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# Global fixed points for nilpotent actions on the torus

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*Abstract*. An isotopic to the identity map of the 2-torus, that has zero rotation vector with respect to an invariant ergodic probability measure, has a fixed point by a theorem of Franks. We give a version of this result for nilpotent subgroups of isotopic to the identity diffeomorphisms of the 2-torus. In such a context we guarantee the existence of global fixed points for nilpotent groups of irrotational diffeomorphisms. In particular, we show that the derived group of a nilpotent group of isotopic to the identity diffeomorphisms of the 2-torus has a global fixed point.

Key words: rotation vector, global fixed point, irrotational homeomorphism, ergodic probability measure, nilpotent group
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## 1. *Introduction*

Abelian groups of isotopic to the identity $C^1$ diffeomorphisms of a closed oriented surface $S$ have global fixed points when the Euler characteristic of $S$ is different from zero. When $S$ is the 2-sphere, it is also assumed that the group is generated by diffeomorphisms $C^0$-close to the identity map of $S$.

In these cases the existence of global fixed points is essentially imposed by the topology of $S$ and its interplay with the dynamics of nilpotent (or abelian) actions. Several papers have dealt with this question such as [**3–5**, **10**, **14**, **15**, **18**] for the abelian cases and [**9**, **12**, **26**] for the nilpotent ones.

The starting point for this kind of problem has been the articles of Lima in 1964 [**21**] and Plante in 1986 [**25**], where they proved the existence of global fixed points for the action of abelian and nilpotent connected Lie groups, respectively, on surfaces with Euler characteristic different from zero.

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When $S$ is the 2-torus $\mathbb{T}^2$, we need to assume more conditions on the group, other than the nilpotent property, to guarantee the existence of global fixed points.

In a recent paper, *Finite orbits for nilpotent actions on the torus* (cf. [**11**]), we studied the existence of finite orbits for the action assuming an hypothesis of topological type on the group. More precisely, we showed that if a nilpotent group of $C^1$ diffeomorphisms of the 2-torus has an element with non-vanishing Lefschetz number, then it has a finite orbit. In this article we follow a different approach, considering hypotheses of dynamical nature on the group that are inspired by a result of Franks (cf. Theorem 1.1).

To approach this question, let us consider a homeomorphism $\phi$ of the torus $\mathbb{T}^2$ that belongs to the connected component of the identity, $\mathrm{Homeo}_0(\mathbb{T}^2)$, of $\mathrm{Homeo}(\mathbb{T}^2)$. Let $\tilde{\phi}$ be a lift of $\phi$ to the universal covering $\mathbb{R}^2$ of $\mathbb{T}^2$. Following [**22**], the rotation vectors of $\tilde{\phi}$ are the limits of sequences of the form

$$\frac{\tilde{\phi}^{n_k}(\tilde{x}_k) - \tilde{x}_k}{n_k},$$

where $(n_k)_{k\geq 1}$ is an increasing sequence of positive integers and $(\tilde{x}_k)_{k\geq 1}$ is a sequence of points in $\mathbb{R}^2$. This set will be denoted by $\rho(\tilde{\phi})$. It follows from [**22**] that it is a non-empty compact and convex subset of $\mathbb{R}^2$.

Since we are working in the torus $\mathbb{T}^2$, we have that $\rho(\tilde{\phi}) = \{\rho_\mu(\tilde{\phi});\ \mu \in \mathcal{P}(\phi)\}$ [**22**], where $\mathcal{P}(\phi)$ is the set of $\phi$-invariant Borel probability measures and

$$\rho_\mu(\tilde{\phi}) := \int_{\mathbb{T}^2} (\tilde{\phi} - \mathrm{Id})\, d\mu.$$

Since $\phi$ belongs to $\mathrm{Homeo}_0(\mathbb{T}^2)$, it follows that $\tilde{\phi}$ commutes with the covering transformations and $\tilde{\phi} - \mathrm{Id}$ is a well-defined map on $\mathbb{T}^2$. The set $\rho(\tilde{\phi})$ depends on the lift $\tilde{\phi}$ of $\phi$ and satisfies $\rho_\mu(T_v \circ \tilde{\phi}) = T_v(\rho_\mu(\tilde{\phi}))$ for any $\mu \in \mathcal{P}(\phi)$, where $T_v$ is the translation in $\mathbb{R}^2$ by the vector $v \in \mathbb{Z}^2$. In particular, the projection $\rho(\phi)$ of $\rho(\tilde{\phi})$ in $\mathbb{R}^2/\mathbb{Z}^2$ does not depend on the lift $\tilde{\phi}$ of $\phi$.

THEOREM 1.1. (Franks [**13**]) *Let $\phi \in \mathrm{Homeo}_0(\mathbb{T}^2)$. Let $\mu$ be an ergodic $\phi$-invariant Borel probability measure such that the rotation vector $\rho_\mu(\phi)$ is the zero element of $\mathbb{R}^2/\mathbb{Z}^2$. Then $\phi$ has a fixed point.*

The next result is a generalization of this theorem in the context of nilpotent groups of diffeomorphisms.

*Conventions*. From now on, we make the following conventions. A homeomorphism $\tilde{\psi} \in \mathrm{Homeo}(\mathbb{R}^2)$ always denotes a lift to the universal covering of $\psi \in \mathrm{Homeo}(\mathbb{T}^2)$ and vice versa. Moreover, $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ denotes the universal covering map and unless explicitly stated otherwise a lift means a lift to the universal covering.

THEOREM 1.2. *Let $G = \langle H, \phi \rangle$ be a nilpotent subgroup of $\mathrm{Diff}^1_0(\mathbb{T}^2)$, where $H$ is a normal subgroup of $G$. Suppose that there exists a $\phi$-invariant ergodic probability measure $\mu$ such that the support of $\mu$ is contained in $\mathrm{Fix}(H)$ and $\rho_\mu(\phi)$ is the zero element in $\mathbb{R}^2/\mathbb{Z}^2$. Then $G$ has a global fixed point.*

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We denote by $\langle H, \phi\rangle$ the group generated by $H$ and $\phi$. Moreover, $\mathrm{Fix}(\phi)$ denotes the set of fixed points of $\phi$. We define $\mathrm{Fix}(L) = \bigcap_{f\in L} \mathrm{Fix}(f)$ as the set of global fixed points of a set $L$ of transformations. By definition, $\mathrm{Diff}_0^1(\mathbb{T}^2)$ is the group of $C^1$ diffeomorphisms of $\mathbb{T}^2$ which are $C^0$-isotopic to the identity.

Let us remark that the fixed point detected by the theorem of Franks is not necessarily contained in the support of the invariant ergodic measure. However, if the homeomorphism has a unique fixed point, then it belongs to the aforementioned support. This follows from the next two remarks. If $\phi$ has a unique fixed point $p \in \mathbb{T}^2$, then the fixed-point index of $\phi$ at $p$ is zero by the Lefschetz fixed-point theorem. In this case, a result of Schmitt in [**27**] guarantees that we can perturb $\phi$ in an arbitrarily small neighborhood of $p$ to obtain $\psi \in \mathrm{Homeo}_0(\mathbb{T}^2)$ with no fixed points.

Of course, from the remarks above, it is simple to conclude the existence of a global fixed point for $\langle H, \phi\rangle$ in Theorem 1.2 when $\phi$ has a unique fixed point even in the case where $\langle H, \phi\rangle \subset \mathrm{Homeo}_0(\mathbb{T}^2)$.

The previous theorem guarantees, in particular, that two commuting elements $\phi$ and $\psi$ of $\mathrm{Diff}_0^1(\mathbb{T}^2)$ share a common fixed point if there exists a $\phi$-invariant ergodic probability measure $\mu$ whose support is contained in $\mathrm{Fix}(\psi)$ and $\rho_\mu(\phi) = 0 \in \mathbb{R}^2/\mathbb{Z}^2$. Notice that the original Franks' theorem holds true for homeomorphisms but we need the differentiability since we use a Thurston decomposition *à la* Franks, Handel and Parwani [**14**].

We say that an element $\phi$ of $\mathrm{Homeo}_0(\mathbb{T}^2)$ is *irrotational* if there exists a lift $\tilde{\phi} \in \mathrm{Homeo}_0(\mathbb{R}^2)$ of $\phi$ that has $(0, 0)$ as its unique rotation vector, i.e. $\rho_\mu(\tilde{\phi}) = (0, 0)$ for any $\mu \in \mathcal{P}(\phi)$. In that case $\tilde{\phi}$ is unique and we say that it is the *irrotational lift* of $\phi$. A subgroup $G$ of $\mathrm{Homeo}_0(\mathbb{T}^2)$ is irrotational if every one of its elements is irrotational.

The result of Franks [**13**] implies, in particular, that the irrotational elements of the group $\mathrm{Homeo}_0(\mathbb{T}^2)$ have fixed points. Following the context of Franks and of Theorem 1.2, it is natural to study the existence of global fixed points for irrotational nilpotent subgroups of $\mathrm{Diff}_0^1(\mathbb{T}^2)$. We solve this problem in the next theorem. It will follow as a consequence of Theorem 1.2 and simple well-known properties of nilpotent groups.

THEOREM 1.3. *Let $G$ be an irrotational nilpotent subgroup of* $\mathrm{Diff}_0^1(\mathbb{T}^2)$. *Then $G$ has a global fixed point.*

Let us remark that the existence of fixed points for each element of a nilpotent subgroup $G$ of $\mathrm{Diff}_0^1(\mathbb{T}^2)$ is not sufficient to guarantee a global fixed point for $G$. Indeed, consider the set $S$ consisting of the real analytic functions $a : \mathbb{R} \to \mathbb{R}$ such that $a(x + 1) \equiv a(x)$ and $\int_0^1 a(x)\,dx = 0$. We define the subgroup

$$\mathcal{G} = \{h(x, y) = (x, y + a(x)) : a \in S\}$$

of the group of real analytic diffeomorphisms of $\mathbb{R}^2$. Since any element of $\mathcal{G}$ commutes with $(x + 1, y)$ and $(x, y + 1)$ by construction we can consider the subgroup $G$ of isotopic to the identity diffeomorphisms of $\mathbb{T}^2$ induced by $\mathcal{G}$. The groups $\mathcal{G}$ and $G$ are abelian. Any element of $G$ (and any element of $\mathcal{G}$) has a fixed point since every function in $S$ has a zero. The orbit of an element of $\mathbb{T}^2$ by the action of the group $G$ is always a circle and hence $G$ has no global fixed points. In Theorem 1.3, we exploit the fixed points given by the irrotational nature of the elements of $G$.

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The dynamics of irrotational homeomorphisms has been studied by Koropecki and Tal (cf. [**20**, Theorem A]) in the area-preserving case.

Motivated by Theorem 1.3, it is natural to search for examples of irrotational groups. The next theorem illustrates that irrotational groups appear naturally in the context of nilpotent groups.

THEOREM 1.4. *If $G$ is a nilpotent subgroup of* $\mathrm{Homeo}(\mathbb{T}^2)$*, then*:
(i) $G' := [G, G]$ *is irrotational when* $G \subset \mathrm{Homeo}_0(\mathbb{T}^2)$;
(ii) $G'' := [G', G']$ *is irrotational when* $G \subset \mathrm{Homeo}_+(\mathbb{T}^2)$;
(iii) $G''' := [G'', G'']$ *is always irrotational.*

The group $\mathrm{Homeo}_+(\mathbb{T}^2)$ is the subgroup of $\mathrm{Homeo}(\mathbb{T}^2)$ formed by the orientation-preserving elements. Theorems 1.4 and 1.3 provide immediately the following result.

THEOREM 1.5. *Let $G$ be a nilpotent subgroup of* $\mathrm{Diff}^1(\mathbb{T}^2)$. *The following subgroups of $G$ have a global fixed point*:
(i) $G'$ *when* $G \subset \mathrm{Diff}^1_0(\mathbb{T}^2)$;
(ii) $G''$ *when* $G \subset \mathrm{Diff}^1_+(\mathbb{T}^2)$;
(iii) $G'''$.

Given a subgroup $G \subset \mathrm{Homeo}(\mathbb{T}^2)$, we always denote by $G_0$ the subgroup of isotopic to the identity elements of $G$ and we define the sets

$$\mathcal{I}_G := \{\phi \in G_0;\ \exists \text{ a lift } \tilde{\phi} \text{ s.t. } \rho_\mu(\tilde{\phi}) = (0, 0) \text{ for all } \mu \in \mathcal{P}(\phi)\}$$

and

$$G_{\mathcal{I}} := \{\phi \in G_0;\ \exists \text{a lift } \tilde{\phi} \text{ s.t. } \rho_\mu(\tilde{\phi}) = (0, 0) \text{ for all } \mu \in \mathcal{P}(G)\},$$

where $\mathcal{P}(G) = \bigcap_{\psi \in G} \mathcal{P}(\psi)$. Of course, we are following our convention that $\tilde{\phi}$ is a lift of $\phi \in \mathrm{Homeo}(\mathbb{T}^2)$ to the universal covering. The set $\mathcal{I}_G$ consists of the irrotational elements of $G$, whereas $G_{\mathcal{I}}$ is the set of elements of $G$ that are irrotational with respect to $G$-invariant measures. It is clear that $\mathcal{I}_G$ is contained in $G_{\mathcal{I}}$. Moreover, $G_{\mathcal{I}}$ is a normal subgroup of $G$ by Lemmas 2.2 and 2.3.

Given a surface $S$, we fix the notation $\mathrm{Homeo}(S)$, $\mathrm{Homeo}_+(S)$ and $\mathrm{Homeo}_0(S)$ with similar meanings as we have done previously for the cases $S = \mathbb{T}^2$ and $S = \mathbb{R}^2$. We do the same with the notation $\mathrm{Diff}^1(S)$, $\mathrm{Diff}^1_+(S)$ and $\mathrm{Diff}^1_0(S)$ for sets of $C^1$ diffeomorphisms of $S$.

Let $G$ be a subgroup of $\mathrm{Homeo}(S)$ and let $\tau : \tilde{S} \to S$ be a covering map. We say that a subgroup $G_\tau$ of $\mathrm{Homeo}(\tilde{S})$ is a *$\tau$-lift* of $G$ if each element $\phi_\tau \in G_\tau$ is a $\tau$-lift of some element $\phi \in G$ and the natural projection $\kappa : G_\tau \to G$ defined by $\kappa(\phi_\tau) = \phi$ is an isomorphism of groups.

In the study of nilpotent groups of homeomorphisms of the 2-torus, the irrotational elements play a similar role as the isotopic to the identity homeomorphisms in closed orientable surfaces $S$ with negative Euler characteristic. In both cases there exists a privileged lift, namely the irrotational lift and the *identity lift*, respectively, for irrotational and isotopic to the identity homeomorphisms.

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Let us remind the reader that the negative Euler characteristic hypothesis on $S$ guarantees that $\mathrm{Homeo}_0(S)$ is homotopically trivial (cf. [**17**]) and we have that the *identity lift* $\tilde{\phi}$ of $\phi \in \mathrm{Homeo}_0(S)$ is the unique lift of $\phi$ to the universal covering of $S$ that commutes with all covering transformations. It is obtained by lifting, to the universal covering $\tilde{S}$ of $S$, any isotopy from $\mathrm{Id}_S$ to $\phi$ starting at $\mathrm{Id}_{\tilde{S}}$. The set $\tilde{\mathcal{I}}_{\mathrm{Homeo}_0(S)}$ of identity lifts of all elements in $\mathrm{Homeo}_0(S)$ is a normal subgroup of the group of all lifts of elements of $\mathrm{Homeo}(S)$. Analogously, the set $\tilde{\mathcal{I}}_{\mathrm{Homeo}_0(\mathbb{T}^2)}$, of irrotational lifts of elements of $\mathcal{I}_{\mathrm{Homeo}_0(\mathbb{T}^2)}$, is not a group but it is a *normal subset* of the group of all lifts of elements of $\mathrm{Homeo}(\mathbb{T}^2)$ in the following sense: $\tilde{\psi} \circ \tilde{\phi} \circ \tilde{\psi}^{-1} \in \tilde{\mathcal{I}}_{\mathrm{Homeo}_0(\mathbb{T}^2)}$ if $\tilde{\phi} \in \tilde{\mathcal{I}}_{\mathrm{Homeo}_0(\mathbb{T}^2)}$ and $\tilde{\psi}$ is a lift of any $\psi \in \mathrm{Homeo}(\mathbb{T}^2)$ to the universal covering of $\mathbb{T}^2$. This property is a consequence of Lemma 2.2. Similarly, the set $\mathcal{I}_{\mathrm{Homeo}_0(\mathbb{T}^2)}$ has the same property.

Anyway, we prove that even if $\mathcal{I}_G$ is not a group for a general subgroup $G$ of the group $\mathrm{Homeo}(\mathbb{T}^2)$, it is always a normal subgroup of $G$ when $G$ is a nilpotent subgroup of $\mathrm{Homeo}(\mathbb{T}^2)$. A natural strategy in order to show this result is to compare $\mathcal{I}_G$ with $G_{\mathcal{I}}$. More precisely, we compare the set of rotation vectors of a lift $\tilde{\phi}$ of $\phi \in G_0$ with the set $\{\rho_\mu(\tilde{\phi});\ \mu \in \mathcal{P}(G)\}$ of rotation vectors with respect to $G$-invariant measures. In such context, we prove the following result.

THEOREM 1.6. *Let $G$ be a nilpotent subgroup of* $\mathrm{Homeo}(\mathbb{T}^2)$. *Then there exists a finite-index normal subgroup $H$ of $G$ containing $G_0$ such that*

$$\{\rho_\mu(\tilde{\phi});\ \mu \in \mathcal{P}(\phi)\} = \{\rho_\nu(\tilde{\phi});\ \nu \in \mathcal{P}(H)\}$$

*for any lift $\tilde{\phi}$ of every $\phi \in G_0$. In particular, the set $\mathcal{I}_G$ of irrotational elements of $G$ is a normal subgroup of $G$. Moreover, if $G = G_0$, then $\mathcal{I}_G = G_{\mathcal{I}}$.*

It is possible to choose the group $H$ such that $G/H$ is equal to a subgroup of $C_6$ or a subgroup of $D_4$, where $C_n$ is the cyclic group with $n$ elements and $D_m$ is the dihedral group with $2m$ elements.

From Theorems 1.6, 1.3 and 1.4, we easily conclude the following corollary.

COROLLARY 1.7. *Let $G$ be a nilpotent subgroup of* $\mathrm{Diff}^1_0(\mathbb{T}^2)$. *The set of irrotational elements of $G$ is a normal subgroup of $G$ that contains $G'$ and has a global fixed point.*

Other results comparing the rotational properties of the elements of a nilpotent subgroup of $\mathrm{Homeo}(\mathbb{T}^2)$ with the rotational properties with respect to measures in $\mathcal{P}(G)$ are introduced in §4 (cf. Propositions 4.7, 4.8 and 4.9).

## 2. *Preliminaries*

Let $G$ be a group and $H$ be a subgroup of $G$. We denote by $[H, G]$ the subgroup of $G$ generated by the elements of the form $[h, g] = hgh^{-1}g^{-1}$, where $h \in H$ and $g \in G$. If $H$ is a normal subgroup of $G$, then $[H, G]$ is a subgroup of $H$ which is normal in $G$.

Given a group $G$, let us consider the *upper central series* $\{Z^{(n)}(G)\}_{n\geq 0}$ of $G$:

$$Z^{(n+1)}(G) := \{g \in G;\ [g, f] \in Z^{(n)}(G) \text{ for all } f \in G\},$$

where $Z^{(0)}(G)$ is the trivial subgroup of $G$. The members of the upper central series are characteristic subgroups of $G$. In particular, they are normal subgroups of $G$ and we have

$$Z^{(0)}(G) \subset Z^{(1)}(G) \subset \cdots \subset Z^{(n)}(G) \subset \cdots \subset G.$$

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If $Z^{(n)}(G) = G$ for some $n \in \mathbb{Z}_{\geq 0}$, we say that $G$ is a *nilpotent group*. The smallest $n \in \mathbb{Z}_{\geq 0}$ such that $Z^{(n)}(G) = G$ is the *nilpotency class* of $G$.

The focus of this paper is the study of nilpotent groups of homeomorphisms and diffeomorphisms of the 2-torus.

*Definition 2.1.* Let $[\,] : \mathrm{Homeo}(\mathbb{T}^2) \to \mathrm{MCG}(\mathbb{T}^2) = \mathrm{GL}(2, \mathbb{Z})$ be the map associating to an element of $\mathrm{Homeo}(\mathbb{T}^2)$ its image in the mapping class group of $\mathbb{T}^2$. Given a subgroup $G$ of $\mathrm{Homeo}(\mathbb{T}^2)$, we denote by $[G]$ the image of $G$ by $[\,]$.

In what follows, we will be using frequently the next two well-known lemmas.

LEMMA 2.2. *Let $\phi \in \mathrm{Homeo}_0(\mathbb{T}^2)$, $\mu \in \mathcal{P}(\phi)$ and $\psi \in \mathrm{Homeo}(\mathbb{T}^2)$. For all lifts to the universal covering $\tilde{\phi}$ and $\tilde{\psi}$ of $\phi$ and $\psi$, respectively, we have*

$$\tilde{\psi} \circ T_v = T_{[\psi](v)} \circ \tilde{\psi} \quad \text{and} \quad [\psi](\rho_\mu(\tilde{\phi})) = \rho_\nu(\tilde{\psi} \circ \tilde{\phi} \circ \tilde{\psi}^{-1}),$$

*where $v \in \mathbb{Z}^2$ and $\nu = \psi_*(\mu)$.*

LEMMA 2.3. *Consider the subgroup $\mathrm{Homeo}_{0,\mu}(\mathbb{T}^2)$ of $\mathrm{Homeo}_0(\mathbb{T}^2)$ whose elements preserve a probability measure $\mu$ and let $\mathrm{Homeo}_{0,\mu}(\mathbb{R}^2)$ be the subgroup of $\mathrm{Homeo}_0(\mathbb{R}^2)$ consisting of all the lifts of elements of $\mathrm{Homeo}_{0,\mu}(\mathbb{T}^2)$. Then the map*

$$\rho_\mu : \mathrm{Homeo}_{0,\mu}(\mathbb{R}^2) \to \mathbb{R}^2$$

*is a morphism of groups.*

Now we introduce three results proved by the second author in [**26**] that will be used to find global fixed points for nilpotent groups of torus diffeomorphisms.

THEOREM 2.4. *Let $G \subset \mathrm{Diff}^1_+(\mathbb{R}^2)$ be a finitely generated nilpotent subgroup that preserves a non-empty compact set. Then $G$ has a global fixed point.*

THEOREM 2.5. *Let $G \subset \mathrm{Diff}^1_+(\mathbb{R}^2)$ be a nilpotent subgroup such that $\mathrm{Fix}(\phi)$ is a non-empty compact set for some $\phi \in G$. Then $G$ has a global fixed point.*

The next result is a Thurston decomposition for a normal subgroup of a nilpotent group. It is inspired by similar results given in [**14**] for the abelian case.

PROPOSITION 2.6. *Let $G$ be a finitely generated nilpotent subgroup of $\mathrm{Diff}_+(S)$, where $S$ is a finitely punctured surface. Consider a normal subgroup $H$ of $G$ and a compact $G$-invariant set $K$ such that $K \subset \mathrm{Fix}(H)$. Denote $M = S \setminus K$ and suppose that the Euler characteristic $\chi(M)$ is negative if $K$ is finite. Then there exist a finite family $\mathcal{R}$ of pairwise disjoint, pairwise non-homotopic simple closed curves in $M$ and a family of pairwise disjoint open annuli $\mathcal{A}$ in $M$ of the same cardinal as the family $\mathcal{R}$ such that every curve in $\mathcal{R}$ is a core curve of exactly an annulus in $\mathcal{A}$ and given $\phi \in G$ there exists $\theta_\phi \in \mathrm{Homeo}_+(S)$ such that $\phi$ is isotopic to $\theta_\phi$ relative to $K$ and $\theta_\phi(\mathcal{A}) = \mathcal{A}$. Moreover, given a connected component $X$ of $S \setminus \mathcal{A}$, we have:*

- $(\theta_\phi)_{|X} \equiv \mathrm{Id}$ *if* $\sharp(X \cap K) = \infty$ *and* $\phi \in H$;
- $\chi(X \setminus K) < 0$ *if $X \cap K$ is finite. Moreover, the set $\{(\theta_\phi)_{|X \setminus K} : \phi \in J\}$, where $J \subset G$ is the stabilizer of $X$ modulo isotopy relative to $K$, is a group of pseudo-Anosov and finite-order elements that is virtually cyclic and nilpotent.*

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The result is obtained by considering the subgroups $H_1 = H$ and $H_2 = G$ and the sets $K_1 = K$ and $K_2 = \emptyset$ in [**26**, Proposition 3.1].

## 3. *Rotational properties*

In the introduction, we say that a subgroup $G$ of $\mathrm{Homeo}_0(\mathbb{T}^2)$ is irrotational when every element $\phi \in G$ is irrotational with respect to $\mathcal{P}(\phi)$, i.e. there exists a lift $\tilde{\phi} \in \mathrm{Homeo}_0(\mathbb{R}^2)$ of $\phi$ such that $\rho_\mu(\tilde{\phi}) = (0, 0)$ for all $\mu \in \mathcal{P}(\phi)$. Following the definition of $\mathcal{I}_G$, we have that a subgroup $G \subset \mathrm{Homeo}_0(\mathbb{T}^2)$ is irrotational when $G = \mathcal{I}_G$.

Now let us introduce a new definition of irrotational subgroups of $\mathrm{Homeo}_0(\mathbb{T}^2)$. We say that a subgroup $G \subset \mathrm{Homeo}_0(\mathbb{T}^2)$ is $\mathcal{P}(G)$-*irrotational* when every element $\phi \in G$ is irrotational with respect to $\mathcal{P}(G)$, i.e. there exists a lift $\tilde{\phi} \in \mathrm{Homeo}_0(\mathbb{R}^2)$ of $\phi$ such that $\rho_\mu(\tilde{\phi}) = (0, 0)$ for all $\mu \in \mathcal{P}(G)$. Following the definition of $G_\mathcal{I}$, we have that a subgroup $G \subset \mathrm{Homeo}_0(\mathbb{T}^2)$ is $\mathcal{P}(G)$-irrotational when $G = G_\mathcal{I}$.

These two concepts of irrotationality could be a priori different since given $\phi \in G$ the set $\mathcal{P}(\phi)$ can be much bigger than $\mathcal{P}(G)$. Later on, we will see that if $G$ is a nilpotent subgroup of $\mathrm{Homeo}_0(\mathbb{T}^2)$, the definitions coincide (cf. Proposition 4.8).

Now we introduce some notation. Let $G$ be a subgroup of $\mathrm{Homeo}(\mathbb{T}^2)$ and suppose that $\mu \in \mathcal{P}(G)$. Following the convention about $\tilde{\phi}$ and $\phi$, we define the following sets:

$$\begin{aligned}
G_\mathcal{I}^\mu &:= \{\phi \in G_0;\ \exists \text{ a lift } \tilde{\phi} \text{ s.t. } \rho_\mu(\tilde{\phi}) = (0, 0)\},\\
\tilde{G}_\mathcal{I}^\mu &:= \{\tilde{\phi} \in \mathrm{Homeo}_0(\mathbb{R}^2);\ \phi \in G_0 \text{ and } \rho_\mu(\tilde{\phi}) = (0, 0)\},\\
\tilde{G}_\mathcal{I} &:= \{\tilde{\phi} \in \mathrm{Homeo}_0(\mathbb{R}^2);\ \phi \in G_0 \text{ and } \rho_\nu(\tilde{\phi}) = (0, 0) \text{ for all } \nu \in \mathcal{P}(G)\}\\
&= \bigcap_{\nu \in \mathcal{P}(G)} \tilde{G}_\mathcal{I}^\nu.
\end{aligned}$$

Since the map $\rho_\mu : \mathrm{Homeo}_{0,\mu}(\mathbb{R}^2) \to \mathbb{R}^2$ is a morphism, it follows that $\tilde{G}_\mathcal{I}^\mu$, $\tilde{G}_\mathcal{I}$ (respectively $G_\mathcal{I}^\mu$) are subgroups of $\mathrm{Homeo}_0(\mathbb{R}^2)$ (respectively $\mathrm{Homeo}_0(\mathbb{T}^2)$). Moreover, we have that $\tilde{G}_\mathcal{I}$ and $\tilde{G}_\mathcal{I}^\mu$ are lifts of $G_\mathcal{I}$ and $G_\mathcal{I}^\mu$, respectively, since the natural projections $\tilde{\phi} \in \tilde{G}_\mathcal{I}^\mu \xrightarrow{\kappa} \phi \in G_\mathcal{I}^\mu$ and $\tilde{\phi} \in \tilde{G}_\mathcal{I} \xrightarrow{\kappa} \phi \in G_\mathcal{I}$ are isomorphisms.

Clearly, we also have $\mathcal{I}_G \subset G_\mathcal{I} \subset G_\mathcal{I}^\mu$ for any $\mu \in \mathcal{P}(G)$. Notice that $\mathcal{P}(G)$ is non-empty if $G$ is an amenable group.

In [**11**], we related the rotational properties of a nilpotent group with the existence of lifts and we proved the following proposition that will be used in the next section. The condition $1 \notin \mathrm{spec}[\psi]$ in the next proposition is equivalent to the non-vanishing of the Lefschetz number $L(\psi)$ (cf. [**11**, Lemma 1.2] or [**6**]).

PROPOSITION 3.1. *Let $G$ be a nilpotent subgroup of* $\mathrm{Homeo}(\mathbb{T}^2)$. *Suppose that there exists $\psi \in G$ with $1 \notin \mathrm{spec}[\psi]$. Then we obtain $\tilde{G}_\mathcal{I} = \tilde{G}_\mathcal{I}^\mu$ and $G_\mathcal{I} = G_\mathcal{I}^\mu$ for any $\mu \in \mathcal{P}(G)$. Moreover $G_\mathcal{I}$ is a finite-index normal subgroup of $G_0$.*

Nilpotent subgroups of $\mathrm{Homeo}(\mathbb{T}^2)$ induce nilpotent subgroups of the mapping class group of $\mathbb{T}^2$, i.e. nilpotent subgroups of $\mathrm{GL}(2, \mathbb{Z})$. We will need a classification of such groups in order to study rotational properties. They are virtually cyclic and metabelian. Moreover, there exists a unique example of a non-abelian group, up to conjugacy.

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LEMMA 3.2. *Let $\mathcal{G}$ be a nilpotent subgroup of* $\mathrm{MCG}(\mathbb{T}^2)$. *Then $\mathcal{G}$ is either of the form $\langle N\rangle$ or $\langle N, -N\rangle$ for some $N \in \mathcal{G}$, or it is conjugated by a matrix in* $\mathrm{GL}(2,\mathbb{Q})$ *to the group*

$$\mathcal{H} := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right\}.$$

The group $\mathcal{H}$ is isomorphic to the dihedral group $D_4$. We admit subgroups of $\mathrm{MCG}(\mathbb{T}^2)$ that contain orientation-reversing classes. Indeed, $\mathcal{H}$ contains orientation-reversing classes. Hence, a nilpotent subgroup of $\mathrm{MCG}(\mathbb{T}^2)$ consisting of orientation-preserving classes is always abelian. We presented a proof of the above lemma in the Appendix of [**11**].

## 4. *Construction of a G-invariant measure*

Let $G \subset \mathrm{Homeo}(\mathbb{T}^2)$ be a nilpotent subgroup. In this section, we want to present a description for the set $\mathcal{I}_G$ of irrotational elements of $G$. In particular, we will show that $\mathcal{I}_G$ is a subgroup of $G$. The main difficulty to prove this is that the sets $\mathcal{P}(f)$ and $\mathcal{P}(g)$ can be very different for $f, g \in G$. In order to overcome this problem, we do the following: given $\phi \in G_0$ and $\mu \in \mathcal{P}(\phi)$, we construct, under certain natural conditions, a measure $\mu' \in \mathcal{P}(G)$ such that $\rho_\mu(\tilde{\phi}) = \rho_{\mu'}(\tilde{\phi})$ for any lift $\tilde{\phi}$ of $\phi$. This property can be interpreted as some rigidity property of nilpotent groups since a property that is satisfied for one element induces an analogous global property.

Let us begin this section with a simple case, corresponding to groups of isotopic to the identity homeomorphisms of the 2-torus. First we prove Theorem 1.4.

*Theorem 1.4. If $G$ is a nilpotent subgroup of* $\mathrm{Homeo}(\mathbb{T}^2)$, *then:*
(i) $G' := [G, G]$ *is irrotational when* $G \subset \mathrm{Homeo}_0(\mathbb{T}^2)$;
(ii) $G'' := [G', G']$ *is irrotational when* $G \subset \mathrm{Homeo}_+(\mathbb{T}^2)$;
(iii) $G''' := [G'', G'']$ *is always irrotational.*

In particular, there are non-trivial irrotational elements in the nilpotent subgroups of $\mathrm{Homeo}(\mathbb{T}^2)$ except in simple cases. For instance, if $G \subset \mathrm{Homeo}_0(\mathbb{T}^2)$ is nilpotent, then either $\mathcal{I}_G$ is non-trivial or $G$ is abelian.

*Proof of Theorem 1.4.* Suppose that $G$ is a subgroup of $\mathrm{Homeo}_0(\mathbb{T}^2)$. Furthermore, suppose that $G$ finitely generated. Let us remark from [**2**, **23**] that the elements $\phi \in G'$ grow sublinearly with respect to the word metric in the following sense: $|\phi^n| = O(n^{1/2})$, where $|\phi^n|$ denotes the distance from $\phi^n$ to the identity with respect to the word metric for a given finite generating set of $G$. In particular, $\phi$ is a distortion element, i.e. the sequence $(|\phi^n|/n)_{n\geq 1}$ goes to zero when $n$ tends to infinity. Moreover, if $\phi^n = h_{i_1} \circ h_{i_2} \circ \cdots \circ h_{i_m}$, where $h_1, \ldots, h_s$ are generators of $G$, we choose lifts $\tilde{h}_1, \ldots, \tilde{h}_s, \tilde{\phi}$ and we define $\tilde{\psi} = \tilde{h}_{i_1} \circ \cdots \circ \tilde{h}_{i_m}$. We obtain

$$\begin{aligned}(\tilde{\phi}^n(\tilde{x}) - \tilde{x}) - (\tilde{\phi}^n(\tilde{y}) - \tilde{y}) &= (\tilde{\psi}(\tilde{x}) - \tilde{x}) - (\tilde{\psi}(\tilde{y}) - \tilde{y}) \\ &= \sum_{j=1}^{m} [(\tilde{h}_{i_j} - \mathrm{Id}) \circ \tilde{h}_{i_{j+1}} \circ \cdots \circ \tilde{h}_{i_m}](\tilde{x}) - \sum_{j=1}^{m} [(\tilde{h}_{i_j} - \mathrm{Id}) \circ \tilde{h}_{i_{j+1}} \circ \cdots \circ \tilde{h}_{i_m}](\tilde{y}).\end{aligned}$$

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We deduce that $\|(\tilde{\phi}^n(\tilde{x}) - \tilde{x}) - (\tilde{\phi}^n(\tilde{y}) - \tilde{y})\| \leq 2M|\phi^n|$ for all $\tilde{x}, \tilde{y} \in \mathbb{R}^2$, where $M = \max_{1\leq i\leq s,\ \tilde{z}\in\mathbb{R}^2} \|\tilde{h}_i(\tilde{z}) - \tilde{z}\|$ and $\|\cdot\|$ denotes the usual norm in $\mathbb{R}^2$. Then we obtain

$$\lim_{n\to\infty} \max_{\tilde{x},\tilde{y}} \frac{\|(\tilde{\phi}^n(\tilde{x}) - \tilde{x}) - (\tilde{\phi}^n(\tilde{y}) - \tilde{y})\|}{n} = 0. \tag{4.0.1}$$

Let $\nu$ be a $\phi$-invariant ergodic Borel probability measure. Birkhoff's ergodic theorem implies that $\lim_{n\to\infty}((\tilde{\phi}^n(\tilde{x}) - \tilde{x})/n) = \rho_\nu(\tilde{\phi})$ $\nu$-almost everywhere. Moreover, $((\tilde{\phi}^n(\tilde{x}) - \tilde{x})/n$ converges uniformly to $\rho_\nu(\tilde{\phi})$ when $n \to \infty$ by equation (4.0.1). In particular, we obtain $\rho(\tilde{\phi}) = \{\rho_\nu(\tilde{\phi})\}$.

Suppose now that $\phi$ is an element of $G'$, where $G$ is not necessarily finitely generated. By definition of $G'$, the diffeomorphism $\phi$ is of the form $(a_1b_1a_1^{-1}b_1^{-1}) \ldots (a_nb_na_n^{-1}b_n^{-1})$, where $a_1, \ldots, a_n, b_1, \ldots, b_n$ are elements of $G$ and hence $\phi$ is in the derived group of the finitely generated subgroup $\langle a_1, \ldots, a_n, b_1, \ldots, b_n\rangle$ of $G$. The previous discussion implies that $\rho(\tilde{\phi})$ is a singleton for any $\phi \in G'$.

Since $G$ is nilpotent, there exists a probability measure $\mu$ that is invariant by all the elements of $G$. Lemma 2.3 implies that the map $\rho_\mu : \mathrm{Homeo}_{0,\mu}(\mathbb{R}^2) \to \mathbb{R}^2$ is a morphism of groups. The restriction of $\rho_\mu$ to the derived group of $\mathrm{Homeo}_{0,\mu}(\mathbb{R}^2)$ is a vanishing map and hence there exists a unique lift $\tilde{\phi}$ such that $\rho_\mu(\tilde{\phi}) = (0, 0)$ for any $\phi \in G'$. Since the rotation set $\rho(\tilde{\phi})$ contains exactly one point, we deduce that $\rho(\tilde{\phi}) = \{(0, 0)\}$. In particular, $\phi$ is irrotational and $\tilde{\phi}$ is its irrotational lift.

Suppose that $G \subset \mathrm{Homeo}_+(\mathbb{T}^2)$. In this case the group $[G]$ is abelian by Lemma 3.2. Hence, $G' = [G, G\,]$ is contained in $\mathrm{Homeo}_0(\mathbb{T}^2)$ and all the elements of $G'' = [G', G']$ are irrotational.

If $G \subset \mathrm{Homeo}(\mathbb{T}^2)$, then $G' \subset \mathrm{Homeo}_+(\mathbb{T}^2)$ and we have that $G''$ is contained in $\mathrm{Homeo}_0(\mathbb{T}^2)$. Consequently, all the elements of $G''' = [G'', G'']$ are irrotational. □

*Proof of Theorem 1.6 for $G \subset \mathrm{Homeo}_0(\mathbb{T}^2)$.* It suffices to show that given a lift $\tilde{\phi}$ of an element $\phi$ of $G$ and a $\phi$-invariant probability measure $\mu$, then there exists a $G$-invariant probability measure $\nu$ such that $\rho_\mu(\tilde{\phi}) = \rho_\nu(\tilde{\phi})$.

We will prove that there exists a $\langle Z^{(j)}(G), \phi\rangle$-invariant probability measure $\nu_j$ such that $\rho_\mu(\tilde{\phi}) = \rho_{\nu_j}(\tilde{\phi})$ by induction on $j$. The result is obvious for $j = 0$ by defining $\nu_0 = \mu$. Suppose that the result holds true for $j$. We have $\rho_{\nu_j}(\tilde{\phi}) = \rho_{\psi_*^{-1}\nu_j}(\tilde{\psi}^{-1} \circ \tilde{\phi} \circ \tilde{\psi})$ for any lift $\tilde{\psi}$ of any element $\psi \in Z^{(j+1)}(G)$ by Lemma 2.2. Notice that since $\psi$ normalizes $\langle Z^{(j)}(G), \phi\rangle$, the measure $\psi_*^{-1}\nu_j$ is $\langle Z^{(j)}(G), \phi\rangle$-invariant. Consider a $G$-invariant Borel probability measure $\nu'$. Since $\rho_{\nu'} : \mathrm{Homeo}_{0,\nu'}(\mathbb{R}^2) \to \mathbb{R}^2$ is a morphism of groups by Lemma 2.3, we obtain $\rho_{\nu'}[\tilde{\psi}^{-1}, \tilde{\phi}] = (0, 0)$. Since $G'$ is irrotational by Theorem 1.4 and $[\psi^{-1}, \phi] \in Z^{(j)}(G)$, we get $\rho_{\psi_*^{-1}\nu_j}[\tilde{\psi}^{-1}, \tilde{\phi}] = \rho_{\nu'}[\tilde{\psi}^{-1}, \tilde{\phi}] = (0, 0)$. We deduce that

$$\rho_{\nu_j}(\tilde{\phi}) = \rho_{\psi_*^{-1}\nu_j}(\tilde{\psi}^{-1} \circ \tilde{\phi} \circ \tilde{\psi}) = \rho_{\psi_*^{-1}\nu_j}[\tilde{\psi}^{-1}, \tilde{\phi}] + \rho_{\psi_*^{-1}\nu_j}(\tilde{\phi}) = \rho_{\psi_*^{-1}\nu_j}(\tilde{\phi}).$$

Hence, $Z^{(j+1)}(G)$ acts affinely on the compact convex set

$$C_j := \{\nu \in \mathcal{P}\langle Z^{(j)}(G), \phi\rangle : \rho_\nu(\tilde{\phi}) = \rho_\mu(\tilde{\phi})\}.$$

The fixed-point property of amenable groups implies that there exists a fixed point $\nu_{j+1}$ of the action of $Z^{(j+1)}(G)$ on $C_j$, i.e. a $\langle Z^{(j+1)}(G), \phi\rangle$-invariant probability measure $\nu_{j+1}$ such that $\rho_{\nu_{j+1}}(\tilde{\phi}) = \rho_\mu(\tilde{\phi})$.

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4.1. *The construction method.* We want to extend the result in Theorem 1.6 to more general subgroups of $\mathrm{Homeo}(\mathbb{T}^2)$. In order to achieve this goal, we construct a $G$-invariant measure $\mu'$ following the classical construction of an invariant measure for a solvable group of homeomorphisms on a compact metrizable space.

Let $\phi$, $\mu$ be as above. We denote $\rho_0 = \rho_\mu(\tilde{\phi})$. Up to replacing $\mu$ with another measure in $\mathcal{P}(\phi)$, we can suppose that $\mu \in \mathcal{P}(G_0)$ and $\rho_\mu(\tilde{\phi}) = \rho_0$ by Theorem 1.6 for $G_0 \subset \mathrm{Homeo}_0(\mathbb{T}^2)$. We construct new measures by using an inductive method. We choose a finite set $\mathcal{S} := \{g_1, \ldots, g_n\}$ such that $\langle G_0, g_1, \ldots, g_n\rangle = G$. If we do not require any other property for the elements of $\mathcal{S}$, then we can choose $n \le 2$ by Lemma 3.2. Let us define $\mu_0 = \mu$. Our goal is constructing a Borel probability measure $\mu_j$ that is $\psi$-invariant for any $\psi \in \langle G^j, \phi\rangle$, where $G^j = \langle G_0, g_0, g_1, \ldots, g_j\rangle$ and $g_0$ denotes the identity map of $\mathbb{T}^2$. In this context, the following condition will play a fundamental role in the construction:

$$\mathrm{spec}[g_j] \cap (e^{2\pi i\mathbb{Q}} \setminus \{1\}) = \emptyset \quad \text{for all } 0 \le j \le n, \tag{$*$}$$

where $e^{2\pi i\mathbb{Q}}$ is the set of roots of unity. We will see that Condition $(*)$ and the construction process imply that $\rho_{\mu_j}(\tilde{\phi}) = \rho_\mu(\tilde{\phi})$ for all lifts $\tilde{\phi}$ of $\phi$ and $0 \le j \le n$. We say that $G$ satisfies Condition $(*)$ if there exists a set $\mathcal{S}$ as described above.

*Remark 4.1.* Let $G$ be a subgroup of $\mathrm{Homeo}(\mathbb{T}^2)$ such that $G/G_0$ is a non-trivial finite group. Then every non-trivial element of $[G]$ has an eigenvalue that is a root of unity different than 1. Since $G \neq \langle \mathrm{Id}\rangle$, the group $G$ does not satisfy Condition $(*)$.

Let $G$ be a subgroup of $\mathrm{Homeo}(\mathbb{T}^2)$ that satisfies Condition $(*)$. Thus, $G/G_0$ is abelian since otherwise $G/G_0$ is conjugated to the non-trivial finite group $\mathcal{H}$ by Lemma 3.2.

Let us suppose that the $\langle G^j, \phi\rangle$-invariant measure $\mu_j$ has been already constructed for $0 \le j < n$. Then we define the measures

$$\mu_{j,\ell} = (g_{j+1}^\ell)_*(\mu_j) \quad \text{for } \ell \in \mathbb{Z} \quad \text{and} \quad \nu_{j,\ell} = \frac{1}{\ell}\sum_{p=0}^{\ell-1} \mu_{j,p} \quad \text{for } \ell \in \mathbb{Z}^+.$$

From the compactness of the space of probability measures on $\mathbb{T}^2$ in the weak$^*$ topology, we know that the sequence $\{\nu_{j,\ell}\}_{\ell\ge1}$ has a subsequence that converges to a probability measure $\mu_{j+1}$ in this topology. Of course, the choice of $\mu_{j+1}$ is not necessarily unique. The measure $\mu_{j+1}$ is $g_{j+1}$-invariant since we have the relation

$$(g_{j+1})_*(\nu_{j,\ell}) - \nu_{j,\ell} = \frac{1}{\ell}(\mu_{j,\ell} - \mu_{j,0}) \quad \text{for all } \ell \in \mathbb{Z}^+.$$

Let us check that $\mu_{j,\ell}$ is $\langle G^j, \phi\rangle$-invariant for any $\ell \in \mathbb{Z}$. For this, let us fix $\psi \in \langle G^j, \phi\rangle$. Since we are interested in groups satisfying Condition $(*)$, we can suppose that $G/G_0$ is abelian by Remark 4.1. We have $\psi^{-1} \circ g_{j+1}^{-\ell} \circ \psi \circ g_{j+1}^\ell = h_j$ for some $h_j \in G_0 \cap G'$ and we conclude that

$$\begin{aligned}\psi_*(\mu_{j,\ell}) &= \psi_*((g_{j+1}^\ell)_*(\mu_j)) = (\psi \circ g_{j+1}^\ell)_*(\mu_j) = (g_{j+1}^\ell \circ \psi \circ h_j)_*(\mu_j)\\ &= ((g_{j+1}^\ell)_* \circ \psi_* \circ (h_j)_*)(\mu_j) = (g_{j+1}^\ell)_*(\psi_*(\mu_j)) = (g_{j+1}^\ell)_*(\mu_j) = \mu_{j,\ell}\end{aligned}$$

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for any $\ell \in \mathbb{Z}$. We obtain that $\nu_{j,\ell}$ is $\langle G^j, \phi\rangle$-invariant for any $\ell \in \mathbb{Z}^+$. Thus, the measure $\mu_{j+1}$ is $g_{j+1}$-invariant and $\langle G^j, \phi\rangle$-invariant. Consequently, it is $\langle G^{j+1}, \phi\rangle$-invariant.

Now let us remember some elementary properties about the eigenvalues of the elements of $\mathrm{GL}(2, \mathbb{Z})$.

*Remark 4.2.* For a given $A \in \mathrm{GL}(2, \mathbb{Z})$, we have:

- the complex (non-real) eigenvalues occur when $\mathrm{tr}(A) = 0, \pm 1$ and $\det(A) = 1$. They are: $\pm i$ (if $\mathrm{tr}(A) = 0$), $\frac{1}{2} \pm (\sqrt{3}/2)i$ (if $\mathrm{tr}(A) = 1$) or $-\frac{1}{2} \pm (\sqrt{3}/2)i$ (if $\mathrm{tr}(A) = -1$);
- the eigenvalue $-1$ occurs when $\mathrm{tr}(A) = -2$ and $\det(A) = 1$ or when $\mathrm{tr}(A) = 0$ and $\det(A) = -1$;
- the eigenvalue 1 occurs when $\mathrm{tr}(A) = 2$ and $\det(A) = 1$ (i.e. when $A$ is either the identity map or a Dehn twist) or when $\mathrm{tr}(A) = 0$ and $\det(A) = -1$;
- in all the other cases $A$ has an eigenvalue of modulus greater than 1 and another of modulus less than 1.

4.2. *Properties of the measure.* The idea of the construction is trying to prove that, defining the measures $\mu_0, \mu_1, \ldots, \mu_j$ as above, we define the subsequent measures in the process preserving rotation numbers of the elements of $G^j$. For this, first we need to obtain a formula for rotation numbers with respect to the new measures as a function of the old ones. Following our convention, $\tilde{\psi}$ denotes a lift of $\psi \in \mathrm{Homeo}(\mathbb{T}^2)$ to the universal covering of $\mathbb{T}^2$.

LEMMA 4.3. *Suppose that $G/G_0$ is abelian. Furthermore, suppose that*

$$\rho_{\mu_{j,p}}(\tilde{h}) = \rho_{\mu_j}(\tilde{h}) = \rho_{\mu_n}(\tilde{h}) \tag{4.3.1}$$

*for all $0 \le j < n$, $p \in \mathbb{Z}$ and $h \in Z^{(k)}(G) \cap G_0$. Then we have*

$$\rho_{\mu_{j,p}}(\tilde{h}) = [g_{j+1}]^p(\rho_{\mu_j}(\tilde{h})) + \sum_{k=1}^{p} [g_{j+1}]^k(\rho_{\mu_j}([\tilde{g}_{j+1}^{-1}, \tilde{h}])) \tag{4.3.2}$$

*for all $0 \le j < n$, $p \ge 1$, $h \in Z^{(k+1)}(G) \cap G_0$ and*

$$\rho_{\mu_{j,p}}(\tilde{h}) = [g_{j+1}]^p(\rho_{\mu_j}(\tilde{h})) - \sum_{k=1}^{-p} [g_{j+1}]^{-(k-1)}(\rho_{\mu_j}([\tilde{g}_{j+1}^{-1}, \tilde{h}])) \tag{4.3.3}$$

*for all $0 \le j < n$, $p \le -1$ and $h \in Z^{(k+1)}(G) \cap G_0$.*

*Proof.* Fix $0 \le j < n$. First we will show equation (4.3.2) by induction on $p$. The case $p = 1$ is a consequence of Lemmas 2.2 and 2.3; indeed, we obtain

$$\begin{aligned}\rho_{\mu_{j,1}}(\tilde{h}) &= \rho_{(g_{j+1})_*(\mu_j)}(\tilde{h}) = [g_{j+1}](\rho_{\mu_j}(\tilde{g}_{j+1}^{-1} \circ \tilde{h} \circ \tilde{g}_{j+1})) \\ &= [g_{j+1}](\rho_{\mu_j}(\tilde{h})) + [g_{j+1}](\rho_{\mu_j}([\tilde{g}_{j+1}^{-1}, \tilde{h}]))\end{aligned}$$

for any $h \in Z^{(k+1)}(G) \cap G_0$. Let us remind the reader that $(g_{j+1}^m)_*(\mu_j)$ is $\langle G_0, \phi\rangle$-invariant for all $m \in \mathbb{Z}$. Moreover, if $h \in Z^{(k+1)}(G) \cap G_0$, then $[g_{j+1}^{-1}, h] \in G_0$ since

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$G/G_0$ is abelian. Suppose that the result holds for some integer $p \geq 1$. Given $h \in Z^{(k+1)}(G) \cap G_0$, we have

$$\begin{aligned}\rho_{\mu_{j,p+1}}(\tilde{h}) &= \rho_{(g_{j+1}^{p+1})_*(\mu_j)}(\tilde{h}) = [g_{j+1}](\rho_{(g_{j+1}^{p})_*(\mu_j)}(\tilde{g}_{j+1}^{-1} \circ \tilde{h} \circ \tilde{g}_{j+1})) \\ &= [g_{j+1}](\rho_{(g_{j+1}^{p})_*(\mu_j)}([\tilde{g}_{j+1}^{-1}, \tilde{h}])) + [g_{j+1}](\rho_{(g_{j+1}^{p})_*(\mu_j)}(\tilde{h})) \\ &= [g_{j+1}](\rho_{\mu_{j,p}}([\tilde{g}_{j+1}^{-1}, \tilde{h}])) + [g_{j+1}](\rho_{\mu_{j,p}}(\tilde{h}))\end{aligned}$$

by Lemmas 2.2 and 2.3. The induction hypothesis gives the equality

$$\begin{aligned}\rho_{\mu_{j,p+1}}(\tilde{h}) &= [g_{j+1}]\left([g_{j+1}^{p}](\rho_{\mu_j}(\tilde{h})) + \sum_{k=1}^{p} [g_{j+1}^{k}](\rho_{\mu_j}([\tilde{g}_{j+1}^{-1}, \tilde{h}]))\right) \\ &\quad + [g_{j+1}](\rho_{\mu_{j,p}}([\tilde{g}_{j+1}^{-1}, \tilde{h}])) \\ &= [g_{j+1}^{p+1}](\rho_{\mu_j}(\tilde{h})) + [g_{j+1}](\rho_{\mu_{j,p}}([\tilde{g}_{j+1}^{-1}, \tilde{h}])) + \sum_{k=2}^{p+1} [g_{j+1}^{k}](\rho_{\mu_j}([\tilde{g}_{j+1}^{-1}, \tilde{h}])).\end{aligned}$$

Since $[g_{j+1}^{-1}, h] \in Z^{(k)}(G) \cap G_0$, it follows from hypothesis (4.3.1) that

$$\rho_{\mu_{j,p}}([\tilde{g}_{j+1}^{-1}, \tilde{h}]) = \rho_{\mu_j}([\tilde{g}_{j+1}^{-1}, \tilde{h}]).$$

Thus, returning to the last expression of $\rho_{\mu_{j,p+1}}(\tilde{h})$, we obtain equation (4.3.2) for the case $p+1$ and the proof of (4.3.2) is finished.

Now we apply equation (4.3.2) to $g_{j+1}^{-1}$ and we obtain

$$\rho_{\mu_{j,p}}(\tilde{h}) = [g_{j+1}]^{p}(\rho_{\mu_j}(\tilde{h})) + \sum_{k=1}^{-p} [g_{j+1}]^{-k}(\rho_{\mu_j}([\tilde{g}_{j+1}, \tilde{h}])) \tag{4.3.4}$$

for all $p \leq -1$ and $h \in Z^{(k+1)}(G) \cap G_0$. Returning to the hypothesis (4.3.1) and remembering that $[g_{j+1}, h] \in Z^{(k)}(G) \cap G_0$, we conclude by using Lemmas 2.2 and 2.3 that

$$\begin{aligned}\rho_{\mu_j}([\tilde{g}_{j+1}, \tilde{h}]) &= \rho_{\mu_n}(\tilde{g}_{j+1} \circ \tilde{h} \circ \tilde{g}_{j+1}^{-1} \circ \tilde{h}^{-1}) = \rho_{\mu_n}(\tilde{g}_{j+1} \circ \tilde{h} \circ \tilde{g}_{j+1}^{-1}) - \rho_{\mu_n}(\tilde{h}) \\ &= -[g_{j+1}]([g_{j+1}^{-1}](\rho_{\mu_n}(\tilde{h})) - [g_{j+1}^{-1}](\rho_{\mu_n}(\tilde{g}_{j+1} \circ \tilde{h} \circ \tilde{g}_{j+1}^{-1}))) \\ &= -[g_{j+1}](\rho_{\mu_n}(\tilde{g}_{j+1}^{-1} \circ \tilde{h} \circ \tilde{g}_{j+1}) - \rho_{\mu_n}(\tilde{h})) \\ &= -[g_{j+1}](\rho_{\mu_n}([\tilde{g}_{j+1}^{-1}, \tilde{h}])) = -[g_{j+1}](\rho_{\mu_j}([\tilde{g}_{j+1}^{-1}, \tilde{h}])).\end{aligned}$$

From the above equality and equation (4.3.4), we conclude equation (4.3.3). □

Rotation numbers can vary along the process if we do not impose any condition. Roughly speaking, Condition (∗) implies that the sequence of rotation numbers is either constant or unbounded. We obtain invariance of rotation numbers since clearly the latter situation is impossible.

LEMMA 4.4. *Consider* $\{g_1, \ldots, g_n\}$ *satisfying Condition* (∗). *Fix* $k \geq 0$ *and suppose that*

$$\rho_{\mu_j}(\tilde{g}) = \rho_{\mu_{j,q}}(\tilde{g}) = \rho_{\nu_{j,p}}(\tilde{g}) = \rho_{\mu_n}(\tilde{g}) \tag{4.4.1}$$

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*for all* $0 \le j < n$, $q \in \mathbb{Z}$, $p \in \mathbb{Z}^+$ *and* $g \in Z^{(k)}(G) \cap G_0$. *Then we obtain*

$$\rho_{\mu_j}(\tilde{h}) = \rho_{\mu_{j,q}}(\tilde{h}) = \rho_{\nu_{j,p}}(\tilde{h}) = \rho_{\mu_n}(\tilde{h}) \tag{4.4.2}$$

*for all* $0 \le j < n$, $q \in \mathbb{Z}$, $p \in \mathbb{Z}^+$ *and* $h \in Z^{(k+1)}(G) \cap G_0$.

*Proof.* Fix $0 \le j < n$ and let $h \in Z^{(k+1)}(G) \cap G_0$. We will prove that $\rho_{\mu_{j,q}}(\tilde{h}) = \rho_{\mu_j}(\tilde{h})$ for any $q \in \mathbb{Z}$. For this, we recall that the hypothesis on $[g_{j+1}]$ implies that the mapping class $[g_{j+1}]$ is trivial, equal to a Dehn twist or $1 \notin \operatorname{spec}[g_{j+1}]$ by Remark 4.2.

Suppose that $[g_{j+1}] = \mathrm{Id}$. Notice that

$$\rho_{\mu_{j,q}}(\tilde{h}) = \rho_{\mu_j}(\tilde{h}) + q\rho_{\mu_j}([\tilde{g}_{j+1}^{-1}, \tilde{h}])$$

for all $h \in Z^{(k+1)}(G) \cap G_0$ and $q \in \mathbb{Z}$ by equations (4.3.2) and (4.3.3). Since the set of rotation vectors of $\tilde{h}$ is bounded, it follows that $\rho_{\mu_j}([\tilde{g}_{j+1}^{-1}, \tilde{h}]) = (0, 0)$. Thus, $\rho_{\mu_{j,q}}(\tilde{h}) = \rho_{\mu_j}(\tilde{h})$ for any $q \in \mathbb{Z}$.

Suppose that $[g_{j+1}]$ is a (non-trivial) Dehn twist. Up to a change of coordinates in $\mathrm{GL}(2, \mathbb{Z})$, we can suppose that

$$[g_{j+1}] = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \quad \text{for some } 0 \ne m \in \mathbb{Z}.$$

Thus, in these new coordinates, $\rho_{\mu_{j,q}}(\tilde{h})$ takes the form

$$\rho_{\mu_{j,q}}(\tilde{h}) = (v_1, mqv_1 + v_2) + (qw_1, (1 + \cdots + q)mw_1 + qw_2) \quad \text{for all } q \in \mathbb{Z}^+,$$

where $\rho_{\mu_j}(\tilde{h}) = (v_1, v_2)$ and $\rho_{\mu_j}([\tilde{g}_{j+1}^{-1}, \tilde{h}]) = (w_1, w_2)$ by Lemma 4.3. Since the set of rotation vectors of $\tilde{h}$ is bounded, we deduce that $w_1 = 0$ and

$$\rho_{\mu_{j,q}}(\tilde{h}) = (v_1, q(mv_1 + w_2) + v_2) \quad \text{for all } q \in \mathbb{Z}^+.$$

Again, the boundedness of the set of rotation vectors of $\tilde{h}$ guarantees that $mv_1 + w_2 = 0$ and then we have $\rho_{\mu_{j,q}}(\tilde{h}) = \rho_{\mu_j}(\tilde{h})$ for any $q \in \mathbb{Z}^+$. Similar arguments with equation (4.3.3) give us $\rho_{\mu_{j,q}}(\tilde{h}) = \rho_{\mu_j}(\tilde{h})$ for any integer $q < 0$ and then we obtain $\rho_{\mu_{j,q}}(\tilde{h}) = \rho_{\mu_j}(\tilde{h})$ for all $q \in \mathbb{Z}$.

Suppose finally that $1 \notin \operatorname{spec}[g_{j+1}]$. From Proposition 3.1, there exists $\tau \in \mathbb{Z}^+$ such that $h^\tau \in G_{\mathcal{I}} = G_{\mathcal{I}}^{\mu_n}$ and hence $\rho_{\mu_n}(\tilde{h}^\tau) \in \mathbb{Z}^2$. On the other hand, $[g_{j+1}^{-1}, h] \in Z^{(k)}(G) \cap G_0$ and we conclude that

$$\begin{aligned} \rho_{\mu_j}([\tilde{g}_{j+1}^{-1}, \tilde{h}]) &= \rho_{\mu_n}(\tilde{g}_{j+1}^{-1} \circ \tilde{h} \circ \tilde{g}_{j+1} \circ \tilde{h}^{-1}) \\ &= [g_{j+1}]^{-1}(\rho_{\mu_n}(\tilde{h})) - \rho_{\mu_n}(\tilde{h}), \end{aligned} \tag{4.4.3}$$

where the first equality comes from (4.4.1), whereas the second is a consequence of Lemmas 2.2 and 2.3. Now let $u = \rho_{\mu_n}(\tilde{h}^\tau)$. Replacing $\tilde{h}$ with $T_{-u} \circ \tilde{h}^\tau$ in (4.4.3), we obtain

$$\begin{aligned} \rho_{\mu_j}([\tilde{g}_{j+1}^{-1}, T_{-u} \circ \tilde{h}^\tau]) &= [g_{j+1}]^{-1}(\rho_{\mu_n}(T_{-u} \circ \tilde{h}^\tau)) - \rho_{\mu_n}(T_{-u} \circ \tilde{h}^\tau) \\ &= [g_{j+1}]^{-1}(-u + \rho_{\mu_n}(\tilde{h}^\tau)) + (u - \rho_{\mu_n}(\tilde{h}^\tau)) = (0, 0). \end{aligned}$$

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By inserting the last expression in equations (4.3.2) and (4.3.3), we obtain

$$\rho_{\mu_{j,q}}(T_{-u}\circ\tilde{h}^{\tau}) = [g_{j+1}]^q(\rho_{\mu_j}(T_{-u}\circ\tilde{h}^{\tau})) \quad \text{for all } q\in\mathbb{Z}.$$

Again, the boundedness of the set of rotation vectors $\rho(T_{-u}\circ\tilde{h}^{\tau})$ allows us to deduce that the set $\{[g_{j+1}]^q(\rho_{\mu_j}(T_{-u}\circ\tilde{h}^{\tau}))\}_{q\in\mathbb{Z}}$ is bounded.

Moreover, Remark 4.2 and the hypothesis on $\operatorname{spec}[g_{j+1}]$ imply that $[g_{j+1}]$ has an eigenvalue of modulus greater than 1 and an eigenvalue of modulus less than 1. These properties are compatible only if $\rho_{\mu_j}(T_{-u}\circ\tilde{h}^{\tau}) = 0$. In such a case, we have $\rho_{\mu_{j,q}}(T_{-u}\circ\tilde{h}^{\tau}) = \rho_{\mu_j}(T_{-u}\circ\tilde{h}^{\tau})$ and then

$$\rho_{\mu_{j,q}}(\tilde{h}) = \rho_{\mu_j}(\tilde{h}) \quad \text{for any } q\in\mathbb{Z}.$$

As a consequence, we obtain $\rho_{v_{j,p}}(\tilde{h}) = \rho_{\mu_j}(\tilde{h})$ for any $p\in\mathbb{Z}^+$ and then $\rho_{\mu_{j+1}}(\tilde{h}) = \rho_{\mu_j}(\tilde{h})$. We are done by making $j$ vary in $\{0,\dots,n-1\}$. □

Next we identify the nilpotent subgroups of $\mathrm{MCG}(\mathbb{T}^2)$ that are compatible with the existence of generator sets satisfying Condition (∗). First we consider the following Condition (∗∗) on $G\subset\mathrm{Homeo}(\mathbb{T}^2)$:

$[G]$ has one of the following forms: $[G]=\{\mathrm{Id}\}$, $[G]=\langle D\rangle$, where $D$ is a (non-trivial) Dehn twist, $[G]=\langle A\rangle$ or $[G]=\langle A,-\mathrm{Id}\rangle$, where $\operatorname{spec}(A)$ does not contain any eigenvalue of modulus 1. (∗∗)

Now we state the following remark.

*Remark 4.5.* Let $G\subset\mathrm{Homeo}(\mathbb{T}^2)$ be a nilpotent group and suppose that $G$ satisfies Condition (∗∗). We can construct a set $\mathcal{S} := \{g_1,\dots,g_n\}$ such that $G=\langle G_0, g_1,\dots,g_n\rangle$ and $\mathcal{S}$ satisfies Condition (∗).

The remark is trivial if $[G]=\{\mathrm{Id}\}$ or $[G]=\langle D\rangle$, where $D$ is a (non-trivial) Dehn twist since then $\operatorname{spec}[g]=\{1\}$ for all $g\in G$.

Let us consider the case $[G]=\langle A\rangle$. It suffices to define $n=1$ and to take $g_1\in G$ such that $[g_1]=A$.

Consider now the case $[G]=\langle A,-\mathrm{Id}\rangle$. We define $n=2$ and elements $g_1, g_2\in G$ such that $[g_1]=A$ and $[g_2]=-A$.

In the next remark, we describe the nilpotent subgroups of $\mathrm{Homeo}(\mathbb{T}^2)$ that do not satisfy Condition (∗∗).

*Remark 4.6.* Let $G$ be a nilpotent subgroup of $\mathrm{Homeo}(\mathbb{T}^2)$. Following Remark 4.2, it is very simple to conclude that if $[G]\neq\{\mathrm{Id}\}$ is finite, then $G$ does not satisfy either Condition (∗) or Condition (∗∗). Suppose that $[G]$ is an infinite group that does not satisfy Condition (∗∗). It is of the form $\langle N\rangle$ or $\langle N,-\mathrm{Id}\rangle$ by Lemma 3.2, where $N\in\mathrm{GL}(2,\mathbb{Z})\setminus\{\mathrm{Id}\}$ has an eigenvalue of modulus equal to 1. Notice that the eigenvalues of $N$ are real numbers, otherwise $[G]$ is finite by Remark 4.2. Analogously, we deduce that $\operatorname{spec}(N)\neq\{1,-1\}$. We obtain:

- if $\operatorname{spec}(N)=\{1\}$, then $N$ is a Dehn twist;
- if $\operatorname{spec}(N)=\{-1\}$, then $N=-D$, where $D$ is a Dehn twist.

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Consequently, the nilpotent subgroups $G$ of $\mathrm{Homeo}(\mathbb{T}^2)$ that do not satisfy Condition $(**)$ have one of the forms: $[G]$ is a non-trivial finite group, $[G] = \langle -D \rangle$ or $[G] = \langle D, -\mathrm{Id} \rangle$ for a (non-trivial) Dehn twist $D$.

In these cases, the set $S$ of elements of $[G]$ that do not have non-trivial roots of unity as eigenvalues is a proper finite-index normal subgroup of $[G]$. Hence, $G$ does not satisfy Condition $(*)$ and our construction cannot be applied. The previous discussion, together with Remark 4.5, imply that Condition $(*)$ and Condition $(**)$ are equivalent.

Notice that $S = [G]$ for all the groups that satisfy Condition $(**)$ except $[G] = \langle A, -\mathrm{Id} \rangle$, where $S = [G] \setminus \{-\mathrm{Id}\}$ and $\langle S \rangle = [G]$.

PROPOSITION 4.7. *Let $G$ be a nilpotent subgroup of* $\mathrm{Homeo}(\mathbb{T}^2)$ *and suppose that $G$ satisfies Condition* $(**)$. *Given $\phi \in G_0$ and $\mu \in \mathcal{P}(\phi)$, there exists $\nu \in \mathcal{P}(G)$ such that $\rho_\mu(\tilde{\phi}) = \rho_\nu(\tilde{\phi})$ for any lift $\tilde{\phi}$ of $\phi$. Moreover, if $\mu$ is $G_0$-invariant, we have $\rho_\mu(\tilde{\psi}) = \rho_\nu(\tilde{\psi})$ for any lift $\tilde{\psi}$ of every $\psi \in G_0$.*

*Proof.* We can suppose that $\mu$ is $G_0$-invariant by Theorem 1.6 for subgroups of $\mathrm{Homeo}_0(\mathbb{T}^2)$.

Since $G$ satisfies Condition $(**)$, it follows that $[G]$ is infinite by Remark 4.6 and thus abelian by Lemma 3.2. Then we can apply the construction of a $G$-invariant measure in § 4.1. There exists a subset $\mathcal{S} := \{g_1, \ldots, g_n\}$ of $G$ such that $G = \langle G_0, g_1, \ldots, g_n \rangle$ and $\mathcal{S}$ satisfies Condition $(*)$ by Remark 4.5. Equation (4.4.1) in Lemma 4.4 holds trivially for $k = 0$. Moreover, if it is satisfied for $k$, then so it is for $k + 1$ by Lemma 4.4. We deduce that $\rho_\mu(\tilde{\psi}) = \rho_\nu(\tilde{\psi})$ for any lift $\tilde{\psi}$ of any $\psi \in G_0$, where $\nu := \mu_n$ is the $G$-invariant probability measure provided by Lemma 4.4. □

Now we compare our definitions of irrotationality for nilpotent groups that satisfy Condition $(*)$.

PROPOSITION 4.8. *Let $G$ be a nilpotent subgroup of* $\mathrm{Homeo}(\mathbb{T}^2)$ *and suppose that $G$ satisfies Condition* $(**)$. *Then we have $\mathcal{I}_G = G_{\mathcal{I}}$.*

*Proof.* It is clear that $\mathcal{I}_G \subset G_{\mathcal{I}}$. Now suppose that there exists $\phi \in G_{\mathcal{I}}$ such that $\phi \notin \mathcal{I}_G$. Fix a lift $\tilde{\phi}$ of $\phi$. Since $\phi$ is non-irrotational, the homeomorphism $\tilde{\phi}$ is not an irrotational lift of $\phi$. Thus, there exists $\mu \in \mathcal{P}(\phi)$ such that $\rho_\mu(\tilde{\phi}) \neq (0, 0)$. There exists $\nu \in \mathcal{P}(G)$ such that $\rho_\nu(\tilde{\phi}) = \rho_\mu(\tilde{\phi}) \neq (0, 0)$ by Proposition 4.7.

Thus, we deduce that $\tilde{\phi} \notin \tilde{G}_{\mathcal{I}}^{\nu}$ and then $\tilde{\phi} \notin \tilde{G}_{\mathcal{I}}$. Since $\tilde{\phi}$ is any lift of $\phi$, we conclude that $\phi \notin G_{\mathcal{I}}$, which is a contradiction. Consequently, we obtain $G_{\mathcal{I}} \subset \mathcal{I}_G$ and then $\mathcal{I}_G = G_{\mathcal{I}}$. □

The next rigidity results are consequences of Propositions 4.7 and 4.8 and Remark 4.6. The next proposition implies Theorem 1.6.

PROPOSITION 4.9. *Let $G$ be a nilpotent subgroup of* $\mathrm{Homeo}(\mathbb{T}^2)$. *Then there exists a finite-index normal subgroup $H$ of $G$ (of index* 1, 2, 3, 4, 6 *or* 8*) containing $G_0$ such that*

$$\{\rho_\mu(\tilde{\phi});\ \mu \in \mathcal{P}(\phi)\} = \{\rho_\nu(\tilde{\phi});\ \nu \in \mathcal{P}(H)\}$$

*for any lift $\tilde{\phi}$ of every $\phi \in G_0$. In particular, we obtain $\mathcal{I}_G = \mathcal{I}_H = H_{\mathcal{I}}$. Moreover, $\mathcal{I}_G$ is a normal subgroup of $G$.*

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*Proof.* If $G$ satisfies Condition $(**)$, then Proposition 4.7 gives us $H = G$. In the case where $G$ does not satisfy Condition $(**)$, we have from Remark 4.6 that:

- if $[G]$ is finite, then we take $H = G_0$;
- if $[G] = \langle -D \rangle$ (respectively $[G] = \langle D, -\mathrm{Id} \rangle$), then we take $H$ as the inverse image of $\langle D^2 \rangle$ (respectively $\langle D \rangle$) by the map $\psi \in G \mapsto [\psi] \in [G]$.

In the latter case, $H$ is an index-2 normal subgroup of $G$. In the former case, we just have to notice that the list of finite nilpotent subgroups of $\mathrm{GL}(2, \mathbb{Z})$ consists of the subgroups of $C_6$ and the subgroups of $D_4$. It can be proved by using Remark 4.2. The group $H$ satisfies Condition $(**)$. We have $\mathcal{I}_G = \mathcal{I}_H$ by definition, whereas $\mathcal{I}_H = H_{\mathcal{I}}$ is a consequence of Proposition 4.8. Since $H_{\mathcal{I}}$ is a subgroup of $H$, it follows that $\mathcal{I}_G$ is a subgroup of $G$. It is normal in $G$ since $\mathcal{I}_{\mathrm{Homeo}_0(\mathbb{T}^2)}$ is a normal subset of $\mathrm{Homeo}_0(\mathbb{T}^2)$. □

*Remark 4.10.* The calculations in Lemma 4.4 and Propositions 4.7–4.9 work analogously if $G$ is a nilpotent subgroup of $\mathrm{Homeo}_0(S)$, where $S$ is a compact orientable surface of genus $g \geq 2$. As a consequence, the obvious adaptation of Theorem 1.6 to the case $g \geq 2$ holds true.

PROPOSITION 4.11. *Let $G$ be a nilpotent subgroup of* $\mathrm{Homeo}(\mathbb{T}^2)$ *such that* $[G]$ *contains an Anosov class, i.e. there exists $\psi \in G$ such that* $\mathrm{spec}[\psi] \cap \mathbb{S}^1 = \emptyset$. *Then $\mathcal{I}_G$ is a finite-index normal subgroup of $G_0$.*

*Proof.* The group $[G]$ is of the form $\langle A \rangle$ or $\langle A, -\mathrm{Id} \rangle$, where $\mathrm{spec}(A) \cap \mathbb{S}^1 = \emptyset$. Therefore, we obtain $\mathcal{I}_G = G_{\mathcal{I}}$ by Proposition 4.8. The group $G_{\mathcal{I}}$ is a finite-index normal subgroup of $G_0$ by Proposition 3.1. □

*Example 1.* The property $G_{\mathcal{I}} = \mathcal{I}_G$ is false if we do not impose any condition on the nilpotent group $[G]$. Let us consider a $C^\infty$ diffeomorphism $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ of the form $\tilde{h}(x, y) = (x + f(x), y + g(x))$, where

$$f(x+1) \equiv f(x), \quad f(x) \equiv -f(-x), \quad g(x+1) \equiv g(x) \quad \text{and} \quad g(x) \equiv -g(-x).$$

We always have $f(0) = f(1/2) = g(0) = g(1/2) = 0$ and hence we obtain $\{0, 1/2\} \times \mathbb{R} \subset \mathrm{Fix}(\tilde{h})$. We can choose $f, g$ such that $x + f(x)$ has fixed-point set $\{0, 1/4, 1/2\}$ at $[0, 1/2]$ and $g(1/4) \neq 0$. The diffeomorphism $\tilde{h}$ is a lift of an element $h$ of $\mathrm{Diff}_0(\mathbb{T}^2)$. We denote by $\phi$ the element of $\mathrm{Diff}(\mathbb{T}^2)$ having the lift $\tilde{\phi}(x, y) = (-x, -y)$. We define the groups

$$\tilde{G} = \langle \tilde{h}, -\mathrm{Id} \rangle, \quad G = \langle h, \phi \rangle.$$

They are both abelian by construction, $G_0 = \langle h \rangle$ and $\tilde{G}$ is a lift of $G$. The group $G_0$ coincides with $G_{\mathcal{I}}$ [**11**, Propositions 3.1 and 3.4]. If $h^k$ belongs to $\mathcal{I}_G$, then $\tilde{h}^k$ is its irrotational lift because $\mathrm{Fix}(\tilde{h}^k) \neq \emptyset$. But the Lebesgue measure Leb in the circle $\{1/4\} \times \mathbb{R}/\mathbb{Z}$ is $h$-invariant and $\rho_{\mathrm{Leb}}(\tilde{h}^k) = (0, kg(1/4))$. We obtain $\mathcal{I}_G = \{\mathrm{Id}\}$ and then $\mathcal{I}_G \neq G_{\mathcal{I}}$.

## 5. *Proof of Theorem 1.2*

In order to show Theorem 1.2, we reduce the general situation to more tractable ones. The main tools are a Thurston decomposition of all elements in the normal subgroup $H$ à la Franks, Handel and Parwani [**14**] and Theorem 2.4, that allows us to find global fixed points by lifting the problem to the universal covering.

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First we deal with the simple case in which $\mathrm{supp}(\mu)$ admits a $\tilde{\phi}$-invariant compact lift.

PROPOSITION 5.1. *Let $G = \langle H, \phi\rangle$ be a nilpotent subgroup of $\mathrm{Diff}^1_0(\mathbb{T}^2)$, where $H$ is a normal subgroup of $G$. Suppose that there exist a lift $\tilde{\phi}$ of $\phi$ to the universal covering and a non-empty compact set $\tilde{C} \subset \mathbb{R}^2$ which is $\tilde{\phi}$-invariant. If $\pi(\tilde{C})$ is contained in $\mathrm{Fix}(H)$, then $G$ has a global fixed point.*

*Proof.* If $\tilde{C}$ is not $\tilde{\phi}$-minimal, we replace it by a $\tilde{\phi}$-minimal set contained in $\tilde{C}$. Let $\nu$ be a $G$-invariant Borel probability measure on $\mathbb{T}^2$ with $\mathrm{supp}(\nu) \subset \pi(\tilde{C})$. Fix $\tilde{p} \in \tilde{C}$ and let $\tilde{H}$ be the following lift of $H$:

$$\tilde{H} := \{\tilde{h} \in \mathrm{Diff}^1_0(\mathbb{R}^2);\ h \in H \text{ and } \tilde{h}(\tilde{p}) = \tilde{p}\}.$$

We claim that $\tilde{H}$ is a normal subgroup of $\langle \tilde{H}, \tilde{\phi}\rangle$. For this, let us denote $H^j = Z^{(j)}(G) \cap H$. We will show by induction on $j \geq 0$ that the lift $\tilde{H}^j \subset \tilde{H}$ of $H^j$ is a normal subgroup of $\langle \tilde{H}, \tilde{\phi}\rangle$.

The result is obviously true for $j = 0$.

Now suppose that the result holds true for some integer $j \geq 0$. Given $\tilde{\psi} \in \langle \tilde{H}, \tilde{\phi}\rangle$, we have that the elements of $\tilde{\psi}\tilde{H}^j\tilde{\psi}^{-1}$ fix $\tilde{\psi}(\tilde{p})$. From the induction hypothesis, we have that $\tilde{\psi}\tilde{H}^j\tilde{\psi}^{-1} = \tilde{H}^j$. Thus, the $\tilde{\phi}$-orbit of $\tilde{p}$ is contained in $\mathrm{Fix}(\tilde{H}^j)$ and then $\tilde{C}$ is contained in $\mathrm{Fix}(\tilde{H}^j)$ since $\tilde{C}$ is $\tilde{\phi}$-minimal. Consequently, we conclude that $\pi^{-1}(\pi(\tilde{C})) \subset \mathrm{Fix}(\tilde{H}^j)$ since the elements of $\tilde{H}^j$ commute with the covering transformations.

Given $h \in H^{j+1}$ and $\tilde{\psi} \in \langle \tilde{H}, \tilde{\phi}\rangle$, consider the element $\psi$ of $\langle H, \phi\rangle$ whose lift is $\tilde{\psi}$. We know that $h' := [\psi, h]$ belongs to $H^j$. Since $\rho_\nu|_{\langle \tilde{H}, \tilde{\phi}\rangle}$ is a morphism of groups by Lemma 2.3, we obtain $\rho_\nu([\tilde{\psi}, \tilde{h}]) = (0, 0)$. Moreover, $\tilde{h}' \in \tilde{H}^j$ satisfies $\rho_\nu(\tilde{h}') = (0, 0)$ too since $\pi^{-1}(\pi(\tilde{C})) \subset \mathrm{Fix}(\tilde{H}^j)$ and $\mathrm{supp}(\nu) \subset \pi(\tilde{C})$. Hence, we obtain $[\tilde{\psi}, \tilde{h}] = \tilde{h}'$ and

$$\tilde{\psi} \circ \tilde{h} \circ \tilde{\psi}^{-1} = \tilde{h}' \circ \tilde{h} \in \tilde{H}^{j+1},$$

proving that $\tilde{H}^{j+1}$ is a normal subgroup of $\langle \tilde{H}, \tilde{\phi}\rangle$. Since $G$ is nilpotent, we conclude that $\tilde{H}$ is normal in $\langle \tilde{H}, \tilde{\phi}\rangle$.

Now, in particular, we have that $\tilde{C}$ is contained in $\mathrm{Fix}(\tilde{H})$ and it is a non-empty compact set which is invariant by $\langle \tilde{H}, \tilde{\phi}\rangle$. Moreover, since $H$ (respectively $\tilde{H}$) is a normal subgroup of $\langle H, \phi\rangle$ (respectively $\langle \tilde{H}, \tilde{\phi}\rangle$), we conclude that $[\langle \tilde{H}, \tilde{\phi}\rangle, \langle \tilde{H}, \tilde{\phi}\rangle] \subset \tilde{H}$ and $[\langle H, \phi\rangle, \langle H, \phi\rangle] \subset H$. Consequently, the natural projection from $[\langle \tilde{H}, \tilde{\phi}\rangle, \langle \tilde{H}, \tilde{\phi}\rangle]$ to $[\langle H, \phi\rangle, \langle H, \phi\rangle]$ is an isomorphism, proving that $\langle \tilde{H}, \tilde{\phi}\rangle$ is also a nilpotent group.

Thus, if $G$ is finitely generated, then $\mathrm{Fix}(\langle \tilde{H}, \tilde{\phi}\rangle) \neq \emptyset$ by Theorem 2.4 and $G$ has a global fixed point. The general case for $G$ follows now from the finite intersection property on $\mathbb{T}^2$. □

Let us return to the statement of Theorem 1.2 and consider the following special case: $\mathrm{supp}(\mu)$ is a finite set. Let $p \in \mathrm{supp}(\mu)$ and $\tilde{p} \in \pi^{-1}(p)$. The ergodic nature of $\mu$ guarantees that the $\phi$-orbit of $p$ coincides with $\mathrm{supp}(\mu)$. Moreover, the property $\rho_\mu(\tilde{\phi}) = (0, 0)$ implies that the $\tilde{\phi}$-orbit of $\tilde{p}$ is finite. Consequently, $\mathrm{supp}(\mu)$ has a $\tilde{\phi}$-invariant compact lift. In particular, we obtain the following result.

COROLLARY 5.2. *Consider the hypotheses of Theorem 1.2. If $\mathrm{supp}(\mu)$ is a finite set, then $G$ has a global fixed point.*

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Later on, we consider the Thurston decomposition of the normal subgroup $H$. The next lemma is used to rule out the existence of non-essential reducing curves. Moreover, let us remark that if $\mathrm{supp}(\mu) = \mathbb{T}^2$ in Theorem 1.2, then $H = \{\mathrm{Id}\}$ and the result follows from the referred to theorem of Franks in [**13**]. Then, in what follows we are assuming that $\mathrm{supp}(\mu) \neq \mathbb{T}^2$.

LEMMA 5.3. *Consider the setting in Theorem 1.2. Suppose that there exists a non-essential simple closed curve $\gamma$ such that*:

- $\gamma \subset \mathbb{T}^2 \setminus \mathrm{supp}(\mu)$;
- *the compact disk $D$ enclosed by $\gamma$ contains points of* $\mathrm{supp}(\mu)$;
- $\phi^k(\gamma)$ *is isotopic to $\gamma$ in $\mathbb{T}^2 \setminus \mathrm{supp}(\mu)$ for some $k \in \mathbb{Z}^+$.*

*Then $G$ has a global fixed point.*

*Proof.* Let us denote $C = D \cap \mathrm{supp}(\mu)$ and let $\tilde{D}$ be a lift of $D$ in $\mathbb{R}^2$. Let $\tilde{\phi}$ be the lift of $\phi$ such that $\rho_\mu(\tilde{\phi}) = (0, 0)$ and let us fix the lift $\tilde{C} = \pi^{-1}(C) \cap \tilde{D}$ of $C$.

Since $\phi^k(\gamma)$ is isotopic to $\gamma$ in $\mathbb{T}^2 \setminus \mathrm{supp}(\mu)$, we obtain $\phi^k(C) = C$. Consequently, the compact set $\bigcup_{j=0}^{k-1} \phi^j(C)$ is $\phi$-invariant and has positive measure. Now it follows from the $\phi$-ergodicity of $\mu$ that

$$\mathrm{supp}(\mu) = \bigcup_{j=0}^{k-1} \phi^j(C).$$

We have $\tilde{\phi}^k(\tilde{C}) = \tilde{C} + (m, n)$ for some $(m, n) \in \mathbb{Z}^2$. The property $\rho_\mu(\tilde{\phi}) = (0, 0)$ implies that $(m, n) = (0, 0)$ and we obtain $\tilde{\phi}^k(\tilde{C}) = \tilde{C}$. Thus, $\bigcup_{j=0}^{k-1} \tilde{\phi}^j(\tilde{C})$ is a non-empty compact $\tilde{\phi}$-invariant set whose projection by $\pi$ is contained in $\mathrm{Fix}(H)$. Then the group $G$ has a global fixed point by Proposition 5.1. □

A common trait in several subcases in the proof of Theorem 1.2 is that if $\phi$ preserves a non-trivial simple closed curve modulo isotopy relative to $\mathrm{supp}(\mu)$, then $G$ has a global fixed point. In particular, such phenomenon forces the Thurston decomposition of $H$ to be trivial.

LEMMA 5.4. *Consider the hypotheses of Theorem 1.2. Then either* $\mathrm{Fix}(G) \neq \emptyset$ *or any element of $H$ is isotopic to the identity relative to* $\mathrm{supp}(\mu)$.

*Proof.* Suppose that $\mathrm{Fix}(G) = \emptyset$. Let us prove that any element of $H$ is isotopic to the identity relative to $\mathrm{supp}(\mu)$.

Since $\mathrm{Fix}(G) = \emptyset$, there exists a finitely generated subgroup $J$ of $G$ containing $\phi$ and such that $\mathrm{Fix}(J) = \emptyset$. Now it suffices to prove that $h$ is isotopic to the identity relative to $\mathrm{supp}(\mu)$ for every finitely generated subgroup $L$ of $G$ containing $J$ and $h \in L \cap H$. Hence, up to replacing $G$ with $L$ and $H$ with $L \cap H$, we can suppose that $G$ is finitely generated and $\mathrm{Fix}(G) = \emptyset$.

Let us denote $K = \mathrm{supp}(\mu)$. Since $K$ is a subset of $\mathrm{Fix}(H)$, $K$ is $G$-invariant. Now we apply Proposition 2.6. We can suppose that the Thurston decomposition is minimal, i.e. that it has a minimal number of reducing curves. It suffices to show that $\mathcal{R} = \emptyset$ and $\sharp K = \infty$. The former property implies that $\mathbb{T}^2$ is the unique connected component of

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$\mathbb{T}^2 \setminus \mathcal{A}$, whereas the latter property guarantees that $h$ is isotopic to the identity relative to $K$ for any $h \in H$.

Moreover, since $\mathcal{R}$ is $\phi$-invariant modulo isotopy relative to $K$, there exists $k \in \mathbb{Z}$ such that $\phi^k(\gamma)$ is isotopic to $\gamma$ in $\mathbb{T}^2 \setminus K$ for any connected component $\gamma$ of $\mathcal{R}$. Hence, we may admit that $\mathcal{R}$ does not contain non-essential reducing curves by Proposition 2.6 and Lemma 5.3.

Suppose that $\mathcal{R} \neq \emptyset$. Since the curves in $\mathcal{R}$ are simple and pairwise disjoint, they are freely homotopic to an essential curve $\kappa$ in $\mathbb{T}^2$. Consider a connected component $U$ of $\mathbb{T}^2 \setminus \mathcal{R}$. Then either $\sharp(U \cap K) = \infty$ or $\sharp(U \cap K) < \infty$ and $\chi(U \setminus K) < 0$ by the properties of the Thurston decomposition. Since $U$ is an annulus ($\chi(U) = 0$), it contains points of $K$. The curve $\kappa$ defines a non-trivial class $(a, b)$ in $H^1(\mathbb{T}^2, \mathbb{Z}) \sim \mathbb{Z}^2$. The property $\rho_\mu(\tilde{\phi}) = (0, 0)$ implies that $\phi(U \cap K) = U \cap K$, otherwise we obtain

$$\liminf_{n\to\infty} \left\| \left\langle \frac{\tilde{\phi}^n(\tilde{z}) - \tilde{z}}{n}, (-b, a) \right\rangle \right\| > 0$$

for all $z \in K \cap U$, where $<, >$ is the canonical inner product. The last property contradicts Birkhoff's ergodic theorem.

Since $\phi(U \cap K) = U \cap K$ and $\mu$ is $\phi$-ergodic, we obtain that $\mathbb{T}^2 \setminus \mathcal{R}$ has one connected component. We can suppose that $K$ is an infinite set, otherwise $\mathrm{Fix}(G)$ is non-empty by Corollary 5.2.

If $\mathcal{R}$ contains exactly one reducing curve, then any $h$ in $H$ is isotopic to a Dehn twist $\theta$ relative to $K$. The diffeomorphism $h$ is isotopic to the identity; thus, the Dehn twist is trivial. We deduce that $h$ is isotopic to the identity relative to $K$. If we remove the unique curve in $\mathcal{R}$, we still have a Thurston decomposition, contradicting the minimality of $\mathcal{R}$. □

Lemma 5.4 implies that we can consider a canonical lift of $H$, namely the identity lift $\tilde{H}$ of $H$ to the universal covering of $\mathbb{T}^2 \setminus \mathrm{supp}(\mu)$. The practical consequence is that from now on we focus on properties of $\phi$ and the role of $H$ is secondary.

*Definition 5.5.* Let $\psi \in \mathrm{Homeo}(\mathbb{T}^2)$ and let $K \subset \mathbb{T}^2$ be a non-empty $\psi$-invariant compact set. Let us fix $z \in \mathrm{Fix}(\psi) \setminus K$ and let $U$ be the connected component of $\mathbb{T}^2 \setminus K$ containing $z$. Fixing a lift $\tilde{z}$ of $z$ in the universal covering $\tilde{U}$ of $U$, we will denote by $\tilde{\psi}_{\tilde{z}}$ the lift of $\psi$ to $\tilde{U}$ such that $\tilde{\psi}_{\tilde{z}}(\tilde{z}) = \tilde{z}$.

Our goal is finding a lift $\tilde{\psi}_{\tilde{z}}$ having a compact fixed-point set and then applying Theorem 2.5. The next definition is useful to study the obstruction to the existence of such a lift.

*Definition 5.6.* Let us define a subset $Q(K, \psi)$ of $\mathrm{Fix}(\psi) \setminus K$. Given $z \in \mathrm{Fix}(\psi) \setminus K$, we say that $z \notin Q(K, \psi)$ if there exists a simple closed curve $\varsigma : [0, 1] \to \mathbb{T}^2 \setminus K$ such that $\varsigma(0) = \varsigma(1) = z$, $\varsigma$ is essential in $\mathbb{T}^2$ and $[\psi \circ \varsigma] = [\varsigma]$, where $[\varsigma]$ denotes the homotopy class defined by $\varsigma$ in $\pi_1(\mathbb{T}^2 \setminus K, z)$.

The property $Q(\mathrm{supp}(\mu), \phi) \neq \emptyset$ is a necessary condition for the existence of a lift of $\phi$ with a non-empty compact fixed-point set. Curiously, it is also a sufficient condition.

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LEMMA 5.7. *Consider the setting in Theorem 1.2. Suppose that there exists* $z \in Q(\mathrm{supp}(\mu), \phi)$ *and let* $U$ *be the connected component of* $\mathbb{T}^2 \setminus \mathrm{supp}(\mu)$ *containing* $z$. *Then either* $\mathrm{Fix}(G) \neq \emptyset$ *or* $\mathrm{Fix}(\tilde{\phi}_{\tilde{z}})$ *is a non-empty compact subset of* $\tilde{U}$.

*Proof.* Suppose that $\mathrm{Fix}(G) = \emptyset$ and let us denote $\tilde{\phi} = \tilde{\phi}_{\tilde{z}}$ following Definition 5.5. Let $\pi' : \tilde{U} \to U$ be the universal covering map. By construction, $\tilde{z}$ belongs to $\mathrm{Fix}(\tilde{\phi})$. Moreover, it is clear that $\mathrm{Fix}(\phi) \cap U$ is a compact set since $\mathrm{Fix}(G) = \emptyset$ implies that $\mathrm{Fix}(\phi) \cap \mathrm{supp}(\mu) = \emptyset$. Hence, it suffices to prove that the map $\pi'|_{\mathrm{Fix}(\tilde{\phi})} : \mathrm{Fix}(\tilde{\phi}) \to \mathrm{Fix}(\phi)$ is injective. We will prove this by contradiction.

Suppose that $\pi'|_{\mathrm{Fix}(\tilde{\phi})}$ is not injective. Then $\tilde{\phi}$ commutes with some covering transformation and there exists a non-trivial homotopy class $[\beta] \in \pi_1(U, z)$ such that $[\phi \circ \beta] = [\beta]$. Since $[\phi \circ \beta] = [\beta]$, it follows from [**14**, Lemma 2.12] that, up to replacing $\beta$ with another representative of $[\beta]$, there exist a finite-type compact and connected subsurface $T$ contained in $U$, containing $\beta$, with no connected component of $\partial T$ bounding a disk in $\mathbb{T}^2 \setminus \mathrm{supp}(\mu)$ and a homeomorphism $\theta : \mathbb{T}^2 \to \mathbb{T}^2$ such that $\theta$ is isotopic to $\phi$ relative to $\mathrm{supp}(\mu)$, $\theta(T) = T$ and $\theta|_T$ has finite order.

If $\partial T$ has a connected component that bounds a compact disk $D$ in $\mathbb{T}^2$, then $G$ has a global fixed point by Lemma 5.3, contradicting the condition $\mathrm{Fix}(G) = \emptyset$. So, we can suppose that all the boundary curves in $\partial T$ are essential in $\mathbb{T}^2$. Since they do not intersect each other, they are all homotopic in $\mathbb{T}^2$ to some essential simple closed curve $\kappa'$ and the surface $T$ is an annulus. The class $[\beta]$ is not null-homotopic in $U$ and hence it is not null-homotopic in the annulus. Consequently, $\beta$ is homotopic to a non-trivial multiple of the core fundamental class $\varsigma$ of the annulus and $[\phi \circ \varsigma] = [\varsigma]$ in $\pi_1(U, z)$. But this also leads to a contradiction since $z \in Q(\mathrm{supp}(\mu), \phi)$ and the proof is complete. □

PROPOSITION 5.8. *Consider the setting in Theorem 1.2. If* $Q(\mathrm{supp}(\mu), \phi)$ *is non-empty, then* $G$ *has a global fixed point.*

*Proof.* Let us denote $K = \mathrm{supp}(\mu)$. If $K \cap \mathrm{Fix}(\phi) \neq \emptyset$, then there is nothing to prove. Now let $z \in Q(K, \phi)$ and suppose that $K \cap \mathrm{Fix}(\phi) = \emptyset$. Consider the connected component $U$ of $\mathbb{T}^2 \setminus K$ containing $z$. An orientation-preserving homeomorphism $\psi$ defined in a connected topological manifold $M$ leaves the connected components of $M \setminus \mathrm{Fix}(\psi)$ invariant [**7**]. Since $K \subset \mathrm{Fix}(H)$, we deduce that $U$ is $H$-invariant.

If $U$ is a topological disk, then the restriction of $G$ to $U$ defines a nilpotent group such that $\mathrm{Fix}(\phi|_U)$ is a non-empty compact set and we obtain $\mathrm{Fix}(G) \neq \emptyset$ by Theorem 2.5.

Suppose that $U$ is an annulus and let $\kappa'$ be a core curve of the annulus. The curve $\kappa'$ is non-essential in $\mathbb{T}^2$, otherwise $z$ does not belong to $Q(K, \phi)$. Thus, we obtain $\mathrm{Fix}(G) \neq \emptyset$ by Lemma 5.3. Hence, we can suppose from now on that $U$ has negative Euler characteristic. Let us denote $\tilde{\phi} = \tilde{\phi}_{\tilde{z}}$ (cf. Definition 5.5). Following Lemma 5.7, we can assume that $\mathrm{Fix}(\tilde{\phi})$ is a non-empty compact set. Moreover, we can admit that $h$ is isotopic to the identity relative to $K$ for any $h \in H$ by Lemma 5.4.

Now let $\tilde{H} \subset \mathrm{Diff}_0^1(\tilde{U})$ be the subgroup composed by the identity lifts of elements of $H$. The identity lift is well defined since $\chi(U) < 0$. Given that the elements of $\tilde{H}$ are the identity lifts of elements in $H$, we have $\tilde{\phi}\tilde{H}\tilde{\phi}^{-1} = \tilde{H}$. Consequently, $\tilde{H}$ is a normal subgroup of $\langle \tilde{H}, \tilde{\phi} \rangle$. Moreover, from $[\langle \tilde{H}, \tilde{\phi} \rangle, \langle \tilde{H}, \tilde{\phi} \rangle] \subset \tilde{H}$, we conclude that $\langle \tilde{H}, \tilde{\phi} \rangle$ is

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a nilpotent group as we have done at the end of the proof of Proposition 5.1. Therefore, $\langle \tilde{H}, \tilde{\phi} \rangle$ has a global fixed point by Theorem 2.5 and we obtain $\mathrm{Fix}(G) \neq \emptyset$. □

In order to complete the proof of Theorem 1.2, it suffices to show that $Q(\mathrm{supp}(\mu), \phi)$ is non-empty. We need the next auxiliary definition.

*Definition 5.9.* Let $\mathfrak{f} \in \mathrm{Homeo}_+(\mathbb{R}^2)$. Consider a $\mathfrak{f}$-invariant compact set $\mathcal{O}$ such that $\mathrm{Fix}(\mathfrak{f}) \cap \mathcal{O} = \emptyset$. Let us define a subset $P(\mathcal{O}, \mathfrak{f})$ of $\mathrm{Fix}(\mathfrak{f})$. We say that $\mathfrak{p} \in \mathrm{Fix}(\mathfrak{f})$ does not belong to $P(\mathcal{O}, \mathfrak{f})$ if there exists a path $\gamma : [0, \infty) \to \mathbb{R}^2 \setminus \mathcal{O}$ such that $\gamma(0) = \mathfrak{p}$, $\lim_{t\to\infty} \|\gamma(t)\| = \infty$ and $\mathfrak{f} \circ \gamma$ is properly homotopic to $\gamma$ in $\mathbb{R}^2 \setminus \mathcal{O}$ relative to 0.

In [**14**, Proposition 5.3], Franks, Handel and Parwani proved the following result.

THEOREM 5.10. *Consider the setting in the above definition. If $\mathcal{O} \neq \emptyset$, then $P(\mathcal{O}, \mathfrak{f}) \neq \emptyset$.*

The next lemma completes the proof of Theorem 1.2. In its proof we are going to use the above theorem only when the compact set is a finite orbit. In such a case, the result is due to Gambaudo [**16**].

LEMMA 5.11. *Let $\phi \in \mathrm{Diff}_0^1(\mathbb{T}^2)$. Suppose that there exist a $\phi$-invariant ergodic probability measure $\mu$ and a lift $\tilde{\phi} \in \mathrm{Diff}_0^1(\mathbb{R}^2)$ such that $\rho_\mu(\tilde{\phi}) = (0, 0)$. If $\mathrm{Fix}(\phi) \cap \mathrm{supp}(\mu) = \emptyset$, then $Q(\mathrm{supp}(\mu), \phi) \neq \emptyset$.*

*Proof.* Since $\rho_\mu(\tilde{\phi}) = (0, 0)$ and $\mu$ is $\phi$-ergodic, we have that $\mathrm{Fix}(\phi) \neq \emptyset$ by Franks' theorem 1.1. Denote $K = \mathrm{supp}(\mu)$ and let $p$ be a fixed point of $\phi$. If $p$ belongs to $Q(K, \phi)$, then there is nothing to prove.

Suppose that $p \notin Q(K, \phi)$ and consider the simple closed curve $\varsigma_0$ in $\mathbb{T}^2 \setminus K$ provided by Definition 5.6. This loop is based on $p \in \mathrm{Fix}(\phi)$ and it is essential in $\mathbb{T}^2$. Moreover, $\varsigma_0$ and $\phi \circ \varsigma_0$ are homotopic in $\mathbb{T}^2 \setminus K$ relative to the base point $p$. Let $v \in \mathbb{Z}^2$ be the non-zero vector such that the translation $T_v$ is the covering transformation for $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ associated to $\varsigma_0$. A result by Koropecki and Tal [**20**, Lemma 4.4], inspired by Atkinson's lemma [**1**], implies that there exists a total measure set $E \subset K \subset \mathbb{T}^2$ such that for any $z \in E$, there exists an increasing sequence $(n_k)_{k\geq 1}$ of positive integers such that

$$\phi^{n_k}(z) \to z \quad \text{and} \quad < \tilde{\phi}^{n_k}(\tilde{z}) - \tilde{z}, v > \to 0, \tag{5.11.1}$$

where $<, >$ is the canonical inner product and $\tilde{z}$ is any lift of $z$. Moreover, since $\varsigma_0$ is essential in $\mathbb{T}^2$ and $[\phi \circ \varsigma_0] = [\varsigma_0]$ in $\pi_1(\mathbb{T}^2 \setminus K, p)$, we conclude that $\tilde{z}$ and $\tilde{\phi}(\tilde{z})$ are in the same connected component of $\pi^{-1}(\mathbb{T}^2 \setminus \varsigma_0([0, 1]))$ for any $\tilde{z} \in \pi^{-1}(K)$. In particular, we obtain from (5.11.1) that $\tilde{\phi}^{n_k}(\tilde{z}) \to \tilde{z}$ whenever $\pi(\tilde{z}) \in E$. Therefore, the set of points in $K$ whose preimages are $\tilde{\phi}$-recurrent has total measure.

We will prove the lemma by using Theorem 5.10. For this, we consider a local small $C^0$ perturbation of $\tilde{\phi}$ without changing the set of fixed points and getting a new non-empty compact set which is invariant by the perturbation. This new invariant compact set for the perturbation will be a periodic orbit contained in $\pi^{-1}(K)$.

Let us fix $\tilde{q} \in \pi^{-1}(K)$, a $\omega$-recurrent point for $\tilde{\phi}$. We will choose $H \in \mathrm{Homeo}_0(\mathbb{R}^2)$ whose support is contained in a small compact disk $\tilde{D}$ containing $\tilde{q}$ in its interior. We are always assuming that $\tilde{D}$ is a lift of a small disk $D$ contained in $\mathbb{T}^2$. Furthermore, $H$ is such that:

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- the orbit $\mathcal{O}$ of $\tilde{q}$ by $H \circ \tilde{\phi}$ is finite and $\mathcal{O} \subset \pi^{-1}(K)$;
- $\mathrm{Fix}(\tilde{\phi}) = \mathrm{Fix}(H \circ \tilde{\phi})$.

Of course, the choice of $H$ depends on the compact disk $D$, which can be made arbitrarily small.

Now let us remind the reader that Franks proved the existence of a fixed point for $\phi$, proving that $\tilde{\phi}$ has a fixed point (cf. the beginning of the proof of Theorem 3.5 in [**13**]). In the same spirit, to finish the proof of the lemma, it suffices to conclude that

$$\mathrm{Fix}(\tilde{\phi}) \setminus \pi^{-1}(Q(K, \phi)) \subset \mathrm{Fix}(H \circ \tilde{\phi}) \setminus P(\mathcal{O}, H \circ \tilde{\phi}). \tag{5.11.2}$$

From the above relation, we obtain $Q(K, \phi) \neq \emptyset$ as required by the lemma since we have $P(\mathcal{O}, H \circ \tilde{\phi}) \neq \emptyset$ by Theorem 5.10.

To prove (5.11.2) for a suitable choice of $H$ and $D$, we do the following. Consider $\tilde{z}_1 \in \mathrm{Fix}(\tilde{\phi}) \setminus \pi^{-1}(Q(K, \phi))$. The point $z_1 = \pi(\tilde{z}_1)$ is in $\mathrm{Fix}(\phi) \setminus Q(K, \phi)$. Consider a simple closed curve $\varsigma$ in $\mathbb{T}^2 \setminus K$ given by Definition 5.6. We recall that $\varsigma$ is based on $z_1$, is essential in $\mathbb{T}^2$ and $[\phi \circ \varsigma] = [\varsigma]$ in $\pi_1(\mathbb{T}^2 \setminus K, z_1)$. Let $\tilde{\varsigma} : [0, 1] \to \mathbb{R}^2 \setminus \pi^{-1}(K)$ be the lift of $\varsigma$ with $\tilde{\varsigma}(0) = \tilde{z}_1$ and let $T_w$ be the translation given by the covering transformation for $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ associated to $\varsigma$.

We extend $\tilde{\varsigma}$ to $[0, \infty)$ by defining $\tilde{\varsigma}(t) = T_w^{[t]}(\tilde{\varsigma}(t - [t]))$, where $[t]$ denotes the greatest integer less than or equal to $t$. It is a continuous path such that $\|\tilde{\varsigma}(t)\| \to \infty$ when $t$ goes to infinity. The loops $\varsigma$ and $\phi \circ \varsigma$ are homotopic in $\mathbb{T}^2 \setminus K$ relative to the base point $z_1$. Thus, the paths $\tilde{\varsigma}|_{[0,1]}$ and $(\tilde{\phi} \circ \tilde{\varsigma})|_{[0,1]}$ are homotopic relative to the ends $\tilde{z}_1$, $T_w(\tilde{z}_1)$ by a homotopy $F : [0, 1] \times [0, 1] \to \mathbb{R}^2 \setminus \pi^{-1}(K)$. We extend $F$ to $[0, \infty) \times [0, 1]$ by defining $F(t, s) = T_w^{[t]}(F(t - [t], s))$. In this way, the extended paths $\tilde{\varsigma}$ and $\tilde{\phi} \circ \tilde{\varsigma}$ are properly homotopic in $\mathbb{R}^2 \setminus \pi^{-1}(K)$ relative to $0 \in [0, \infty)$.

Furthermore, since the loops $\varsigma$ and $\phi \circ \varsigma$ are homotopic in $\mathbb{T}^2 \setminus K$, we conclude that the extended homotopy $F$ between the extended curves $\tilde{\varsigma}$ and $\tilde{\phi} \circ \tilde{\varsigma}$ does not intersect the set $\pi^{-1}(\pi(\tilde{D}))$ if $D$ is sufficiently small. Thus, $\tilde{\phi} \circ T_u \circ \tilde{\varsigma} = (H \circ \tilde{\phi}) \circ T_u \circ \tilde{\varsigma}$ and the extended homotopy $T_u \circ F$ is a proper homotopy in $\mathbb{R}^2 \setminus \pi^{-1}(K)$ between the extended curves $T_u \circ \tilde{\varsigma}$ and $(H \circ \tilde{\phi}) \circ T_u \circ \tilde{\varsigma}$ for every $u \in \mathbb{Z}^2$. Since $\mathrm{Fix}(\tilde{\phi}) = \mathrm{Fix}(H \circ \tilde{\phi})$ by construction, we obtain

$$\pi^{-1}(z_1) \subset \mathrm{Fix}(H \circ \tilde{\phi}) \setminus P(\mathcal{O}, H \circ \tilde{\phi})$$

following Definition 5.9.

We remind the reader that $\tilde{D}$ is the lift containing $\tilde{q}$ of a convenient small disk in $\mathbb{T}^2$ containing $q = \pi(\tilde{q})$ in its interior.

Now we need to make a choice of $D$ that does not depend on $z \in \mathrm{Fix}(\phi) \setminus Q(K, \phi)$. For this, we remark that if the above construction holds true for some $z \in \mathrm{Fix}(\phi) \setminus Q(K, \phi)$, then the same is true for all the points of $\mathrm{Fix}(\phi) \setminus Q(K, \phi)$ in a small neighborhood of $z$ even without changing the compact disk $D$. Moreover, we have that $\mathrm{Fix}(\phi) \setminus Q(K, \phi)$ is a compact set since $\mathrm{Fix}(\phi) \cap K = \emptyset$. The existence of a common choice of $D$ for any $z \in \mathrm{Fix}(\phi) \setminus Q(K, \phi)$ is a consequence of the compactness of $\mathrm{Fix}(\phi) \setminus Q(K, \phi)$, completing the proof of the lemma. □

Now we prove Theorems 1.2, 1.3 and 1.5.

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*Proof of Theorem 1.2.* If $\mathrm{Fix}(\phi) \cap \mathrm{supp}(\mu) \neq \emptyset$, then $G$ has a global fixed point since $\mathrm{supp}(\mu) \subset \mathrm{Fix}(H)$ and $G = \langle H, \phi \rangle$. If $\mathrm{Fix}(\phi) \cap \mathrm{supp}(\mu) = \emptyset$, then Theorem 1.2 follows from Lemma 5.11 and Proposition 5.8.

*Proof of Theorem 1.3.* We can suppose that $G$ is finitely generated. Let us show that $\mathrm{Fix}(Z^{(k)}(G)) \neq \emptyset$ by induction on $k$; it is obvious for $k = 0$. Suppose that $\mathrm{Fix}(Z^{(k)}(G)) \neq \emptyset$. Since $G$ is a finitely generated nilpotent group, it is polycyclic (cf. [**19**, Theorem 17.2.2]) and as a consequence every subgroup of $G$ is finitely generated. Let $\{g_1, \ldots, g_n\}$ be a generator set of $Z^{(k+1)}(G)$. We denote $H^0 = Z^{(k)}(G)$ and $H^j = \langle Z^{(k)}(G), g_1, \ldots, g_j \rangle$ for any $0 < j \leq n$. Notice that $H^j$ is a normal subgroup of $H^{j+1} = \langle H^j, g_{j+1} \rangle$ for any $0 \leq j < n$.

Let us show that $\mathrm{Fix}(H^j)$ is a non-empty set for any $j \geq 0$ by induction on $j$. The result is clear for $j = 0$. Since $H^j$ is normal in $H^{j+1}$, the set $\mathrm{Fix}(H^j)$ is $g_{j+1}$-invariant. Let $\mu$ be a $g_{j+1}$-invariant ergodic measure with $\mathrm{supp}(\mu) \subset \mathrm{Fix}(H^j)$. The diffeomorphism $g_{j+1}$ is irrotational; hence, we deduce that $\mathrm{Fix}(H^{j+1}) \neq \emptyset$ by Theorem 1.2. We obtain $\mathrm{Fix}(H^j) \neq \emptyset$ for any $0 \leq j \leq n$ and in particular $\mathrm{Fix}(Z^{(k+1)}(G)) \neq \emptyset$. As a consequence, $\mathrm{Fix}(Z^{(k)}(G))$ is a non-empty set for any $k \in \mathbb{Z}_{\geq 0}$. Since $G$ is nilpotent, we deduce that $\mathrm{Fix}(G) \neq \emptyset$.

*Proof of Theorem 1.5.* Let us prove the result for $G \subset \mathrm{Diff}^1(\mathbb{T}^2)$. The remaining cases are analogous. The group $G'''$ is irrotational by Theorem 1.4. Thus, $\mathrm{Fix}(G''')$ is non-empty by Theorem 1.3.

## 6. *Replacing $\mathbb{T}^2$ with the Klein bottle, the compact annulus or the Möbius strip*

In this section, we remark that almost all the theorems in this article have versions for the Klein bottle $\mathbb{K}^2$, the compact annulus $\mathbb{S}^1 \times [0, 1]$ and the compact Möbius strip.

First, let us remark that an isotopic to the identity $\mathbb{S}^1$-homeomorphism has a trivial rotation number if and only if it has a fixed point. In this context, the following result can be interpreted as a version of Theorems 1.2 and 1.3. Notice that solvable (and in particular nilpotent) groups are amenable.

PROPOSITION 6.1. *Let $G$ be an amenable subgroup of $\mathrm{Homeo}_+(\mathbb{S}^1)$ such that every element of $G$ has a non-empty fixed-point set. Then $G$ has a global fixed point.*

*Proof.* Since $G$ is amenable, there exists a $G$-invariant Borel probability measure $\mu$. Given $\phi \in G$, the dynamics of $\phi$ in every connected component of $\mathbb{S}^1 \setminus \mathrm{Fix}(\phi)$ is conjugated to a translation. Thus, no point of $\mathbb{S}^1 \setminus \mathrm{Fix}(\phi)$ can be in the support of a $\phi$-invariant Borel probability measure. As a consequence, we obtain $\mathrm{supp}(\mu) \subset \mathrm{Fix}(G)$ and hence $\mathrm{Fix}(G)$ is non-empty. □

The versions of Theorems 1.4 and 1.5 for the circle are very elementary. In this context, given an amenable subgroup $G$ of $\mathrm{Homeo}(\mathbb{S}^1)$, then:

(i) $G'$ is irrotational when $G \subset \mathrm{Homeo}_+(\mathbb{S}^1)$;

(ii) $G''$ is irrotational when $G \subset \mathrm{Homeo}(\mathbb{S}^1)$.

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Indeed, the rotation number for elements of $G$ coincides with the rotation number with respect to a $G$-invariant measure. Thus, the map sending each element of an amenable subgroup $G$ of $\mathrm{Homeo}_+(\mathbb{S}^1)$ to its Poincaré's rotation number is a morphism of groups by an analogue of Lemma 2.3 for the circle. Thus, every element of $G'$ has vanishing rotation number. Consequently, in cases (i) and (ii) above, there exist global fixed points by Proposition 6.1.

In what follows, let $\tau : \mathbb{T}^2 \to \mathbb{K}^2$ be the 2-fold orientation covering map of the Klein bottle by the 2-torus and let us denote by $\sigma$ the non-trivial $\tau$-lift of the identity map. Consider the covering transformations

$$T_1(x, y) = (x + 1/2, -y) \quad \text{and} \quad T_2(x, y) = (x, y + 1)$$

generating the fundamental group of $\mathbb{K}^2 = \mathbb{R}^2/\langle T_1, T_2\rangle$. Moreover, we have $\mathbb{T}^2 = \mathbb{R}^2/\langle T_1^2, T_2\rangle$.

Given a subgroup $G \subset \mathrm{Homeo}(\mathbb{K}^2)$, let us denote by $G_\tau \subset \mathrm{Homeo}(\mathbb{T}^2)$ the subgroup of all $\tau$-lifts of elements of $G$. As remarked in [**11**, §4], given $f \in G$, its distinct $\tau$-lifts are $\hat{f}$ and $\sigma \circ \hat{f}$. Moreover, $\hat{f}$ and $\sigma$ commute and, consequently, $G_\tau$ is nilpotent when $G$ has this property.

On the other hand, for each $f \in \mathrm{Homeo}_0(\mathbb{K}^2)$, there exists a unique $\tau$-lift $f_\tau$ of $f$ in $\mathrm{Homeo}_0(\mathbb{T}^2)$. It can be obtained by lifting via $\tau$ any isotopy from the identity of $\mathbb{K}^2$ to $f$ starting at the identity of $\mathbb{T}^2$. Let us remark that $\sigma \circ f_\tau$ cannot be obtained by this process since $\sigma$ is orientation reversing. In this way, we construct a natural $\tau$-lift $G_{\tau,0} \subset \mathrm{Homeo}_0(\mathbb{T}^2)$ of any subgroup $G \subset \mathrm{Homeo}_0(\mathbb{K}^2)$.

For a given $f \in \mathrm{Homeo}(\mathbb{K}^2)$ and $\mu \in \mathcal{P}(f)$, we denote by $\mu_\tau$ the Borel probability measure on $\mathbb{T}^2$ obtained by normalizing the lift of $\mu$ via the local homeomorphism $\tau$. In this case, the measure $\mu_\tau$ is the unique probability measure invariant by $f_\tau$ and $\sigma$ (and then also by $\sigma \circ f_\tau$), but we lost the ergodicity for the measure $\mu_\tau$ if $\mu$ is ergodic. Let us address this issue. There exists a natural map

$$\begin{aligned} \tau_* : \mathcal{P}(f_\tau) &\to \mathcal{P}(f) \\ \nu &\mapsto \tau_*\nu, \end{aligned}$$

where $(\tau_*\nu)(B) = \nu(\tau^{-1}(B))$ for any Borel subset $B$ of $\mathbb{T}^2$. Let $\mu$ be an $f$-ergodic Borel probability measure. The measure $\mu_\tau$ is the unique Borel probability measure in $(\tau_*)^{-1}(\mu)$ such that $\sigma_*\mu_\tau = \mu_\tau$. The subset $(\tau_*)^{-1}(\mu)$ of $\mathcal{P}(f_\tau)$ is non-empty (it contains $\mu_\tau$), compact in the weak* topology and convex. Hence, $(\tau_*)^{-1}(\mu)$ has an extreme point $\nu$ by the Bauer maximum principle (cf. [**8**, Corollary 25.10]). We claim that $\nu$ is $f_\tau$-ergodic. Otherwise, $\nu$ is not an extreme point among $f_\tau$-invariant measures, i.e. there exist $f_\tau$-invariant Borel probability measures $\nu_1$, $\nu_2$ and $0 < t < 1$ such that $\nu_1 \neq \nu_2$ and $\nu = t\nu_1 + (1-t)\nu_2$. We deduce that $\tau_*\nu = t\tau_*\nu_1 + (1-t)\tau_*\nu_2$. Since $\tau_*\nu = \mu$ is $f$-ergodic, it follows that $\tau_*\nu$ is an extreme point of the set of $f$-invariant Borel probability measures. We obtain $\tau_*\nu = \tau_*\nu_1 = \tau_*\nu_2 = \mu$. Now the equality $\nu = t\nu_1 + (1-t)\nu_2$ (with $0 < t < 1$ and $\nu_1 \neq \nu_2$) contradicts that $\nu$ is an extreme point of $(\tau_*)^{-1}(\mu)$.

Since $\nu$ is ergodic, there exists a Borel set $A$ such that $\nu(A) = 1$ and

$$\nu = \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \delta_{f_\tau^j(p)}}{n}$$

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in the weak* topology for any $p \in A$, where $\delta_q$ is the Dirac delta at $q$. Moreover, the equality $f_\tau \circ \sigma = \sigma \circ f_\tau$ implies that

$$\sigma_* \nu = \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \delta_{f_\tau^j(p)}}{n}$$

for any $p \in \sigma(A)$. By construction, we have $\mu(\tau(A)) = 1$ and

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \delta_{f_\tau^j(p)}}{n} \in \{\nu, \sigma_*\nu\}$$

for any $p \in \tau^{-1}(\tau(A))$. Since $\nu'(\tau^{-1}(\tau(A))) = 1$ for any probability measure in $(\tau_*)^{-1}(\mu)$, we deduce that $\nu$ and $\sigma_*\nu$ are the unique ergodic $f_\tau$-invariant probability measures by Birkhoff's ergodic theorem. The measure $(\nu + \sigma_*\nu)/2$ belongs to $(\tau_*)^{-1}(\mu)$ and is $\sigma$-invariant and hence it is equal to $\mu_\tau$. Notice that $\mu_\tau$ is ergodic if and only if $\sigma_*\nu = \nu$ and then $(\tau_*)^{-1}(\mu) = \{\mu_\tau\}$.

Let $\tilde{f}_\tau$ be an identity lift of $f_\tau$ to $\mathbb{R}^2$. We denote $\rho_\nu(\tilde{f}_\tau) = (a, b)$. We have

$$\lim_{n\to\infty} \frac{\tilde{f}_\tau^n(x, y) - (x, y)}{n} = \rho_\nu(\tilde{f}_\tau) \quad \text{and} \quad \lim_{n\to\infty} \frac{\tilde{f}_\tau^n(T_1(x, y)) - T_1(x, y)}{n} = (a, -b)$$

for any $(x, y) \in \pi^{-1}(p)$ and $\nu$-almost every $p \in \mathbb{T}^2$ by Birkhoff's ergodic theorem and simple calculations. We deduce that $\rho_{\sigma_*\nu}(\tilde{f}_\tau) = (a, -b)$ and then $\rho_{\mu_\tau}(\tilde{f}_\tau) = (a, 0)$. Thus, the vector $(a, |b|)$ does not depend on the choice of the ergodic measure in $(\tau_*)^{-1}(\mu)$ and the following concept is well defined.

*Definition 6.2.* Let $\mu$ be an ergodic $f$-invariant measure and $\tilde{f}_\tau$ be an identity lift of $f_\tau$ to $\mathbb{R}^2$. We define $\overline{\rho}_\mu(\tilde{f}_\tau)$ as the vector $(a, |b|)$, where $(a, b) = \rho_\nu(\tilde{f}_\tau)$ for some ergodic $f_\tau$-invariant measure $\nu$ with $\tau_*\nu = \mu$.

*Remark 6.3.* Notice that any other identity lift $\hat{f}_\tau$ of $f_\tau$ to $\mathbb{R}^2$ is of the form $T \circ \tilde{f}_\tau$, where $T$ is a covering transformation of $\mathbb{T}^2$ that commutes with every map in $\langle T_1, T_2\rangle$. As a consequence, $T$ is of the form $T(x, y) = (x + m, y)$ for some $m \in \mathbb{Z}$. We obtain $\overline{\rho}_\mu(\hat{f}_\tau) = (a + m, |b|)$.

*Definition 6.4.* Let $\mu$ be an ergodic $f$-invariant measure. We define $\overline{\rho}_\mu(f_\tau)$ as the class of the vector $(a, |b|)$ in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$, where $(a, b) = \overline{\rho}_\mu(\tilde{f}_\tau)$ for some identity lift $\tilde{f}_\tau$ to $\mathbb{R}^2$.

The following lemma is a trivial consequence of the previous discussion.

LEMMA 6.5. *Let $f \in \mathrm{Homeo}_0(\mathbb{K}^2)$. Consider an ergodic $f$-invariant Borel probability measure $\mu$. Then the following conditions are equivalent:*

- $\overline{\rho}_\mu(f_\tau) \in \{0\} \times \mathbb{Z} \subset \mathbb{R}/\mathbb{Z} \times \mathbb{R}$;
- $\rho_\nu(f_\tau) = 0 \in \mathbb{R}^2/\mathbb{Z}^2$ *for some ergodic $f_\tau$-invariant probability measure* $\nu \in (\tau^*)^{-1}(\mu)$;
- $\rho_\nu(f_\tau) = 0 \in \mathbb{R}^2/\mathbb{Z}^2$ *for any ergodic $f_\tau$-invariant probability measure* $\nu \in (\tau^*)^{-1}(\mu)$.

Now we can introduce the analogue of Theorem 1.2 for the Klein bottle.

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COROLLARY 6.6. *Let $G = \langle H, \phi\rangle$ be a nilpotent subgroup of $\mathrm{Diff}_0^1(\mathbb{K}^2)$, where $H$ is a normal subgroup of $G$. Suppose that there exists a $\phi$-invariant ergodic probability measure $\mu$ such that the support of $\mu$ is contained in $\tau(\mathrm{Fix}(H_{\tau,0}))$ and $\overline{\rho}_\mu(\phi_\tau) \in \{0\} \times \mathbb{Z}$. Then $G$ has a global fixed point.*

*Proof.* Let $\nu$ be an ergodic $\phi_\tau$-invariant Borel probability measure $\nu \in (\tau^*)^{-1}(\mu)$. Given a point $p \in \mathbb{K}^2$ in $\mathrm{supp}(\mu)$, there exists $p_\tau \in \mathbb{T}^2 \cap \tau^{-1}(p) \cap \mathrm{Fix}(H_{\tau,0})$ by hypothesis. Since $\sigma$ commutes with $g_\tau$ for any $g \in H$, we obtain $\mathrm{supp}(\nu) \subset \tau^{-1}(\mathrm{supp}(\mu)) \subset \mathrm{Fix}(H_{\tau,0})$. Since $H_{\tau,0}$ is normal in $\langle H_{\tau,0}, \phi_\tau\rangle$ and $\rho_\nu(\phi_\tau) = 0 \in \mathbb{R}^2/\mathbb{Z}^2$ (Lemma 6.5), it follows that $\langle H_{\tau,0}, \phi_\tau\rangle$ has a global fixed point by Theorem 1.2. Thus, $\langle H, \phi\rangle$ has a global fixed point in $\mathbb{K}^2$. □

We can refine the previous result by replacing the condition $\tau^{-1}(\mathrm{supp}(\mu)) \subset \mathrm{Fix}(H_{\tau,0})$ with $\mathrm{supp}(\mu) \subset \mathrm{Fix}(H)$.

PROPOSITION 6.7. *Let $G = \langle H, \phi\rangle$ be a nilpotent subgroup of $\mathrm{Diff}_0^1(\mathbb{K}^2)$, where $H$ is a normal subgroup of $G$. Suppose that there exists a $\phi$-invariant ergodic probability measure $\mu$ such that $\mathrm{supp}(\mu) \subset \mathrm{Fix}(H)$ and $\overline{\rho}_\mu(\phi_\tau) \in \{0\} \times \mathbb{Z}$. Then $G$ has a finite orbit with $2^m$ elements, where $m$ is at most the nilpotency class of $G$.*

*Proof.* Let $\nu$ be an ergodic $\phi_\tau$-invariant Borel probability measure $\nu$ with $\tau_*\nu = \mu$. We denote $\psi_j = \phi_\tau^{2^j}$ for $j \geq 0$. We define $\nu_0 = \nu$. Given $j \geq 0$, we consider an ergodic $\phi_{j+1} = \phi_j^2$-invariant Borel probability measure $\nu_{j+1}$ such that

$$\nu_j = \frac{\nu_{j+1} + \phi_j^*\nu_{j+1}}{2}.$$

There exists a lift $\tilde{\psi}_0$ of $\psi_0$ to $\mathbb{R}^2$ such that $\rho_{\nu_0}(\tilde{\psi}_0) = (0, 0)$ by Lemma 6.5. We denote $\tilde{\psi}_{j+1} = \tilde{\psi}_j^2$ for any $j \geq 0$. We have $\rho_{\nu_{j+1}}(\tilde{\psi}_{j+1}) = 2\rho_{\nu_j}(\tilde{\psi}_j)$ for any $j \geq 0$ by construction.

Let us show, by induction on $j$, that $[\psi_j, \eta_\tau]$ fixes $\tau^{-1}(\mathrm{supp}(\nu_j))$ pointwise for all $j \geq 0$ and $\eta \in Z^{(j+1)}(G) \cap H$. The result is obvious for $j = 0$ since $[\psi_0, \eta_\tau] = \mathrm{Id}$ for any $\eta \in Z^{(1)}(G) \cap H$. Suppose that the result holds for $j \geq 0$. Given $\eta \in Z^{(j+2)}(G) \cap H$, we have $\psi_j \circ \eta_\tau \circ \psi_j^{-1} = h_\tau \circ \eta_\tau$ for some $h \in Z^{(j+1)}(G) \cap H$. We get

$$\psi_j^2 \circ \eta_\tau \circ \psi_j^{-2} = (\psi_j \circ h_\tau \circ \psi_j^{-1}) \circ h_\tau \circ \eta_\tau.$$

Since $[\psi_j, h_\tau]$ fixes $\tau^{-1}(\mathrm{supp}(\nu_j))$ by the induction hypothesis and $\mathrm{supp}(\nu_{j+1}) \subset \mathrm{supp}(\nu_j)$, it follows that $\psi_j \circ h_\tau \circ \psi_j^{-1} = h'_\tau \circ h_\tau$ for some $h' \in Z^{(j)}(G) \cap H$ such that $h'_\tau$ fixes $\tau^{-1}(\mathrm{supp}(\nu_{j+1}))$ pointwise. Since $[\psi_{j+1}, \eta_\tau] = h'_\tau \circ h_\tau^2$, it follows that $[\psi_{j+1}, \eta_\tau]$ fixes $\tau^{-1}(\mathrm{supp}(\nu_{j+1}))$ pointwise for any $\eta \in Z^{(j+2)}(G) \cap H$.

Let $n$ be the nilpotency class of $G$ and $r = 2^{n-1}$. We denote $\psi = \phi^r$. Given $\eta \in H$, we define the subset $A_\eta$ of $\tau(\mathrm{sup}(\nu_{n-1}))$ of points $p$ such that $\tau^{-1}(p) \subset \mathrm{Fix}(\eta_\tau)$. The property $\mathrm{sup}(\nu_{n-1}) \subset \mathrm{Fix}[\psi_\tau, \eta_\tau]$ implies that $A_\eta$ is $\psi$-invariant. Now consider a $\nu_{n-1}$ generic point $p \in \mathbb{T}^2$ and its $\psi_\tau$-orbit $\mathcal{O}$. We have that $\tau(\mathcal{O})$ is contained in either $A_\eta$ or its complement. Since $\overline{\mathcal{O}} = \mathrm{sup}(\nu_{n-1})$ and both sets $A_\eta$ and $\tau(\mathrm{sup}(\nu_{n-1})) \setminus A_\eta$ are open and closed in $\tau(\mathrm{sup}(\nu_{n-1}))$, we deduce that $A_\eta = \emptyset$ or $A_\eta = \tau(\mathrm{sup}(\nu_{n-1}))$. We define

$$H_+ = \{\eta \in H : \mathrm{sup}(\nu_{n-1}) \subset \mathrm{Fix}(\eta_\tau)\}.$$

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Since $H$ is a normal subgroup of $G$, it follows that $H_+$ is also a normal subgroup of $\langle H_+, \psi\rangle$ of index at most 2 in $H$. By construction, we have $\sup(\nu_{n-1}) \subset \mathrm{Fix}(H_+)_{\tau,0}$. This condition together with $\rho_{\nu_{n-1}}(\tilde{\psi}_{n-1}) = (0, 0)$ allows to apply Corollary 6.6 to obtain a global fixed point of $\langle (H_+)_{\tau,0}, \psi_\tau\rangle$ and then a global fixed point $p$ of $\langle H_+, \psi\rangle$. It is clear that $|\langle H, \phi\rangle : \langle H, \psi\rangle|$ is of the form $2^a$ for some $0 \le a \le n-1$ since $\phi^r = \psi$ and $r = 2^{n-1}$. Since $H$ is a normal subgroup of $G$, it follows that $|\langle H, \psi\rangle : \langle H_+, \psi\rangle|$ is either 1 or 2. Thus, the index $|\langle H, \phi\rangle : \langle H_+, \psi\rangle|$ is of the form $2^b$ with $0 \le b \le n$. Consider the subgroup $J$ of $G$ such that $J = \{\eta \in G : \eta(p) = p\}$. The group $J$ contains $\langle H_+, \psi\rangle$ and thus $|G : J|$ is of the form $2^m$ with $0 \le m \le n$. Since $|G : J|$ is the cardinal of the $G$-orbit of $p$, the result is proved. □

Let us focus now on the generalization of Theorem 1.3 for the Klein bottle.

*Definition 6.8.* We say that $f \in \mathrm{Homeo}_0(\mathbb{K}^2)$ is *irrotational* when the $\tau$-lift $f_\tau \in \mathrm{Homeo}_0(\mathbb{T}^2)$ is irrotational.

*Remark 6.9.* An alternative definition of an irrotational element of $\mathrm{Homeo}(\mathbb{K}^2)$ is the following: an element $f$ of $\mathrm{Homeo}(\mathbb{K}^2)$ is irrotational if its orientation-preserving lift $f_+$ to $\mathbb{T}^2$ is irrotational. A priori, the definitions could be different since a non-isotopic to the identity $f \in \mathrm{Homeo}(\mathbb{K}^2)$ can have an isotopic to the identity lift $f_+$. An example is the homeomorphism induced by $\tilde{f}(x, y) = (x, y + 1/2)$ in the Klein bottle. Anyway, we claim that both definitions are equivalent. It suffices to show that if $f$ satisfies the second definition, then $f$ belongs to $\mathrm{Homeo}_0(\mathbb{K}^2)$. Let $\tilde{f}$ be the irrotational lift of $f_+$. The sequence of maps $(\tilde{f}^n - \mathrm{Id}/n)$ converges uniformly to the constant map $(0, 0)$ when $n \to \infty$. Since $f_+$ is isotopic to the identity, it follows that $\tilde{f} \circ T = T \circ \tilde{f}$ for any $T \in \langle T_1^2, T_2\rangle$.

Moreover, since $f_+ \circ \sigma = \sigma \circ f_+$, we obtain

$$\tilde{f} \circ T_1 \circ \tilde{f}^{-1} = (x + a, y + b) \circ T_1$$

for some $(a, b) \in \mathbb{Z}^2$. We deduce that

$$\tilde{f}^n \circ T_1 \circ \tilde{f}^{-n} = (x + na, y + nb) \circ T_1$$

for any $n \in \mathbb{Z}$. This leads us to

$$\frac{\tilde{f}^n - \mathrm{Id}}{n} \circ T_1 = \frac{(x + na, y + nb) \circ T_1 \circ \tilde{f}^n - T_1}{n} = (a, b) + (x, -y) \circ \frac{\tilde{f}^n - \mathrm{Id}}{n}.$$

The left-hand side of the previous expression converges uniformly to $(0, 0)$, whereas the right-hand side converges uniformly to $(a, b)$ when $n \to \infty$. Hence, we get $(a, b) = (0, 0)$ and thus $\tilde{f}$ commutes with every covering transformation in $\langle T_1, T_2\rangle$. As a consequence, $f$ is isotopic to the identity map.

We have the following immediate corollary of Theorem 1.3.

COROLLARY 6.10. *Let $G$ be an irrotational nilpotent subgroup of* $\mathrm{Diff}_0^1(\mathbb{K}^2)$. *Then $G$ has a global fixed point.*

Before we announce the corollaries of Theorems 1.4 and 1.5, let us recall another property of the group $G_\tau$.

Given a subgroup $G \subset \mathrm{Homeo}(\mathbb{K}^2)$, let us denote by $G_{\tau,+}$ the subgroup of all the elements of $G_\tau$ that are orientation preserving. It is not difficult to see that $G_{\tau,+}$ is

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a $\tau$-lift of $G$. Since $f_\tau$ and $\sigma$ commute for all $f \in G$, we can also conclude that $[G_\tau, G_\tau] = [G_{\tau,+}, G_{\tau,+}]$.

COROLLARY 6.11. *If $G$ is a nilpotent subgroup of* $\mathrm{Homeo}(\mathbb{K}^2)$*, then*:
(i) *$G'$ is irrotational when $G \subset \mathrm{Homeo}_0(\mathbb{K}^2)$*;
(iI) *$G''$ is irrotational when $G \subset \mathrm{Homeo}(\mathbb{K}^2)$*.

COROLLARY 6.12. *Let $G$ be a nilpotent subgroup of* $\mathrm{Diff}^1(\mathbb{K}^2)$*. The following subgroups of $G$ have a global fixed point*:
(i) *$G'$ when $G \subset \mathrm{Diff}^1_0(\mathbb{K}^2)$*;
(ii) *$G''$ when $G \subset \mathrm{Diff}^1(\mathbb{K}^2)$*.

For the compact annulus, we have the following version for Theorem 1.2.

COROLLARY 6.13. *Let $\mathcal{N} = \langle \mathcal{H}, \psi \rangle$ be a nilpotent subgroup of* $\mathrm{Diff}^1_0(\mathbb{S}^1 \times [0, 1])$*, where $\mathcal{H}$ is a normal subgroup of $\mathcal{N}$. Suppose that there exists a $\psi$-invariant ergodic probability measure $\mu$ such that the support of $\mu$ is contained in* $\mathrm{Fix}(\mathcal{H})$ *and $\rho_\mu(\psi)$ is the zero element in $\mathbb{R}/\mathbb{Z}$. Then $\mathcal{N}$ has a global fixed point.*

*Proof.* To detect a global fixed point for $\mathcal{N}$, we consider the double $\mathbb{T}^2$ of the annulus and the double of each element of $\mathcal{N}$. This construction give us a nilpotent subgroup $G = \langle H, \phi \rangle$ of $\mathrm{Homeo}_0(\mathbb{T}^2)$, where $H$ is a normal subgroup of $G$ and $\phi$ is the double of $\psi$. Moreover, let us fix one of the copies of the annulus in the double $\mathbb{T}^2$ and let us consider the measure $\mu$ on this copy as a measure on the double $\mathbb{T}^2$. We have that $\mu$ is a $\phi$-invariant ergodic probability measure on $\mathbb{T}^2$ whose support is contained in $\mathrm{Fix}(H)$ and $\rho_\mu(\phi)$ is the null element of $\mathbb{R}^2/\mathbb{Z}^2$.

Let us recall that in the above construction the group $G$ is a subgroup of $\mathrm{Homeo}_0(\mathbb{T}^2)$. At this point, to apply Theorem 1.2, we can smooth the action by using a result of Parkhe in [**24**]. He proved (cf. Theorem 5 and Remark 6 in §2) that there exists a homeomorphism of the annulus, supported in a small neighborhood of its boundary, such that the conjugate of each element of $\mathcal{N}$ by this homeomorphism gives a $C^1$ diffeomorphism that can be glued with itself to obtain a $C^1$ diffeomorphism in the double $\mathbb{T}^2$ of the annulus.

Of course, the $C^1$ diffeomorphism corresponding to the map $\phi$ and the measure corresponding to the measure $\mu$ via the conjugation stay with the same properties as above. Now we can apply Theorem 1.2 to conclude the existence of a global fixed point in the annulus. □

Notice that if $\mathrm{supp}(\mu)$ is contained in the annulus boundary $\mathcal{C}$, then $\mathcal{N}$ has a global fixed point in $\mathcal{C}$ even when $\mathcal{N}$ is contained in $\mathrm{Homeo}_0(\mathbb{S}^1 \times [0, 1])$ by Proposition 6.1.

Following the strategy of passing to the double of the annulus as in the proof of Corollary 6.13, we can easily adapt the statements of Theorems 1.3–1.5 for the case $\mathbb{S}^1 \times [0, 1]$. Since $G' \subset \mathrm{Homeo}_0(\mathbb{S}^1 \times [0, 1])$, the second derived group $G''$ always has a global fixed point. In a similar way, we can give versions of Corollaries 6.6, 6.10–6.12 and Proposition 6.7 for the compact Möbius strip.

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