CONJUGATE BANACH SPACES

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The purpose of this note is to present two characterizations of conjugate Banach spaces. More precisely, we present two conditions, each necessary and sufficient for a (real or complex) Banach space $\mathfrak{B}$ to be isomorphic to the conjugate space of a Banach space, and two corresponding conditions for $\mathfrak{B}$ to be equivalent to the conjugate space of a Banach space. Other characterizations, in terms of weak topologies, have been given by Alaoglu ((1), Theorem 2:1, p. 256, and Corollary 2:1, p. 257) and Bourbaki ((4), Chap. IV, § 5, exerc. 15*c*, p. 122). Here, by the *conjugate space* $\mathfrak{X}^*$ of a Banach space $\mathfrak{X}$ we mean ((2), p. 188) the space of continuous linear functionals over $\mathfrak{X}$. Two Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$ are said to be *isomorphic* if there is a one-one continuous linear mapping of $\mathfrak{X}$ onto $\mathfrak{Y}$ (its inverse is necessarily continuous by the inversion theorem ((2), Théorème 5, p. 41; (6), Theorem 2·13·7, Corollary, p. 29)); they are said to be *equivalent* if there is a norm-preserving linear mapping of $\mathfrak{X}$ onto $\mathfrak{Y}$ ((2), p. 180).

Since I submitted this paper to the Society, Dr Bonsall has drawn my attention to a paper by Dixmier, which (to avoid renumbering my list of references) I shall call (4′). In the course of that paper, Dixmier presented (p. 1069) characterizations of conjugate Banach spaces which are equivalent to the principal results of this paper (cf. (4′), p. 1061). His characterizations, however, are expressed in terms of weak topologies, and his proofs are different from mine. I apologize for overlooking (4′) when working on this paper.

We first recall that, if $\mathfrak{X}$ and $\mathfrak{Y}$ are two Banach spaces, then there is a natural equivalence (cf. (9), p. 48) between the Banach space of continuous linear mappings of $\mathfrak{X}$ into $\mathfrak{Y}^*$ and the Banach space of continuous linear mappings of $\mathfrak{Y}$ into $\mathfrak{X}^*$, namely, that given by the rule $(Tx)(y) = (T'y)(x)$. We shall call T' the *dual* of T. (It can be regarded as a restriction of the adjoint of T if $\mathfrak{Y}$ is identified with its canonical image in $\mathfrak{Y}^{**}$.) In particular, if $\mathfrak{Y}$ is a closed subspace of $\mathfrak{X}^*$, then there is a natural mapping of $\mathfrak{X}$ into $\mathfrak{Y}^*$, namely, the dual of the identity mapping of $\mathfrak{Y}$ into $\mathfrak{X}^*$ (and so given by $(Ux)(f) = f(x)$ for all $f \in \mathfrak{Y}$). We call this mapping the *canonical mapping* of $\mathfrak{X}$ into $\mathfrak{Y}^*$, and we call the image of $\mathfrak{X}$ in $\mathfrak{Y}^*$ under this mapping the *canonical image* of $\mathfrak{X}$ in $\mathfrak{Y}^*$. If the canonical mapping sets up an isomorphism, or an equivalence, between $\mathfrak{X}$ and $\mathfrak{Y}^*$, we shall say that $\mathfrak{X}$ and $\mathfrak{Y}^*$ are *canonically isomorphic*, or *canonically equivalent*, respectively.

THEOREM 1. *The Banach space $\mathfrak{B}$ is isomorphic to the conjugate space of a Banach space if and only if there is a minimal total† closed subspace $\mathfrak{M}$ of the conjugate $\mathfrak{B}^*$ of $\mathfrak{B}$. Moreover, $\mathfrak{B}$ is canonically isomorphic to the conjugate $\mathfrak{M}^*$ of a closed subspace $\mathfrak{M}$ of $\mathfrak{B}^*$ if and only if $\mathfrak{M}$ is a minimal total closed subspace of $\mathfrak{B}^*$.*

† $\mathfrak{M}$ is said to be *total* if $f(x) = 0$ for all $f \in \mathfrak{M}$ only if $x = \Theta$ ((2), p. 42).

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Let T be an isomorphism between $\mathfrak{B}$ and the conjugate $\mathfrak{E}^*$ of a Banach space $\mathfrak{E}$. The dual mapping T' maps $\mathfrak{E}$ onto a subspace $\mathfrak{M}$ of $\mathfrak{B}^*$. Since

$$\|T'y\| = \sup_{x \neq \Theta} \frac{|(Tx)(y)|}{\|Tx\|} \frac{\|Tx\|}{\|x\|},$$

it follows that† $\|T^{-1}\|^{-1}\|y\| \leqslant \|T'y\| \leqslant \|T\|\|y\|$ for all $y \in \mathfrak{E}$, and so that T' is an isomorphism between $\mathfrak{E}$ and $\mathfrak{M}$. In particular $\mathfrak{M}$ (being complete) is a closed subspace of $\mathfrak{B}^*$. Moreover, $\mathfrak{M}$ is total (cf. (1), p. 257), since $(T'y)(x) = (Tx)(y) = 0$ for all y of $\mathfrak{E}$ only if $Tx = \Theta$, and so $x = \Theta$. Finally, any proper closed subspace $\mathfrak{N}$ of $\mathfrak{M}$ is the image under T' of a proper closed subspace $\mathfrak{F}$ of $\mathfrak{E}$. There is, by a well-known corollary of the Hahn–Banach theorem ((2), Lemma, p. 57; (6), Theorem 2·9·4, p. 19), a non-zero continuous linear functional Tx_0 over $\mathfrak{E}$ which annihilates $\mathfrak{F}$. Hence $(T'y)(x_0) = (Tx_0)(y) = 0$ for every element $T'y$ of $\mathfrak{N}$, whilst $x_0 \neq \Theta$, and so $\mathfrak{N}$ is not total. Thus we have proved that $\mathfrak{M}$ is a minimal total closed subspace of $\mathfrak{B}^*$, and incidentally that it is isomorphic to $\mathfrak{E}$. This proves the necessity of the first condition. The necessity of the second condition follows since, if T is the canonical mapping of $\mathfrak{B}$ into $\mathfrak{M}^*$, then T' is the identity mapping of $\mathfrak{M}$ into $\mathfrak{B}^*$.

The sufficiency of the first condition follows from that of the second, which we now prove.

Let $\mathfrak{M}$ be a minimal total closed subspace of $\mathfrak{B}^*$, and let U be the canonical mapping of $\mathfrak{B}$ into $\mathfrak{M}^*$. Since $\mathfrak{M}$ is total, $(Ux)(f) = f(x) = 0$ for all $f \in \mathfrak{M}$ only if $x = \Theta$, and so U is one-one. On the other hand, let X_0 be any element of $\mathfrak{M}^*$. We shall find an element x_0 of $\mathfrak{B}$ such that $Ux_0 = X_0$. We can dismiss at once the trivial case $X_0 = \Theta$ and assume $X_0 \neq \Theta$. The subspace $\mathfrak{N}$ of $\mathfrak{M}$ annihilated by X_0 is then a proper closed subspace of $\mathfrak{M}$, and so (by hypothesis) $\mathfrak{N}$ is not total, that is to say, there is a non-zero element y_0 of $\mathfrak{B}$ such that $f(y_0) = 0$ for all $f \in \mathfrak{N}$. $\mathfrak{M}$, on the other hand, is total, and so there is an element g_0 of $\mathfrak{M}$ (which cannot belong to $\mathfrak{N}$) such that $g_0(y_0) \neq 0$. By definition of $\mathfrak{N}$, $X_0(g_0) \neq 0$. We put $f_0 = g_0/X_0(g_0)$ and $x_0 = y_0/f_0(y_0)$, so that

$$X_0(f_0) = f_0(x_0) = 1.$$

Then $X_0 = Ux_0$, for (cf. (2), p. 131), given any $f \in \mathfrak{M}$, we have

$$X_0[f - X_0(f)f_0] = X_0(f) - X_0(f)\,X_0(f_0) = 0,$$

which means $f - X_0(f)f_0 \in \mathfrak{N}$, and so

$$[f - X_0(f)f_0](x_0) = 0,$$

that is, $f(x_0) = X_0(f)$. Thus U is a one-one continuous linear mapping of $\mathfrak{B}$ onto $\mathfrak{M}^*$, and so $\mathfrak{B}$ and $\mathfrak{M}^*$ are canonically isomorphic.

COROLLARY. *If $\mathfrak{M}$ and $\mathfrak{N}$ are two closed subspaces of $\mathfrak{B}^*$ whose conjugates $\mathfrak{M}^*$ and $\mathfrak{N}^*$ are canonically isomorphic to $\mathfrak{B}$, then neither of $\mathfrak{M}$ and $\mathfrak{N}$ can be properly contained in the other.*

† We have, in fact, $\|T^{-1}\|^{-1} \leqslant \|(T')^{-1}\|^{-1} \leqslant \|T'\| = \|T\|$. We do not necessarily have equality between $\|T^{-1}\|^{-1}$ and $\|(T')^{-1}\|^{-1}$. For example, let $\mathfrak{E} = \mathfrak{M}$ be a minimal total closed subspace of $\mathfrak{B}^*$ which is not duxial (examples are given after Theorem 3); then we have

$$\|T^{-1}\|^{-1} < \|(T')^{-1}\|^{-1} = \|T'\| = \|T\| = 1.$$

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Note the importance of the isomorphism being canonical. Both† (c) and (c_0) are subspaces of (m), which can be identified with $(l)^*$, and $(c)^*$ and $(c_0)^*$ are both equivalent to (l). Yet (c_0) is properly contained in (c). In fact, the equivalence between (l) and $(c_0)^*$ is canonical, but not that between (l) and $(c)^*$. The subspace of $(m) = (l)^*$ corresponding to (c) in the sense of Theorem 1 is the space (c_1) of sequences $\{\eta_n\}$ such that $\eta_n \to \eta_1$ as $n \to \infty$, and this neither contains nor is contained in (c_0).

THEOREM 2. *The Banach space $\mathfrak{B}$ is equivalent to the conjugate space of a Banach space if and only if there is a minimal total closed subspace $\mathfrak{M}$ of the conjugate $\mathfrak{B}^*$ of $\mathfrak{B}$ which is also duxial‡. Moreover, $\mathfrak{B}$ is canonically equivalent to the conjugate $\mathfrak{M}^*$ of a closed subspace $\mathfrak{M}$ of $\mathfrak{B}^*$ if and only if $\mathfrak{M}$ is a duxial minimal total closed subspace of $\mathfrak{B}^*$.*

Let T be an equivalence between $\mathfrak{B}$ and the conjugate $\mathfrak{E}^*$ of a Banach space $\mathfrak{E}$, and suppose (as in the proof of Theorem 1) that its dual T' maps $\mathfrak{E}$ onto the subspace $\mathfrak{M}$ of $\mathfrak{B}^*$. Then

$$\| T'y \| = \sup_{\|x\|=1} |(T'y)(x)| = \sup_{\|Tx\|=1} |(Tx)(y)| = \|y\|,$$

and so T' is an equivalence between $\mathfrak{E}$ and $\mathfrak{M}$. By Theorem 1, $\mathfrak{M}$ is a minimal total closed subspace of $\mathfrak{B}^*$. Finally,

$$\sup_{f \in \mathfrak{M}-(\Theta)} \frac{|f(x)|}{\|f\|} = \sup_{y \in \mathfrak{E}-(\Theta)} \frac{|(T'y)(x)|}{\|T'y\|} = \sup_{y \in \mathfrak{E}-(\Theta)} \frac{|(Tx)(y)|}{\|y\|} = \|Tx\| = \|x\|$$

for every $x \in \mathfrak{B}$, and so $\mathfrak{M}$ is duxial. This proves the necessity of the first condition. The necessity of the second condition follows as in Theorem 1.

As in Theorem 1, the sufficiency of the first condition follows from that of the second. Let $\mathfrak{M}$ be any closed subspace of $\mathfrak{B}^*$, and let U be the canonical mapping of $\mathfrak{B}$ into $\mathfrak{M}^*$. Then

$$\| Ux \| = \sup_{f \in \mathfrak{M}-(\Theta)} \frac{|(Ux)(f)|}{\|f\|} = \sup_{f \in \mathfrak{M}-(\Theta)} \frac{|f(x)|}{\|f\|},$$

and so U is norm-preserving if and only if $\mathfrak{M}$ is duxial. It follows, by Theorem 1, that if $\mathfrak{M}$ is a duxial minimal total closed subspace of $\mathfrak{B}^*$, then $\mathfrak{B}$ is canonically equivalent to $\mathfrak{M}^*$.

It also follows (since $\mathfrak{B}^*$ is duxial) that $\mathfrak{B}$ is canonically equivalent to its canonical image $\mathfrak{B}_0$ in $\mathfrak{B}^{**}$ (cf. (7)), and so that $\mathfrak{B}$ is reflexive if and only if it is canonically equivalent to its second conjugate $\mathfrak{B}^{**}$. We thus have the following corollary to Theorem 2.

THEOREM 3. *The Banach space $\mathfrak{B}$ is reflexive if and only if $\mathfrak{B}^*$ has no proper total closed subspace.*

Note that the condition of duxiality in Theorem 2 is not redundant. The space (c_t) of convergent sequences $\{\eta_n\}$ such that $\eta_n \to t\eta_1$ as $n \to \infty$ (where t is a fixed scalar) is a minimal total closed subspace of $(m) = (l)^*$ which is duxial if and only if $|t| \leqslant 1$. If $|t| \geqslant 1$, we have

$$\sup_{f \in (c_t)-(\Theta)} \frac{|f(x)|}{\|f\|} = \frac{|\xi_1|}{|t|} + \sum_{n=2}^{\infty} |\xi_n|$$

† For definitions of the spaces mentioned in this example, see (2), pp. 11, 181.

‡ $\mathfrak{M}$ is said to be *duxial* ((8), Definition 2·3·3, p. 344) if $\sup_{f \in \mathfrak{M}-(\Theta)} |f(x)|/\|f\| = \|x\|$ for every $x \in \mathfrak{B}$.

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for $x = \{\xi_n\} \in (l)$. This example is clearly capable of generalization. Another example of a total closed subspace which is not duxial is given by Bonsall ((3), § 4, Example 1, p.109).

A slight modification of Bonsall's example shows that we cannot replace 'duxial minimal total closed subspace' by 'minimal duxial closed subspace' in Theorem 2, since no proper closed subspace of $(c_0)^*$ is duxial (any proper closed subspace $\mathfrak{M}$ of $(c_0)^*$ is annihilated by an element $X = \{\xi_n\} \in (m) = (c_0)^{**}$; if ν is so chosen that $|\xi_\nu| > \frac{3}{4}\|X\|$, and $e_\nu = \{\delta_{\nu n}\} \in (c_0)$, then $\|e_\nu\| = 1$, but

$$|f(e_\nu)| = |[Ve_\nu - X/(2\xi_\nu)](f)| \leqslant \tfrac{2}{3}\|f\| \quad \text{for} \quad f \in \mathfrak{M},$$

where V is the canonical mapping of (c_0) into $(c_0)^{**}$), but (c_0) is not canonically equivalent to $(c_0)^{**}$.

We have seen above that there is a non-denumerable infinity of duxial minimal total closed subspaces of $(l)^*$. On the other hand, the reader may like to be reminded of a result of Grothendieck ((5), Remarque 4, p. 557). If $C_0(K)$ is the space of continuous functions on a locally compact Hausdorff space K which vanish at infinity, then its conjugate space contains at most one duxial minimal total closed subspace.

Our second characterization of conjugate spaces concerns the existence of a projection of $\mathfrak{B}^{**}$ onto $\mathfrak{B}_0$.

THEOREM 4. *The Banach space $\mathfrak{B}$ is isomorphic to the conjugate space of a Banach space if and only if there is a projection of $\mathfrak{B}^{**}$ onto $\mathfrak{B}_0$ which annihilates a regularly closed† subspace of $\mathfrak{B}^{**}$.*

Let $\mathfrak{M}$ be a minimal total closed subspace of $\mathfrak{B}^*$ (cf. Theorem 1). Any element X of $\mathfrak{B}^{**}$ induces a (unique) continuous linear functional Y_X on $\mathfrak{M}$ with $\|Y_X\| \leqslant \|X\|$. If U denotes the canonical isomorphism between $\mathfrak{B}$ and $\mathfrak{M}^*$, and V the canonical equivalence between $\mathfrak{B}$ and $\mathfrak{B}_0$, we define the operator P on $\mathfrak{B}^{**}$ by the equation $PX = VU^{-1}Y_X$. Clearly PX is the unique element of $\mathfrak{B}_0$ which induces the same continuous linear functional Y_X on $\mathfrak{M}$ as X does, that is to say $(PX)(f) = X(f)$ for all $f \in \mathfrak{M}$. P is clearly continuous and linear, and $\|P\| = \|U^{-1}\|$ (since, on the one hand,

$$\|PX\| = \|U^{-1}Y_X\| \leqslant \|U^{-1}\|\,\|Y_X\| \leqslant \|U^{-1}\|\,\|X\|$$

and, on the other hand, each continuous linear functional Y over $\mathfrak{M}$ has an extension X over $\mathfrak{B}^*$ with the same norm, and so

$$\|U^{-1}Y\| = \|PX\| \leqslant \|P\|\,\|X\| = \|P\|\,\|Y\|).$$

Moreover, if $X \in \mathfrak{B}_0$, then $PX = X$. It follows that P is a projection of $\mathfrak{B}^{**}$ onto $\mathfrak{B}_0$. Finally, let $\mathfrak{Z}$ be the subspace of $\mathfrak{B}^{**}$ annihilated by P, and let X_0 be any element of $\mathfrak{B}^{**}$ which does not belong to $\mathfrak{Z}$, so that $PX_0 \neq \Theta$. Since $\mathfrak{M}$ is total, there is an element $f_0 \in \mathfrak{M}$ such that $f_0(V^{-1}PX_0) = 1$. Then $X_0(f_0) = (PX_0)(f_0) = f_0(V^{-1}PX_0) = 1$, but $X(f_0) = (PX)(f_0) = 0$ for every $X \in \mathfrak{Z}$. Thus $\mathfrak{Z}$ is regularly closed.

Conversely, let P be a projection of $\mathfrak{B}^{**}$ on $\mathfrak{B}_0$ annihilating the regularly closed subspace $\mathfrak{Z}$ of $\mathfrak{B}^{**}$, and let $\mathfrak{M}$ be the (closed) subspace of $\mathfrak{B}^*$ orthogonal to $\mathfrak{Z}$: Then

† A subspace $\mathfrak{Y}$ of a conjugate space $\mathfrak{X}^*$ is said to be *regularly closed* if, given any element f_0 of $\mathfrak{X}^*$ which does not belong to $\mathfrak{Y}$, there is an element x_0 of $\mathfrak{X}$ such that $f(x_0) = 0$ for every f of $\mathfrak{Y}$ but $f_0(x_0) = 1$ ((2), pp. 116–17).

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$X(f) = (PX)(f)$ for all $f \in \mathfrak{M}$, since $X - PX \in \mathfrak{Z}$. As before, let U be the canonical mapping of $\mathfrak{B}$ into $\mathfrak{M}^*$, and let V be the canonical equivalence between $\mathfrak{B}$ and $\mathfrak{B}_0$. If x_0 is any non-zero element of $\mathfrak{B}$, then $P(Vx_0) = Vx_0 \neq \Theta$, and so $Vx_0 \notin \mathfrak{Z}$. Since $\mathfrak{Z}$ is regularly closed, there is an element $f_0 \in \mathfrak{B}^*$ such that $(Vx_0)(f_0) = 1$ but $X(f_0) = 0$ for all $X \in \mathfrak{Z}$. Thus $f_0 \in \mathfrak{M}$ and $(Ux_0)(f_0) = f_0(x_0) = (Vx_0)(f_0) = 1$. It follows that $\mathfrak{M}$ is total and that U is one-one. On the other hand, let X_0 be any element of $\mathfrak{M}^*$. This has an extension $X \in \mathfrak{B}^{**}$ with the same norm. Then

$$X_0(f) = X(f) = (PX)(f) = f(V^{-1}PX) = (UV^{-1}PX)(f)$$

for every $f \in \mathfrak{M}$, and so U maps $V^{-1}PX$ onto X_0. It follows that U maps $\mathfrak{B}$ onto $\mathfrak{M}^*$, and so that $\mathfrak{B}$ and $\mathfrak{M}^*$ are canonically isomorphic.

THEOREM 5. *The Banach space $\mathfrak{B}$ is equivalent to the conjugate space of a Banach space if and only if there is a projection of $\mathfrak{B}^{**}$ onto $\mathfrak{B}_0$ of bound 1 which annihilates a regularly closed subspace of $\mathfrak{B}^{**}$.*

In view of Theorems 2 and 4, it will suffice to prove that $\|P\| = 1$ if and only if $\|U^{-1}\| = 1$, and this follows from the fact that $\|P\| = \|U^{-1}\|$ (cf. (4), Chap. IV, § 5, exerc. 16*a*, p. 122).

The existence of a projection of $\mathfrak{B}^{**}$ onto $\mathfrak{B}_0$ of bound 1 is not sufficient to ensure that $\mathfrak{B}$ is equivalent to a conjugate space. For instance†, we can define a projection of bound 1 of $(L)^{**}$ onto $(L)_0$ as follows. Any element of $(L)^{**}$ (which can be regarded as an element of $(M)^*$) induces an element of $(C)^*$ of no greater norm. This in turn gives rise to an element of (L) (namely, the derivative of the corresponding function of bounded variation), again without increase of norm. But it is known‡ that (L) is not equivalent to a conjugate space. The subspace of $(L)^*$ orthogonal to the subspace $\mathfrak{Z}$ of $(L)^{**}$ annihilated by this projection in fact consists of Θ alone.

REFERENCES

(1) ALAOGLU, L. Weak topologies of normed linear spaces. *Ann. Math., Princeton*, (2) 41 (1940), 252–67.

(2) BANACH, S. *Théorie des Opérations Linéaires* (Warsaw, 1932).

(3) BONSALL, F. F. Dual extremum problems in the theory of functions. *J. Lond. Math. Soc.* 31 (1956), 105–10.

(4) BOURBAKI, N. *Eléments de Mathématique, Fascicule* XVIII [*Espaces Vectoriels Topologiques, Chap.* III–V *et Dictionnaire*] (Paris, 1955).

(4′) DIXMIER, J. Sur un théorème de Banach. *Duke Math. J.* 15 (1948), 1057–71.

(5) GROTHENDIECK, A. Une caractérisation vectorielle-métrique des espaces L^1. *Canad. J. Math.* 7 (1955), 552–61.

(6) HILLE, E. *Functional analysis and semi-groups* (New York), 1948.

(7) RUSTON, A. F. A short proof of a theorem on reflexive spaces. *Proc. Camb. Phil. Soc.* 45 (1949), 674.

(8) RUSTON, A. F. Direct products of Banach spaces and linear functional equations. *Proc. Lond. Math. Soc.* (3) 1 (1951), 327–84.

(9) SCHATTEN, R. *A theory of cross-spaces* (Princeton, 1950).

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† See (2), pp. 10–12, for definitions of the spaces in this example.

‡ I am indebted to Prof. Northcott for drawing my attention to this fact.

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