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Elliptic functions, theta function and hypersurfaces satisfying a basic equality

By BANG-YEN CHEN and JIE YANG

Department of Mathematics, Michigan State University, East Lansing, MI 48824, *U.S.A.*

e-mail: bychen@math.msu.edu & jyang@math.msu.edu

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Abstract

In previous papers [**4**, **6**], B.-Y. Chen introduced a Riemannian invariant δ_M for a Riemannian n-manifold M^n, namely take the scalar curvature and subtract at each point the smallest sectional curvature. He proved that every submanifold M^n in a Riemannian space form $R^m(\epsilon)$ satisfies: $\delta_M \leqslant [n^2(n-2)]/2(n-1)H^2 + \frac{1}{2}(n+1)(n-2)\epsilon$. In this paper, first we classify constant mean curvature hypersurfaces in a Riemannian space form which satisfy the equality case of the inequality. Next, by utilizing Jacobi's elliptic functions and theta function we obtain the complete classification of conformally flat hypersurfaces in Riemannian space forms which satisfy the equality.

1. *Introduction*

For a submanifold M^n in an m-dimensional Riemannian space form $R^m(\epsilon)$ of constant sectional curvature ϵ, the first author proves in [**4**] a sharp inequality involving the sectional curvature K of M^n, the scalar curvature τ of M^n and the squared mean curvature H^2. It is convenient to introduce a Riemannian invariant δ_M of M^n by $\delta_M(p) = \tau(p) - \inf K(p)$, where $\inf K$ is the function assigning to each $p \in M^n$ the infimum of $K(\pi)$, where π runs over all 2-planes in T_pM and τ is defined by $\tau := \sum_{i<j} K(e_i \wedge e_j)$, and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM^n. The inequality proved in [**4**] can be written as follows:

$$\delta_M \leqslant \frac{n^2(n-2)}{2(n-1)} H^2 + \tfrac{1}{2}(n+1)(n-2)\epsilon. \tag{1·1}$$

The equality case of (1·1) holds identically for $n = 2$ since both sides of (1·1) vanish trivially.

Inequality (1·1) has some important applications, for example, it gives rise to the second Riemannian obstruction for a Riemannian manifold to admit a minimal isometric immersion into a Euclidean space. It also gives rise to an obstruction to Lagrangian isometric immersions from compact Riemannian manifolds with finite fundamental group π_1 into complex space forms (see [**6**] for details).

Since (1·1) is a very general and sharp inequality, it is natural and interesting to

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investigate and to understand submanifolds in a Riemannian space form which verify the equality of this inequality, i.e. the equality

$$\delta_M = \frac{n^2(n-2)}{2(n-1)}H^2 + \tfrac{1}{2}(n+1)(n-2)\,\epsilon. \tag{1·2}$$

When M is a hypersurface with dimension $\geqslant 3$, equality (1·2) holds at a point $p \in M$ if and only if, up to permutations, the principal curvatures $\lambda_1, \ldots, \lambda_n$ of M at p satisfy the condition $\lambda_1 + \lambda_2 = \lambda_3 = \cdots = \lambda_n$ (cf. [**4**]). Therefore, if M satisfies (1·2) identically, M has a principal curvature μ with multiplicity $\geqslant n-2$. When $n \geqslant 4$, the multiplicity of μ is at least 2. In particular, if the principal curvature μ has constant multiplicity then $d\mu$ vanishes along the corresponding principal directions and, moreover, in this case the leaves of the corresponding foliation of M are known to be extrinsic spheres of the ambient space $R^m(\epsilon)$ (cf. [**18**, **21**–**23**]). In Section 5 we also prove that a similar phenomenon holds for conformally flat hypersurfaces in 4-dimensional Riemannian space forms which satisfy equality (1·2).

Submanifolds satisfying this basic equality were studied recently in several papers (see for instance [**4**–**6**, **8**, **9**, **12**, **15**, **16**]). In this respect we point out that 3-dimensional totally real submanifolds satisfying this basic equality in the nearly Kähler 6-sphere S^6 (1) have been completely classified in [**16**] by F. Dillen and L. Vrancken.

In this paper we investigate the most fundamental case; namely hypersurfaces satisfying the basic equality.

In the first part of this article we completely classify CMC hypersurfaces satisfying the basic equality. More precisely, we prove the following theorems.

THEOREM 1. *A hypersurface M^n $(n > 2)$ of a Euclidean $(n+1)$-space $\mathbb{E}^{n+1}$ with constant mean curvature satisfies equality* (1·2) *if and only if either M^n is a minimal hypersurface with relative nullity $\geqslant n-2$ or M^n is an open portion of a spherical hypercylinder $\mathbb{R} \times S^{n-1}(r)$.*

THEOREM 2. *Let M^n $(n > 2)$ be a hypersurface with constant mean curvature in the sphere S^{n+1} (1). Then M^n satisfies equality* (1·2) *if and only if one of the following two cases occurs*:

(1) *M^n is a totally geodesic hypersurface*;

(2) *there is an open dense subset U of M^n and a non-totally geodesic, isometric, minimal immersion $\phi: B^2 \to S^{n+1}(1)$ from a surface B^2 into $S^{n+1}(1)$ such that U is an open subset of $NB^2 \subset S^{n+1}(1)$ defined by*

$$N_p B^2 = \{\xi \in T_{\phi(p)} S^{n+1}(1) \,|\, \langle \xi, \xi \rangle = 1 \quad \text{and} \quad \langle \xi, \phi_*(T_p B^2) \rangle = 0\}.$$

Let $\mathbb{E}_1^{n+2}$ denote the $(n+2)$-dimensional Minkowski space-time with the Lorentzian metric $g = -dx_1^2 + dx_2^2 + \cdots + dx_{n+2}^2$. Recall that the unit hyperbolic space $H^{n+1}(-1)$ and the unit de Sitter space-time $S_1^{n+1}(1)$ are isometrically embedded in $\mathbb{E}_1^{n+2}$ respectively in the following standard ways:

$$H^{n+1}(-1) = \{\mathbf{x} = (x_1, \ldots, x_{n+2}) \in \mathbb{E}_1^{n+2} \,|\, \langle \mathbf{x}, \mathbf{x} \rangle = -1\},$$
$$S_1^{n+1}(1) = \{x \in \mathbb{E}_1^{n+2} \,|\, \langle x, x \rangle = 1\}.$$

THEOREM 3. *Let M^n $(n > 2)$ be a hypersurface with constant mean curvature in the hyperbolic space $H^{n+1}(-1)$. Then M^n satisfies equality* (1·2) *if and only if one of the following three cases occurs*:

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(1) *M^n is a totally geodesic hypersurface*;

(2) *M^n is a tubular hypersurface with radius $r = \coth^{-1}(\sqrt{2})$ about a 2-dimensional totally geodesic surface of $H^{n+1}(-1)$*;

(3) *there is an open dense subset U of M^n and a non-totally geodesic, isometric, minimal immersion $\phi: B^2 \to S_1^{n+1}$ from a surface B^2 into the de Sitter space-time $S_1^{n+1}(1)$ such that U is an open subset of the unit normal bundle NB^2 of B^2 defined by*

$$N_p B^2 = \{\xi \in T_{\phi(p)} S_1^{n+1}(1) \,|\, \langle \xi, \xi \rangle = -1 \quad \text{and} \quad \langle \xi, \phi_*(T_p B^2) \rangle = 0\}.$$

In order to state our next results we recall the three families of Riemannian manifolds P_a^n $(a > 1)$, $C_a^n (a > 1)$, $D_a^n (0 < a < 1)$ and the two exceptional spaces F^n, L^n, first introduced in [**7**].

Let $\mathrm{cn}\,(u, k)$, $\mathrm{dn}\,(u, k)$ and $\mathrm{sn}\,(u, k)$ denote the three main Jacobi's elliptic functions with modulus k. The nine other elliptic functions $\mathrm{nd}\,(u, k)$, $\mathrm{nc}\,(u, k)$, $\mathrm{ns}\,(u, k)$, $\mathrm{sc}\,(u, k)$, $\mathrm{cd}\,(u, k)$, $\mathrm{ds}\,(u, k)$, $\mathrm{cs}\,(u, k)$, $\mathrm{dc}\,(u, k)$, $\mathrm{sd}\,(u, k)$ are defined by taking reciprocals and quotients; for example, $\mathrm{sd}\,(u, k) = \mathrm{sn}\,(u, k)/\mathrm{dn}\,(u, k)$, $\mathrm{nd}\,(u, k) = 1/\mathrm{dn}\,(u, k)$ (cf. [**17**, **19**] and Section 2 for details).

We put

$$\mu_a = ak\,\mathrm{cn}\,(ax, k), \quad k = \frac{\sqrt{a^2-1}}{\sqrt{2}a}, \quad a > 1, \tag{1·3}$$

$$\eta_a = \frac{a}{k}\mathrm{dn}\left(\frac{a}{k}x, k\right), \quad k = \frac{\sqrt{2}a}{\sqrt{a^2+1}}, \quad 0 < a < 1, \tag{1·4}$$

$$\rho_a = ak\,\mathrm{cn}\,(ax, k), \quad k = \frac{\sqrt{a^2+1}}{\sqrt{2}a}, \quad a > 1. \tag{1·5}$$

Let $S^n(c)$ and $H^n(-c)$ denote the n-sphere with constant sectional curvature c and the hyperbolic n-space with constant sectional curvature $-c$, respectively. For $n > 2$, P_a^n, D_a^n and C_a^n are the Riemannian n-manifolds given by the warped product manifolds $I \times_{\mu_a} S^{n-1}[(a^4-1)/4]$, $\mathbb{R} \times_{\eta_a} H^{n-1}[(a^4-1)/4]$ and $I \times_{\rho_a} S^{n-1}[(a^4-1)/4]$ with warped functions μ_a, η_a and ρ_a, respectively, where I are the open intervals on which the corresponding warped functions are positive. The two exceptional spaces F^n and L^n are the warped product manifolds $\mathbb{R} \times_{1/\sqrt{2}} H^{n-1}(-\frac{1}{4})$ and $\mathbb{R} \times_{\mathrm{sech}\,(x)} \mathbb{R}^{n-1}$, respectively.

D_a^n, F^n, L^n are complete Riemannian n-manifolds but P_a^n and C_a^n are not complete. Topologically, S^n is the two point compactification of both P_a^n and C_a^n. From [**10**] we know that the Riemannian metrics defined on P_a^n and C_a^n can be extended smoothly to their two point compactifications S^n. We denote by $\hat{P}_a^n$ and $\hat{C}_a^n$ the S^n together with the Riemannian metrics given by the (smooth) extensions of the metrics on P_a^n and C_a^n to S^n, respectively. We remark that P_a^n, D_a^n, C_a^n are indeed isometric to the warped products n-manifolds $I \times_{f_1} S^{n-1}(1)$, $I \times_{f_2} H^{n-1}(-1)$, $I \times_{f_3} S^{n-1}(1)$ with warped functions $2\rho_a/(a^2+1)$, $2\mu_a/(a^2-1)$, $2\eta_a/(a^2+1)$, respectively.

Let $A_a^n (a > 1)$, $B_a^n (0 < a < 1)$, G^n, $H_a^n (a > 0)$ and $Y_a^n (0 < a < 1)$ denote the warped product manifolds

$$\mathbb{R} \times_{\sqrt{a^2-1}\cosh x} S^{n-1}(1), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times_{\sqrt{1-a^2}\cos x} S^{n-1}(1),$$

$$\mathbb{R} \times_{\cosh x} \mathbb{E}^{n-1}, \quad \mathbb{R} \times_{\sqrt{a^2+1}\cosh x} H^{n-1}(-1), \quad (0, \infty) \times_{\sqrt{1-a^2}\sinh x} S^{n-1}(1),$$

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respectively. When $n=2$, the second factor S^{n-1} or H^{n-1} in each of the warped product manifolds shall be replaced either by $S^1(1)$ or by $\mathbb{R}$. The geometry of A_a^2, G^2 and H_a^2 are similar in the sense that one can be obtained from the others by applying some suitable scalings on the first factor $\mathbb{R}$.

Clearly, A_a^n, G^n, H_a^n are complete Riemannian manifolds. Topologically, S^n is the two point compactification of B_a^n. As for P_a^n and C_a^n, the warped metric on B_n^a can be extended smoothly to its two point compactification; a fact that follows from (1·8). We denote by $\hat{B}_a^n$ the S^n together with the Riemannian metrics on S^n extended from the metric on B_a^n.

For $n \geqslant 2$ and any real number $a>0$, there is a well-known Lagrangian immersion from the unit n-sphere S^n into a complex Euclidean n-space $\mathbb{C}^n$ defined by

$$w_a(y_0, y_1, \ldots, y_n) = \frac{a}{1+y_0^2}(y_1, \ldots, y_n, y_0 y_1, \ldots, y_0 y_n), \tag{1·6}$$

where $y_0^2+y_1^2+\cdots+y_n^2=1$. This Whitney's immersion w_a has a unique self-intersection point $w_a(-1,0,\ldots,0)=w_a(1,0,\ldots,0)$. The S^n together with the metric induced from Whitney's immersion w_a, denoted by W_a^n, is called a Whitney n-sphere.

In the second part of this article we completely classify conformally flat hypersurfaces satisfying the basic equality. In this respect first we sharpen proposition 2 of [**15**] to the following.

THEOREM 4. *Let $\mathbf{x}: M^n \to \mathbb{E}^{n+1}(n>2)$ be an isometric immersion of a conformally flat n-manifold into a Euclidean $(n+1)$-space. Then it satisfies equality* (1·2) *if and only if one of the following four cases occurs*:

(1) *M^n is totally geodesic*;

(2) *M^n is an open portion of a spherical hypercylinder $S^{n-1}\times\mathbb{R}$*;

(3) *M^n is an open portion of a round hypercone*;

(4) *$n=3$ and M^3 is an open portion of a Whitney 3-sphere W_a^3 for some $a>0$ and, up to rigid motions, the immersion $\mathbf{x}: W_a^3 \to \mathbb{E}^4$ is given by*

$$\mathbf{x}(x,y_1,y_2,y_3) = \left(\frac{1}{2}\int_0^x \mathrm{sd}^2\left(\frac{\sqrt{2}}{a}x\right)dx, ay_1\,\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right), ay_2\,\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right), ay_3\,\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right)\right), \tag{1·7}$$

where $y_1^2+y_2^2+y_3^2=\frac{1}{2}$ and $k=1/\sqrt{2}$ is the modulus of the Jacobi's elliptic functions.

Our next main results are those following.

THEOREM 5. *Let $\mathbf{x}: M^n \to S^{n+1}(1) \subset \mathbb{E}^{n+2}(n>2)$ be an isometric immersion of a conformally flat n-manifold. Then it satisfies equality* (1·2) *if and only if one of the following three cases occurs*:

(i) *M^n is an open portion of $S^n(1)$ and the immersion $\mathbf{x}: M^n \to S^{n+1}(1)$ is totally geodesic*;

(ii) *M^n is an open portion of $\hat{B}_a^n$ for some $a\in(0,1)$ and, up to rigid motions, the immersion $\mathbf{x}: \hat{B}_a^n \to S^{n+1}(1) \subset \mathbb{E}^{n+2}$ is given by*

$$\mathbf{x}(x,y_1,\ldots,y_n) = (\sin x, a\cos x, \sqrt{1-a^2}\,y_1\cos x, \ldots, \sqrt{1-a^2}\,y_n\cos x) \tag{1·8}$$

with $y_1^2+y_2^2+\cdots+y_n^2=1$;

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(iii) *$n=3$, M^3 is an open portion of $\hat{P}^3_a$ for some $a>1$ and, up to rigid motions, the immersion $\mathbf{x}\colon \hat{P}^3_a \to S^4(1)\subset \mathbb{E}^5$ is given by*

$$\left.\begin{aligned}\mathbf{x}(x,y_1,y_2,y_3) = \frac{1}{ak'}\Bigg(&y_1\,\mathrm{cn}\,(ax), y_2\,\mathrm{cn}\,(ax), y_3\,\mathrm{cn}\,(ax),\\ &\sqrt{a^2k'^2-\mathrm{cn}^2\,(ax)}\cos\left(\frac{k'}{k}x+\frac{i}{2}\left(\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)}+2aZ(\gamma)\,x\right)\right)\\ &\sqrt{a^2k'^2-\mathrm{cn}^2(ax)}\sin\left(\frac{k'}{k}x+\frac{i}{2}\left(\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)}+2aZ(\gamma)\,x\right)\right)\Bigg),\\ y_1^2+y_2^2+y_3^2=1,\quad &\gamma=\mathrm{sn}^{-1}\left(\frac{i}{ak^2}\right),\quad i=\sqrt{-1},\end{aligned}\right\} \tag{1·9}$$

where $k=\sqrt{a^2-1}/(\sqrt{2}a)$, $k'=\sqrt{a^2+1}/(\sqrt{2}a)$ are the modulus and the complementary modulus of Jacobi's elliptic functions respectively, $\Theta(u)=\Theta(u,k)$ is the theta function and $Z(u)=Z(u,k)$ the zeta function.

THEOREM 6. *Let $\mathbf{x}\colon M^n\to H^{n+1}(-1)\subset \mathbb{E}_1^{n+2}$ $(n>2)$ be an isometric immersion of a conformally flat n-manifold. Then it satisfies equality (1·2) if and only if one of the following nine cases occurs.*

(i) *M^n is an open portion of $H^n(-1)$ and the immersion $\mathbf{x}\colon M^n\to H^{n+1}(-1)$ is totally geodesic.*

(ii) *M^n is an open portion of A^n_a for some $a>1$ and, up to rigid motions, the immersion $\mathbf{x}\colon A^n_a\to H^{n+1}(-1)\subset\mathbb{E}_1^{n+2}$ is given by*

$$\mathbf{x}(x,y_1,\dots,y_n)=(a\cosh x,\sinh x,\sqrt{a^2-1}\,y_1\cosh x,\dots,\sqrt{a^2-1}\,y_n\cosh x) \tag{1·10}$$

with $y_1^2+y_2^2+\cdots+y_n^2=1$.

(iii) *M^n is an open portion of G^n and, up to rigid motions, the immersion $\mathbf{x}\colon G^n\to H^{n+1}(-1)\subset\mathbb{E}_1^{n+2}$ is given by*

$$\begin{aligned}&\mathbf{x}(x,u_2,\dots,u_n)\\ &=\left(\left(1+\frac{u_2^2+\cdots+u_n^2}{2}\right)\cosh x,\frac{u_2^2+\cdots+u_n^2}{2}\cosh x,\sinh x,u_2\cosh x,\dots,u_n\cosh x\right).\end{aligned} \tag{1·11}$$

(iv) *M^n is an open portion of H^n_a for some $a>0$ and, up to rigid motions, the immersion $\mathbf{x}\colon H^n_a\to H^{n+1}(-1)\subset\mathbb{E}_1^{n+2}$ is given by*

$$\mathbf{x}(x,y_1,\dots,y_n)=(\sqrt{a^2+1}\,y_1\cosh x,\dots,\sqrt{a^2+1}\,y_n\cosh x,a\cosh x,\sinh x) \tag{1·12}$$

with $y_1^2-y_2^2+\cdots-y_n^2=1$.

(v) *M^n is an open portion of Y^n_a for some a, $0<a<1$ and, up to rigid motions, the immersion $\mathbf{x}\colon Y^n_a\to H^{n+1}(-1)\subset\mathbb{E}_1^{n+2}$ is given by*

$$\mathbf{x}(x,y_1,\dots,y_n)=(\cosh x,a\sinh x,\sqrt{1-a^2}\,y_1\sinh x,\dots,\sqrt{1-a^2}\,y_n\sinh x) \tag{1·13}$$

with $y_1^2+y_2^2+\cdots+y_n^2=1$.

(vi) *$n=3$, M^3 is an open portion of F^3 and, up to rigid motions, the immersion $\mathbf{x}\colon F^3\to H^4(-1)\subset\mathbb{E}_1^5$ is given by*

$$\mathbf{x}(x,u,v)=(\sqrt{2}\,\cosh u\cosh v,\sqrt{2}\,\cosh u\sinh v,\sqrt{2}\sinh u,\cos x,\sin x). \tag{1·14}$$

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(vii) $n=3$, M^3 *is an open portion of* L^3 *and, up to rigid motions, the immersion* $\mathbf{x}\colon L^3\to H^4(-1)\subset\mathbb{E}_1^5$ *is given by*

$$\begin{aligned}\mathbf{x}(x,u,v)&=\operatorname{sech}x\\&\quad\times(x^2+u^2+v^2+\cosh^2x+\tfrac14,x^2+u^2+v^2+\cosh^2x-\tfrac14,x,u,v)\end{aligned}\tag{1·15}$$

(viii) $n=3$, M^3 *is an open portion of* $\hat{C}_a^3$ *for some* $a>1$ *and, up to rigid motions, the immersion* $\mathbf{x}\colon\hat{C}_a^3\to H^4(-1)\subset\mathbb{E}_1^5$ *is given by*

$$\left.\begin{aligned}&\mathbf{x}(x,y_1,y_2,y_3)\\&\quad=\frac{1}{ak'}\Bigg(\sqrt{a^2k'^2+\operatorname{cn}^2(ax)}\cosh\left(\frac{k'}{k}x-\frac12\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)}-aZ(\gamma)x\right),\\&\qquad\sqrt{a^2k'^2+\operatorname{cn}^2(ax)}\sinh\left(\frac{k'}{k}x-\frac12\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)}-aZ(\gamma)x\right),\\&\qquad y_1\operatorname{cn}(ax),y_2\operatorname{cn}(ax),y_3\operatorname{cn}(ax)\Bigg),\quad y_1^2+y_2^2+y_3^2=1,\end{aligned}\right\}\tag{1·16}$$

where $\gamma=\operatorname{sn}^{-1}(1/(ak^2))$, $k=\sqrt{a^2+1}/(\sqrt2a)$ *and* $k'=\sqrt{a^2-1}/(\sqrt2a)$.

(ix) $n=3$, M^3 *is an open portion of* D_a^3 *for some* $0<a<1$ *and, up to rigid motions, the immersion* $\mathbf{x}\colon D_a^3\to H^4(-1)\subset\mathbb{E}_1^5$ *is given by*

$$\left.\begin{aligned}&\mathbf{x}(x,u,v)\\&=\frac{1}{ak'}\Bigg(k\operatorname{dn}\left(\frac{a}{k}x\right)\cosh u\cosh v,\\&\quad k\operatorname{dn}\left(\frac{a}{k}x\right)\cosh u\sinh v,k\operatorname{dn}\left(\frac{a}{k}x\right)\sinh u,\\&\quad\sqrt{k^2\operatorname{dn}^2\left(\frac{a}{k}x\right)-a^2k'^2}\cos\left(k'x-\frac{i}{2}\ln\frac{\Theta((a/k)x-\gamma)}{\Theta((a/k)x+\gamma)}-i\frac{a}{k}Z(\gamma)x\right)\\&\quad\sqrt{k^2\operatorname{dn}^2\left(\frac{a}{k}x\right)-a^2k'^2}\sin\left(k'x-\frac{i}{2}\ln\frac{\Theta((a/k)x-\gamma)}{\Theta((a/k)x+\gamma)}-i\frac{a}{k}Z(\gamma)x\right)\Bigg)\end{aligned}\right\}\tag{1·17}$$

where $\gamma=\operatorname{sn}^{-1}(k/a)$, $k=\sqrt2a/\sqrt{1+a^2}$ *and* $k'=\sqrt{1-a^2}/\sqrt{1+a^2}$.

2. *Jacobi's elliptic functions, theta function and zeta function*

We review very briefly some known facts on Jacobi's elliptic functions, theta function and zeta function for later use (for details see, e.g. [**1**, **17**, **20**]).

Let θ be the temperature at time t at any point in a solid material whose conducting properties are uniform and isotropic. If ρ is the material's density, s its specific heat and k its thermal conductivity, θ satisfies the heat conduction equation: $\kappa\nabla^2\theta=\partial\theta/\partial t$, where $\kappa=k/s\rho$ is the diffusivity. In the special case where there is no variation of temperature in the x- and y-directions the heat flow is everywhere parallel to the z-axis and the heat equation reduced to the form:

$$\kappa\frac{\partial^2\theta}{\partial s^2}=\frac{\partial\theta}{\partial t},\quad\theta=\theta(z,t).\tag{2·1}$$

Consider the boundary conditions: $\theta(0,t)=\theta(\pi,t)=0$ and $\theta(z,0)=\pi\delta(z-\pi/2)$ for

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$0 < z < \pi$, where $\delta(z)$ is Dirac's unit impulse function. Then the solution of the boundary value problem is given by

$$\theta(z,t) = 2\sum_{n=0}^{\infty}(-1)^n e^{-(2n+1)^2\kappa t}\sin(2n+1)z \tag{2·2}$$

By writing $e^{-4\kappa t} = q$, the solution (2·2) assumes the form:

$$\theta_1(z,q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin(2n+1)z, \tag{2·3}$$

which is the first of the four theta functions. When the precise value of q is not important we shall suppress the dependence upon q.

If one changes the boundary conditions to $\partial\theta/\partial z = 0$ on $z = 0$ and $z = \pi$ with $\theta(z,0) = \pi\delta(z-\pi/2)$ for $0 < z < \pi$, then the corresponding solution of the boundary value problem of the heat equation (2·1) is given by

$$\theta_4(z) = \theta_4(z,q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz. \tag{2·4}$$

The theta function $\theta_1(z)$ of (2·3) is periodic with period 2π. Incrementing z by $\frac{1}{2}\pi$ yields the second theta function:

$$\theta_2(z) = \theta_2(z,q) = \theta_1\left(z+\frac{\pi}{2},q\right) = 2\sum_{n=0}^{\infty}q^{(n+\frac{1}{2})^2}\cos(2n+1)z. \tag{2·5}$$

Similarly, incrementing z by $\frac{1}{2}\pi$ for θ_4 yields the third theta function:

$$\theta_3(z) = \theta_3(z,q) = \theta_4\left(z+\frac{\pi}{2},q\right) = 1 + 2\sum_{n=1}^{\infty}q^{n^2}\cos 2nz. \tag{2·6}$$

The four theta functions θ_1, θ_2, θ_3, θ_4 can be extended to complex values for z and q such that $|q| < 1$.

The elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$ are defined as ratios of theta functions:

$$\operatorname{sn} u = \frac{\theta_3(0)\,\theta_1(z)}{\theta_2(0)\,\theta_4(z)}, \quad \operatorname{cn} u = \frac{\theta_4(0)\,\theta_2(z)}{\theta_2(0)\,\theta_4(z)}, \quad \operatorname{dn} u = \frac{\theta_4(0)\,\theta_3(z)}{\theta_3(0)\,\theta_4(z)}, \tag{2·7}$$

where $z = u/\theta_3^2(0)$. Define parameters k and k' by

$$k = \theta_2^2(0)/\theta_3^2(0), \quad k' = \theta_4^2(0)/\theta_3^2(0)$$

which are called the modulus and the complementary modulus of the elliptic functions; k and k' satisfy $k^2 + k'^2 = 1$. When it is required to state the modulus explicitly the elliptic functions of Jacobi will be written $\operatorname{sn}(u,k)$, $\operatorname{cn}(u,k)$, $\operatorname{dn}(u,k)$.

The elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$ satisfy the following relations:

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1, \quad \operatorname{dn}^2 u + k^2\operatorname{sn}^2 u = 1, \quad k^2\operatorname{cn}^2 u + k'^2 = \operatorname{dn}^2 u, \tag{2·8}$$

$$\operatorname{sn}(u+v) + \operatorname{sn}(u-v) = \frac{2\operatorname{sn} u\operatorname{cn} v\operatorname{dn} v}{1-k^2\operatorname{sn}^2 u\operatorname{sn}^2 v}, \tag{2·9}$$

$$\operatorname{sn}'(u) = \operatorname{cn}(u)\operatorname{dn}(u), \operatorname{cn}'(u) = -\operatorname{sn}(u)\operatorname{dn}(u), \operatorname{dn}'(u) = -k^2\operatorname{sn}(u)\operatorname{cn}(u). \tag{2·10}$$

The theta function $\Theta(u)$ and the zeta function $Z(u)$ are defined by

$$\Theta(u) = \theta_4\left(\frac{\pi u}{2K}\right), \quad Z(u) = \frac{d}{du}(\ln\theta_4), \quad K = \frac{\pi}{2}\theta_3^2(0), \tag{2·11}$$

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which satisfy the following identities:

$$Z(u+v) = Z(u)+Z(v)-k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u+v), \tag{2·12}$$

$$Z(u) = \frac{\Theta'(u)}{\Theta(u)}, \tag{2·13}$$

where $\Theta(u)$ is an even function. From (2·12) we have

$$k^2 \operatorname{sn} u \operatorname{sn} v\,[\operatorname{sn}(u+v)+\operatorname{sn}(u-v)] = Z(u-v)-Z(u+v)+2Z(v). \tag{2·14}$$

3. *Some lemmas*

Assume that M^n is a hypersurface in a real-space-form $R^{n+1}(\epsilon)$, $\epsilon = 1, 0$ or -1. We shall make use of the following convention on the ranges of indices:

$$1 \leqslant A, B, C \leqslant n+1; \quad 1 \leqslant i, j, k \leqslant n; \quad 3 \leqslant \alpha, \beta, \gamma \leqslant n.$$

Denote by $A = A_\xi$ the shape operator of M^n in $R^{n+1}(\epsilon)$ with respect to a unit normal vector ξ and by h the second fundamental form of M^n in $R^{n+1}(\epsilon)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A with associated orthonormal eigenvectors $e_1, \ldots, e_n$. Then we have

$$Ae_i = \lambda_i e_i. \tag{3·1}$$

If M^n satisfies the basic equality (1·2) then, by rearranging $e_1, \ldots, e_n$ if necessary, we have [**4**, **5**]

$$\lambda_1 = a, \quad \lambda_2 = \mu - a, \quad \lambda_\alpha = \mu. \tag{3·2}$$

In the following, let V denote the open subset of M^n on which M has exactly three distinct principal curvatures, i.e.

$$V = \left\{x \in U \,|\, a(x) \neq 0, \quad a(x) \neq \mu, \quad \text{and} \quad a(x) \neq \frac{\mu}{2}\right\}, \tag{3·3}$$

where a and μ are given by (3·2). Then there exists a frame $(e_1, \ldots, e_n)$ of orthonormal eigenvector fields on V. Let $\omega^1, \omega^2, \ldots, \omega^n$ denote the dual frame of $e_1, e_2, \ldots, e_n$. Then Cartan's structure equations give

$$d\omega^i = -\omega^i_j \wedge \omega^j, \quad d\omega^i_j = -\omega^i_k \wedge \omega^k_j + (\epsilon + \lambda_i\lambda_j)\,\omega^i \wedge \omega^j, \tag{3·4}$$

where (ω^A_B) are the connection forms. From (3·4) and Codazzi's equation, we have

$$\left.\begin{aligned} e_i\lambda_j &= (\lambda_i - \lambda_j)\,\omega^j_i(e_j) \\ (\lambda_j - \lambda_k)\,\omega^k_j(e_i) &= (\lambda_i - \lambda_k)\,\omega^k_i(e_j) \end{aligned}\right\} \tag{3·5}$$

for distinct i, j, k.

Let Γ^k_{ij} be defined by $\Gamma^k_{ij} := \langle \nabla_{e_j} e_i, e_k \rangle$ where ∇ is the Levi–Civita connection on M^n. Then we have

$$\omega^k_i = \sum_k \Gamma^k_{ij}\,\omega^j, \quad \Gamma^k_{ij} = \omega^k_i(e_j), \quad \Gamma^k_{ij} = -\Gamma^i_{kj}$$

In this way, (3.5) becomes

$$\left.\begin{aligned} e_i\lambda_j &= (\lambda_i - \lambda_j)\,\Gamma^j_{ij} \\ (\lambda_j - \lambda_k)\,\Gamma^k_{ji} &= (\lambda_i - \lambda_k)\,\Gamma^k_{ij} \end{aligned}\right\} \tag{3·6}$$

for distinct i, j, k.

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Let H denote the mean curvature of M^n in $R^{n+1}(\epsilon)$. Then $H = ((n-1)/n)\mu$.

From (3·2) and (3·6) we obtain

LEMMA 1. *Let M^n be a hypersurface in a Riemannian space form $R^{n+1}(\epsilon)$. If M^n satisfies equality* (1·2) *then on V we have*

$$\left.\begin{aligned}
&\Gamma^2_{1\alpha} = \frac{a}{2a-\mu}\Lambda_\alpha, \quad \Gamma^1_{\alpha 2} = \frac{a}{\mu-a}\Lambda_\alpha, \quad \Gamma^\beta_{1\alpha} = \Gamma^\beta_{2\alpha} = 0, \quad \alpha \neq \beta \\
&\Gamma^\alpha_{1\alpha} = \frac{e_1\mu}{a-\mu}, \quad \Gamma^\alpha_{2\alpha} = -\frac{e_2\mu}{a}, \quad \Gamma^2_{12} = -\frac{e_1(\mu-a)}{\mu-2a}, \\
&\Gamma^1_{21} = \frac{e_2 a}{\mu-2a}, \quad \Gamma^2_{\alpha 2} = \frac{e_\alpha(\mu-a)}{a}, \quad \Gamma^1_{\alpha 1} = \frac{e_\alpha a}{\mu-a}.
\end{aligned}\right\} \tag{3·7}$$

$$\left.\begin{aligned}
\omega^1_\alpha &= \frac{e_\alpha a}{\mu-a}\omega^1 + \frac{a\Lambda_\alpha}{\mu-a}\omega^2 + \frac{e_1\mu}{\mu-a}\omega^\alpha, \\
\omega^1_2 &= \frac{e_2 a}{\mu-2a}\omega^1 + \frac{e_1(\mu-a)}{\mu-2a}\omega^2 + \frac{a}{\mu-2a}\sum_{\alpha=3}^n \Lambda_\alpha\omega^\alpha, \\
\omega^2_\alpha &= \Lambda_\alpha\omega^1 + \frac{e_\alpha(\mu-a)}{a}\omega^2 + \frac{e_2\mu}{a}\omega^\alpha,
\end{aligned}\right\} \tag{3·8}$$

where $\Lambda_\alpha = \Gamma^2_{\alpha 1}$.

We also need

LEMMA 2. *Let M^n be a hypersurface in $R^{n+1}(\epsilon)$. If M^n satisfies equality* (1·2) *then on V we have*

$$e_1\left(\frac{e_1\mu}{\mu-a}\right) - e_\alpha\left(\frac{e_\alpha a}{\mu-a}\right) - \frac{(e_\alpha a)^2}{(\mu-a)^2} - \frac{2a\Lambda_\alpha^2}{\mu-2a} - \frac{(e_1\mu)^2}{(\mu-a)^2}$$
$$= -\frac{(e_2 a)(e_2\mu)}{(\mu-2a)a} - \frac{1}{\mu-a}\sum_{\beta\neq\alpha}^n (e_\beta a)\Gamma^\beta_{\alpha\alpha} + a\mu + \epsilon, \tag{3·9}$$

$$e_2\left(\frac{e_1\mu}{\mu-a}\right) - e_\alpha\left(\frac{a\Lambda_\alpha}{\mu-a}\right) + \frac{a^2\Lambda_\alpha(e_\alpha a)}{(\mu-a)^2(\mu-2a)} + \frac{e_\alpha(a-\mu)\Lambda_\alpha}{\mu-a}$$
$$-\frac{e_\alpha(\mu-a)}{\mu-2a}\Lambda_\alpha - \frac{(e_1\mu)(e_2\mu)}{(\mu-a)a} = \frac{e_1(a-\mu)(e_2\mu)}{(\mu-2a)a} - \frac{1}{\mu-a}\sum_{\beta\neq\alpha}^n a\Lambda_\beta\Gamma^\beta_{\alpha\alpha}, \tag{3·10}$$

$$e_1\left(\frac{a\Lambda_\alpha}{\mu-a}\right) - e_2\left(\frac{e_\alpha a}{\mu-a}\right) - \frac{(e_\alpha a)(e_2 a)}{(\mu-a)(\mu-2a)} - \frac{\mu\Lambda_\alpha e_1(\mu-a)}{(\mu-a)(\mu-2a)}$$
$$+\frac{(\mu-2a)\Lambda_\alpha e_1\mu}{(\mu-a)^2} = -\frac{(e_2 a)e_\alpha(\mu-a)}{a(\mu-2a)} - \sum_{\beta\neq\alpha}^n\left(\frac{e_\beta a}{\mu-a}\Gamma^\beta_{\alpha 2} - \frac{a\Lambda_\beta}{\mu-a}\Gamma^\beta_{\alpha 1}\right). \tag{3·11}$$

Proof. By applying the second Cartan's structure equation and by using Lemma 1 we may compute $d\omega^1_\alpha$ in two different ways. Formulas (3·9), (3·10) and (3·11) are obtained by comparing the coefficients of $\omega^1\wedge\omega^\alpha$, $\omega^2\wedge\omega^\alpha$, $\omega^1\wedge\omega^2$ of $d\omega^1_\alpha$, respectively. □

Similarly, by computing $d\omega^\alpha_2$ in two different ways and by comparing the coefficients of $\omega^1\wedge\omega^\alpha$, $\omega^2\wedge\omega^\alpha$, $\omega^1\wedge\omega^2$ of $d\omega^2_\alpha$, respectively, we obtain

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LEMMA 3. *Let M^n be a hypersurface in a Riemannian space form $R^{n+1}(\epsilon)$. If M^n satisfies equality* (1·2) *then on V we have*

$$e_2\left(\frac{e_2\mu}{a}\right)+e_\alpha\left(\frac{e_\alpha(a-\mu)}{a}\right)+\frac{2a^2\Lambda_\alpha^2}{(\mu-a)(\mu-2a)}-\frac{(e_\alpha(\mu-a))^2}{a^2}-\frac{(e_2\mu)^2}{a^2}$$
$$=\frac{e_1(\mu-a)(e_1\mu)}{(\mu-a)(\mu-2a)}-\sum_{\beta\neq\alpha}^{n}\frac{e_\beta(\mu-a)}{a}\Gamma_{\alpha\alpha}^{\mathrm{B}}+(\mu-a)\mu+\epsilon, \tag{3·12}$$

$$e_1\left(\frac{e_2\mu}{a}\right)-e_\alpha\Lambda_\alpha-\frac{\Lambda_\alpha e_\alpha a}{\mu-a}-\frac{\mu-a}{a(\mu-2a)}\Lambda_\alpha e_\alpha(\mu-a)-\frac{(e_2\mu)(e_1\mu)}{a(\mu-a)}$$
$$=\frac{(e_2a)(e_1\mu)}{(\mu-a)(\mu-2a)}-\frac{a\Lambda_\alpha e_\alpha a}{(\mu-a)(\mu-2a)}-\sum_{\beta\neq\alpha}\Lambda_\beta\Gamma_{\alpha\alpha}^{\beta}, \tag{3·13}$$

$$e_1\left(\frac{a_\alpha(\mu-a)}{a}\right)-e_2\Lambda_\alpha-\frac{\mu\Lambda_\alpha e_2a}{(\mu-a)(\mu-2a)}-\frac{(e_1(\mu-a))(e_\alpha(\mu-a))}{a(\mu-2a)}$$
$$+\frac{(\mu-2a)\Lambda_\alpha(e_2\mu)}{a(\mu-a)}=-\frac{(e_1(\mu-a))(e_\alpha a)}{(\mu-a)(\mu-2a)}-\sum_{\beta\neq\alpha}^{n}\left(\Lambda_\beta\Gamma_{\alpha2}^{\beta}-\frac{e_\beta(\mu-\alpha)}{\alpha}\Gamma_{\alpha1}^{\beta}\right). \tag{3·14}$$

We also need

LEMMA 4. *Let $\tilde{W}_a^n$ denote the warped product manifold $I\times_{(a/\sqrt{2})\,\mathrm{sd}\,(\sqrt{2}x/a,\,1/\sqrt{2})}S^{n-1}(1)$, with the warped product metric given by*

$$g=dx^2+\frac{a^2}{2}\mathrm{sd}^2\left(\frac{\sqrt{2}}{a}x,\frac{1}{\sqrt{s}}\right)g_0, \tag{3·15}$$

where I denotes the largest open interval containing 0 such that $\mathrm{sd}\,((\sqrt{2}/a)\,x)$ is nowhere zero on I and g_0 is the standard metric on the unit $(n-1)$-sphere. Then the Whitney n-sphere W_a^n is topologically the two point compactification of W_a^n; moreover, the metric on W_a^n is the smooth extension of the warped product metric on $\tilde{W}_a^n$ to its two point compactification.

Proof. Let $\tilde{S}^n$ denote the unit n-sphere with the north and south poles, $\{N,S\}$, being removed and let $\{u_1,\dots,u_n\}$ denote the spherical coordinate system on $\tilde{S}_n$ given by

$$\left.\begin{aligned}y_0=\cos u_1,\quad y_1=\sin u_1\cos u_2,\dots,\\ y_{n-1}=\sin u_1\cdots\sin u_{n-1}\cos u_n,\quad y_n=\sin u_1\cdots\sin u_{n-1}\sin u_n.\end{aligned}\right\} \tag{3·16}$$

From (1·6) and (3·16) we know that the metric induced from Whitney's immersion w_a on $\tilde{S}^n$ is given by

$$g=\left(\frac{a^2}{1+\cos^2u_1}\right)du_1^2+\left(\frac{a^2\sin^2u_1}{1+\cos^2u_1}\right)g_0. \tag{3·17}$$

Put

$$x(u_1)=\int_0^{u_1}\frac{a}{\sqrt{1+\cos^2t}}dt. \tag{3.18}$$

Then

$$x(u_1)=\frac{a}{\sqrt{2}}\int_0^{u_1}\frac{a}{\sqrt{1-\frac{1}{2}\sin^2t}}dt=\frac{a}{\sqrt{2}}\mathrm{sn}^{-1}\left(\sin u_1,\frac{1}{\sqrt{2}}\right). \tag{3·19}$$

Thus

$$\sin u_1=\mathrm{sn}\left(\frac{\sqrt{2}}{a}x,\frac{1}{\sqrt{2}}\right). \tag{3.20}$$

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From (2·8), (3·17), (3·18) and (3·20) we obtain (3·15). This shows that $\tilde{W}_a^n$ is the $\tilde{S}^n$ endowed with the metric induced from Whitney's immersion w_a. Hence, topologically, the Whitney n-sphere W_a^n is the two point compactification of $\tilde{W}_a^n$ and, moreover, the metric on W_a^n is the smooth extension of the warped product metric on $\tilde{W}_a^n$ to its two point compactification.

4. *Proofs of Theorems* 1, 2 *and* 3

Let M^n be a hypersurface with constant mean curvature H in a Riemannian space form $R^{n+1}(\epsilon)$ ($n>2$, $\epsilon=1$, 0, or -1) which satisfies equality (1·2). If M^n is not minimal, then $\mu=(n/(n-1))H$ is a nonzero constant. In this case, (3·9) and (3·12) reduce respectively to

$$(a-\mu)(e_\alpha e_\alpha a)-2(e_\alpha a)^2=\frac{2a\Lambda_\alpha^2(\mu-a)^2}{\mu-2a}+(\epsilon+\mu a)(\mu-a)^2-\sum_{\beta\neq\alpha}^{n}(e_\beta a)\,\Gamma_{\alpha\alpha}^{\beta}(\mu-a), \tag{4·1}$$

$$a(e_\alpha e_\alpha a)-2(e_\alpha a)^2=-\frac{2a^4\Lambda_\alpha^2}{(\mu-a)(\mu-2a)}+(\epsilon+(\mu-a)\mu)\,a^2+a\sum_{\beta\neq\alpha}^{n}(e_\beta a)\,\Gamma_{\alpha\alpha}^{\beta}. \tag{4·2}$$

From (4·1) and (4·2) we obtain

$$\left.\begin{aligned} e_\alpha e_\alpha a&=-\frac{2a\Lambda_\alpha^2(\mu^2-3\mu a+3a^2)}{(\mu-2a)(\mu-a)}+(\mu-2a)(a^2-a\mu-\epsilon)+\sum_{\beta\neq\alpha}^{n}(e_\beta a)\,\Gamma_{\alpha\alpha}^{\beta},\\ (e_\alpha a)^2&=-a^2(\Lambda_\alpha^2+(\mu-a)^2)-\frac{\epsilon}{2}a(\mu-a),\quad \text{on } V. \end{aligned}\right\} \tag{4·3}$$

Since μ is constant, (3·13) reduces to

$$e_\alpha\Lambda_\alpha=\frac{\Lambda_\alpha(e_\alpha a)(\mu^2-3\mu a+4a^2)}{(\mu-a)(\mu-2a)}+\sum_{\beta\neq\alpha}^{n}\Lambda_\beta\,\Gamma_{\alpha\alpha}^{\beta}. \tag{4·4}$$

Now, by applying Lemma 1 and Cartan's structure equations, we can compute $d\omega_\alpha^1$ and $d\omega_\alpha^2$ in two different ways. After that, by comparing the coefficients of $\omega^1\wedge\omega^\beta$ and of $\omega^2\wedge\omega^\beta$ with $\beta\neq\alpha$ in the formulas of $d\omega_\alpha^1$ and of $d\omega_\alpha^2$ so obtained, we find respectively the following two formulas:

$$e_\beta e_\alpha a+\frac{2(e_\alpha a)(e_\beta a)}{\mu-a}+\frac{2a(\mu-a)\Lambda_\alpha\Lambda_\beta}{\mu-2a}=\sum_{\gamma=3}^{n}(e_\gamma a)\,\Gamma_{\alpha\beta}^{\gamma},$$

$$e_\beta e_\alpha a-\frac{2(e_\alpha a)(e_\beta a)}{a}+\frac{2a^3\Lambda_\alpha\Lambda_\beta}{(\mu-a)(\mu-2a)}=\sum_{\gamma=3}^{n}(e_\gamma a)\,\Gamma_{\alpha\beta}^{\gamma}.$$

By taking the difference of these two equations, we obtain

$$(e_\alpha a)(e_\beta a)=-a^2\Lambda_\alpha\Lambda_\beta,\quad \alpha\neq\beta, \tag{4·5}$$

on V.

Now, by applying (4·3), we may also obtain

$$-(e_\alpha a)^2=a^2(\Lambda_\alpha^2+(\mu-a)^2)+\frac{\epsilon}{2}a(\mu-a),$$

$$-(e_\beta a)^2=a^2(\Lambda_\beta^2+(\mu-a)^2)+\frac{\epsilon}{2}a(\mu-a).$$

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By taking the difference of these two equations we get

$$(e_\alpha a)^2-(e_\beta a)^2 = a^2(\Lambda_\beta^2-\Lambda_\alpha^2). \tag{4·6}$$

Combining (4·5) and (4·6) we obtain, on V,

$$(e_\alpha a)^2 = a^2\Lambda_\beta^2, \quad \alpha \neq \beta. \tag{4·7}$$

Case (i). $\epsilon = 1$, $R^{n+1}(1) = S^{n+1}(1)$.

If M^n is non-minimal, then the second equation of (4·3) implies that V is an empty set. Thus, M^n is an isoparametric hypersurface of $R^{n+1}(1)$ with at most two distinct principal curvatures given either by $\lambda_1 = 0$, $\lambda_2 = \cdots = \lambda_n = \mu$ or by $\lambda_1 = \lambda_2 = \frac{1}{2}\mu$, $\lambda_3 = \cdots = \lambda_n = \mu$. Both cases are impossible according to a well-known result of Cartan [**2**]. Consequently, M^n is minimal in $S^{n+1}(1)$. Let U denote the open subset of M^n consisting of non-totally geodesic points. Then U is an open dense subset of M^n. Now, by a result of [**4**], U has relative nullity $n-2$. Thus, by applying a result of Dajczer and Gromoll (lemma 2·2 of [**11**]), the Gauss image B^2 of U is a minimal surface in the unit sphere. Consequently, U is an open subset of NB^2 defined in Theorem 2.

Case (ii). $\epsilon = 0$, $R^{n+1}(0) = \mathbb{E}^{n+1}$.

If M^n is non-minimal, then the second equation in (4·3) implies $a = 0$ or $a = \mu$ on V which contradicts the definition of V. Thus V is an empty set. Hence, M is a non-minimal isoparametric hypersurface with at most two distinct principal curvatures given either by $\lambda_1 = 0$, $\lambda_2 = \cdots = \lambda_n = \mu$ or by $\lambda_1 = \lambda_2 = \frac{1}{2}\mu$, $\lambda_3 = \cdots = \lambda_n = \mu$. It is well-known that the first case occurs if and only if M^n is an open portion of a spherical hypercylinder $\mathbb{R}\times S^{n-1}(r)$ for some $r > 0$. It is also known that the later case cannot occur.

Case (iii). $\epsilon = -1$, $R^{n+1}(-1) = H^{n+1}(-1)$.

First we assume that M^n is non-minimal. We claim that the function a is constant on each component of V. We divide the proof of this claim into three cases.

Case (iii.1). $n \geqslant 5$

From (4·7) we have

$$(e_3a)^2 = (e_4a)^2 = \cdots = (e_na)^2 = a^2\Lambda_\alpha^2, \quad \alpha = 3,\ldots,n. \tag{4·8}$$

Without loss of generality, we may assume $e_3a = a\Lambda_3$. By using (4·5), we have $e_4a = -a\Lambda_4$ and $e_5a = -a\Lambda_5$ which imply

$$(e_4a)(e_5a) = a^2\Lambda_4\Lambda_5.$$

Thus $e_aa = \Lambda_\alpha = 0$ on V by (4·5) and (4·8). Hence, by applying the first equation in (4·3), we obtain $a^2-a\mu-\epsilon = 0$ on V. Since μ is constant, this implies that a is constant on each component of V.

Case (iii.2). $n = 4$

From (4·5) and (4·7) we may assume, without loss of generality, that $e_3a = a\Lambda_4$ and $e_4a = -a\Lambda_3$. By differentiating the second equation of (4·3) with respect to e_3, we get

$$\begin{aligned} &-2(e_3a)(e_3e_3a) \\ &\quad = 2(e_3a)\,a(\Lambda_3^2+(\mu-a)^2)+a^2(2\Lambda_3(e_3\Lambda_3)-2(\mu-a)\,e_3a)+\frac{\epsilon}{2}(\mu-2a)\,e_3a. \end{aligned} \tag{4·9}$$

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On the other hand, since $\Lambda_4 = e_3(\ln a)$, (4·4) yields

$$e_3\Lambda_3 = \frac{(e_3a)\Lambda_3(\mu^2-3\mu a+4a^2)}{(\mu-2a)(\mu-a)a}+\frac{e_3a}{a}\Gamma^4_{33}.$$

Substituting this into (4.9), we find

$$\begin{aligned}&-(e_3a)(e_3e_3a)\\&=(e_3a)\left(\frac{2a\Lambda_3^2(\mu^2-3\mu a+3a^2)}{(\mu-a)(\mu-2a)}+a(\mu-a)(\mu-2a)+\frac{\epsilon}{4}(\mu-2a)-(e_4a)\,\Gamma^4_{33}\right).\end{aligned}\tag{4·10}$$

If e_3a does not vanish identically on V, there exists an open set $W \subset V$ such that e_3a is nowhere zero on W. Thus, on W, (4·8) becomes

$$-e_3e_3a = \frac{2a\Lambda_3^2(\mu^2-3\mu a+3a^2)}{(\mu-a)(\mu-2a)}+a(\mu-2a)(\mu-a)+\frac{\epsilon}{4}(\mu-2a)-(e_4a)\Gamma^4_{33}.\tag{4·11}$$

Combining (4·3) and (4·11) we obtain $\mu = 2a$ which implies that a is constant on W. This contradicts the assumption that $e_3a \neq 0$ on W. So $e_3a \equiv 0$ on V. A similar argument yields $e_4a \equiv 0$ on V. Thus, by applying (4·5), we know that Λ_3 vanishes identically on V. Hence, by applying (4·3) again, we have $a^2-a\mu+\epsilon/2 = 0$ on V. Consequently, a is constant on each component of V.

Case (iii.3). $n = 3$

From (4·3) we have

$$-2(e_3a)(e_3e_3a) = a^2(2\Lambda_3(e_3\Lambda_3)-2(e_3a)(\mu-a))+2a(e_3a)(\Lambda_3^2+(\mu-a)^2)+\frac{\epsilon}{2}(e_3a)(\mu-2a).$$

By using (4·4) this equation becomes

$$\begin{aligned}-2(e_3a)(e_3e_3a) &= (e_3a)\\&\quad\times\left\{\frac{4a\Lambda_3^2(\mu^2-3\mu a+3a^2)}{(\mu-a)(\mu-2a)}+2a(\mu-a)(\mu-2a)+\frac{\epsilon}{2}(\mu-2a)\right\}.\end{aligned}\tag{4·12}$$

We claim that $e_3a \equiv 0$ on V. Since, otherwise, there exists an open subset W of V such that e_3a is nowhere vanished on W. Then, on W, (4·12) becomes

$$-2(e_3e_3a) = \frac{4a\Lambda_3^2(\mu^2-3\mu a+3a^2)}{(\mu-a)(\mu-2a)}+2a(\mu-a)(\mu-2a)+\frac{\epsilon}{2}(\mu-2a).\tag{4·13}$$

Combining this with the first equation in (4·3) we conclude that a is constant on W which is a contradiction. Therefore, e_3a vanishes identically on W. Consequently, from the second equation of (4·3), we obtain

$$a^2\Lambda_3^2 = -a^2(\mu-a)^2-\frac{\epsilon}{2}a(\mu-a).\tag{4·14}$$

On the other hand, from the first equation in (4·3), we have

$$\frac{-2a\Lambda_3^2(\mu^2-3\mu a+3a^2)}{(\mu-2a)(\mu-a)}+(\mu-2a)(a^2-a\mu-\epsilon) = 0.\tag{4·15}$$

(4.14) and (4·15) imply that a satisfies a polynomial equation of degree 4 with constant coefficients on V. Hence, a is constant on each component of V.

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Therefore, we know that, for any dimension $n > 2$, every component of V is an isoparametric hypersurface of the hyperbolic space. On the other hand, according to a well-known result of Cartan, every isoparametric hypersurface of $H^{n+1}(-1)$ has at most two distinct principal curvatures. Therefore, we conclude that each component of V has at most two distinct principal curvatures. This contradicts the definition of V. Thus, V must be an empty set. Consequently, M^n is an isoparametric hypersurface of $H^{n+1}(-1)$ with exactly two distinct principal curvatures. Therefore, by applying Cartan's classification theorem of isoparametric hypersurfaces in hyperbolic spaces, M^n is an open set of the Riemannian product of $H^2(-\frac{1}{2})$ and $S^{n-2}(1)$ isometric embedded in the hyperbolic $(n+1)$-space in a standard way. Such a hypersurface is a tubular hypersurface with radius $r = \coth^{-1}(\sqrt{2})$ about a 2-dimensional totally geodesic surface (cf. [**5**]).

Now, assume that M^n is a non-totally geodesic minimal hypersurface. Let U denote the open subset of M^n consisting of non-totally geodesic points. Then U is an open dense subset of M^n. Now, by a result of [**4**], U has relative nullity $n-2$. In this case, by applying an argument similar to spherical case, we may conclude that the Gauss image B^2 of U is a minimal surface in the unit de Sitter space-time $S_1^{n+1}(1)$. Consequently, U is an open subset of the unit bundle NB^2 defined in Theorem 3.

The converses are easy to verify.

Remark 1. Given a non-totally geodesic minimal hypersurface M^n of $S^{n+1}(1)$ satisfying equality (1·2), let U be the open dense subset consisting of non-totally geodesic points as before. Then the distribution $\mathscr{D}$ given by the eigenvectors with eigenvalue 0 is integrable and its leaves are totally geodesic in $S^{n+1}(1)$. Hence, there are local coordinates $(u, v, x_1, \ldots, x_{n-2})$ such that $\partial/\partial x_1, \ldots, \partial/\partial x_{n-2}$ span $\mathscr{D}$. The surface B^2 of Theorem 2 can be chosen to be $B^2 = \{q \in U \,|\, x_1 = 0, \ldots, x_{n-2} = 0\}$ and $\phi : B^2 \to S^{n+1} \subset \mathbb{R}^{n+2}$ is given by $(u, v) \mapsto \xi(u, v, 0, \ldots, 0)$, where ξ is the unit normal vector field of M in $S^{n+1}(1)$.

The surface $\phi : B^2 \to S^{n+1}(1)$ can also be considered as the set of corresponding focal points on U in the sense of [**22**], which in Theorem 3 is no longer possible. However, in the hyperbolic case, $\phi(B^2)$ can be interpreted as the focal points beyond the horizon of the hyperbolic space.

5. *Proof of Theorem* 4

First we prove

LEMMA 5. *Let $M^3 \subset R^4(\epsilon)$ be a conformally flat hypersurface of a Riemannian space form $R^4(\epsilon)$ satisfying equality* (1·2), *then M^3 has at most two distinct principal curvatures.*

This Lemma can be proved as follows. Assume M^3 is a conformally flat hypersurface in a 4-dimensional Riemannian space form $R^4(\epsilon)$ satisfying equality (1·2).

Let L be the symmetric 2-tensor defined by

$$L = -\mathrm{Ric} + \frac{\tau}{2} g, \tag{5·1}$$

where Ric, τ and g denote respectively the Ricci tensor, the scalar curvature and the metric tensor of M^3. Then by a result of H. Weyl we have

$$(\nabla_Y L)(Z, W) = (\nabla_Z L)(Y, W), \tag{5·2}$$

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for vectors Y, Z, W tangent to M^3. Let $\lambda_1 = a$, $\lambda_2 = \mu - a$, $\lambda_3 = \mu$ be the principal curvatures of M^3 with their corresponding principal directions e_1, e_2, e_3 given as before. Then, from the equations of Gauss and (5·2), we have

$$\left.\begin{aligned}(\lambda_j^2-\lambda_i^2)\,\Gamma_{ij}^j &= 3(e_iH)\lambda_j-\tfrac{1}{2}e_i\tau-e_i\lambda_j^2,\\ (\lambda_j^2-\lambda_k^2)\Gamma_{ji}^k &= (\lambda_i^2-\lambda_k^2)\Gamma_{ij}^k,\end{aligned}\right\} \tag{5·3}$$

$$\tau = 3\epsilon+\mu^2+a\mu-a^2, \tag{5·4}$$

for distinct i, j, k (i, j, $k = 1, 2, 3$), where $H = \frac{2}{3}\mu$ is the mean curvature function.

Equations (3·4) and (5·3) imply

$$\begin{aligned}2(e_i\mu)\,\lambda_j-\tfrac{1}{2}e_i(\mu^2+a\mu-a^2)-e_i\lambda_j^2 &= (\lambda_j+\lambda_i)(\lambda_j-\lambda_i)\,\Gamma_{ij}^j\\ &= -(\lambda_i+\lambda_j)\,e_i\lambda_j\,.\end{aligned}$$

Thus

$$(\lambda_i-\lambda_j)e_i\lambda_j = (-2\lambda_j+\mu+\tfrac{1}{2}a)\,e_i\mu+(\tfrac{1}{2}\mu-a)\,e_ia.$$

By taking (i,j) being (1, 2), (2, 3) and (3, 1) respectively, we obtain

$$(\mu-2a)\,e_1a = ae_1\mu, \tag{5·5}$$

$$(\mu-2a)e_2a = (2\mu-3a)\,e_2\mu, \tag{5·6}$$

$$\mu e_3a = (2\mu-3a)\,e_3\mu. \tag{5·7}$$

Let V denote the open subset of M^3 on which M^3 has exactly three distinct principal curvatures, i.e., V is given by (3·6). Suppose that V is not empty. In the remaining part of this section we shall work on this non-empty subset to obtain a contradiction.

From (3·5) and (5·3) we obtain

$$\Gamma_{jk}^i = 0, \tag{5·8}$$

for distinct i, j, k. Equation (5·8) (or Lemma 1) implies

$$\omega_2^1 = \frac{e_2a}{\mu-2a}\omega_1+\frac{e_1(\mu-a)}{\mu-2a}\omega_2\,. \tag{5·9}$$

By taking the exterior derivative of (5·9) and by applying Cartan's structure equations, we obtain the following formulas on V by comparing the corresponding coefficients in the resulting formula of $d\omega_2^1$.

$$\begin{aligned}e_1\left(\frac{e_1(\mu-a)}{\mu-2a}\right)-e_2\left(\frac{e_2a}{\mu-2a}\right)-\frac{(e_2a)^2}{(\mu-2a)^2}-\frac{(e_1(\mu-a))^2}{(\mu-2a)^2}&\\ = \frac{(e_3a)\,e_3(\mu-a)}{a(\mu-a)}+a(\mu-a)+\epsilon,&\end{aligned} \tag{5·10}$$

$$e_3\left(\frac{e_2a}{\mu-2a}\right)+\frac{(e_2a)(e_3a)}{(\mu-a)(\mu-2a)} = -\frac{(e_3a)(e_2\mu)}{a(\mu-a)}, \tag{5·11}$$

$$e_3\left(\frac{e_1(\mu-a)}{\mu-2a}\right)+\frac{e_1(\mu-a)\,e_3(\mu-a)}{a(\mu-2a)} = \frac{(e_1\mu)\,e_3(\mu-a)}{a(\mu-a)}. \tag{5·12}$$

By (5·7) we have $\Lambda_3 = \Gamma_{31}^2 = 0$. Thus, by applying (3·10), (3·13), (5·5), (5·6) and (5·7), we find

$$e_1e_2\mu = \frac{2\mu-3a}{(\mu-2a)^2}(e_1\mu)(e_2\mu), \tag{5·13}$$

$$e_2e_1\mu = \frac{3a-\mu}{(\mu-2a)^2}(e_1\mu)(e_2\mu). \tag{5·14}$$

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On the other hand, by Lemma 1, (5·5) and (5·6), we have

$$[e_1,e_2]\mu=-\Gamma_{21}^1 e_1\mu+\Gamma_{12}^2 e_2\mu=-\frac{e_2 a}{\mu-2a}e_1\mu-\frac{e_1(\mu-a)}{\mu-2a}e_2\mu=-\frac{3}{\mu-2a}(e_1\mu)(e_2\mu).$$

Combining this with (5·13) and (5·14) we find

$$(e_1\mu)(e_2\mu)=0. \tag{5·15}$$

Therefore $(e_1\mu)(e_2\mu)=0$ on V.

If $e_1\mu\not\equiv 0$ on V, there exists an open subset $W\subset V$ on which $e_2\mu\neq 0$. On W, we have $e_1\mu=0$ identically. Moreover, from (3·11) and (5·11) we have

$$(2\mu-3a)\,e_2e_3\mu=-\frac{3a^2-3a\mu+2\mu^2}{a\mu}(e_2\mu)(e_3\mu), \tag{5·16}$$

$$(2\mu-3a)\,e_3e_2\mu=\frac{6a^2-3a\mu-2\mu^2}{a\mu}(e_2\mu)(e_3\mu). \tag{5·17}$$

Combining (5·16) and (5·17) we obtain

$$(2\mu-3a)\,[e_2,e_3]\mu=\frac{3(2\mu-3a)}{\mu}(e_2\mu)(e_3\mu). \tag{5·18}$$

On the other hand, we also have

$$[e_2,e_3]\mu=\Gamma_{23}^2 e_2\mu-\Gamma_{32}^3 e_3\mu=-\frac{e_3(\mu-a)}{a}e_2\mu-\frac{e_2\mu}{a}e_3\mu$$

$$=-\frac{3}{\mu}(e_2\mu)(e_3\mu).$$

Therefore, we find

$$(2\mu-3a)(e_2\mu)(e_3\mu)=0. \tag{5·19}$$

If $(e_2\mu)(e_3\mu)\not\equiv 0$ on W, then on the open subset W' of W on which $(e_2\mu)(e_3\mu)\neq 0$ we have

$$2\mu-3a=0. \tag{5·20}$$

Since $2\mu-3a=0$ on W', (5·6) and (5·7) imply $e_2a=e_3a=0$ on W'. On the other hand, since $e_1\mu=0$ on W, (5·5) yields $e_1a=0$. Therefore, we obtain $e_1\mu=e_2\mu=e_3\mu=0$ which is a contradiction. Consequently, we must have $e_3\mu\equiv 0$ on W, from which we obtain $e_1a=e_3a\equiv 0$ on W by virtue of (5·7). Thus, by (3·9) and (5·6), we have

$$(2\mu-3a)(e_2\mu)^2=a(\mu-2a)^2(a\mu+\epsilon). \tag{5·21}$$

On the other hand, from (5·6), we know that $2\mu-3a\neq 0$ on W. Thus, there is an open subset $W_1\subset W$ on which $2\mu-3a\neq 0$. On W_1, we have (5·21) which yields

$$(e_2\mu)^2=\frac{a(a\mu+\epsilon)(\mu-2a)^2}{2\mu-3a}. \tag{5·22}$$

By using (5·6) and (5·22) we obtain

$$e_2e_2\mu=\frac{1}{(2\mu-3a)^2}\{-6a^5-15a^4\mu+38a^3\mu^2-23a^2\mu^3+4a\mu^4$$

$$+\epsilon(-16a^3+27a^2\mu-14a\mu^2+2\mu^3)\}. \tag{5·23}$$

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On the other hand, (5·6) and (5·10) imply on W_1 we have

$$\frac{2\mu-3a}{(\mu-2a)^2}e_2e_2\mu+\frac{12\mu^2-33\mu a+23a^2}{(\mu-2a)^4}(e_2\mu)^2=-a(\mu-a)-\epsilon.$$

By using (5·22), the above equation becomes

$$-e_2e_2\mu=\frac{1}{(2\mu-3a)^2}\{12a^5-9a^4\mu-2a^3\mu^2-a^2\mu^3+2a\mu^4 \\ +\epsilon(11a^3-13a^2\mu+a\mu^2+2\mu^3)\}. \quad (5\cdot24)$$

Combining (5·23) and (5·24) we get

$$6a^3-12a^2\mu+6a\mu^2+\epsilon(4\mu-5a)=0. \quad (5\cdot25)$$

Differentiating (5·25) and applying (5·6) we find

$$-30a^3+72a^2\mu-54a\mu^2+12\mu^3+\epsilon(7a-6\mu)=0. \quad (5\cdot26)$$

Also, by combining (5·25) and (5·26) we get

$$6a^2\mu+6\mu^3-12a\mu^2+\epsilon(7\mu-9a)=0. \quad (5\cdot27)$$

Equation (5·27) implies

$$a=\frac{9\epsilon+12\mu^2\mp\sqrt{81+48\epsilon\mu^2}}{12\mu}. \quad (5\cdot28)$$

By substituting (5·28) into (5·26) we conclude that μ must satisfy the following polynomial equation with constant coefficients

$$162+120\mu^2+88\epsilon\mu^4+32\mu^6=\pm\sqrt{3}(9+8\epsilon\mu+8\mu^4)\sqrt{27+16\epsilon\mu^2}. \quad (5\cdot29)$$

which is impossible since otherwise μ is locally a constant function. Thus we must have $e_2\mu\equiv0$.

Similarly, we may prove that $e_1\mu\equiv0$, too. Hence, by applying (5·5), (5·6) and (5·7), we obtain

$$e_1a=e_2a=e_1\mu=e_2\mu=0. \quad (5\cdot30)$$

Now we claim that $e_3\mu\equiv0$ on V, too. In fact, otherwise there exists an open subset $O_1\subset V$ on which $e_3\mu\neq0$.

If $2\mu-3a=0$ on V, then (5·7) implies $e_3a=0$. Therefore, by (5·10), we get $a(\mu-a)+\epsilon=0$. Thus, $2\mu^2+9\epsilon=0$ which is impossible.

If $\mu=3a$, then (5·7) yields $\mu e_3\mu=0$, which is also a contradiction. Hence, we have $2\mu\neq3a$ and $\mu\neq3a$ on V.

Now, from (5·10) we get

$$\frac{(2\mu-3a)(\mu-3a)}{\mu^2a(\mu-a)}(e_3\mu)^2=a(\mu-a)+\epsilon. \quad (5\cdot31)$$

Thus, there exists an open subset $O_2\subset O_1$ on which

$$(e_3\mu)^2=\frac{(a(\mu-a)+\epsilon)\mu^2a(\mu-a)}{(2\mu-3a)(\mu-3a)}. \quad (5\cdot32)$$

On the other hand, (3·9) and (5·7) imply

$$-e_3\left(\frac{(2\mu-3a)\,e_3\mu}{\mu(\mu-a)}\right)-\frac{(e_3\mu)^2(2\mu-3a)^2}{(\mu-a)^2\mu^2}=a\mu+\epsilon. \quad (5\cdot33)$$

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Equations (5·32) and (5·33) yield

$$e_3e_3\mu = \frac{\mu}{(\mu-3a)(2\mu-3a)^2} \times\{6a^5-15a^3\mu^2+11a^2\mu^3-2a\mu^4+\epsilon(3a^3-15a^2\mu+11a\mu^2-2\mu^3)\}. \quad (5{\cdot}34)$$

Similarly, by applying (3·12), (5·7) and (5·32), we have

$$e_3e_3\mu = \frac{\mu}{(2\mu-3a)(\mu-3a)^2} \times\{6a^5-30a^4\mu+45a^3\mu^2+26a^2\mu^3+5a\mu^4+\epsilon(3a^3+6a^2\mu-10a\mu^2+3\mu^3)\}. \quad (5{\cdot}35)$$

Summing up (5·34) and (5·35) yield

$$\mu^2(\mu-2a)\{-24a^4+48a^3\mu-30a^2\mu^2+6a\mu^3+\epsilon(15a^2-15a\mu+4\mu^2)\}=0. \quad (5{\cdot}36)$$

On the other hand, from (5·8), we know that the distribution $\mathscr{F}^\perp$ spanned by e_1, e_2 is integrable. Also, the distribution $\mathscr{F}$ spanned by e_3 is clearly integrable. Therefore, there exists a local coordinate system $\{x_1, x_2, x_3\}$ such that $e_3 = \partial/\partial t$, where $t = x_3$. Hence, by applying (5·30), we know that both a and μ depend only on t. Therefore, (5·7) yields

$$\mu\frac{da}{d\mu} = 2\mu-3a. \quad (5{\cdot}37)$$

By solving (5·37) we obtain

$$a = \frac{\mu}{2}+C\mu^{-3}, \quad (5{\cdot}38)$$

for some constant C. By substituting (5·38) into (5·36), we know that μ must satisfy a polynomial equation with constant coefficients. Therefore, $e_3\mu = 0$ on O_2 which is a contradiction.

Consequently, both μ and a are constant on each component of V. Hence, by (3·9), (3·12) and (5·10), we get $a\mu = (\mu-a)\mu = -\epsilon$ which is clearly impossible. Consequently, we know that V is an empty set. Hence, M^3 has at most two distinct principal curvatures. This completes the proof of Lemma 5.

Now, we go back to the proof of Theorem 4. If $\mathbf{x}: M^n \to \mathbb{E}^{n+1}$ is an isometric immersion of a conformally flat n-manifold with $n > 2$ which satisfies equality (1·2). Then, by Lemma 5 and a well-known result of Cartan and Schouten on conformally flat hypersurfaces (cf. p. 154 of [**3**]), we know that M^n has a principal curvature with multiplicity at least $n-1$ for $n = 3$ as well as for $n \geqslant 4$. Thus, by applying (3·2), we know that either (i) the principal curvatures of M^n are given by $\lambda_1 = 0$, $\lambda_2 = \cdots = \lambda_n = \mu$, or (ii) $n = 3$ and the principal curvatures of M^3 are given by $\lambda_1 = \lambda_2 = \frac{1}{2}\mu$, $\lambda_3 = \mu \neq 0$.

If (i) occurs, M^n is either totally geodesic, or an open portion of spherical hypercylinder, or an open portion of a round hypercone (cf. [**15**]).

Now, we assume (ii) occurs. Denote by U the open subset of M^3 on which the mean curvature function is nonzero. Then U is a non-empty open subset of M^3. We shall work on U unless mentioned otherwise.

From (3·5) we obtain

$$\Gamma^2_{31} = \Lambda_3 = 0, \quad e_1\mu = u_2\mu = 0, \quad \Gamma^1_{33} = \Gamma^2_{33} = 0. \quad (5{\cdot}39)$$

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Denote by $\mathscr{F}$ and $\mathscr{F}^{\perp}$ the distributions spanned by $\{e_1, e_2\}$ and $\{e_3\}$, respectively. By (5·39) we know that the integral curves of $\mathscr{F}^{\perp}$ are geodesics and the distribution $\mathscr{F}$ is integrable. Consequently, there exist local coordinate systems $\{x, u, v\}$ such that $\mathscr{F}$ is spanned by $\{\partial/\partial u, \partial/\partial v\}$ and $e_3 = \partial/\partial x$.

From (5·39) we know that μ depends only on x, i.e., $\mu = \mu(x)$. Also, from (3·8) and (5·39), we have

$$\omega_3^1 = \frac{\mu'(x)}{\mu(x)}\omega^1, \quad \omega_3^2 = \frac{\mu'(x)}{\mu(x)}\omega^2. \tag{5·40}$$

Using (5·40) we obtain

$$\nabla_{e_j} e_3 = \frac{\mu'(x)}{\mu(x)} e_j, \quad j = 1, 2. \tag{5·41}$$

Therefore, each integral submanifold of $\mathscr{F}$ is an extrinsic sphere of $\mathbb{E}^4$. Hence, by applying a result of Hiepko [**14**] (cf. also [**13**, remark 2·1]), we know that M^3 is locally the warped product $I \times_{f(x)} S^2(1)$, where $f(x)$ is a suitable warped function. So, the metric of M^3 is given by

$$g = dx^2 + f(x)^2 g_0, \tag{5·42}$$

where g_0 is the standard metric of $S^2(1)$. In particular, if we choose the spherical coordinate system $\{\theta, \phi\}$ for $S^2(1)$, we have

$$g = dx^2 + f^2(x)\,(d\phi^2 + \cos^2\phi d\theta^2). \tag{5·43}$$

Applying (5·43) we obtain

$$\left.\begin{aligned} &\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = 0, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial \theta} = \frac{f'}{f}\frac{\partial}{\partial \theta}, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial \phi} = \frac{f'}{f}\frac{\partial}{\partial \phi} \\ &\nabla_{\frac{\partial}{\partial \phi}}\frac{\partial}{\partial \phi} = -ff'\frac{\partial}{\partial x}, \quad \nabla_{\frac{\partial}{\partial \phi}}\frac{\partial}{\partial \theta} = -\tan\phi\frac{\partial}{\partial \theta} \\ &\nabla_{\frac{\partial}{\partial \theta}}\frac{\partial}{\partial \theta} = -ff'\cos^2\phi\frac{\partial}{\partial x} + \sin\phi\cos\phi\frac{\partial}{\partial \phi}. \end{aligned}\right\} \tag{5·44}$$

By computing $d\omega_3^1$ and by using (5·40) and Cartan's structure equations, we find

$$\mu''(x) + \frac{\mu^3(x)}{2} = 0. \tag{5·45}$$

Solving (5·45) yields

$$4\mu'^2 + \mu^4 = \frac{4}{a^4}, \tag{5·46}$$

for some real number $a > 0$.

Now, we claim that U is dense in M^3. If it is not, then $M^3 - U$ has non-empty interior. From the definition of U, we know that the squared mean curvature function, μ and μ' vanish identically on the interior of $M^3 - U$. On the other hand, (5·46) says that this is impossible due to the continuity of the squared mean curvature function. Thus, U must be an open dense subset of M^3.

Solving (5·46) yields $\mu(x) = (\sqrt{2}/a)\,\mathrm{sd}\,((\sqrt{2}/a)\,x + b, 1/\sqrt{2})$ for some constant b. By applying a translation in x if necessary, we obtain

$$\mu(x) = \frac{\sqrt{2}}{a}\mathrm{sd}\left(\frac{\sqrt{2}}{a}x, k\right), \quad k = \frac{1}{\sqrt{2}}. \tag{5·47}$$

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From (5·43) and from our assumption on the principal curvatures, we know that the second fundamental form h of M^3 on $\mathbb{E}^4$ satisfies

$$\left.\begin{aligned} &h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right)=\mu\xi,\quad h\left(\frac{\partial}{\partial\phi},\frac{\partial}{\partial\phi}\right)=\tfrac{1}{2}\mu f^{\,2}\xi,\quad h\left(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\right)=\tfrac{1}{2}\mu f^2\cos^2\phi\,\xi\\ &h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial\phi}\right)=h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial\theta}\right)=h\left(\frac{\partial}{\partial\phi},\frac{\partial}{\partial\theta}\right)=0,\end{aligned}\right\} \tag{5·48}$$

where ξ is a unit normal vector field of M^3 in $\mathbb{E}^4$ and μ is given by (5·47).

Applying (5·44), (5·48) and the equation of Codazzi, we may obtain $\mu' f=\mu f'$. Therefore,

$$f(x)=c\mu(x), \tag{5·49}$$

for some nonzero constant c.

On the other hand, using (5·44) we can compute the sectional curvature K_{23} of the plane section spanned by $\{\partial/\partial\phi,\partial/\partial\theta\}$. On the other hand, we may also compute K_{23} by using the equation of Gauss. By comparing these two different expressions of K_{23}, we find $\mu^4 f^2=4(1-f'^2)$. Thus, by using (5·47) and (5·49), we find

$$f(x)=\frac{\sqrt{a}}{2}\,\mathrm{sd}\left(\frac{\sqrt{2}}{a}x,k\right),\quad k=\frac{1}{\sqrt{2}}. \tag{5·50}$$

From (5·43), (5·50) and Lemma 4 we know that M^3 is an open portion of the Whitney 3-sphere W^3_a for some $a>0$.

(5·43), (5·44), (5·47), (5·50) and the formula of Gauss imply that the immersion $\mathbf{x}$ satisfies the following system of partial differential equations:

$$\frac{\partial^2\mathbf{x}}{\partial x^2}=\frac{\sqrt{2}}{a}\,\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right)\xi, \tag{5·51}$$

$$\frac{\partial^2\mathbf{x}}{\partial\phi^2}=-\frac{a}{\sqrt{2}}\,\mathrm{cd}\left(\frac{\sqrt{2}}{a}x\right)\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right)\mathrm{nd}\left(\frac{\sqrt{2}}{a}x\right)\frac{\partial\mathbf{x}}{\partial x}+\frac{a}{2\sqrt{2}}\,\mathrm{sd}^3\left(\frac{\sqrt{2}}{a}x\right)\xi, \tag{5·52}$$

$$\frac{\partial^2\mathbf{x}}{\partial\theta^2}=\cos^2\phi\,\frac{\partial^2\mathbf{x}}{\partial\phi^2}+\sin\phi\cos\phi\,\frac{\partial\mathbf{x}}{\partial\phi}, \tag{5·53}$$

$$\frac{\partial^2\mathbf{x}}{\partial x\,\partial\phi}=\frac{\sqrt{2}}{a}\,\mathrm{cs}\left(\frac{\sqrt{2}}{a}x\right)\mathrm{nd}\left(\frac{\sqrt{2}}{a}x\right)\frac{\partial\mathbf{x}}{\partial\phi}, \tag{5·54}$$

$$\frac{\partial^2\mathbf{x}}{\partial x\,\partial\theta}=\frac{\sqrt{2}}{a}\,\mathrm{cs}\left(\frac{\sqrt{2}}{a}x\right)\mathrm{nd}\left(\frac{\sqrt{2}}{a}x\right)\frac{\partial\mathbf{x}}{\partial\theta}, \tag{5·55}$$

$$\frac{\partial^2\mathbf{x}}{\partial\theta\,\partial\phi}=-\tan\phi\,\frac{\partial\mathbf{x}}{\partial\theta}. \tag{5·56}$$

Solving (5·56) yields

$$\mathbf{x}(x,\phi,\theta)=B(x,\theta)\cos\phi+C(x,\phi), \tag{5·57}$$

for some functions $B(x,\theta)$, $C(x,\phi)$ of two variables. Substituting (5·57) into (5·54) yields

$$\left.\begin{aligned}\frac{\partial B}{\partial x}&=\frac{\sqrt{2}}{a}\,\mathrm{cs}\left(\frac{\sqrt{2}}{a}x\right)\mathrm{nd}\left(\frac{\sqrt{2}}{a}x\right)B,\\ \frac{\partial^2 C}{\partial x\,\partial\phi}&=\frac{\sqrt{2}}{a}\,\mathrm{cs}\left(\frac{\sqrt{2}}{a}x\right)\mathrm{nd}\left(\frac{\sqrt{2}}{a}x\right)\frac{\partial C}{\partial\phi}.\end{aligned}\right\} \tag{5·58}$$

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Solving (5·58) yields

$$B(x,\theta)=G(\theta)\,\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right),\quad C(x,\phi)=F(\phi)\,\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right)+H(x). \tag{5·59}$$

Combining (5·57) and (5·59) yields

$$\mathbf{x}(x,\phi,\theta)=G(\theta)\,\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right)\cos\phi+F(\phi)\,\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right)+H(x). \tag{5·60}$$

By taking the partial derivatives of (5·52) and (5·53) with respect to ϕ and θ respectively, we find

$$F'''(\phi)+F'(\phi)=0,\quad G'''(\theta)+G'(\theta)=0. \tag{5·61}$$

From (5·60) and (5·61) we find

$$\mathbf{x}(x,\phi,\theta)=\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right)(c_1\cos\phi\cos\theta+c_2\cos\phi\sin\theta+c_3\cos\phi+c_4\sin\phi)+A(x). \tag{5·62}$$

where $c_1,\dots,c_4$ are constant vectors.

Substituting (5·62) into (5·52) and using (5·51) yields

$$A''(x)=\frac{2\sqrt{2}}{a}\,\mathrm{cs}\left(\frac{\sqrt{2}}{a}x\right)\mathrm{nd}\left(\frac{\sqrt{2}}{a}x\right)A'(x) \tag{5·63}$$

After solving (5·63) we obtain from (5·62) that

$$\begin{aligned}\mathbf{x}(x,\phi,\theta)&=\mathrm{sd}\left(\frac{\sqrt{2}}{a}x\right)\\&\quad\times(c_1\cos\phi\cos\theta+c_2\cos\phi\sin\theta+c_3\cos\phi+c_4\sin\phi)+c_5\int\mathrm{sd}^3\left(\frac{\sqrt{2}}{a}x\right)dx.\end{aligned} \tag{5·64}$$

where $c_1,\dots,c_5$ are constant vectors in $\mathbb{E}^4$.

By choosing suitable initial conditions we will obtain (1·7) from (5·64). Consequently, up to rigid motions of $\mathbb{E}^4$, the immersion $\mathbf{x}$ is given by (1·7).

The converse can be verified by straightforward computations.

6. *Exact solutions of differential equations of Picard type*

For the proof of Theorems 5 and 6, we need the exact solutions of some differential equations with Jacobi's elliptic functions in their coefficients. The results obtained in this section seem to be of independent interest by themselves.

Proposition 1. *For any real numbers $a>1$ and $\beta>0$, the general solution of the second order differential equation*:

$$y''(x)+2a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)\,y'(x)-a^2\beta^2y(x)=0 \tag{6·1}$$

is given by $y(x)=c_1y_1(x)+c_2y_2(x)$ with

$$y_1=\sqrt{k'^2-\beta^2\,\mathrm{cn}^2(ax)}\cos\left(\left(\frac{a\beta\sqrt{k^2+\beta^2}}{\sqrt{k'^2-\beta^2}}\right)x+\frac{i}{2}\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)}+iaZ(\gamma)\,x\right)$$

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and

$$y_2 = \sqrt{k'^2-\beta^2\,\mathrm{cn}^2(ax)}\,\sin\left(\left(\frac{a\beta\sqrt{k^2+\beta^2}}{\sqrt{k'^2-\beta^2}}\right)x+\frac{i}{2}\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)}+iaZ(\gamma)\,x\right),$$

where $k=\sqrt{a^2-1}/(\sqrt{2}a)$ *and* $k'=\sqrt{a^2+1}/(\sqrt{2}a)$ *are the modulus and the complementary modules of the elliptic functions, and* $\gamma=\mathrm{sn}^{-1}(i\beta/(k\sqrt{k'^2-\beta^2}))$.

Proof. The trick to solve (6·1) is to make two key transformations. First we make the transformation:

$$y(x)=f(x)\exp(\mathrm{i}\,g(x)), \tag{6·2}$$

where $f(x)$ and $g(x)$ are real-valued functions. Then

$$y'=(f'+\mathrm{i}fg')\exp(\mathrm{i}g) \tag{6·3}$$

$$y''=(f''-f(g')^2+\mathrm{i}(2f'g'+fg''))\exp(\mathrm{i}g). \tag{6·4}$$

Substituting (6·3) and (6·4) into (6·1) we get

$$f(x)\,g''(x)+2(f'(x)+a\,\mathrm{sc}(ax)\,\mathrm{dn}(ax)f(x))\,g'(x)=0, \tag{6·5}$$

$$f''(x)-f(x)(g'(x))^2+2a\,\mathrm{sc}(ax)\,\mathrm{dn}(ax)f'(x)-a^2\beta^2f(x)=0. \tag{6·6}$$

Equation (6·5) can be written as

$$(\ln g'(x))'=-2\,(\ln f(x))'+2\,(\ln\mathrm{cn}(ax))' \tag{6·7}$$

which yields

$$g'(x)=\frac{\alpha\,\mathrm{cn}(ax)^2}{f(x)^2}, \tag{6·8}$$

for some constant α.

Substituting (6·8) into (6·6) we obtain a second order nonlinear equation:

$$f''(x)+2a\,\mathrm{sc}(ax)\,\mathrm{dn}(ax)f'(x)-a^2\beta^2f(x)=\frac{\alpha^2\,\mathrm{cn}^4(ax)}{f(x)^3}. \tag{6·9}$$

We make the second transformation by putting

$$f(x)=\sqrt{h(u)},\quad u=\mathrm{cn}^2(ax). \tag{6·10}$$

From (6·9) and (6·10) we obtain another nonlinear equation:

$$a^2\,\mathrm{cn}^2(ax)\,\mathrm{dn}^2(ax)\,\mathrm{sn}^2(ax)\,(2h(u)\,h''(u)-(h'(u))^2)+a^2(k^2\,\mathrm{cn}^2(ax)\,\mathrm{sn}^2(ax)$$
$$-\mathrm{dn}^2(ax))\,h(u)\,h'(u)=a^2\beta^2h(u)^2+\alpha^2\,\mathrm{cn}^4(ax), \tag{6·11}$$

which, by (2·8), is equivalent to the following nonlinear equation:

$$2a^2u(1-u)\,(k^2u+k'^2)\,h(u)\,h''(u)-a^2(k^2u^2+k'^2)\,h(u)\,h'(u)$$
$$-a^2(k^2u+k'^2)\,(u-u^2)\,h'(u)^2=a^2\beta^2h(u)^2+\alpha^2u^2. \tag{6·12}$$

If $h=b-cu$ is a linear function in u with constant coefficients, then (6·12) becomes

$$a^2b(\beta^2b-ck'^2)+2a^2c(ck'^2-\beta^2b)\,u+(\alpha^2+a^2\beta^2c^2-a^2bck^2+a^2c^2(k^2-k'^2))\,u^2=0. \tag{6·13}$$

It is straightforward to verify that (6·13) holds if and only if $b=ck'^2/\beta^2$ and $\alpha^2=c^2a^2(k^2+\beta^2)(k'^2-\beta^2)/\beta^2$. This shows that $h(u)=c(k'^2/\beta^2-u)$ is a solution of (6·12).

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If we choose $c = 1$ and $\alpha = a\sqrt{(k^2+\beta^2)(k'^2-\beta^2)}/\beta$, then we obtain

$$f(x) = \frac{1}{\beta}\sqrt{k'^2-\beta^2\,\mathrm{cn}^2\,(ax)}. \tag{6·14}$$

By combining (6·8) and (6·14) we obtain

$$g(x) = \int_0^x \frac{\alpha\beta\sqrt{(k^2+\beta^2)(k'^2-\beta^2)}\,\mathrm{cn}^2\,(ax)}{k'^2-\beta^2\,\mathrm{cn}^2\,(ax)}dx. \tag{6·15}$$

On the other hand, from (2·8) and (2·9) we have

$$\left.\begin{aligned}
&\frac{\alpha\beta\sqrt{(k^2+\beta^2)(k'^2-\beta^2)}\,\mathrm{cn}^2\,(ax)}{k'^2-\beta^2\,\mathrm{cn}^2\,(ax)}\\
&\quad=\frac{\alpha\beta\sqrt{k^2+\beta^2}}{\sqrt{k'^2-\beta^2}}-\frac{a\beta k'^2\sqrt{k^2+\beta^2}\,\mathrm{sn}^2\,(ax)}{\sqrt{k'^2-\beta^2}\,(k'^2-\beta^2+\beta^2\,\mathrm{sn}^2\,(ax))}\\
&\quad=\frac{a\beta\sqrt{k^2+\beta^2}}{\sqrt{k'^2-\beta^2}}-\frac{a\beta k}{\sqrt{k'^2-\beta^2}}\left(\frac{\mathrm{cn}\,(\gamma)\,\mathrm{dn}\,(\gamma)\,\mathrm{sn}^2\,(ax)}{1-k^2\,\mathrm{sn}^2\,(\gamma)\,\mathrm{sn}^2\,(ax)}\right)\\
&\quad=\frac{a\beta\sqrt{k^2+\beta^2}}{\sqrt{k'^2-\beta^2}}-\frac{a\beta k}{2\sqrt{k'^2-\beta^2}}(\mathrm{sn}\,(ax+\gamma)+\mathrm{sn}\,(ax-\gamma))\,\mathrm{sn}\,(ax)\\
&\quad=\frac{a\beta\sqrt{k^2+\beta^2}}{\sqrt{k'^2-\beta^2}}+\frac{a\mathrm{i}}{2}(k^2\,(\mathrm{sn}\,(ax+\gamma)+\mathrm{sn}\,(ax-\gamma))\,\mathrm{sn}\,(\gamma)\,\mathrm{sn}\,(ax)),
\end{aligned}\right\} \tag{6·16}$$

where

$$\mathrm{sn}\,(\gamma) = \frac{\mathrm{i}\beta}{k\sqrt{k'^2-\beta^2}},\quad \mathrm{dn}\,(\gamma) = \frac{k'}{\sqrt{k'^2-\beta^2}},\quad \mathrm{cn}\,(\gamma) = \frac{k'\sqrt{k^2+\beta^2}}{k\sqrt{k'^2-\beta^2}}.$$

Combining (2·14) and (6·16), we obtain

$$\begin{aligned}
&\frac{a\beta\sqrt{(k^2+\beta^2)(k'^2-\beta^2)}\,\mathrm{cn}^2\,(ax)}{k'^2-\beta^2\,\mathrm{cn}^2\,(ax)}\\
&\qquad=\frac{a\beta\sqrt{k^2+\beta^2}}{\sqrt{k'^2-\beta^2}}+\frac{a\mathrm{i}}{2}(Z\,(ax-\gamma)-Z\,(ax+\gamma)+2Z(\gamma)).
\end{aligned} \tag{6·17}$$

From (6·15), (6·16) and (6·17) we find

$$\left.\begin{aligned}
g(x) &= \frac{a\beta\sqrt{k^2+\beta^2}}{\sqrt{k'^2-\beta^2}}x+\frac{a\mathrm{i}}{2}\int_0^x (Z\,(ax-\gamma)-Z\,(ax+\gamma)+2Z(\gamma))\,dx\\
&= \frac{a\beta\sqrt{k^2+\beta^2}}{\sqrt{k'^2-\beta^2}}x+\frac{\mathrm{i}}{2}\left(\ln\frac{\Theta\,(ax-\gamma)}{\Theta\,(ax+\gamma)}+2aZ(\gamma)\,x\right),
\end{aligned}\right\} \tag{6·18}$$

where we applied (2·13) and the fact that Θ is an even function. Therefore, by (6·2), (6·14) and (6·18), we conclude that the functions y_1, y_2 defined in Proposition 1 are independent solutions of (6·1). Consequently, the general solution of (6·1) is given by the linear combination of y_1, y_2.

Proposition 2. *For any real numbers $a > 1$ and $\beta > 0$, the general solution of*

$$y''(x)+2a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)\,y'(x)+a^2\beta^2y(x) = 0 \tag{6·19}$$

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is given by $y(x)=c_1y_1(x)+c_2y_2(x)$ *with*

$$y_1=\sqrt{k'^2+\beta^2\,\mathrm{cn}^2\,(ax)}\cosh\left(-\frac{a\beta\sqrt{k^2-\beta^2}}{\sqrt{k'^2+\beta^2}}x+\tfrac{1}{2}\ln\frac{\Theta\,(ax-\gamma)}{\Theta\,(ax+\gamma)}+aZ(\gamma)\,x\right)$$

and

$$y_2=\sqrt{k'^2+\beta^2\,\mathrm{cn}^2\,(ax)}\sinh\left(\frac{a\beta\sqrt{k^2-\beta^2}}{\sqrt{k'^2+\beta^2}}x-\tfrac{1}{2}\ln\frac{\Theta\,(ax-\gamma)}{\Theta\,(ax+\gamma)}-aZ(\gamma)\,x\right),$$

where $k=\sqrt{a^2+1}/(\sqrt{2}a)$ *and* $k'=\sqrt{a^2-1}/(\sqrt{2}a)$ *are the modulus and the complementary modules of the elliptic functions and* $\gamma=\mathrm{sn}^{-1}(\beta/(k\sqrt{k'^2+\beta^2}))$.

Proof. This can be proved by using the same trick given in the proof of Proposition 1. After making the key transformation (6·2) for (6·19), we obtain (6·5) and

$$f''(x)-f(x)(g'(x))^2+2a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)f'(x)+a^2\beta^2f(x)=0. \tag{6·20}$$

Solving (6·5) yields

$$g'(x)=\frac{\alpha\,\mathrm{cn}\,(ax)^2}{f(x)^2}, \tag{6·21}$$

for some constant α. By substituting (6·21) into (6·20) we obtain

$$f''(x)+2a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)f'(x)+a^2\beta^2f(x)=\frac{\alpha^2\,\mathrm{cn}^4\,(ax)}{f(x)^3}. \tag{6·22}$$

After making the second key transformation $f(x)=\sqrt{h(u)}$ with $u=\mathrm{cn}^2\,(ax)$ for (6·22), we find

$$\begin{aligned}2a^2u(1-u)(k^2u+k'^2)\,h(u)\,h''(u)-a^2(k^2u^2+k'^2)\,h(u)\,h'(u)\\ -a^2(k^2u+k'^2)(u-u^2)\,h'(u)^2+a^2\beta^2h(u)^2=\alpha^2u^2.\end{aligned} \tag{6·23}$$

It is then straightforward to verify that a linear function $h=b+cu$ is a solution of (6·23) if and only if $b=ck'^2/\beta^2$ and $\alpha^2=a^2c^2(\beta^2-k^2)(\beta^2+k'^2)/\beta^2$. In particular, if we choose $c=1$ and $\alpha=a\sqrt{(\beta^2-k^2)(\beta^2+k'^2)}/\beta$, then

$$f(x)=\frac{1}{\beta}\sqrt{k'^2+\beta^2\,\mathrm{cn}^2\,(ax)},\quad g'(x)=\frac{a\beta\,\sqrt{(\beta^2-k^2)(\beta^2+k'^2)}\,\mathrm{cn}^2\,(ax)}{k'^2+\beta^2\,\mathrm{cn}^2\,(ax)}. \tag{6·24}$$

On the other hand, from (2·8), (2·9) we have

$$\left.\begin{aligned}&\frac{a\beta\sqrt{(\beta^2-k^2)(\beta^2+k'^2)}\,\mathrm{cn}^2\,(ax)}{k'^2+\beta^2\,\mathrm{cn}^2\,(ax)}\\ &=\frac{a\beta\sqrt{\beta^2-k^2}}{\sqrt{\beta^2+k'^2}}-\frac{a\beta k'^2\sqrt{\beta^2-k^2}\,\mathrm{sn}^2\,(ax)}{\sqrt{\beta^2+k'^2}\,(k'^2+\beta^2-\beta^2\,\mathrm{sn}^2\,(ax))}\\ &=\frac{a\beta\,\sqrt{\beta^2-k^2}}{\sqrt{\beta^2+k'^2}}-\frac{\mathrm{i}\,a\beta k\,\mathrm{cn}\,(\gamma)\,\mathrm{dn}\,(\gamma)\,\mathrm{sn}^2\,(ax)}{\sqrt{\beta^2+k'^2}\,(1-k^2\,\mathrm{sn}^2(\gamma)\,\mathrm{sn}^2\,(ax))}\\ &=\frac{a\beta\sqrt{\beta^2-k^2}}{\sqrt{\beta^2+k'^2}}-\frac{\mathrm{i}\,a\beta k}{2\sqrt{\beta^2+k'^2}}(\mathrm{sn}\,(ax+\gamma)+\mathrm{sn}\,(ax-\gamma))\,\mathrm{sn}\,(ax)\\ &=\frac{a\beta\sqrt{\beta^2-k^2}}{\sqrt{\beta^2+k'^2}}-\frac{\mathrm{i}a}{2}(k^2\,(\mathrm{sn}\,(ax+\gamma)+\mathrm{sn}\,(ax-\gamma))\,\mathrm{sn}\,(\gamma)\,\mathrm{sn}\,(ax)),\end{aligned}\right\} \tag{6·25}$$

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where

$$\mathrm{sn}\,(\gamma)=\frac{\beta}{k\sqrt{\beta^2+k'^2}},\quad \mathrm{dn}\,(\gamma)=\frac{k'}{\sqrt{k'^2+\beta^2}},\quad \mathrm{cn}\,(\gamma)=\frac{k'\sqrt{k^2-\beta^2}}{k\sqrt{k'^2+\beta^2}}.$$

Combining (2·14) and (6·25), we obtain

$$\frac{a\beta\sqrt{(\beta^2-k^2)(\beta^2+k'^2)}\,\mathrm{cn}^2\,(ax)}{k'^2+\beta^2\,\mathrm{cn}^2\,(ax)}$$

$$=\frac{a\beta\sqrt{\beta^2-k^2}}{\sqrt{\beta^2+k'^2}}-\frac{\mathrm{i}a}{2}(Z\,(ax-\gamma)-Z\,(ax+\gamma)+2Z(\gamma)). \quad (6\cdot26)$$

From (6·24) and (6·26), we have

$$\left.\begin{aligned} g(x)&=\frac{a\beta\,\sqrt{\beta^2-k^2}}{\sqrt{\beta^2+k'^2}}x-\frac{\mathrm{i}a}{2}\int_0^x(Z\,(ax-\gamma)-Z\,(ax+\gamma)+2Z\,(\gamma))\,dx\\ &=\mathrm{i}\frac{a\beta\sqrt{k^2-\beta^2}}{\sqrt{\beta^2+k'^2}}x-\frac{\mathrm{i}}{2}\ln\frac{\Theta\,(ax-\gamma)}{\Theta\,(ax+\gamma)}-\mathrm{i}\,aZ(\gamma)\,x,\end{aligned}\right\} \quad (6\cdot27)$$

where we applied (2·13) and the fact that Θ is an even function. Therefore, by (6·2), (6·24) and (6·27) we conclude that

$$z_1(x)=\sqrt{k'^2+\beta^2\,\mathrm{cn}^2\,(ax)}\exp\left(-\frac{a\beta\,\sqrt{k^2-\beta^2}}{\sqrt{\beta^2+k'^2}}x+\tfrac{1}{2}\ln\frac{\Theta\,(ax-\gamma)}{\Theta\,(ax+\gamma)}+aZ(\gamma)\,x\right)$$

is a solution of (6·19). By applying the method of reduction of order, we know that

$$z_2(x)=\sqrt{k'^2+\beta^2\,\mathrm{cn}^2\,(ax)}\exp\left(\frac{a\beta\,\sqrt{k^2-\beta^2}}{\sqrt{\beta^2+k'^2}}x-\tfrac{1}{2}\ln\frac{\Theta\,(ax-\gamma)}{\Theta\,(ax+\gamma)}-aZ(\gamma)\,x\right)$$

is a second independent solution of (6·19). Consequently, the functions y_1, y_2 defined in Proposition 2 are two independent solutions of (6·19). Hence, the general solution of (6·19) is given by the linear combination of y_1, y_2.

Proposition 3. *For any real numbers $0<a<1$ and $\beta>0$, the general solution of*

$$y''(x)+2ak\,\mathrm{cn}\left(\frac{a}{k}x\right)\mathrm{sd}\left(\frac{a}{k}x\right)y'(x)+\frac{a^2\beta^2}{k^2}y(x)=0 \quad (6\cdot28)$$

is given by $y(x)=c_1y_1(x)+c_2y_2(x)$ with

$$y_1=\sqrt{\beta^2\,\mathrm{dn}^2\left(\frac{a}{k}x\right)-k'^2}\cos\left(\frac{a\beta\,\sqrt{\beta^2-1}}{k\sqrt{\beta^2-k'^2}}x-\frac{i}{2}\ln\frac{\Theta((a/k)\,x-\gamma)}{\Theta((a/k)\,x+\gamma)}-i\frac{a}{k}Z(\gamma)\,x\right)$$

and

$$y_2=\sqrt{\beta^2\,\mathrm{dn}^2\left(\frac{a}{k}x\right)-k'^2}\sin\left(\frac{a\beta\,\sqrt{\beta^2-1}}{k\sqrt{\beta^2-k'^2}}x-\frac{i}{2}\ln\frac{\Theta((a/k)\,x-\gamma)}{\Theta((a/k)\,x+\gamma)}-i\frac{a}{k}Z(\gamma)\,x\right),$$

where $k=\sqrt{2}a/\sqrt{1+a^2}$ and $k'=\sqrt{1-a^2}/\sqrt{1+a^2}$ are the modulus and the complementary modules of the elliptic functions and $\gamma=\mathrm{sn}^{-1}(\beta/\sqrt{\beta^2-k'^2})$.

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Proof. This can also be proved by using the same trick. After making the key transformation (6·2) for (6·28), we obtain

$$f(x)\,g''(x)+2\left(f'(x)+ak\,\mathrm{cn}\left(\frac{a}{k}x\right)\mathrm{sd}\left(\frac{a}{k}x\right)f(x)\right)g'(x)=0, \tag{6·29}$$

$$f''(x)-f(x)(g'(x))^2+2ak\,\mathrm{cn}\left(\frac{a}{k}x\right)\mathrm{sd}\left(\frac{a}{k}x\right)f'(x)+\frac{a^2\beta^2}{k^2}f(x)=0. \tag{6·30}$$

Solving (6·29) yields

$$g'(x)=\frac{\alpha\,\mathrm{dn}^2((a/k)\,x)}{f^2}, \tag{6·31}$$

for some constant α. By substituting (6·31) into (6·30) we obtain

$$f''(x)+2ak\,\mathrm{cn}\left(\frac{a}{k}x\right)\mathrm{sd}\left(\frac{a}{k}x\right)f'(x)+\frac{a^2\beta^2}{k^2}f(x)=\frac{\alpha^2\,\mathrm{dn}^4(ax)}{f(x)^3}. \tag{6·32}$$

After making the second key transformation $f(x)=\sqrt{h(u)}$ with $u=\mathrm{dn}^2(ax/k)$ for (6·30), we find

$$\begin{aligned}2a^2u(1-u)(u-k'^2)\,h(u)\,h''(u)-a^2(u^2-k'^2)\,h(u)\,h'(u)\\ -a^2(u-k'^2)(u-u^2)\,h'(u)^2+a^2\beta^2h(u)^2=\alpha^2k^2u^2.\end{aligned} \tag{6·33}$$

It is then straightforward to verify that a linear function $h=b+cu$ is a solution of (6·33) if and only if $b=-ck'^2/\beta^2$ and $\alpha^2=a^2c^2(\beta^2-1)(\beta^2-k'^2)/(\beta^2k^2)$. In particular, if we choose $c=1$, $\alpha=a\sqrt{(\beta^2-1)(\beta^2-k'^2)}/(\beta k)$, we obtain

$$f(x)=\frac{1}{\beta}\quad\overline{\beta^2\,\mathrm{dn}^2\left(\frac{a}{k}x\right)-k'^2},\quad g'(x)=\frac{a\beta\sqrt{(\beta^2-1)(\beta^2-k'^2)}\,\mathrm{dn}^2((a/k)\,x)}{k(\beta^2\,\mathrm{dn}^2((a/k)\,x)-k'^2)}. \tag{6·34}$$

On the other hand, from (2·8), (2·9) we have

$$\left.\begin{aligned}&\frac{a\beta\sqrt{(\beta^2-1)(\beta^2-k'^2)}\,\mathrm{dn}^2((a/k)\,x)}{k(\beta^2\,\mathrm{dn}^2((a/k)\,x)-k'^2}\\ &=\frac{a\beta\sqrt{\beta^2-1}}{k\sqrt{\beta^2-k'^2}}-\frac{\mathrm{i}a\beta k\,\mathrm{cn}\,(\gamma)\,\mathrm{dn}\,(\gamma)\,\mathrm{sn}^2((a/k)\,x)}{\sqrt{\beta^2-k'^2}\,(1-k^2\,\mathrm{sn}^2\,(\gamma)\,\mathrm{sn}^2((a/k)\,x)}\\ &=\frac{a\beta\sqrt{\beta^2-1}}{k\sqrt{\beta^2-k'^2}}-i\frac{a}{2k}\left(k^2\left(\mathrm{sn}\left(\frac{a}{k}x+\gamma\right)+\mathrm{sn}\left(\frac{a}{k}x-\gamma\right)\right)\mathrm{sn}\,(\gamma)\,\mathrm{sn}\left(\frac{a}{k}x\right)\right),\end{aligned}\right\} \tag{6·35}$$

where

$$\mathrm{sn}\,(\gamma)=\frac{\beta}{\sqrt{\beta^2-k'^2}},\quad \mathrm{cn}\,(\gamma)=\frac{\mathrm{i}k'}{\sqrt{\beta^2-k'^2}},\quad \mathrm{dn}\,(\gamma)=\frac{k'\sqrt{\beta^2-1}}{\sqrt{\beta^2-k'^2}}.$$

From (2·14), (6·34), (6·35), we obtain

$$\left.\begin{aligned}g(x)&=\frac{\alpha\beta\sqrt{\beta^2-1}}{k\sqrt{\beta^2-k'^2}}x-\mathrm{i}\frac{a}{2k}\int_0^x\left(Z\left(\frac{a}{k}x-\gamma\right)-Z\left(\frac{a}{k}x+\gamma\right)+2Z(\gamma)\,x\right)dx\\ &=\frac{a\beta\sqrt{\beta^2-1}}{k\sqrt{\beta^2-k'^2}}x-\frac{\mathrm{i}}{2}\ln\frac{\Theta((a/k)\,x-\gamma)}{\Theta((a/k)\,x+\gamma)}-\mathrm{i}\frac{a}{k}Z(\gamma)\,x.\end{aligned}\right\} \tag{6·36}$$

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Therefore, the functions y_1, y_2 defined in Proposition 3 are independent solutions of (6·28).

Corollary 1. *For any real numbers $a > 1$ and $\beta > 0$, the general solution of*

$$z''(x) = a^2(1+\beta^2)\,\mathrm{nc}^2(ax,k)\,z, \quad k = \frac{\sqrt{a^2-1}}{\sqrt{2}a} \tag{6·37}$$

is given by $z(x) = c_1\,\mathrm{cn}(ax,k)\,y_1(x) + c_2\,\mathrm{cn}(ax,k)\,y_2(x)$ where $y_1(x)$, $y_2(x)$ are defined in Proposition 1.

Proof. Follows from Proposition 1 and the fact that equation (6·37) can be obtained from (6·1) by making the transformation: $y(x) = \mathrm{cn}(ax,k)\,z(x)$.

Corollary 2. *For any real numbers $a > 1$ and $\beta > 0$, the general solution of*

$$z''(x) = a^2(1-\beta^2)\,\mathrm{nc}^2(ax,k)\,z, \quad k = \frac{\sqrt{a^2+1}}{\sqrt{2}a} \tag{6·38}$$

is given by $z(x) = c_1\,\mathrm{cn}(ax,k)\,y_1(x) + c_2\,\mathrm{cn}(ax,k)\,y_2(x)$ where $y_1(x)$, $y_2(x)$ are defined in Proposition 2.

Proof. Follows from Proposition 2 and the fact that equation (6·38) can be obtained from (6·19) by making the transformation: $y(x) = \mathrm{cn}(ax,k)\,z(x)$.

Corollary 3. *For any real numbers $0 < a < 1$ and $\beta > 0$, the general solution of*

$$z''(x) = a^2\left(1-\frac{\beta^2}{k^2}\right)\mathrm{nd}^2\left(\frac{a}{k}x,k\right)z, \quad k = \frac{\sqrt{2}a}{\sqrt{1+a^2}}, \quad k' = \frac{\sqrt{1-a^2}}{\sqrt{1+a^2}} \tag{6·39}$$

is given by $z(x) = c_1\,\mathrm{dn}(ax/k,\ k)\,y_1(x) + c_2\,\mathrm{dn}(ax/k,k)\,y_2(x)$ where $y_1(x)$, $y_2(x)$ are defined in Proposition 3.

Proof. Follows from Proposition 3 and the fact that equation (6·39) can be obtained from (6·28) by making the transformation: $y(x) = \mathrm{dn}(ax/k,k)\,z(x)$.

Remark 2. Conversely, since

$$\left.\begin{aligned} &\frac{d}{du}\left(\ln\frac{\Theta(u-\gamma)}{\Theta(u+\gamma)}\right) = Z(u-\gamma) - Z(u+\gamma) \\ &= k^2\,\mathrm{sn}(u)\,\mathrm{sn}(\gamma)\,(\mathrm{sn}(u-\gamma) + \mathrm{sn}(u+\gamma)) - 2Z(\gamma) \\ &= \frac{2k^2\,\mathrm{cn}(\gamma)\,\mathrm{dn}(\gamma)\,\mathrm{sn}(\gamma)\,\mathrm{sn}^2(u)}{1-k^2\,\mathrm{sn}^2(\gamma)\,\mathrm{sn}^2(u)} - 2Z(\gamma), \end{aligned}\right\} \tag{6·40}$$

it is straightforward to verify that the two functions y_1, y_2 defined in Proposition 1 (respectively, in Propositions 2 and 3) are indeed independent solutions of (6·1) (respectively, of (6·19) and of (6·28)).

Remark 3. In 1879, Picard [**19**] discovered a method for solving the differential equation

$$y''(x) + nk^2\,\mathrm{cn}(x)\,\mathrm{sd}(x)\,y'(x) + \alpha y(x) = 0, \tag{6·41}$$

where n is a positive integer and α a constant. Although (6·41) is similar to our differential equations (6·1) and (6·19). Picard's method does not apply to them.

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7. *Proof of Theorem* 5

Let $\mathbf{x}: M^n \to S^{n+1}(1) \subset \mathbb{E}^{n+2}$ be an isometric immersion of a conformally flat n-manifold with $n > 2$ which satisfies equality (1·2). Then, by Lemma 5 and a well-known result of Cartan and Schouten on conformally flat hypersurfaces (cf. p. 154 of [**3**]), we know that M^n has a principal curvature with multiplicity at least $n-1$ for $n = 3$ as well as for $n \geqslant 4$. Thus, by applying (3·2), we know that either (i) the principal curvatures of M^n are given by $\lambda_1 = 0, \lambda_2 = \cdots = \lambda_n = \mu$, or (ii) $n = 3$ and the principal curvatures of M^3 are given by $\lambda_1 = \lambda_2 = \frac{1}{2}\mu, \lambda_3 = \mu \neq 0$. We treat these two cases separately.

Case (i). $\lambda_1 = 0$ and $\lambda_2 = \cdots = \lambda_n = \mu$.

If $\mu = 0$ identically, then M^n is totally geodesic in $S^{n+1}(1)$.

Now, suppose that M^n is not totally geodesic in $S^{n+1}(1)$. Denote by U the open subset of M^n on which the mean curvature function is nonzero. Then U is non-empty. We will prove in the following that U is the whole manifold M^n in this case.

We denote by $\mathscr{D}$ and $\mathscr{D}^\perp$ the distributions on the open subset U spanned by $\{e_1\}$ and $\{e_2, e_3, \ldots, e_n\}$, respectively.

Now, by putting $j = 1$, $i = 2, \ldots, n$ in (3·6), we get $\omega_1^i(e_1) = 0$ which implies that integral curves of $\mathscr{D}$ are geodesics.

Also by using the first equation in (3·6), we find

$$e_2\mu = \cdots = e_n\mu = 0. \tag{7·1}$$

It is easy to see from (3·5) that

$$\omega_1^i(e_j) = 0, \quad 2 \leqslant i \neq j \leqslant n, \tag{7·2}$$

which implies that $\mathscr{D}^\perp$ is integrable. Consequently, there exist local coordinate systems $\{x_1, x_2, \ldots, x_n\}$ such that $\partial/\partial x_2, \ldots, \partial/\partial x_n$ span $\mathscr{D}^\perp$ and $e_1 = \partial/\partial x$ with $x = x_1$. From (7·1), we know that μ depends only on x, i.e. $\mu = \mu(x)$.

If we choose $i = 1$ for the first equation in (3·5), we get $\mu'(x) = -\mu(x)\,\omega_1^j(e_j)$ for any $j \geqslant 2$. Thus

$$\nabla_{e_j} e_1 = \sum_{k=2}^{n} \omega_1^k(e_j)\, e_k = \omega_1^j(e_j)\, e_j = -(\ln\mu)'\, e_j\,. \tag{7·3}$$

Using (7·3) we know that each integral submanifold of $\mathscr{D}^\perp$ is an extrinsic sphere of $S^{n+1}(1)$, i.e., it is a totally umbilical submanifold with nonzero parallel mean curvature vector in $S^{n+1}(1)$. Hence, the distribution $\mathscr{D}^\perp$ is a spherical distribution. Therefore, by applying a result of Hiepko [**14**] (cf. also [**13**, remark 2·1]), we know that U is locally the warped product $I \times_{f(x)} S^{n-1}(1)$, where $f(x)$ is a suitable warped function. Therefore, the metric of U is given by

$$g = dx^2 + f(x)^2\, g_0, \tag{7·4}$$

where g_0 is the metric of $S^{n-1}(1)$. In particular, if we choose the spherical coordinate system $\{u_2, \ldots, u_n\}$ on $S^{n-1}(1)$, then we have

$$g = dx^2 + f^2(x)\,(du_2^2 + \cos^2 u_2\, du_3^2 + \cdots + \cos^2 u_2 \cdots \cos^2 u_{n-1}\, du_n^2). \tag{7·5}$$

By (7·5) we obtain

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$$\left.\begin{aligned}
&\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}=0, \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial u_k}=\frac{f'}{f}\frac{\partial}{\partial u_k}, \quad \nabla_{\frac{\partial}{\partial u_2}}\frac{\partial}{\partial u_2}=-ff'\frac{\partial}{\partial x},\\
&\nabla_{\frac{\partial}{\partial u_i}}\frac{\partial}{\partial u_j}=-(\tan u_i)\frac{\partial}{\partial u_j}, \quad 2\leqslant i<j,\\
&\nabla_{\frac{\partial}{\partial u_j}}\frac{\partial}{\partial u_j}=-\left(ff'\prod_{l=2}^{j-1}\cos^2 u_l\right)\frac{\partial}{\partial x}+\sum_{k=2}^{j-1}\left(\frac{\sin 2u_k}{2}\prod_{l=k+1}^{j-1}\cos^2 u_l\right)\frac{\partial}{\partial u_k},\\
&2\leqslant i,j,k\leqslant n.
\end{aligned}\right\} \tag{7·6}$$

From (7·5) and the assumption on the principal curvatures, we know that the second fundamental form h of M^n in $S^{n+1}(1)$ satisfies

$$\left.\begin{aligned}
&h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right)=0, \quad h\left(\frac{\partial}{\partial u_2},\frac{\partial}{\partial u_2}\right)=\mu f^2\xi,\\
&h\left(\frac{\partial}{\partial u_3},\frac{\partial}{\partial u_3}\right)=\mu f^2\cos^2 u_2\,\xi,\ldots, \quad h\left(\frac{\partial}{\partial u_n},\frac{\partial}{\partial u_n}\right)=\mu f^2\prod_{k=2}^{n-1}\cos^2 u_k\,\xi,\\
&h\left(\frac{\partial}{\partial x},\frac{\partial}{\partial u_j}\right)=0, \quad h\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right)=0, \quad 2\leqslant i\neq j\leqslant n,
\end{aligned}\right\} \tag{7·7}$$

where ξ is a unit normal vector field of M^n in $S^{n+1}(1)$.

Let $\bar{\nabla}h$ denote the covariant derivative of the second fundamental form h. Then by the equation of Codazzi, we have

$$(\bar{\nabla}_{\frac{\partial}{\partial x}}h)\left(\frac{\partial}{\partial u_2},\frac{\partial}{\partial u_2}\right)=(\bar{\nabla}_{\frac{\partial}{\partial u_2}}h)\left(\frac{\partial}{\partial x},\frac{\partial}{\partial u_2}\right). \tag{7·8}$$

From (7·6), (7·7) and (7·8) we obtain $\mu' f=-\mu f'$. Therefore

$$\mu(x)=\frac{a}{f(x)}, \tag{7·9}$$

for some nonzero constant a.

Now, by applying (7·6), we know that the sectional curvatures K_{12} and K_{23} of the plane sections spanned by $\partial/\partial x$, $\partial/\partial u_2$ and $\partial/\partial u_2$, $\partial/\partial u_3$ are equal to $-f''/f$ and $(1-f'^2)/f^2$, respectively. On the other hand, from the equation of Gauss and our assumption on principal curvatures, we have $K_{12}=1$ and $K_{23}=1+\mu^2$. Therefore, by combining these facts with (7·9), we obtain

$$f''+f=0, \quad f'^2+f^2=1-a^2, \tag{7·10}$$

which implies in particular that $0<a<1$.

Solving the first differential equation in (7·10) yields $f(x)=C\cos(x+b)$ for some constant b, C. By applying a translation in x if necessary, we have

$$f(x)=C\cos x. \tag{7·11}$$

Substituting (7·11) into the second differential equation in (7·10), we obtain $C=\sqrt{1-a^2}$. Consequently, we obtain

$$f(x)=\sqrt{1-a^2}\cos x, \quad \mu(x)=\frac{a}{\sqrt{1-a^2}}\sec x, \tag{7·12}$$

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where a is a constant satisfying $0 < a < 1$.

(7·12) implies $\mu \geqslant a/\sqrt{1-a^2} > 0$ on U. Hence, by the continuity of the squared mean curvature function $H^2 = (n-1)^2\mu^2/n^2$, we know that the mean curvature function is nowhere zero. Hence, U is the whole manifold M^n. Consequently, (7·4) and (7·12) imply that M^n is isometric to an open portion of Riemannian manifold $\hat{B}^n_a$ which was defined in the introduction.

By applying (7·5), (7·6), (7·7), (7·12) and the formula of Gauss, we conclude that the isometric immersion $\mathbf{x}: M^n \to S^{n-1}(1) \subset \mathbb{E}^{n+2}$ satisfies the following system of partial differential equations:

$$\frac{\partial^2 \mathbf{x}}{\partial x^2} = -\mathbf{x}, \tag{7·13}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x\, \partial u_j} = -(\tan x)\frac{\partial \mathbf{x}}{\partial u_j}, \quad j = 2, \dots, n, \tag{7·14}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_i\, \partial u_j} = -(\tan u_i)\frac{\partial \mathbf{x}}{\partial u_j}, \quad 2 \leqslant i < j, \tag{7·15}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_2^2} = \tfrac{1}{2}(1-a^2)\sin 2x \frac{\partial \mathbf{x}}{\partial x} + a\sqrt{1-a^2}\cos x\xi - ((1-a^2)\cos^2 x)\,\mathbf{x}, \tag{7·16}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_{j+1}^2} = (\cos^2 u_j)\frac{\partial^2 \mathbf{x}}{\partial u_j^2} + (\tfrac{1}{2}\sin 2u_j)\frac{\partial \mathbf{x}}{\partial u_j}, \quad j = 2, \dots, n-1. \tag{7·17}$$

Solving equation (7·13) yields

$$\mathbf{x} = P(u_2, \dots, u_n)\sin x + Q(u_2, \dots, u_n)\cos x, \tag{7·18}$$

for some $\mathbb{E}^{n+1}$-valued functions $P = P(u_2, \dots, u_n)$ and $Q = Q(u_2, \dots, u_n)$. Substituting (7·18) into (7·14) we know that P is a constant vector, we denote it by c_1. Thus

$$\mathbf{x} = c_1 \sin x + Q(u_2, \dots, u_n)\cos x. \tag{7·19}$$

Substituting (7·19) into (7·15) with $i = 2$, we obtain

$$\frac{\partial^2 Q}{\partial u_j\, \partial u_2} + (\tan u_2)\frac{\partial Q}{\partial u_j} = 0, \quad j = 3, \dots, n \tag{7·20}$$

which implies

$$\frac{\partial Q}{\partial u_2} + (\tan u_2)\, Q = \tilde{\phi}_2(u_2), \tag{7·21}$$

for some function $\tilde{\phi}_2 = \tilde{\phi}_2(u_2)$. Therefore, by solving (7·21), we obtain

$$Q = \phi_2(u_2) + Q_3(u_3, \dots, u_n)\cos u_2 \tag{7·22}$$

for some functions $\phi_2 = \phi_2(u_2)$ and $Q_3 = Q_3(u_3, \dots, u_n)$.

Similarly, by substituting (7·19) and (7·22) into (7·15) with $i = 3$ and $j > 3$, we find

$$Q_3 = \phi_3(u_3) + Q_4(u_4, \dots, u_n)\cos u_3, \tag{7·23}$$

for some function $\phi_3 = \phi_3(u_3)$ and $Q_4 = Q_4(u_4, \dots, u_n)$. Repeating such procedure $n-2$ times, we obtain

$$\left.\begin{aligned} Q &= \phi_2(u_2) + Q_3(u_3, \dots, u_n)\cos u_2 \\ Q_3 &= \phi_3(u_3) + Q_4(u_4, \dots, u_n)\cos u_3, \\ Q_4 &= \phi_4(u_4) + Q_5(u_5, \dots, u_n)\cos u_4, \\ &\vdots \\ Q_{n-1} &= \phi_{n-1}(u_{n-1}) + \phi_n(u_n)\cos u_{n-1}, \end{aligned}\right\} \tag{7·24}$$

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with $\phi_n(u_n) = Q_n(u_n)$. Substituting (7·24) into (7·19) we get

$$\begin{aligned}\mathbf{x} = {} & c_1 \sin x + \phi_2(u_2)\cos x + \phi_3(u_3)\cos u_2 \cos x + \cdots \\ & + \phi_{n-1}(u_{n-1})\cos u_2 \cdots \cos u_{n-2}\cos x + \phi_n(u_n)\cos u_2 \cdots \cos u_{n-1}\cos x. \end{aligned} \tag{7·25}$$

Substituting (7·25) into (7·17) with $j = n-1$, we obtain

$$\phi_n''(u_n) + \phi_n(u_n) = \cos u_{n-1}\,\phi_{n-1}''(u_{n-1}) + \sin u_{n-1}\,\phi_{n-1}'(u_{n-1}), \tag{7·26}$$

which implies that

$$\phi_n''(u_n) + \phi_n(u_n) = k_n, \tag{7·27}$$

$$\cos u_{n-1}\,\phi_{n-1}''(u_{n-1}) + \sin u_{n-1}\,\phi_{n-1}'(u_{n-1}) = k_n, \tag{7·28}$$

for some constant vector k_n. Solving (7·27) yields

$$\phi_n = c_{n+1}\sin u_n + c_{n+2}\cos u_n + k_n, \tag{7·29}$$

for some constant vectors c_{n+1}, c_{n+2}. Combining (7·25) and (7·29) we obtain

$$\begin{aligned}\mathbf{x} = {} & c_1 \sin x + \phi_2(u_2)\cos x + \phi_3(u_3)\cos u_2 \cos x + \cdots \\ & + \phi_{n-1}(u_{n-1})\cos u_2 \cdots \cos u_{n-2}\cos x \\ & + c_{n+1}\cos u_2 \cdots cos\, u_{n-1}\sin u_n \cos x + c_{n+2}\cos u_2 \cdots \cos u_n \cos x, \end{aligned} \tag{7·30}$$

where we rename $\phi_{n-1}(u_{n-1}) + k_n \cos u_{n-1}$ as $\phi_{n-1}(u_{n-1})$. By applying (7·17) and by repeating such procedure $n-2$ times, we may obtain

$$\begin{aligned}\mathbf{x} = {} & c_1 \sin x + c_2 \cos x + c_3 \sin u_2 \cos x + c_4 \cos u_2 \sin u_3 \cos x + \cdots \\ & + c_{n+1}\cos u_2 \cdots \cos u_{n-1}\sin u_n \cos x + c_{n+2}\cos u_2 \cdots \cos u_n \cos x, \end{aligned} \tag{7·31}$$

for some constant vectors $c_1, \ldots, c_{n+2}$.

Now, we choose the initial conditions at $0 = (0, \ldots, 0)$ as follows:

$$\left.\begin{aligned} \mathbf{x}(0) = a\epsilon_2 + \sqrt{1-a^2}\,\epsilon_{n+2}, \quad \frac{\partial \mathbf{x}}{\partial x}(0) = \epsilon_1, \\ \frac{\partial \mathbf{x}}{\partial u_2}(0) = \sqrt{1-a^2}\,\epsilon_3, \ldots, \frac{\partial \mathbf{x}}{\partial u_n}(0) = \sqrt{1-a^2}\,\epsilon_{n+1}, \end{aligned}\right\} \tag{7·32}$$

where $\{\epsilon_1, \ldots, \epsilon_{n+2}\}$ is the natural coordinate basis of $\mathbb{E}^{n+2}$. Then, by applying (7·31) and (7·32), we obtain

$$c_1 = \epsilon_1, \quad c_2 = a\epsilon_2, \quad c_3 = \sqrt{1-a^2}\,\epsilon_3, \ldots, c_{n+2} = \sqrt{1-a^2}\,\epsilon_{n+2}. \tag{7·33}$$

Consequently, by (7·31) and (7·33) we conclude that, up to rigid motions of $\mathbb{E}^{n+2}$, the immersion $\mathbf{x}$ is given by (1·8). It is clear from (1·8) that the immersion $\mathbf{x}$ can be extended to the two point compactification $\hat{B}_a^n$ of B_a^n.

Case (ii). $\lambda_1 = \lambda_2 = \frac{1}{2}\mu$ and $\lambda_3 = \mu \neq 0$.

In this case, (3·6) yields

$$\Gamma_{31}^2 = \Lambda_3 = 0, \quad e_1\mu = e_2\mu = 0, \quad \Gamma_{33}^1 = \Gamma_{33}^2 = 0. \tag{7·34}$$

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Denote by $\mathscr{F}$ and $\mathscr{F}^{\perp}$ the distributions spanned by $\{e_1, e_2\}$ and $\{e_3\}$, respectively. By (7·34) we know that the integral curves of $\mathscr{F}^{\perp}$ are geodesics and the distribution $\mathscr{F}$ is integrable. Consequently, there exist local coordinate systems $\{x, u, v\}$ such that $\mathscr{F}$ is spanned by $\{\partial/\partial u, \partial/\partial v\}$ and $e_3 = \partial/\partial x$.

From (7·34) we know that μ depends only on x, i.e. $\mu = \mu(x)$. Also, from (3·8) and (7·34), we have

$$\omega_3^1 = \frac{\mu'(x)}{\mu(x)}\omega^1, \quad \omega_3^2 = \frac{\mu'(x)}{\mu(x)}\omega^2. \tag{7·35}$$

Using (7·35) we obtain

$$\nabla_{e_j} e_3 = \frac{\mu'(x)}{\mu(x)} e_j, \quad j = 1, 2. \tag{7·36}$$

Therefore, each integral submanifold of $\mathscr{F}$ is an extrinsic sphere of $S^4(1)$. Hence, by applying a result of Hiepko, we know that M^3 is locally the warped product $I \times_{f(x)} S^2(1)$, where $f(x)$ is a suitable warped function. In particular, if we choose the spherical coordinate system $\{\theta, \phi\}$ for $S^2(1)$, we have

$$g = dx^2 + f^2(x)\,(d\phi^2 + \cos^2\phi d\theta^2). \tag{7·37}$$

By computing $d\omega_3^1$ and by using (7·35) and Cartan's structure equations, we find

$$\mu''(x) + \frac{\mu^3(x)}{2} + \mu(x) = 0. \tag{7·38}$$

Put $\psi(x) = 2f(x)$. Then (7·38) becomes $\psi''(x) + 2\psi^3(x) + \psi(x) = 0$. Hence, by applying lemma 5·3 of [**7**], we obtain

$$\mu(x) = \sqrt{2(a^2-1)}\,\mathrm{cn}\,(ax, k), \quad k = \frac{\sqrt{a^2-1}}{\sqrt{2}a} \tag{7·39}$$

where a is real number > 1.

Now, by applying (7·37), (7·39) and the equation of Codazzi, we may obtain $\mu' f = \mu f'$. Therefore,

$$f(x) = c\mu(x), \tag{7·40}$$

for some nonzero constant c.

On the other hand, using (7·37) we can compute the sectional curvature K_{23} of the plane section spanned by $\{\partial/\partial\phi, \partial/\partial\theta\}$. On the other hand, we may also compute K_{23} by using the equation of Gauss. Comparing these two different expressions of K_{23} so obtained yields

$$c^2\mu'^2 = 1 - c^2\mu^2 - \tfrac{1}{4}c^2\mu^4. \tag{7·41}$$

Substituting (7·39) into (7·41) we have $c^2 = 1/(a^4-1)$. Therefore

$$f = \frac{\sqrt{2}}{\sqrt{a^2+1}}\,\mathrm{cn}\,(ax, k), \quad \mu = \sqrt{2(a^2-1)}\,\mathrm{cn}\,(ax, k), \quad k = \frac{\sqrt{a^2-1}}{\sqrt{2}a}. \tag{7·42}$$

(7·37) and (7·42) imply that M^3 is an open portion of the warped product manifold $I \times (\sqrt{2}/\sqrt{a^2+1})\,\mathrm{cn}\,(ax)\,S^2(1)$ which is isometric to P_a^3; first introduced in [**7**] (see also Introduction and [**10**]).

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(7·37), (7·39), (7·42) and the formula of Gauss imply that the immersion $\mathbf{x}$ satisfies the following system of partial differential equations:

$$\frac{\partial^2\mathbf{x}}{\partial x^2} = \sqrt{2(a^2-1)}\,\mathrm{cn}\,(ax)\,\xi - \mathbf{x}, \tag{7·43}$$

$$\frac{\partial^2\mathbf{x}}{\partial\phi^2} = \frac{2a}{a^2+1}\,\mathrm{cn}\,(ax)\,\mathrm{dn}\,(ax)\,\mathrm{sn}\,(ax)\frac{\partial\mathbf{x}}{\partial x} + \frac{\sqrt{2(a^2-1)}}{a^2+1}\,\mathrm{cn}^3\,(ax)\,\xi - \frac{2}{a^2+1}\,\mathrm{cn}^2\,(ax)\,\mathbf{x}, \tag{7·44}$$

$$\frac{\partial^2\mathbf{x}}{\partial\theta^2} = \frac{2a}{a^2+1}\cos^2\phi\,\mathrm{cn}\,(ax)\,\mathrm{dn}\,(ax)\,\mathrm{sn}\,(ax)\frac{\partial\mathbf{x}}{\partial x} + \sin\phi\cos\phi\frac{\partial\mathbf{x}}{\partial\phi} + \frac{\sqrt{2(a^2-1)}}{a^2+1}\cos^2\phi\,\mathrm{cn}^3\,(ax)\,\xi - \frac{2}{a^2+1}\cos^2\phi\,\mathrm{cn}^2\,(ax)\,\mathbf{x}, \tag{7·45}$$

$$\frac{\partial^2\mathbf{x}}{\partial x\,\partial\phi} = -a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)\frac{\partial\mathbf{x}}{\partial\phi}, \tag{7·46}$$

$$\frac{\partial^2\mathbf{x}}{\partial x\,\partial\theta} = -a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)\frac{\partial\mathbf{x}}{\partial\theta}, \tag{7·47}$$

$$\frac{\partial^2\mathbf{x}}{\partial\theta\,\partial\phi} = -\tan\phi\frac{\partial\mathbf{x}}{\partial\theta}. \tag{7·48}$$

By taking the partial derivative of (7·44) with respect to ϕ and by applying the equation of Weingarten, we obtain

$$\frac{\partial^3\mathbf{x}}{\partial\phi^3} = -\frac{\partial\mathbf{x}}{\partial\phi}, \tag{7·49}$$

by virtue of (2·8), (2·10) and

$$k^2 = \frac{a^2-1}{2a^2}, \quad k'^2 = \frac{a^2+1}{2a^2}. \tag{7·50}$$

Solving (7·49) yields

$$\mathbf{x}(x,\phi,\theta) = A(x,\theta)\sin\phi + B(x,\theta)\cos\phi + C(x,\theta), \tag{7·51}$$

for some functions A, B, C of two variables. Substituting (7·51) into (7·46) yields

$$\frac{\partial A}{\partial x} = -a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)\,A, \quad \frac{\partial B}{\partial x} = -a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)\,B. \tag{7·52}$$

Solving (7·52) yields

$$A(x,\theta) = E(\theta)\,\mathrm{cn}\,(ax), \quad B(x,\theta) = D(\theta)\,\mathrm{cn}\,(ax) \tag{7·53}$$

for some functions E, D. Thus,

$$\mathbf{x}(x,\phi,\theta) = E(\theta)\,\mathrm{cn}\,(ax)\sin\phi - D(\theta)\,\mathrm{cn}\,(ax)\cos\phi + C(x,\theta). \tag{7·54}$$

Substituting (7·54) into (7·47) yields

$$\frac{\partial^2 C}{\partial x\,\partial\theta} = -a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)\frac{\partial C}{\partial\theta}. \tag{7·55}$$

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By solving (7·55) we find $C(x,\theta) = G(\theta)\,\mathrm{cn}\,(ax) + K(x)$ for some functions $G(\theta)$ and $K(x)$. Thus, (7·54) gives

$$\mathbf{x}(x,\phi,\theta) = E(\theta)\,\mathrm{cn}\,(ax)\sin\phi + D(\theta)\,\mathrm{cn}\,(ax)\cos\phi + G(\theta)\,\mathrm{cn}\,(ax) + K(x). \tag{7·56}$$

Substituting (7·56) into (7·48) yields $E'(\theta) = G'(\theta) = 0$. Thus, E and G are constant vectors in $\mathbb{E}^5$. Consequently, from (7·56) we know that $\mathbf{x}$ takes the form:

$$\mathbf{x}(x,\phi,\theta) = c_1\,\mathrm{cn}\,(ax)\sin\phi + D(\theta)\,\mathrm{cn}\,(ax)\cos\phi + F(x). \tag{7·57}$$

where c_1 is a constant vector. From (7·43) we have

$$\xi = \frac{1}{\sqrt{2(a^2-1)}\,\mathrm{cn}\,(ax)}\left(\frac{\partial^2\mathbf{x}}{\partial x^2} + \mathbf{x}\right). \tag{7·58}$$

By (7·45), (7·57), (7·58) and a long computation, we obtain

$$F''(x) + 2a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)\,F'(x) - F(x) = 0, \tag{7·59}$$

$$D''(\theta) + D(\theta) = 0, \tag{7·60}$$

by virtue of (2·8), (2·10) and (7·50). Solving (7·60) yields

$$D(\theta) = c_2\cos\theta + c_3\sin\theta, \tag{7·61}$$

for some constant vectors c_2, c_3 in $\mathbb{E}^5$. Therefore, by applying Proposition 1, (7·57) and (7·61), we obtain

$$\left.\begin{aligned}
&\mathbf{x}(x,\phi,\theta) = c_1\sin\phi\,\mathrm{cn}\,(ax) + c_2\cos\phi\cos\theta\,\mathrm{cn}\,(ax) + c_3\cos\phi\sin\theta\,\mathrm{cn}\,(ax)\\
&+ c_4\sqrt{a^2k'^2 - \mathrm{cn}^2\,(ax)}\cos\left(\frac{k'}{k}x + \frac{\mathrm{i}\,ak'^2}{2\sqrt{a^2k^4+1}}\left(\ln\frac{\Theta(ax-\gamma)}{\Theta(ax+\gamma)} + 2a\,Z(\gamma)\,x\right)\right)\\
&+ c_5\sqrt{a^2k'^2 - \mathrm{cn}^2\,(ax)}\sin\left(\frac{k'}{k}x + \frac{\mathrm{i}\,ak'^2}{2\sqrt{a^2k^4+1}}\left(\ln\frac{\Theta\,(ax-\gamma)}{\theta(ax+\gamma)} + 2a\,Z(\gamma)\,x\right)\right),
\end{aligned}\right\} \tag{7·62}$$

where Θ and Z are the theta and zeta functions.

Now, if we choose the initial conditions at $0 = (0,0,0)$ as follows:

$$\mathbf{x}(0) = \frac{1}{ak'}(\epsilon_2 + \epsilon_4), \quad \frac{\partial\mathbf{x}}{\partial x}(0) = \epsilon_5, \quad \frac{\partial\mathbf{x}}{\partial\phi}(0) = \frac{1}{ak'}\epsilon_1, \quad \frac{\partial\mathbf{x}}{\partial\theta}(0) = \frac{1}{ak'}\epsilon_3, \tag{7·63}$$

where $\{\epsilon_1,\ldots,\epsilon_5\}$ is the standard basis of $\mathbb{E}^5$, then we obtain

$$c_\alpha = \frac{1}{ak'}\epsilon_\alpha, \quad \alpha = 1,\ldots,5. \tag{7·64}$$

Therefore, up to rigid motions, the immersion $\mathbf{x}$ is given by (1·9) in Theorem 5.

The converse can be verified by straightforward but long computations.

8. *Proof of Theorem* 6

Let $\mathbf{x}: M^n \to H^{n+1}(-1) \subset \mathbb{E}_1^{n+2}$ be an isometric immersion of a conformally flat n-manifold with $n > 2$ which satisfies equality (1·2). By Lemma 5 and a result of Cartan and Schouten, either (a) the principal curvatures of M^n are given by $\lambda_1 = 0$, $\lambda_2 = \cdots = \lambda_n = \mu$, or (b) $n = 3$ and the principal curvatures of M^3 are given by $\lambda_1 = \lambda_2 = \frac{1}{2}\mu$, $\lambda_3 = \mu \neq 0$. We treat these two cases separately.

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Case (*a*). $\lambda_1 = 0$ and $\lambda_2 = \cdots = \lambda_n = \mu$.

If $\mu = 0$ identically, then M^n is totally geodesic. This yields Case (i) of Theorem 6.

Now, suppose that M^n is not totally geodesic in $H^{n+1}(-1)$. Denote by U the open subset of M^n on which the mean curvature function is nonzero. We shall work on this open subset U unless mentioned otherwise. Clearly, $\mu \neq 0$ on U.

We denote by $\mathscr{D}$ and $\mathscr{D}^\perp$ the distributions on the open subset U spanned by $\{e_1\}$ and $\{e_2, e_3, \ldots, e_n\}$, respectively. Then, as in the proof of Theorem 5, we can prove that integral curves of $\mathscr{D}$ are geodesics and $\mathscr{D}^\perp$ is integrable. Thus, there exist local coordinate systems $\{x_1, x_2, \ldots, x_n\}$ such that $\partial/\partial x_2, \ldots, \partial/\partial x_n$ span $\mathscr{D}^\perp$ and $e_1 = \partial/\partial x$ with $x = x_1$. Also, using (3·6), we can prove that μ depend, only on x, i.e., $\mu = \mu(x)$.

If we choose $i = 1$ for the first equation in (3·5), we get $\mu'(x) = -\mu(x)\,\omega_1^j(e_j)$ for any $j \geqslant 2$. Thus

$$\nabla_{e_j} e_1 = \sum_{k=2}^{n} \omega_1^k(e_j)\, e_k = \omega_1^j(e_j)\, e_j = -(\ln \mu)'\, e_j. \tag{8·1}$$

Hence, each integral submanifold of $\mathscr{D}^\perp$ is an extrinsic sphere of $H^{n+1}(-1)$, i.e., the distribution $\mathscr{D}^\perp$ is spherical. Therefore, U is locally a warped product manifold $I \times_{f(x)} N^{n-1}(\bar{c})$, where $f(x)$, $(f(x) > 0)$, is the warped function and $N^{n-1}(\bar{c})$ is a Riemannian space form of constant sectional curvature $\bar{c}$.

Since $N^{n-1}(\bar{c})$ is of constant curvature, it is conformally flat. Thus, there exists a local coordinate system $\{u_2, \ldots, u_n\}$ such that metric tensor of $N^{n-1}(\bar{c})$ is given by

$$g_0 = E^2(du_2^2 + du_3^2 + \cdots + du_n^2). \tag{8·2}$$

With respect to coordinate system $\{x, u_2, \ldots, u_n\}$ on $I \times_f N^{n-1}(\bar{c})$, we have

$$g = dx^2 + f^2 E^2(du_2^2 + \cdots + du_n^2). \tag{8·3}$$

From (8·3) we have

$$\left.\begin{aligned} &\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial u_i} = \frac{f'}{f} \frac{\partial}{\partial u_i}, \quad \nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = \frac{E_i}{E} \frac{\partial}{\partial u_j} + \frac{E_j}{E} \frac{\partial}{\partial u_i}, \\ &\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_i} = -ff' E^2 \frac{\partial}{\partial x} + \frac{E_i}{E} \frac{\partial}{\partial u_i} - \sum_{k \neq i} \frac{E_k}{E} \frac{\partial}{\partial u_k}, \end{aligned}\right\} \tag{8·4}$$

for distinct i, j, k $(2 \leqslant i, j, k \leqslant n)$, where $E_i = \partial E/\partial u_i$.

Codazzi's equation, (8·4) and our assumption on principal curvatures imply

$$\mu = \frac{a}{f} \tag{8·5}$$

for some constant $a > 0$. Also, from (8·4), it follows that the sectional curvatures K_{12} and K_{23} of M^n associated with the plane sections spanned by $\partial/\partial x$, $\partial/\partial u_2$ and $\partial/\partial u_2$, $\partial/\partial u_3$ are given by $-f''/f$ and $(\bar{c} - f'^2)/f^2$, respectively. On the other hand, the equation of Gauss yields $K_{12} = -1$ and $K_{23} = -1 + \mu^2$. Combining these facts with (8·5), we get

$$f'' - f = 0, \quad f^2 - f'^2 = a^2 - \bar{c}. \tag{8·6}$$

Solving the first equation of (8·6) yields $f(x) = c_1 \cosh x + c_2 \sinh x$, where c_1, c_2 are constants. From (8·6) we have either $f(x) = \alpha \sinh(x + b)$ or $f(x) = \alpha \cosh(x + b)$, for some constants b, α. Thus, by applying a translation in x if necessary, we have $f(x) = \alpha \sinh x$ or $f(x) = \alpha \cosh x$. We consider these two cases separately.

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Case 1. $f(x) = \alpha \cosh x$.

In this case, the second equation of (8·6) yields $\alpha^2 = a^2 - \bar{c}$. Thus, after applying a scaling on E if necessary, we have

$$\bar{c} = 1, \quad f(x) = \sqrt{a^2-1}\cosh x, \quad \mu = \frac{a}{\sqrt{a^2-1}}\operatorname{sech} x, \quad a > 1, \quad \text{or} \tag{8·7}$$

$$\bar{c} = 0, \quad f(x) = \cosh x, \quad \mu = \operatorname{sech} x, \quad \text{or} \tag{8·8}$$

$$\bar{c} = -1, \quad f(x) = \sqrt{a^2+1}\cosh x, \quad \mu = \frac{a}{\sqrt{a^2+1}}\operatorname{sech} x, \quad a > 0. \tag{8·9}$$

Case 1*a*. $\bar{c} = 1$, $f(x) = \sqrt{a^2-1}\,\cosh x$, $\mu = (a/\sqrt{a^2-1})\,\operatorname{sech}\, x$, $a > 1$.

In this case the open subset U is the whole manifold M^n. Therefore, M^n is an open portion of the warped product manifold $A_a^n = \mathbb{R} \times_{\sqrt{a^2-1}\cosh x} S^{n-1}(1)$. By choosing the spherical coordinates $\{u_2, \dots, u_n\}$ on $S^{n-1}(1)$, we obtain (7·5) and (7·6) with

$$f = \sqrt{a^2-1}\cosh x, \quad \mu = \frac{a}{\sqrt{a^2-1}}\operatorname{sech} x. \tag{8·10}$$

Therefore, the equation of Gauss implies that the isometric immersion $\mathbf{x}$ satisfies the following system of partial differential equations:

$$\frac{\partial^2 \mathbf{x}}{\partial x^2} = \mathbf{x}, \tag{8·11}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x\, \partial u_j} = \tanh x \frac{\partial \mathbf{x}}{\partial u_j}, \quad j = 2, \dots, n, \tag{8·12}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_i\, \partial u_j} = -\tan u_i \frac{\partial \mathbf{x}}{\partial u_j}, \quad 2 \leqslant i < j, \tag{8·13}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_2^2} = \frac{1-a^2}{2}\sinh 2x \frac{\partial \mathbf{x}}{\partial x} + a\sqrt{a^2-1}\cosh x \xi + ((a^2-1)\cosh^2 x)\,\mathbf{x}, \tag{8·14}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_{j+1}^2} = \cos^2 u_j \frac{\partial^2 \mathbf{x}}{\partial u_j^2} + \tfrac{1}{2}\sin 2u_j \frac{\partial \mathbf{x}}{\partial u_j}, \quad j = 2, \dots, n-1. \tag{8·15}$$

where ξ is a unit normal vector field of M^n in $H^{n+1}(-1)$.

Solving equation (8·11) yields

$$\mathbf{x} = P(u_2, \dots, u_n)\sinh x + Q(u_2, \dots, u_n)\cosh x, \tag{8·16}$$

for some $\mathbb{E}_1^{n+1}$-valued functions $P = P(u_2, \dots, u_n)$ and $Q = Q(u_2, \dots, u_n)$. By substituting (8·16) into (8·12), we know that P is a constant vector, denoted by c_1. Thus

$$\mathbf{x} = c_1 \sinh x + Q(u_2, \dots, u_n)\cosh x, \tag{8·17}$$

Substituting (8·17) into (8·13) with $i = 2$ yields

$$\frac{\partial Q}{\partial u_2} + (\tan u_2)\, Q = \tilde{\phi}_2(u_2), \tag{8·18}$$

for some function $\tilde{\phi}_2 = \tilde{\phi}_2(u_2)$. Therefore, after solving (8·18), we obtain

$$Q = \phi_2(u_2) + Q_3(u_3, \dots, u_n)\cos u_2 \tag{8·19}$$

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for some functions $\phi_2 = \phi_2(u_2)$ and $Q_3 = Q_3(u_3, \dots, u_n)$. Repeating such procedure $n-2$ times, we obtain

$$\left.\begin{aligned} Q &= \phi_2(u_2) + Q_3(u_3, \dots, u_n)\cos u_2 \\ Q_3 &= \phi_3(u_3) + Q_4(u_4, \dots, u_n)\cos u_3, \dots, \\ Q_{n-1} &= \phi_{n-1}(u_{n-1}) + \phi_n(u_n)\cos u_{n-1}, \end{aligned}\right\} \tag{8·20}$$

where $\phi_n(u_n) = Q_n(u_n)$.

Substituting (8·20) into (8·17) yields

$$\begin{aligned} \mathbf{x} = c_1 \sinh x + \phi_2(u_2)\cosh x + \phi_3(u_3)\cos u_2 \cosh x + \cdots + \phi_{n-1}(u_{n-1}) \\ \times \cos u_2 \cdots \cos u_{n-2} \cosh x + \phi_n(u_n)\cos u_2 \cdots \cos u_{n-1}\cosh x. \end{aligned} \tag{8·21}$$

Now, by applying (8·21) and (8·15), we may obtain in the same way as given in the proof of Theorem 4 that

$$\begin{aligned} \mathbf{x} = c_1 \sinh x + c_2 \cosh x + c_3 \sin u_2 \cosh x + \cdots + c_{n+1}\cos u_2 \cdots \\ \times \cos u_{n-1} \sin u_n \cosh x + c_{n+2}\cos u_2 \cdots \cos u_n \cosh x, \end{aligned} \tag{8·22}$$

for some constant vectors $c_1, \dots, c_{n+2}$.

If we choose suitable initial conditions for $\mathbf{x}$, $\partial\mathbf{x}/\partial x$, $\partial\mathbf{x}/\partial u_2, \dots, \partial\mathbf{x}/\partial u_n$ at $0 = (0, \dots, 0)$, we will obtain (1·10) from (8·22). Consequently, up to rigid motions, the immersion $\mathbf{x}$ is given by (1·10).

Case 1*b*. $\bar{c} = 0$, $f(x) = \cosh x$, $\mu = \operatorname{sech} x$.

Again, the open subset U is the whole manifold M^n. Thus, M^n is an open portion of the warped product manifold $G^n = \mathbb{R} \times_{\cosh x} \mathbb{E}^{n-1}$. Hence, the metric tensor of M^n is given by

$$g = dx^2 + \cosh^2 x\,(du_2^2 + \cdots + du_n^2). \tag{8·23}$$

In this case, the equation of Gauss implies that the isometric immersion $\mathbf{x}$ satisfies the following system:

$$\frac{\partial^2 \mathbf{x}}{\partial x^2} = \mathbf{x}, \tag{8·24}$$

$$\frac{\partial^2 \mathbf{x}}{\partial x \partial u_j} = \tanh x \frac{\partial \mathbf{x}}{\partial u_j}, \quad j = 2, \dots, n, \tag{8·25}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_i \partial u_j} = 0, \quad 2 \leqslant i < j, \tag{8·26}$$

$$\frac{\partial^2 \mathbf{x}}{\partial u_j^2} = \sinh x \cosh x \frac{\partial \mathbf{x}}{\partial x} + \cosh x \xi + (\cosh^2 x)\,\mathbf{x}, \quad j = 2, \dots, n. \tag{8·27}$$

After solving this system, we obtain

$$\begin{aligned} \mathbf{x}(x, u_2, \dots, u_n) = c_1 \sinh x \\ + (\alpha_2 u_2^2 + \cdots + \alpha_n^2 u_n^2 + \beta_2 u_2 + \cdots + \beta_n u_n + \gamma)\cosh x, \end{aligned} \tag{8·28}$$

for some constant vectors c_1, $\alpha_2, \dots, \alpha_n, \beta_2, \dots, \beta_n, \gamma$.

If we choose suitable initial conditions for $\mathbf{x}$, $\partial\mathbf{x}/\partial x$, $\partial\mathbf{x}/\partial u_2, \dots, \partial\mathbf{x}/\partial u_n$, we may obtain (1·11) from (8·28). Thus, up to rigid motions, the immersion is given by (1·11).

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Case 1*c*. $\bar{c}=-1$, $f(x)=\sqrt{a^2+1}\cosh x$, $\mu=(a/\sqrt{a^2+1})\operatorname{sech} x$, $a>0$.

Again, the open subset U is the whole manifold M^n and M^n is an open portion of the warped product manifold $H^n_a=\mathbb{R}\times_{\sqrt{a^2+1}\cosh x}H^{n-1}(-1)$. Thus, the metric tensor of M^n is given by

$$g=dx^2+(a^2+1)(\cosh^2 x)\,g_0, \tag{8·29}$$

where

$$g_0=du_2^2+\sinh^2 u_2(du_3^2+\cos^2 u_3\,du_4^2+\cdots+\cos^2 u_3\cdots\cos^2 u_{n-1}\,du_n^2). \tag{8·30}$$

If we apply (8·29), (8·30), our assumption on the second fundamental form and the equation of Gauss, we know that the isometric immersion $\mathbf{x}$ satisfies the following system:

$$\frac{\partial^2\mathbf{x}}{\partial x^2}=\mathbf{x}, \tag{8·31}$$

$$\frac{\partial^2\mathbf{x}}{\partial x\,\partial u_j}=\tanh x\frac{\partial\mathbf{x}}{\partial u_j},\quad j=2,\dots,n, \tag{8·32}$$

$$\frac{\partial^2\mathbf{x}}{\partial u_2\,\partial u_j}=\coth u_2\frac{\partial\mathbf{x}}{\partial u_j},\quad j=3,\dots,n, \tag{8·33}$$

$$\frac{\partial^2\mathbf{x}}{\partial u_i\,\partial u_j}=-\tan u_j\frac{\partial\mathbf{x}}{\partial u_k},\quad 3\leqslant i<j\leqslant n, \tag{8·34}$$

$$\frac{\partial^2\mathbf{x}}{\partial u_2^2}=-\frac{a^2+1}{2}\sinh 2x\frac{\partial\mathbf{x}}{\partial x}+a\sqrt{a^2+1}\cosh x\xi+(a^2+1)\cosh^2 x\,\mathbf{x}, \tag{8·35}$$

$$\frac{\partial^2\mathbf{x}}{\partial u_3^2}=\sinh^2 u_2\frac{\partial^2\mathbf{x}}{\partial u_2^2}-\sinh x\cosh x\frac{\partial\mathbf{x}}{\partial u_2}, \tag{8·36}$$

$$\frac{\partial^2\mathbf{x}}{\partial u_{j+1}^2}=\cos^2 u_j\frac{\partial^2\mathbf{x}}{\partial u_j^2}-\sin u_j\cos u_j\frac{\partial\mathbf{x}}{\partial u_j},\quad j=2,\dots,n-1. \tag{8·37}$$

After solving this system in the same way as Case 1*a* we obtain

$$\begin{aligned}\mathbf{x}=c_1\sinh x+c_2\cosh x+c_3\cosh u_2\cosh x+c_4\sinh u_2\cos u_3\sinh x\\ +\cdots+c_{n+1}\sinh u_2\cos u_3\cdots\cos u_{n-1}\sin u_n\cosh x\\ +c_{n+2}\sinh u_2\cos u_3\cdots\cos u_n\cosh x,\end{aligned} \tag{8·38}$$

for some constant vectors $c_1,\dots,c_{n+2}$.

If we choose suitable initial conditions for $\mathbf{x}$, $\partial\mathbf{x}/\partial x$, $\partial\mathbf{x}/\partial u_2,\dots,\partial\mathbf{x}/\partial u_n$, we will obtain (1·12) from (8·38).

Case 2. $f(x)=\alpha\sinh x$

In this case, (8·6) yields $\bar{c}=\alpha^2+a^2$. Thus, $\bar{c}>0$. By applying a scaling on E if necessary, we have $\bar{c}=1$ and $\alpha=\sqrt{1-a^2}$. In summary, we have

$$\bar{c}=1,\quad f(x)=\sqrt{1-a^2}\sinh x,\quad \mu=\frac{a}{\sqrt{1-a^2}}\operatorname{csch} x,\quad 0<a<1. \tag{8·39}$$

From (8·39) and continuity, we know that U is a dense open subset of M^n. Moreover, locally, U is an open subset of the warped product manifold Y^n_a. Thus, the metric tensor of M^n is given by

$$g=dx^2+(1-a^2)\sinh^2 x\left(du_2^2+\cos^2 u_2\,du_3^2+\cdots+\left(\prod_{j=2}^{n-1}\cos^2 u_j\right)du_n^2\right). \tag{8·40}$$

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Therefore, by applying the equation of Gauss, we know that the isometric immersion $\mathbf{x}$ satisfies the following system:

$$\frac{\partial^2\mathbf{x}}{\partial x^2}=\mathbf{x}, \tag{8·41}$$

$$\frac{\partial^2\mathbf{x}}{\partial x\,\partial u_j}=\coth x\frac{\partial\mathbf{x}}{\partial u_j},\quad j=2,\dots,n, \tag{8·42}$$

$$\frac{\partial^2\mathbf{x}}{\partial u_i\,\partial u_j}=-\tan u_i\frac{\partial\mathbf{x}}{\partial u_j},\quad 2\leqslant i<j, \tag{8·43}$$

$$\frac{\partial^2\mathbf{x}}{\partial u_2^2}=\frac{a^2-1}{2}\sinh 2x\frac{\partial\mathbf{x}}{\partial x}+a\sqrt{1-a^2}\sinh x\xi+(1-a^2)\sinh^2 x\,\mathbf{x}, \tag{8·44}$$

$$\frac{\partial^2\mathbf{x}}{\partial u_{j+1}^2}=(\cos^2 u_j)\frac{\partial^2\mathbf{x}}{\partial u_j^2}+\tfrac{1}{2}\sin 2u_j\frac{\partial\mathbf{x}}{\partial u_j},\quad j=2,\dots,n-1. \tag{8·45}$$

After solving this system in the same way as Case 1a we obtain

$$\begin{aligned}\mathbf{x}=c_1\cos x+c_2\sinh x+c_3\sin u_2\sinh x+\cdots+c_{n+1}\cos u_2\\ \times\cdots\cos u_{n-1}\sin u_n\sinh x+c_{n+2}\cos u_2\cdots\cos u_n\sinh x,\end{aligned} \tag{8·46}$$

for some constant vectors $c_1,\dots,c_{n+2}$.

If we choose suitable initial conditions for $\mathbf{x}$, $\partial\mathbf{x}/\partial x$, $\partial\mathbf{x}/\partial u_2,\dots,\partial\mathbf{x}/\partial u_n$ at $0=(0,\dots,0)$, we will obtain (1·13) from (8·46). Therefore, up to rigid motions, the immersion is given by (1·13).

Case (b). $n=3$, $\lambda_1=\lambda_2=\frac{1}{2}\mu$, $\lambda_3=\mu\neq 0$.

Let $\mathscr{F}$ and $\mathscr{F}^{\perp}$ denote the distributions spanned by $\{e_1, e_2\}$ and $\{e_3\}$, respectively. Then, as Case (ii) in the proof of Theorem 4, we can prove that the integral curves of $\mathscr{F}^{\perp}$ are geodesics and the distribution $\mathscr{F}$ is a spherical integral distribution. Thus, there exist a local coordinate system $\{x,u,v\}$ such that $\mathscr{F}$ is spanned by $\{\partial/\partial u,\partial/\partial v\}$ and $e_3=\partial/\partial x$. As in the proof of Theorem 4, we may also prove that $\mu=\mu(x)$ depends only on x.

Again, according to a result of Hiepko, M^3 is locally a warped product manifold $I\times_{f(x)}N^2(\bar{c})$, where $f(x)>0$ is the warped function and $N^2(\bar{c})$ is a surface of constant curvature $\bar{c}$. So, the metric of M^3 can be written as

$$g=dx^2+f(x)^2E^2(du^2+dv^2), \tag{8·47}$$

where $\{u,v\}$ is an isothermal coordinate system on $N^2(\bar{c})$. Applying (8·47) we obtain

$$\left.\begin{aligned}&\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}=0,\quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial u}=\frac{f'}{f}\frac{\partial}{\partial u},\quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial v}=\frac{f'}{f}\frac{\partial}{\partial v}\\ &\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial v}=\frac{E_v}{E}\frac{\partial}{\partial u}+\frac{E_u}{E}\frac{\partial}{\partial v}\\ &\nabla_{\frac{\partial}{\partial u}}\frac{\partial}{\partial u}=-ff'E^2\frac{\partial}{\partial x}+\frac{E_u}{E}\frac{\partial}{\partial u}-\frac{E_v}{E}\frac{\partial}{\partial v},\\ &\nabla_{\frac{\partial}{\partial v}}\frac{\partial}{\partial v}=-ff'E^2\frac{\partial}{\partial x}-\frac{E_u}{E}\frac{\partial}{\partial u}+\frac{E_v}{E}\frac{\partial}{\partial v}.\end{aligned}\right\} \tag{8·48}$$

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From (8·47) and the hypothesis on the principal curvatures, we know that the second fundamental form h of M^3 in $H^4(-1)$ satisfies

$$\left.\begin{aligned} h\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\Big)=\mu\xi,\quad h\Big(\frac{\partial}{\partial u},\frac{\partial}{\partial u}\Big)=\tfrac{1}{2}\mu f^2E^2\,\xi,\quad h\Big(\frac{\partial}{\partial v},\frac{\partial}{\partial v}\Big)=\tfrac{1}{2}\mu f^2E^2\,\xi,\\ h\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial u}\Big)=h\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial v}\Big)=h\Big(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\Big)=0,\end{aligned}\right\} \tag{8·49}$$

where ξ is a unit normal field of M^3 in $H^4(-1)$.

From (8·48), (8·49) and the equations of Gauss and Codazzi, we find

$$f(x)=\alpha\mu(x) \tag{8·50}$$

$$\mu''(x)+\frac{\mu^3(x)}{2}-\mu(x)=0, \tag{8·51}$$

$$K_{13}=-1+\tfrac{1}{2}\mu^2=-\frac{f''}{f},\quad K_{12}=-1+\tfrac{1}{4}\mu^2=\frac{\bar{c}-f'^2}{f^2}, \tag{8·52}$$

where α is a nonzero real number and K_{12}, K_{13} are the sectional curvatures of M^3 of the plane sections spanned by $\{e_1,e_2\}$, $\{e_1,e_3\}$, respectively.

Put $\psi(x)=2f(x)$. Then (8·51) becomes $\psi''(x)+2\psi^3(x)-\psi(x)=0$. Hence, by applying lemma 5·4 of [7], we know that $\mu=\mu(x)$ is one of the following functions:

(1) $\mu=\sqrt{2}$;

(2) $\mu=2\operatorname{sech}(x)$;

(3) $\mu(x)=\sqrt{2(a^2+1)}\,\operatorname{cn}\Big(ax,\dfrac{\sqrt{a^2+1}}{\sqrt{2}a}\Big),\quad a>1$;

(4) $\mu(x)=\sqrt{2(a^2+1)}\,\operatorname{dn}\Big(\dfrac{\sqrt{a^2+1}}{\sqrt{2}}x,\dfrac{\sqrt{2}a}{\sqrt{a^2+1}}\Big),\quad 0<a<1.$

We consider these four cases separately.

*Case b*1. $\mu=\sqrt{2}$, $f=\sqrt{2}\alpha$.

In this case, (8·52) yields $\bar{c}=-\alpha^2<0$. By choosing $\bar{c}=-1$, we obtain $\alpha=1$, $f=\sqrt{2}$. Therefore, $U=M^3$ and M^3 is an open portion of the warped product manifold $\mathbb{R}\times_{\sqrt{2}}H^2(-1)$ which is isometric to F^3. Therefore, if we choose the hyperbolic coordinate system $\{\phi,\theta\}$ on $H^2(-1)$, we obtain

$$g=dx^2+2(d\phi^2+\cosh^2\phi\,d\theta^2). \tag{8·53}$$

By (8·53) $\mu=\sqrt{2}$ and the formula of Gauss, we know that the immersion satisfies the differential system:

$$\left.\begin{aligned} &\frac{\partial^2\mathbf{x}}{\partial x^2}=\sqrt{2}\,\xi+\mathbf{x},\quad \frac{\partial^2\mathbf{x}}{\partial\phi^2}=\sqrt{2}\,\xi+2\mathbf{x};\\ &\frac{\partial^2\mathbf{x}}{\partial\theta^2}=-\sinh\phi\cosh\phi\frac{\partial\mathbf{x}}{\partial\phi}+\sqrt{2}\cosh^2\phi\,\xi+2\cosh^2\phi\,\mathbf{x};\\ &\frac{\partial^2\mathbf{x}}{\partial x\,\partial\theta}=\frac{\partial^2\mathbf{x}}{\partial x\,\partial\phi}=0,\quad \frac{\partial^2\mathbf{x}}{\partial\phi\,\partial\theta}=\tanh\phi\frac{\partial\mathbf{x}}{\partial\theta}.\end{aligned}\right\} \tag{8·54}$$

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After solving (8·54) we obtain

$$\mathbf{x}(u,\phi,\theta) = c_1\cos x + c_2\sin x + c_3\sinh\phi + c_4\cosh\phi\cosh\theta + c_5\cosh\phi\sinh\theta$$

for some constants $c_1,\ldots,c_5$. Thus, by choosing suitable initial conditions, we will obtain (1·14).

*Case b*2. $\mu = 2\operatorname{sech}(x), f = 2\alpha\operatorname{sech}(x)$.

In this case, (8·52) yields $\bar{c} = 0$. By choosing $\alpha = \frac{1}{2}$ we obtain $f = \operatorname{sech} x$. Therefore, M^2 is an open portion of the warped product manifold $\mathbb{R}\times_{\operatorname{sech} x}\mathbb{E}^2$ which is isometric to F^3. Thus

$$g = dx^2 + \operatorname{sech}^2 x(du^2+dv^2). \tag{8·55}$$

By (8·55), $\mu = 2\operatorname{sech} x$ and the formula of Gauss, we know that the immersion satisfies the following system:

$$\frac{\partial^2\mathbf{x}}{\partial x^2} = 2\operatorname{sech} x\,\xi + \mathbf{x}; \tag{8·56}$$

$$\frac{\partial^2\mathbf{x}}{\partial u^2} = \operatorname{sech}^2 x\tanh x\frac{\partial\mathbf{x}}{\partial x} + \operatorname{sech}^3 x\,\xi + \operatorname{sech}^2 x\,\mathbf{x}; \tag{8·57}$$

$$\frac{\partial^2\mathbf{x}}{\partial v^2} = \operatorname{sech}^2 x\tanh x\frac{\partial\mathbf{x}}{\partial x} + \operatorname{sech}^3 x\,\xi + \operatorname{sech}^2 x\,\mathbf{x}; \tag{8·58}$$

$$\frac{\partial^2\mathbf{x}}{\partial x\,\partial u} = -\tanh x\frac{\partial\mathbf{x}}{\partial u},\quad \frac{\partial^2\mathbf{x}}{\partial x\,\partial v} = -\tanh x\frac{\partial\mathbf{x}}{\partial v}; \tag{8·59}$$

$$\frac{\partial^2\mathbf{x}}{\partial u\,\partial v} = 0. \tag{8·60}$$

Solving (8·59) yields

$$\mathbf{x}(x,u,v) = A(x) + B(u,v)\operatorname{sech} x, \tag{8·61}$$

for some $\mathbb{E}^5_1$-valued functions $A(x)$, $B(u,v)$. Substituting (8·61) into (8·60) we obtain $B(u,v) = C(u)+D(v)$ for some functions $C(u)$, $D(v)$. Thus

$$\mathbf{x}(x,u,v) = A(x) + (C(u)+D(v))\operatorname{sech} x. \tag{8·62}$$

Substituting (8·62) into (8·57) and (8·58) yields

$$2C''(u) = \operatorname{sech} x(A''(x) + 2(\tanh x)A'(x) + A(x)), \tag{8·63}$$

$$2D''(v) = \operatorname{sech} x(A''(x) + 2(\tanh x)A'(x) + A(x)). \tag{8·64}$$

(8·63) and (8·64) imply that there is a constant vector c_1 such that

$$\left.\begin{aligned} C''(u) = D''(v) = 2c_1,\\ A''(x) + 2(\tanh x)A'(x) + A(x) = 4c_1\cosh x.\end{aligned}\right\} \tag{8·65}$$

Solving (8·65) yields

$$\left.\begin{aligned} C(u) = c_1u^2 + c_2u + b_1,\quad D(v) = c_1v^2 + c_3v + b_2,\\ A(x) = \operatorname{sech} x(c_1(x^2 - \tfrac{1}{2}\cosh 2x) + c_4x + b_3)\end{aligned}\right\} \tag{8·66}$$

for some constants c_2, c_3, c_4, b_1, b_2, b_3. Combining (8·62) and (8·66) yields

$$\mathbf{x}(u,\phi,\theta) = \operatorname{sech} x\{c_1(x^2+u^2+v^2-\cosh^2 x) + c_2u + c_3v + c_4x + c_5\} \tag{8·67}$$

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for some constant c_5. Thus, by choosing suitable initial conditions, we will obtain (1·15) from (8·67). Consequently, up to rigid motions, the immersion $\mathbf{x}$ is given by (1·15).

*Case b*3. $\mu(x) = \sqrt{2(a^2+1)}\,\mathrm{cn}\,(ax, (\sqrt{a^2+1})/\sqrt{2}a), a > 1$.

In this case, (8·52) yields $\bar{c} = \alpha^2(\alpha^4-1) > 0$. By choosing $\bar{c} = 1$, we obtain

$$\mu(x) = 2ak\,\mathrm{cn}\,(ax,k), \quad f = \frac{1}{ak'}\mathrm{cn}\,(ax,k), \quad k = \frac{\sqrt{a^2+1}}{\sqrt{2}a}, \quad k' = \frac{\sqrt{a^2-1}}{\sqrt{2}a}. \tag{8·68}$$

Thus, locally, U is an open portion of the warped product manifold C_a^3. If we choose the spherical coordinate system $\{\phi,\theta\}$ on $S^2(1)$, we have

$$g = dx^2 + \frac{\mathrm{cn}^2\,(ax,k)}{a^2k'^2}(d\phi^2 + \cos^2\phi d\theta^2). \tag{8·69}$$

By (8·68), (8·69) and the formula of Gauss, we know that the immersion satisfies the following differential system:

$$\frac{\partial^2\mathbf{x}}{\partial x^2} = 2ak\,\mathrm{cn}\,(ax)\,\xi + \mathbf{x}; \tag{8·70}$$

$$\frac{\partial^2\mathbf{x}}{\partial\phi^2} = \frac{1}{ak'^2}\mathrm{cn}\,(ax)\,\mathrm{dn}\,(ax)\,\mathrm{sn}\,(ax)\frac{\partial\mathbf{x}}{\partial x} + \frac{k\,\mathrm{cn}^3\,(ax)}{ak'^2}\xi + \frac{\mathrm{cn}^2\,(ax)}{a^2k'^2}\mathbf{x}; \tag{8·71}$$

$$\frac{\partial^2\mathbf{x}}{\partial\theta^2} = \cos^2\phi\frac{\partial^2\mathbf{x}}{\partial\phi^2} + \sin\phi\cos\phi\frac{\partial\mathbf{x}}{\partial\phi}; \tag{8·72}$$

$$\frac{\partial^2\mathbf{x}}{\partial x\,\partial\phi} = -a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)\frac{\partial\mathbf{x}}{\partial\phi}; \tag{8·73}$$

$$\frac{\partial^2\mathbf{x}}{\partial x\,\partial\theta} = -a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)\frac{\partial\mathbf{x}}{\partial\theta}; \tag{8·74}$$

$$\frac{\partial^2\mathbf{x}}{\partial\phi\,\partial\theta} = -\tan\phi\frac{\partial\mathbf{x}}{\partial\theta}. \tag{8·75}$$

By taking the partial derivative of (8·71) with respect to ϕ by applying (2·8), (8·73) and the Weingarten equation, we obtain $\partial^3\mathbf{x}/\partial\phi^3 + \partial\mathbf{x}/\partial\phi = 0$, from which we get

$$\mathbf{x}(x,\phi,\theta) = A(x,\theta)\sin\phi + B(x,\theta)\cos\phi + C(x,\theta), \tag{8·76}$$

for some $\mathbb{E}_1^5$-valued functions A, B, C. Substituting (8·76) into (8·73) and (8·74) yields

$$A(x,\theta) = \mathrm{cn}\,(ax)\,D(\theta), \quad B(x,\theta) = \mathrm{cn}\,(ax)\,E(\theta), \quad C(\mathbf{x},\theta) = \mathrm{cn}\,(ax)\,F(\theta) + G(x)$$

for some functions D, E, F and G. Thus, we obtain

$$\mathbf{x} = D(\theta)\,\mathrm{cn}\,(ax)\sin\phi + E(\theta)\,\mathrm{cn}\,(ax)\cos\phi + F(\theta)\,\mathrm{cn}\,(ax) + G(x). \tag{8·77}$$

Substituting (8·77) into (8·75) yields $D'(\theta) = F'(\theta) = 0$. Thus, (8·77) implies

$$\mathbf{x} = c_1\,\mathrm{cn}\,(ax)\sin\phi + E(\theta)\,\mathrm{cn}\,(ax)\cos\phi + K(x) \tag{8·78}$$

for some constant vector c_1 and function $K(x)$.

By substituting (8·78) into (8·72) and by applying (2·8) and (8·70), we find

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$E''(\theta)+E(\theta)=0$. Thus, $E(\theta)=c_2\cos\theta+c_3\sin\theta$ for some constants c_2, c_3. Substituting this into (8·77) yields

$$\mathbf{x}=c_1\,\mathrm{cn}\,(ax)\sin\phi+c_2\,\mathrm{cn}\,(ax)\cos\phi\cos\theta+c_3\,\mathrm{cn}\,(ax)\sin\phi\cos\theta+K(x). \tag{8·79}$$

Substituting (8·79) into (8·70) yields

$$K''(x)+2a\,\mathrm{sc}\,(ax)\,\mathrm{dn}\,(ax)\,K'(x)+K(x)=0. \tag{8·80}$$

Therefore, by applying (8·79), (8·80) and Proposition 2, we know that $\mathbf{x}$ takes the form:

$$\left.\begin{aligned}\mathbf{x}(x,\phi,\theta)=\;&c_1\,\mathrm{cn}\,(ax)\sin\phi+c_2\,\mathrm{cn}\,(ax)\cos\phi\cos\theta+c_3\,\mathrm{cn}\,(ax)\sin\phi\cos\theta\\&+c_4\sqrt{a^2k'^2+\mathrm{cn}^2\,(ax)}\cosh\left(\frac{k'}{k}x-\tfrac{1}{2}\ln\frac{\Theta\,(ax-\gamma)}{\Theta\,(ax+\gamma)}-aZ(\gamma)\,x\right)\\&+c_5\sqrt{a^2k'^2+\mathrm{cn}^2\,(ax)}\sinh\left(\frac{k'}{k}x-\tfrac{1}{2}\ln\frac{\Theta\,(ax-\gamma)}{\Theta\,(ax+\gamma)}-aZ(\gamma)x\right),\end{aligned}\right\} \tag{8·81}$$

where $k=\sqrt{a^2+1}/(\sqrt{2}a)$ and $k'=\sqrt{a^2-1}/(\sqrt{2}a)$ are the modulus and the complementary modules of the Jacobi's elliptic functions and $\gamma=\mathrm{sn}^{-1}(1/(ak^2))$. Now, by choosing suitable initial conditions, we will obtain (1·16) from (8·81). Therefore, up to rigid motions, the immersion is given by (1·16).

*Case b*4. $\mu(x)=\sqrt{2(1+a^2)}\,\mathrm{cn}\,((\sqrt{1+a^2}/\sqrt{2})\,x,\sqrt{2}a/\sqrt{1+a^2})$, $0<a<1$

In this case, (8·52) yields $\bar{c}=-4a^2\alpha^2k'^2/k^2<0$. By choosing $\bar{c}=-1$, we obtain

$$\mu(x)=\frac{2a}{k}\,\mathrm{dn}\left(\frac{a}{k}x,k\right),\quad f=\frac{k}{ak'}\,\mathrm{dn}\left(\frac{a}{k}x,k\right),\quad k=\frac{\sqrt{2}a}{\sqrt{1+a^2}},\quad k'=\frac{\sqrt{1-a^2}}{\sqrt{1+a^2}}. \tag{8·82}$$

Thus, $U=M^3$ which is an open portion of the warped product manifold $D^3_a=\mathbb{R}\times_{(k/(ak'))\,\mathrm{dn}\,(ax/k)}H^2(-1)$. If we choose the hyperbolic coordinate system $\{\phi,\theta\}$ on $H^2(-1)$, we get

$$g=dx^2+\frac{k^2\,\mathrm{dn}^2\,((a/k)\,x)}{a^2k'^2}(d\phi^2+\cosh^2\phi d\theta^2). \tag{8·83}$$

By (8·68), (8·69) and the formula of Gauss, we know that the immersion satisfies the following differential system:

$$\frac{\partial^2\mathbf{x}}{\partial x^2}=\frac{2a}{k}\,\mathrm{dn}\left(\frac{a}{k}x\right)\xi+\mathbf{x}; \tag{8·84}$$

$$\frac{\partial^2\mathbf{x}}{\partial\phi^2}=\frac{k^3}{ak'^2}\,\mathrm{cn}\left(\frac{a}{k}x\right)\mathrm{dn}\left(\frac{a}{k}x\right)\mathrm{sn}\left(\frac{a}{k}x\right)\frac{\partial\mathbf{x}}{\partial x}+\frac{k\,\mathrm{dn}^3\,((a/k)\,x)}{ak'^2}\xi+\frac{k^2\,\mathrm{dn}^2\,((a/k)\,x)}{a^2k'^2}\mathbf{x}; \tag{8·85}$$

$$\frac{\partial^2\mathbf{x}}{\partial\theta^2}=\cosh^2\phi\frac{\partial^2\mathbf{x}}{\partial\phi^2}-\sinh\phi\cosh\phi\frac{\partial\mathbf{x}}{\partial\phi}; \tag{8·86}$$

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$$\frac{\partial^2\mathbf{x}}{\partial x\,\partial\phi}=-ak\,\mathrm{cn}\left(\frac{a}{k}x\right)\mathrm{sd}\left(\frac{a}{k}x\right)\frac{\partial\mathbf{x}}{\partial\phi};\tag{8·87}$$

$$\frac{\partial^2\mathbf{x}}{\partial x\,\partial\theta}=-ak\,\mathrm{cn}\left(\frac{a}{k}x\right)\mathrm{sd}\left(\frac{a}{k}x\right)\frac{\partial\mathbf{x}}{\partial\theta};\tag{8·88}$$

$$\frac{\partial^2\mathbf{x}}{\partial\phi\,\partial\theta}=\tanh\phi\,\frac{\partial\mathbf{x}}{\partial\theta}.\tag{8·89}$$

Solving (8·89) yields

$$\mathbf{x}(x,\phi,\theta)=B(x,\theta)\cosh\phi+C(x,\phi),\tag{8·90}$$

for some $\mathbb{E}_1^5$-valued functions $B(x,\theta)$, $C(x,\phi)$. Substituting (8·90) into (8·87) yields

$$\frac{\partial B}{\partial x}=-ak\,\mathrm{cn}\left(\frac{a}{k}x\right)\mathrm{sd}\left(\frac{a}{k}x\right)B,\tag{8·91}$$

$$\frac{\partial^2 C}{\partial x\,\partial\phi}=-ak\,\mathrm{cn}\left(\frac{a}{k}x\right)\mathrm{sd}\left(\frac{a}{k}x\right)\frac{\partial\mathbf{x}}{\partial\phi}.\tag{8·92}$$

Solving (8·91) and (8·92) yields

$$B(x,\theta)=G(\theta)\,\mathrm{dn}\left(\frac{a}{k}x\right),\quad C(x,\phi)=K(\phi)\,\mathrm{dn}\left(\frac{a}{k}x\right)+W(x),\tag{8·93}$$

for some functions $G(\theta)$, $K(\phi)$, $W(x)$. Thus, we obtain

$$\mathbf{x}(x,\phi,\theta)=G(\theta)\,\mathrm{dn}\left(\frac{a}{k}x\right)\cosh\phi+K(\phi)\,\mathrm{dn}\left(\frac{a}{k}x\right)+W(x).\tag{8·94}$$

By substituting (8·94) into (8·85) and (8·86) and by applying (2·8) and (8·84), we obtain

$$G''(\theta)-G(\theta)=0,\quad K''(\phi)-K(\phi)=0,\tag{8·95}$$

$$W''(x)+2ak\,\mathrm{cn}\left(\frac{a}{k}x\right)\mathrm{sd}\left(\frac{a}{k}x\right)W'(x)+W(x)=0.\tag{8·96}$$

Therefore, by applying (8·94), (8·95), (8·96) and Proposition 3, we know that $\mathbf{x}$ takes the form:

$$\left.\begin{aligned}\mathbf{x}=\;&c_1\,\mathrm{dn}\left(\frac{a}{k}x\right)\cosh\phi\cosh\theta+c_2\,\mathrm{dn}\left(\frac{a}{k}x\right)\cosh\phi\sinh\theta\\&+c_3\,\mathrm{dn}\left(\frac{a}{k}x\right)\cosh\phi+c_4\,\mathrm{dn}\left(\frac{a}{k}x\right)\sinh\phi\\&+c_5\quad\sqrt{k^2\,\mathrm{dn}^2\left(\frac{a}{k}x\right)-a^2k'^2}\cos\left(k'x-\frac{\mathrm{i}}{2}\ln\frac{\Theta((a/k)\,x-\gamma)}{\Theta((a/k)\,x+\gamma)}-\mathrm{i}\frac{a}{k}Z(\gamma)\,x\right)\\&+c_6\quad\sqrt{k^2\,\mathrm{dn}^2\left(\frac{a}{k}x\right)-a^2k'^2}\sin\left(k'x-\frac{\mathrm{i}}{2}\ln\frac{\Theta((a/k)\,x-\gamma)}{\Theta((a/k)\,x+\gamma)}-\mathrm{i}\frac{a}{k}Z(\gamma)\,x\right),\end{aligned}\right\}\tag{8·97}$$

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where $k = \sqrt{2}a/\sqrt{1+a^2}$ and $k' = \sqrt{1-a^2}/\sqrt{1+a^2}$ are the modulus and the complementary modules of Jacobi's elliptic functions and $\gamma = \mathrm{sn}^{-1}(k/a)$. Now, by choosing suitable initial conditions, we will obtain (1·17) from (8·97). Therefore, up to rigid motions, the immersion is given by (1·17).

The converse can be verified by straightforward long computations.

9. *Some remarks*

Remark 4. The surfaces A_a^2, B_a^2, C_a^2, D_a^2, G^2, H_a^2, L^2, P_a^2, W_a^2 and Y_a^2 can be realized as the Riemannian universal covering spaces of some surfaces of revolution in Euclidean 3-space. For examples, surfaces $A^2_{\sqrt{5}/2}$, $B^2_{\sqrt{5}}$, $C^2_{\sqrt{2}}$, $D^2_{\frac{1}{2}}$, G^2, H^2_1, L^2, $P^2_{\sqrt{2}}$, $W^2_{\sqrt{2}}$ and $Y^2_{\sqrt{3}/2}$ are the universal covering spaces of the surfaces of revolution pictured in Figs 1–10, respectively.

Remark 5. Although the shapes of A_a^n, G^n and H_a^n are similar for $n = 2$, their shapes are quite different for $n \geqslant 3$. This can be seen from their definitions given in the introduction.

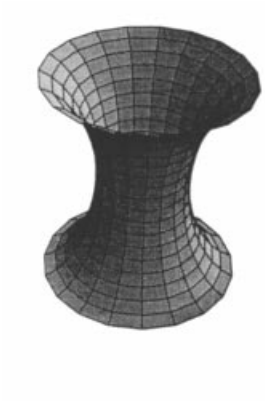


Fig. 1. $A^2_{\sqrt{5}/2}$.



Fig. 2. $B^2_{\sqrt{5}}$.

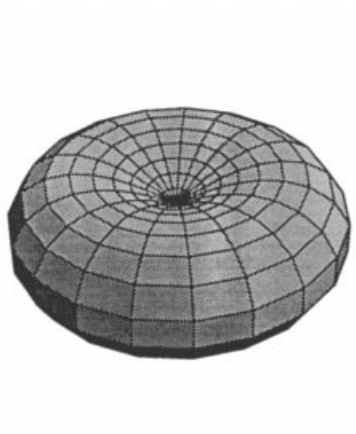


Fig. 3. $C^2_{\sqrt{2}}$.

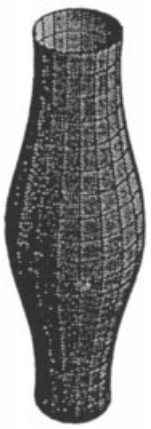


Fig. 4. $D^2_{\frac{1}{2}}$.

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Fig. 5. G^2.



Fig. 6. H_1^2.



Fig. 7. L^2.

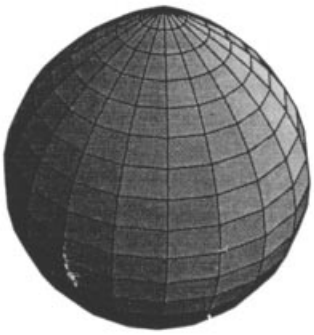


Fig. 8. $P_{\sqrt{2}}^2$.

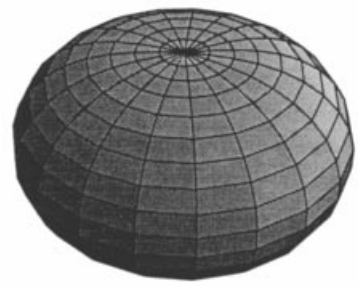


Fig. 9. $W_{\sqrt{2}}^2$.

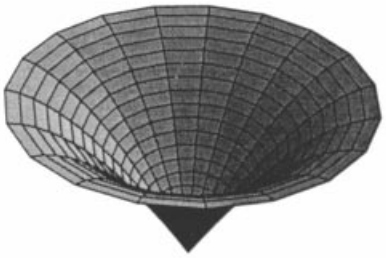


Fig. 10. $Y_{\sqrt{3}/2}^2$.

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