only the first part of Proposition 4.1 as the second part is verified exactly as in [**4–6**]. We can express ψ as

$$\psi(x, y) = 1 + P(x, y) + \sum_{n=1}^{\infty} P_n(x) + e^{-q_n'} \cos(2\pi q_n' y),$$

where P_n is either equal to $e^{-q_n} \cos(2\pi q_n x)$ or to some polynomial P_{μ_n,q_n} of degree less than q_{n+1} given by Proposition 4.5 for some $\mu_n > (1/n)$. Define I_n as the set of $x \in \mathbb{T}$ such that

$$\{q_n x\} \in \left[\frac{1}{n}, \frac{1}{4} - r_n\right] \cup \left[\frac{1}{4} + r_n, \frac{1}{2} - \frac{1}{n}\right] \cup \left[\frac{1}{2} + \frac{1}{n}, \frac{3}{4} - r_n\right] \cup \left[\frac{3}{4} + r_n, 1 - \frac{1}{n}\right],$$

where

$$r_n = \begin{cases} 5\mu_n & \text{if } P_n = P_{\mu_n,q_n}, \\ n^{-1} & \text{otherwise.} \end{cases}$$

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Notice that, by definition of μ_n, we have $\mu_n \to 0$ and thus I_n satisfies the hypotheses of Proposition 4.1. Let $m \in [e^{2q_n}/2, 2e^{2q_n'}]$ and suppose that $\deg(P) < q_{n-1}$. Denote

$$L_n(x, y) = P(x, y) + \sum_{k=1}^{n-1} P_k(x), \quad H_n(x) = \sum_{k>n} P_k(x).$$

Then

$$\partial_x S_m^{R_\omega} \psi = \partial_x S_m^{R_\omega}(L_n + P_n + H_n).$$

By Corollary A.2 applied to L_n, there exists a positive constant C such that

$$\|\partial_x S_m^{R_\omega}(\psi - P_n)\|_{C^0} \le C q_n + e^{-q_n/2} \le \frac{m}{q_n}$$

for sufficiently large n. If $P_n = e^{-q_n} \cos(2\pi q_n x)$, it follows from [6] that

$$|\partial_x S_m^{R_\omega} P_n(x)| \ge 16 \frac{m q_n}{n e^{q_n}}$$

for all $x \in I_n$. If $P_n = P_{\mu_n, q_n}$, then, for all $x \in I_n$ and all $0 \le l \le m$,

$$\left| \{q_n(x + l\Omega)\} + \frac{1}{2} \pm \frac{1}{4} \right| > 5\mu - \frac{l}{q_{n+1}} > 4\mu$$

for sufficiently large n. Thus, by condition (4) of Proposition 4.5,

$$|\partial_x S_m^{R_\omega} P_n(x)| \ge \frac{m q_n^2}{e^{q_n}}$$

for all $x \in I_n$. Hence, in both cases, the first part of Proposition 4.1 is verified. Therefore $\psi \in \mathcal{C}_{\text{Mix}}$. □

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