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# Exact Solutions for Fractional Differential-Difference Equations by $(G'/G)$-Expansion Method with Modified Riemann-Liouville Derivative

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**Abstract.** In this paper, the $(G'/G)$-expansion method is suggested to establish new exact solutions for fractional differential-difference equations in the sense of modified Riemann–Liouville derivative. The fractional complex transform is proposed to convert a fractional partial differential difference equation into its differential difference equation of integer order. With the aid of symbolic computation, we choose nonlinear lattice equations to illustrate the validity and advantages of the algorithm. It is shown that the proposed algorithm is effective and can be used for many other nonlinear lattice equations in mathematical physics and applied mathematics.

**AMS subject classifications**: 34A08, 35R11, 37H10, 26A33, 35Q68

**Key words**: $(G'/G)$-expansion method, fractional Hybrid Lattice equation, fractional modified Korteweg-de Vries equation, modified Riemann-Liouville derivative.

## 1 Introduction

Since the study of Fermi, Pasta, and Ulam in the 1960s [1], nonlinear differential-difference equations (NLDDEs) have been the focus of many nonlinear studies and much attention have been paid to the research of the theory of differential-difference equations during the last decades [2–5]. Among these research works, the investigation of exact solutions of nonlinear differential difference equations plays an important role in the study of

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nonlinear physical phenomena. It is well known that NLDDEs are often presented to describe the motion of the isolated waves, localized in a small part of space, in many fields such as hydrodynamic, physics, biophysics, plasma physics, molecular crystals and nonlinear optic. And in the past several decades, many effective methods for obtaining exact solutions of NLDDEs have been presented [6–11].

Fractional differential equations are generalizations of classical differential equations of integer order. In recent decades, fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, signal processing, systems identification, electrochemistry, finance and fractional dynamics. The fractional differential equations have been investigated by many researchers [12, 13]. Many powerful methods for solving nonlinear fractional ordinary differential equations and fractional partial differential equations were appeared in open literature [14–23].

In the literature, on the basis of the problem under investigation, there exist vast different definitions of fractional derivatives such as Grünwald–Letnikov derivative, Riemann–Liouville derivative, Caputo derivative, modified Riemann–Liouville derivative and these derivatives are widely used. The fractional complex transform [24, 25] is used to analytically deal with fractional differential equations. This method extremely simple but effective for solving fractional differential equations.

In this article, based on a fractional complex transformation, a given fractional differential difference equation in the sense of modified Riemann–Liouville derivative can be turned into differential difference equation of integer order, and the reduced equations can be solved by symbolic computation. The manuscript suggests the $(G'/G)$-expansion method and fractional complex transform to find the exact solutions of nonlinear differential difference equations with the modified Riemann-Liouville derivative by Jumarie [26]. The Jumarie's modified Riemann–Liouville derivative of order $\alpha$ is defined by the following expression:

$$D_t^\alpha f(t)=\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dt}\displaystyle\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n\le\alpha<n+1, \quad n\ge 1. \end{cases} \tag{1.1}$$

Modified Riemann–Liouville derivative has many important properties, one of them is the $\alpha$ order derivative constant is zero and four famous formulas of them are [27, 28]

$$D_t^\alpha x^\gamma=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}x^{\gamma-\alpha}, \quad \gamma>0, \tag{1.2a}$$

$$D_t^\alpha(cf(t))=cD_t^\alpha f(t), \quad c=\text{constant}, \tag{1.2b}$$

$$D_t^\alpha\{af(t)+bg(t)\}=aD_t^\alpha f(t)+bD_t^\alpha g(t), \tag{1.2c}$$

$$D_x^\alpha c=0, \quad c=\text{constant}, \tag{1.2d}$$

which are direct consequences of the equality

$$d^\alpha x(t)=\Gamma(1+\alpha)dx(t), \tag{1.3}$$

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which holds for non-differentiable functions.

## 2 Methodology

In this section we describe the $(G'/G)$-expansion method to solve fractional NLDDEs. Let us consider a system of $M$ fractional NLDDEs in the form

$$P(u_{n+p_1}(x),\cdots,u_{n+p_k}(x),\cdots,u^{\alpha}_{n+p_1}(x),\cdots,u^{\alpha}_{n+p_k}(x),\cdots,u^{(r\alpha)}_{n+p_1}(x),\\ \cdots,u^{(r\alpha)}_{n+p_k}(x))=0,\quad 0<\alpha\leq 1, \tag{2.1}$$

where the dependent variable $u_n$ has $M$ components $u_{i,n}$, the continuous variable $x$ has $N$ components $x_j$, the discrete variable $n$ has $Q$ components $n_i$, the $k$ shift vectors $p_s\in\mathbb{Z}^Q$ and $u^{(r\alpha)}(x)$ denotes the collection of modified Riemann-Liouville derivative terms of order $r\alpha$. Using a fractional complex transformation

$$u_{n+p_s}(x)=U_{n+p_s}(\xi_n),\quad \xi_n=\sum_{i=1}^{Q}d_i n_i+\sum_{j=1}^{N}\frac{c_j}{\Gamma(1+\alpha)}x_j^{\alpha}+\zeta,\quad s=1,2,\cdots,k, \tag{2.2}$$

where the coefficients $d_i$, $c_j$ and the phase $\zeta$ are all constants.

By using the chain rule

$$D_t^{\alpha}u=\sigma_t'\frac{du}{d\xi}D_t^{\alpha}\xi,\quad D_x^{\alpha}u=\sigma_x'\frac{du}{d\xi}D_x^{\alpha}\xi, \tag{2.3}$$

where $\sigma_t'$ and $\sigma_x'$ are called the sigma indexes see [29], without loss of generality we can take $\sigma_t'=\sigma_x'=l$, where $l$ is a constant.

Substituting (2.2) with (1.2a) and (2.3) into (2.1), we can rewrite Eq. (2.1) as a nonlinear differential difference equation (NLDDE) of integer order in the form

$$P(U_{n+p_1}(\xi_n),\cdots,U_{n+p_k}(\xi_n),\cdots,U'_{n+p_1}(\xi_n),\cdots,U'_{n+p_k}(\xi_n),\cdots,U^{(r)}_{n+p_1}(\xi_n),\\ \cdots,U^{(r)}_{n+p_k}(\xi_n))=0. \tag{2.4}$$

Step 1 We assume that the following series expansion as a solution of Eq. (2.4):

$$U_n(\xi_n)=\sum_{l=0}^{m}\alpha_l\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^l,\quad \alpha_m\neq 0, \tag{2.5}$$

where $m$ and $\alpha_i$'s are constants to be determined later, $G(\xi_n)$ satisfies a second order linear ordinary differential equation (LODE):

$$\frac{d^2G(\xi_n)}{d\xi_n^2}+\lambda\frac{dG(\xi_n)}{d\xi_n}+\mu G(\xi_n)=0, \tag{2.6}$$

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where $\lambda$ and $\mu$ are arbitrary constants. Using the general solutions of Eq. (2.6), we get following three cases:

$$\frac{G'(\xi_n)}{G(\xi_n)}\begin{cases}\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}\right)-\frac{\lambda}{2}, & \lambda^2-4\mu>0,\\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)+C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}\right)-\frac{\lambda}{2}, & \lambda^2-4\mu<0,\\ \frac{C_1}{C_1\xi_n+C_2}-\frac{\lambda}{2}, & \lambda^2-4\mu=0,\end{cases} \tag{2.7}$$

where $C_1$ and $C_2$ are arbitrary constants.

Step 2 We derive from (2.7) uniform formulas:

$$\frac{G'(\xi_{n\pm y})}{G(\xi_{n\pm y})}=\frac{\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}\pm\eta\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}f\left(\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}y\right)}{1\pm\frac{2}{\sqrt{\lambda^2-4\mu}}\left(\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}\right)f\left(\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}y\right)}-\frac{\lambda}{2}, \tag{2.8}$$

where $\eta=\pm1$ and

$$f\left(\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}y\right)=\begin{cases}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}y\right), & \eta=1,\\ \tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}y\right), & \eta=-1.\end{cases} \tag{2.9}$$

By a simple computation, we can get the identity

$$\xi_{n+p_s}=\xi_n+\varphi_s,\quad \varphi_s=p_{s1}d_1+p_{s2}d_2+\cdots+p_{sQ}d_Q, \tag{2.10}$$

where $p_{sj}$ is the $j$th component of the shift vector $p_s$. Thus, considering trigonometric/hyperbolic function identities and using the expressions (2.8)-(2.10) we obtain:

$$U_{n+p_s}(\xi_n)=\sum_{l=0}^{m}\alpha_l\left[\frac{\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}\pm\eta\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}f\left(\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}\varphi_s\right)}{1\pm\frac{2}{\sqrt{\lambda^2-4\mu}}\left(\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}\right)f\left(\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}\varphi_s\right)}-\frac{\lambda}{2}\right]^l. \tag{2.11}$$

Step 3 By using the homogeneous balance principle for the highest order nonlinear term(s) and the highest order partial derivative of $U_n(\xi_n)$ in Eq. (2.4) we can easily determine the degree $m$ of Eq. (2.5). It should be noted that the leading terms of $U_{n+p_s}(\xi_n)$, $(p_s\neq0)$ will not affect the balancing procedure. Because $U_{n+p_s}(\xi_n)$ can be interpreted as being of degree zero in $(\frac{G'(\xi_n)}{G(\xi_n)})$ and we are interested in balancing the terms of $\frac{G'(\xi_n)}{G(\xi_n)}$.

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Step 4 Then, substituting Eqs. (2.5) and (2.11) together with (2.6) into Eq. (2.4) and equating the coefficients of $(\frac{G'(\xi_n)}{G(\xi_n)})^i$ $(i=0,1,2,3,\cdots)$ to zero, we obtain a system of nonlinear algebraic equations, from which the constants $\alpha_l$, $d_i$, $c_j$ can be explicitly determined. Lastly we substitute these values into Eq. (2.5) and get various kinds of discrete exact solutions to Eq. (2.1).

# 3 Applications

In this section, we apply $(\frac{G'(\xi_n)}{G(\xi_n)})$-expansion method summarized in Section 2 to two time fractional DDEs.

## 3.1 The fractional hybrid lattice equation:

In this section we will apply the described method to fractional Hybrid lattice equation [30]:

$$D_t^{\gamma}u_n=(1+\alpha u_n+\beta u_n^2)(u_{n+1}-u_{n-1}), \tag{3.1}$$

where $0<\gamma\leq 1$, $u_n(t)=u(n,t)$ and $n\in\mathbb{Z}$.

Using a fractional complex transformation

$$u_{n\pm1}=U_{n\pm1}(\xi_n),\quad \xi_n=d_1n+c_1\frac{t^{\gamma}}{\Gamma(1+\gamma)}+\zeta, \tag{3.2}$$

where $d_1$ and $c_1$ non-zero constant, whereas $\zeta$ denotes the phase shift. On substituting Eq. (3.2) with (1.2a) and (2.3) into Eq. (3.1), we can know that Eq. (3.1) turns into,

$$c_1U_n'-(1+\alpha U_n+\beta U_n^2)(U_{n+1}-U_{n-1})=0. \tag{3.3}$$

In this case $p_s=1$, $\varphi_s=d_1$. According the homogeneous balance procedure, in Eq. (3.3) balancing the highest order nonlinear term $U_n^2$ whose degree is $2m$ and the highest order derivative term $U_n'$ whose degree is $m+1$, we obtain

$$2m=m+1. \tag{3.4}$$

Hence we find

$$m=1. \tag{3.5}$$

Thus we get

$$U_n(\xi_n)=\alpha_0+\alpha_1\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big),\quad \alpha_1\neq 0, \tag{3.6}$$

where $G(\xi_n)$ satisfies Eq. (2.6), whereas $\alpha_0$ and $\alpha_1$ are arbitrary constants to be determined later

$$U_{n\pm1}(\xi_n)=\alpha_0+\alpha_1\left[\frac{\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}\pm\eta\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}f\big(\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}d_1\big)}{1\pm\frac{2}{\sqrt{\lambda^2-4\mu}}\big(\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}\big)f\big(\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}d_1\big)}-\frac{\lambda}{2}\right], \tag{3.7}$$

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for the hyperbolic/trigonometric function solutions of Eq. (3.3). Substituting (3.6) and (3.7) together with (2.6) into Eq. (3.3), clearing the denominator and setting the coefficients of $(\frac{G'(\xi_n)}{G(\xi_n)})^i$ $(0\le i\le 4)$ to zero, we have algebraic system for $\alpha_0$, $\alpha_1$, $d_1$, $c_1$. When $\lambda^2-4\mu>0$, we find the following algebraic system:

Let $\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)=\theta$ and we find coefficients of $(\frac{G'(\xi_n)}{G(\xi_n)})^i$ $(0\le i\le 4)$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^0:\ -\mu\alpha_1\Big(c_1\lambda^2(\theta^2-1)+4c_1\mu+4\theta\sqrt{\lambda^2-4\mu}(\beta\alpha_0^2+\alpha\alpha_0+1)\Big)=0, \tag{3.8a}$$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^1:\ -\alpha_1(c_1\lambda(-\lambda^2+\lambda^2\theta^2+4\mu(\theta^2+1)) +4\theta\sqrt{(\lambda^2-4\mu)}(\beta\lambda\alpha_0^2+2\alpha_1\beta\mu\alpha_0+\alpha\alpha_0\lambda+\alpha\mu\alpha_1+\lambda))=0, \tag{3.8b}$$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^2:\ -\alpha_1\Big(-c_1\lambda^2(5\theta^2-1)+4c_1\mu(\theta^2+1) +4\theta\sqrt{(\lambda^2-4\mu)}(\beta\alpha_0^2+2\alpha_1\beta\lambda\alpha_0+\alpha\alpha_0+\beta\alpha_1^2\mu+\alpha\lambda\alpha_1+1)\Big)=0, \tag{3.8c}$$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^3:\ -4\theta\alpha_1\Big(2\theta\lambda c_1+\sqrt{(\lambda^2-4\mu)}\alpha_1(2\beta\alpha_0+\beta\lambda\alpha_1+\alpha)\Big)=0, \tag{3.8d}$$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^4:\ -4\theta\alpha_1\Big(\theta c_1+\sqrt{(\lambda^2-4\mu)}\beta\alpha_1^2\Big)=0. \tag{3.8e}$$

Solving this algebraic system for $\alpha_0$, $\alpha_1$, $d_1$, $c_1$ with the aid of Maple we get,

$$\alpha_0=\frac{\pm\frac{\lambda}{2}\sqrt{\frac{\alpha^2-4\beta}{\lambda^2-4\mu}}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)-\frac{\alpha}{2}}{\beta}, \qquad \alpha_1=\pm\frac{\sqrt{\frac{\alpha^2-4\beta}{\lambda^2-4\mu}}\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\beta}, \tag{3.9a}$$

$$c_1=-\frac{(\alpha^2-4\beta)\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\beta\sqrt{\lambda^2-4\mu}}, \qquad d_1=d_1,\quad \eta=1,\quad \alpha^2-4\beta>0. \tag{3.9b}$$

Substituting (3.9) into (3.6), we have hyperbolic function solution of Eq. (3.3):

$$u_{n,1}(t)=-\frac{\alpha}{2\beta} \pm\frac{\sqrt{\alpha^2-4\beta}}{2\beta}\tanh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1\Big)\Big(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}\Big), \tag{3.10}$$

where

$$\xi_n=d_1n-\frac{(\alpha^2-4\beta)\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\beta\Gamma(1+\gamma)\sqrt{\lambda^2-4\mu}}t^\gamma+\zeta,$$

$C_1$ and $C_2$ are arbitrary constants.

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As special when we take $\lambda^2-4\mu=4$ and $C_1=0$, the solution (3.10) becomes

$$u_{n,2}(t)=-\frac{\alpha}{2\beta}\pm\frac{\sqrt{\alpha^2-4\beta}\tanh(d_1)}{2\beta}\coth\Big(d_1n-\frac{(\alpha^2-4\beta)\tanh(d_1)}{2\beta\Gamma(1+\gamma)}t^\gamma+\zeta\Big), \tag{3.11}$$

and when we take $\lambda^2-4\mu=4$ and $C_2=0$, the solution (3.10) becomes

$$u_{n,3}(t)=-\frac{\alpha}{2\beta}\pm\frac{\sqrt{\alpha^2-4\beta}\tanh(d_1)}{2\beta}\tanh\Big(d_1n-\frac{(\alpha^2-4\beta)\tanh(d_1)}{2\beta\Gamma(1+\gamma)}t^\gamma+\zeta\Big). \tag{3.12}$$

When $\lambda^2-4\mu<0$. Let

$$\tan\Big(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1\Big)=\theta$$

and we find coefficients of $(\frac{G'(\xi_n)}{G(\xi_n)})^i$ $(0\le i\le 4)$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^0:\ -\mu\alpha_1(c_1\lambda^2\Big(\theta^2+1)-4c_1\mu+4\sqrt{4\mu-\lambda^2}\theta(\beta\alpha_0^2+\alpha\alpha_0+1)\Big)=0, \tag{3.13a}$$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^1:\ -\alpha_1\Big(-c_1\lambda(\lambda^2+\lambda^2\theta^2+4\mu(\theta^2-1)$$
$$+4\theta\sqrt{4\mu-\lambda^2}(\beta\lambda\alpha_0^2+2\alpha_1\beta\mu\alpha_0+\alpha\alpha_0\lambda+\alpha\mu\alpha_1+\lambda)\Big)=0, \tag{3.13b}$$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^2:\ -\alpha_1\Big(-c_1\lambda^2(5\theta^2+1)+4c_1\mu(\theta^2-1)$$
$$+4\theta\sqrt{4\mu-\lambda^2}(\beta\alpha_0^2+2\alpha_1\beta\lambda\alpha_0+\alpha\alpha_0+\beta\alpha_1^2\mu+\alpha\lambda\alpha_1+1)\Big)=0, \tag{3.13c}$$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^3:\ -4\theta\alpha_1\Big(2\theta\lambda c_1+\sqrt{4\mu-\lambda^2}\alpha_1(2\beta\alpha_0+\beta\lambda\alpha_1+\alpha)\Big)=0, \tag{3.13d}$$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^4:\ -4\theta\alpha_1\Big(\theta c_1+\sqrt{4\mu-\lambda^2}\beta\alpha_1^2\Big)=0. \tag{3.13e}$$

Solving this algebraic system for $\alpha_0$, $\alpha_1$, $d_1$, $c_1$ by use of Maple we obtain,

$$\alpha_0=\frac{\pm\frac{\lambda}{2}\sqrt{\frac{\alpha^2-4\beta}{4\mu-\lambda^2}}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)-\frac{\alpha}{2}}{\beta},\qquad \alpha_1=\pm\frac{\sqrt{\frac{\alpha^2-4\beta}{4\mu-\lambda^2}}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\beta}, \tag{3.14a}$$

$$c_1=-\frac{(\alpha^2-4\beta)\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\beta\sqrt{4\mu-\lambda^2}},\qquad d_1=d_1,\quad \eta=-1,\quad \alpha^2-4\beta>0. \tag{3.14b}$$

Substituting (3.14) into (3.6) we get trigonometric function solution of Eq. (3.3):

$$u_{n,4}(t)=\frac{-\alpha\pm\sqrt{\alpha^2-4\beta}\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{2\beta}\Big(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)+C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}\Big), \tag{3.15}$$

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where

$$\xi_n = d_1 n - \frac{(\alpha^2-4\beta)\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\beta\Gamma(1+\gamma)\sqrt{4\mu-\lambda^2}}t^\gamma+\zeta,$$

$C_1$ and $C_2$ are arbitrary constants.

If we take $4\mu-\lambda^2=4$ and $C_1=0$, the solution (3.15) becomes

$$u_{n,5}(t) = -\frac{\alpha}{2\beta} \pm \frac{\sqrt{\alpha^2-4\beta}\tan(d_1)}{2\beta}\cot\Big(d_1 n - \frac{(\alpha^2-4\beta)\tan(d_1)}{2\beta\Gamma(1+\gamma)}t^\gamma+\zeta\Big), \tag{3.16}$$

and when we take $4\mu-\lambda^2=4$ and $C_2=0$, the solution (3.15) becomes

$$u_{n,6}(t) = -\frac{\alpha}{2\beta} \pm \frac{\sqrt{\alpha^2-4\beta}\tan(d_1)}{2\beta}\tan\Big(d_1 n - \frac{(\alpha^2-4\beta)\tan(d_1)}{2\beta\Gamma(1+\gamma)}t^\gamma+\zeta\Big). \tag{3.17}$$

### 3.2 Time fractional discrete modified Korteweg-de Vries (ΔmKdV) equation

Now we consider the time fractional ΔmKdV equation as [31]:

$$\frac{\partial^\alpha u_n}{\partial t^\alpha} = (1+u_n^2)(u_{n+1}-u_{n-1}), \tag{3.18}$$

where $0<\alpha\leq 1$ and $n\in\mathbb{Z}$.

Similarly, using a fractional complex transformation

$$u_{n\pm1} = U_{n\pm1}(\xi_n), \quad \xi_n = d_1 n + c_1\frac{t^\alpha}{\Gamma(1+\alpha)}+\zeta, \tag{3.19}$$

Eq. (3.18) turns into

$$c_1 U_n' - (1+U_n^2)(U_{n+1}-U_{n-1}) = 0. \tag{3.20}$$

In this case $p_s=1$, $\varphi_s=d_1$. Considering the homogeneous balance between $U_n'$ and $U_n^2$ we have,

$$U_n(\xi_n) = \alpha_0+\alpha_1\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big), \quad \alpha_1\neq 0, \tag{3.21a}$$

$$U_{n+1}(\xi_n) = \alpha_0+\alpha_1\left[\frac{\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}+\eta\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}f(\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}d_1)}{1+\frac{2}{\sqrt{\lambda^2-4\mu}}(\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)})f(\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}d_1)}-\frac{\lambda}{2}\right], \tag{3.21b}$$

$$U_{n-1}(\xi_n) = \alpha_0+\alpha_1\left[\frac{\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)}-\eta\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}f(\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}d_1)}{1-\frac{2}{\sqrt{\lambda^2-4\mu}}(\frac{\lambda}{2}+\frac{G'(\xi_n)}{G(\xi_n)})\,f(\frac{\sqrt{\eta(\lambda^2-4\mu)}}{2}d_1)}-\frac{\lambda}{2}\right], \tag{3.21c}$$

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for the solutions of Eq. (3.20). Substituting Eqs. (3.21a)-(3.21c) with (2.6) into Eq. (3.20), clearing the denominator and setting the coefficients of $(\frac{G'(\xi_n)}{G(\xi_n)})^i$ $(0\leq i\leq 4)$ to zero, we have algebraic system for $\alpha_0$, $\alpha_1$, $d_1$, $c_1$. Solving this system by use of Maple, we obtain solutions for Eq. (3.20) as follows:

When $\lambda^2-4\mu>0$. Let

$$\tan\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1\Big)=\theta$$

and we find coefficients of $(\frac{G'(\xi_n)}{G(\xi_n)})^i$ $(0\leq i\leq 4)$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^0:\ -\mu\alpha_1\Big(c_1\lambda^2(\theta^2-1)+4c_1\mu+4\sqrt{\lambda^2-4\mu}\theta(\alpha_0^2+1)\Big)=0, \tag{3.22a}$$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^1:\ -\alpha_1\Big(\lambda c_1(-\lambda^2+4\mu+4\mu\theta^2+\lambda^2\theta^2) +4\sqrt{\lambda^2-4\mu}\theta(\alpha_0^2\lambda+2\alpha_1\mu\alpha_0+\lambda)\Big)=0, \tag{3.22b}$$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^2:\ -\alpha_1\Big(c_1\lambda^2(5\theta^2-1)+4c_1\mu(\theta^2+1) +4\sqrt{\lambda^2-4\mu}\theta(\alpha_0^2+2\alpha_1\lambda\alpha_0+\mu\alpha_1^2+1)\Big)=0, \tag{3.22c}$$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^3:\ -4\theta\alpha_1\Big(2\theta\lambda c_1+\alpha_1\sqrt{\lambda^2-4\mu}(2\alpha_0+\alpha_1\lambda)\Big)=0, \tag{3.22d}$$

$$\Big(\frac{G'(\xi_n)}{G(\xi_n)}\Big)^4:\ -4\theta\alpha_1\Big(\theta c_1+\sqrt{\lambda^2-4\mu}\alpha_1^2\Big)=0. \tag{3.22e}$$

Solving this algebraic system for $\alpha_0$, $\alpha_1$, $d_1$, $c_1$ by use of Maple we get,

$$\alpha_0=\pm\lambda\sqrt{\frac{1}{4\mu-\lambda^2}}\tanh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1\Big),\quad \alpha_1=\pm2\sqrt{\frac{1}{4\mu-\lambda^2}}\tanh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1\Big), \tag{3.23a}$$

$$c_1=\frac{4\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\sqrt{\lambda^2-4\mu}},\qquad d_1=d_1,\quad \eta=1. \tag{3.23b}$$

Substituting (3.23) into (3.21a), we have soliton solutions of Eq. (3.20):

$$u_{n,1}(t)=\pm i\tanh\Big(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1\Big)\Big(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}\Big), \tag{3.24}$$

where

$$\xi_n=d_1n+\frac{4\tanh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\Gamma(1+\alpha)\sqrt{\lambda^2-4\mu}}t^\alpha+\zeta,$$

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$C_1$ and $C_2$ are arbitrary constants.

If we take $\lambda^2-4\mu=4$ and $C_1=0$, the solution (3.24) becomes

$$u_{n,2}(t)=\pm i\tanh(d_1)\coth\left(d_1 n+2\tanh(d_1)\frac{t^\alpha}{\Gamma(1+\alpha)}+\zeta\right), \tag{3.25}$$

and when we take $\lambda^2-4\mu=4$ and $C_2=0$, the solution (3.24) becomes

$$u_{n,3}(t)=\pm i\tanh(d_1)\tanh\left(d_1 n+2\tanh(d_1)\frac{t^\alpha}{\Gamma(1+\alpha)}+\zeta\right). \tag{3.26}$$

When $\lambda^2-4\mu<0$. Let

$$\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1\right)=\theta$$

and we find coefficients of $(\frac{G'(\xi_n)}{G(\xi_n)})^i$ $(0\le i\le 4)$

$$\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^0: \ -\mu\alpha_1\left(c_1\lambda^2(\theta^2+1)-4c_1\mu+4\sqrt{4\mu-\lambda^2}\theta(\alpha_0^2+1)\right)=0, \tag{3.27a}$$

$$\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^1: \ -\alpha_1\Big(\lambda c_1(\lambda^2-4\mu+4\mu\theta^2+\lambda^2\theta^2) +4\sqrt{4\mu-\lambda^2}\theta(\alpha_0^2\lambda+2\alpha_1\mu\alpha_0+\lambda)\Big)=0, \tag{3.27b}$$

$$\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^2: \ -\alpha_1\Big(c_1\lambda^2(5\theta^2-1)+4c_1\mu(\theta^2-1) +4\sqrt{4\mu-\lambda^2}\theta(\alpha_0^2+2\alpha_1\lambda\alpha_0+\mu\alpha_1^2+1)\Big)=0, \tag{3.27c}$$

$$\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^3: \ -4\theta\alpha_1\left(2\theta\lambda c_1+\alpha_1\sqrt{4\mu-\lambda^2}(2\alpha_0+\alpha_1\lambda)\right)=0, \tag{3.27d}$$

$$\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)^4: \ -4\theta\alpha_1\left(\theta c_1+\sqrt{4\mu-\lambda^2}\alpha_1^2\right)=0. \tag{3.27e}$$

Solving the system for $\alpha_0$, $\alpha_1$, $d_1$, $c_1$ by use of Maple we get the following solution sets,

$$\alpha_0=\pm\lambda\sqrt{\frac{1}{\lambda^2-4\mu}}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1\right), \quad \alpha_1=\pm 2\sqrt{\frac{1}{\lambda^2-4\mu}}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1\right), \tag{3.28a}$$

$$c_1=\frac{4\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{4\mu-\lambda^2}}, \qquad d_1=d_1, \quad \eta=-1. \tag{3.28b}$$

Substituting (3.28) into (3.21a) we get trigonometric solutions of Eq. (3.20):

$$u_{n,4}(t)=\pm i\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1\right)\left(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)+C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}\right), \tag{3.29}$$

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where

$$\xi_n = d_1 n + \frac{4\tan(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\Gamma(1+\alpha)\sqrt{4\mu-\lambda^2}}t^\alpha + \zeta,$$

$C_1$ and $C_2$ are arbitrary constants

If we take $4\mu-\lambda^2=4$ and $C_1=0$, the solution (3.29) becomes

$$u_{n,5}(t) = \pm i\tan(d_1)\cot\left(d_1 n + 2\tan(d_1)\frac{t^\alpha}{\Gamma(1+\alpha)} + \zeta\right), \tag{3.30}$$

and if we take $4\mu-\lambda^2=4$ and $C_2=0$, the solution (3.29) becomes

$$u_{n,6}(t) = \pm i\tan(d_1)\tan\left(d_1 n + 2\tan(d_1)\frac{t^\alpha}{\Gamma(1+\alpha)} + \zeta\right). \tag{3.31}$$

## 4 Conclusions

In summary, we have illustrated the $(G'/G)$-expansion method for solving fractional differential difference equations in the sense of modified Riemann-Liouville derivative, and applied it to find exact solutions of the fractional Hybrid Lattice and fractional modified Korteweg-de Vries (ΔmKdV) equation. As a result, some generalized exact solutions for them have been successfully found. Hyperbolic function and trigonometric function solutions with parameters are obtained, from which some known solutions including kink-type solitary wave solution are recovered by setting the parameters as special values. By the way the fractional complex transform is extremely simple but effective for solving fractional differential equations and this transformation is very important, which ensures that a fractional differential difference equation can be turned into differential difference equation of integer order.

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