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Some new generating functions for Jacobi polynomials

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1. *Introduction.* The object of this paper is to obtain some new generating functions for Jacobi polynomials. The results obtained are of general character.

2. In our investigation we require the following results (6) and (7).

$$F_E[\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_2;\ \rho_1,\rho_2,\rho_3;\ x,y,z] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\alpha_1)_{m+n+p}(\beta_1)_m(\beta_2)_{n+p}}{(\rho_1)_m(\rho_2)_n(\rho_3)_p\,m!\,n!\,p}x^m y^n z^p. \tag{1}$$

$$F_G[\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_3;\rho_1,\rho_2,\rho_2;\ x,y,z] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\alpha_1)_{m+n+p}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\rho_1)_m(\rho_2)_{n+p}\,m!\,n!\,p}x^m y^n z^p. \tag{2}$$

$$F_K[\alpha_1,\alpha_2,\alpha_2,\beta_1\beta_2,\beta_1;\ \rho_1,\rho_2,\rho_3;\ x,y,z] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\rho_1)_m(\rho_2)_n(\rho_3)_p\,m!\,n!\,p}x^m y^n z^p. \tag{3}$$

$$F_M[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_1;\ \rho_1,\rho_2,\rho_2;\ x,y,z] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_{m+p}(\beta_2)_n}{(\rho_1)_m(\rho_2)_{n+p}\,m!\,n!\,p}x^m y^n z^p. \tag{4}$$

$$F_S[\alpha_1,\alpha_2,\alpha_2,\beta_1,\beta_2,\beta_3;\rho_1,\rho_1,\rho_1;\ x,y,z] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\alpha_1)_m(\alpha_2)_{n+p}(\beta_1)_m(\beta_2)_n(\beta_3)_p}{(\rho_1)_{m+n+p}\,m!\,n!\,p}x^m y^n z^p. \tag{5}$$

$$\sum_{n=0}^{\infty}\frac{(\delta)_n}{(\lambda)_n}\frac{(m+n)!}{n!}P_{n+m}^{(\alpha-m-n,\beta-m-n)}(x)\,t^n = (-\alpha-\beta)_m(1+x)^{\alpha-m}(x-1)^{m-\alpha} \times F_2\left[m-\alpha-\beta,\ -\alpha,\delta;\ -\alpha-\beta,\lambda;\ \frac{2}{1+x},\ \frac{1-x}{2}t\right]. \tag{6}$$

In case we take $\lambda=\delta$ in (6) it reduces to the following formula

$$\sum_{n=0}^{\infty}\binom{m+n}{m}P_{m+n}^{(\alpha-n,\beta-n)}(x)\,t^n = [1+\tfrac{1}{2}(x+1)t]^{\alpha}[1+\tfrac{1}{2}(x-1)t]^{\beta}P_m^{(\alpha,\beta)}[x+\tfrac{1}{2}(x^2-1)t], \tag{7}$$

(7) is due to Manocha and Sharma (4). If we take $\lambda=-\alpha-\beta$ and $m=0$ in (6), we get

$$\sum_{n=0}^{\infty}\frac{(\delta)_n}{(-\alpha-\beta)_n}P_n^{(\alpha-n,\beta-n)}(x)\,t^n = [1-\tfrac{1}{2}(1-x)t]^{-\delta}\,{}_2F_1\left[\begin{matrix}\delta,-\alpha;\\ -\alpha-\beta,\end{matrix}\ \frac{-t}{1-\frac{1}{2}(1-x)t}\right]. \tag{8}$$

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3. The first formula to be proved is

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{n!} \frac{(\lambda)_n}{(\mu)_n} P_{n+m}^{(\alpha-m-n,\,\beta-m-n)}(x)\, {}_2F_1(\lambda+n,\gamma,\delta;\,y)\, t^n$$
$$= (-\alpha-\beta)_m (1+x)^{\alpha-m} (x-1)^{m-\alpha},$$
$$\times F_K\left[-\alpha,\lambda,\lambda,m-\alpha-\beta,\gamma,m-\alpha-\beta;\ -\alpha-\beta,\delta,\mu;\frac{2}{1+x},\frac{1-x}{2}t,y\right]. \tag{9}$$

Proof. To prove (9), we start with the left-hand side

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{n!} \frac{(\lambda)_n}{(\mu)_n} P_{n+m}^{(\alpha-m-n,\,\beta-m-n)}(x)\, {}_2F_1(\lambda+n,\gamma;\,\delta;\,y)\, t^n$$
$$= \sum_{p=0}^{\infty} \frac{(\lambda)_p (\gamma)_p}{p!\,(\delta)_p} y^p \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} \frac{(\lambda+p)_n}{(\mu)_n} P_{m+n}^{(\alpha-m-n,\,\beta-m-n)}(x)\, t^n$$
$$= (-\alpha-\beta)_m (1+x)^{\alpha-m} (x-1)^{m-\alpha} \sum_{p=0}^{\infty} \frac{(\lambda)_p (\gamma)_p}{p!\,(\delta)_p} y^p,$$

$$F_2\left[m-\alpha-\beta,\,-\alpha,\lambda+p;\,-\alpha-\beta,\mu;\,\frac{2}{1+x},\frac{1-x}{2}t\right] \quad \text{by (6)}$$
$$= (-\alpha-\beta)_m (1+x)^{\alpha-m} (x-1)^{m-\alpha}$$
$$\times F_K\left[-\alpha,\lambda,\lambda,m-\alpha-\beta,\gamma,m-\alpha-\beta;\ -\alpha-\beta,\delta,\mu;\frac{2}{1+x},\frac{1-x}{2}t,y\right] \quad \text{by (3).}$$

This completes the proof of (9). Proceeding in the same way and using the results (8), (1) and (2) we can obtain the following results.

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\,\beta-n)}(x)\, F_4[\lambda+n,\gamma;\,\delta,\rho;\,y,z]\, t^n$$
$$= \left[1-\frac{(1-x)t}{2}\right]^{-\lambda} F_E\left[\lambda,\lambda,\lambda,-\alpha,\gamma,\gamma,-\alpha-\beta,\delta,\rho;\right.$$
$$\left.\times \frac{-2t}{2-(1-x)t}, \frac{2y}{2-(1-x)t}, \frac{2z}{2-(1-x)t}\right]. \tag{10}$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(-\alpha-\beta)_n} P_n^{(\alpha-n,\,\beta-n)}(x)\, F_1[\lambda+n,\gamma;\,\rho,\delta;\,y,z]\, t^n$$
$$= \left[1-\frac{(1-x)t}{2}\right]^{-\lambda} F_G\left[\lambda,\lambda,\lambda,-\alpha,\gamma,\rho;\ -\alpha-\beta,\delta,\delta;\right.$$
$$\left.\times \frac{-2t}{2-(1-x)t}, \frac{2y}{2-(1-x)t}, \frac{2z}{2-(1-x)t}\right]. \tag{11}$$

4. The second formula to be proved is

$$\sum_{n=0}^{\infty} \frac{(n+m)!}{n!} P_{n+m}^{(\alpha-m-n,\,\beta-n-m)}(x)\, F_1[\lambda,-n,\mu;\,\rho;\,y,z]\, t^n$$
$$= (-\alpha-\beta)_m \left[\frac{2+(x-1)t}{2}\right]^{\alpha+\beta-2m} \left(\frac{x+1}{x-1}\right)^{\alpha-m}$$
$$\times F_M\left[-\alpha,\lambda,\lambda,m-\alpha-\beta,\mu,m-\alpha-\beta;\ -\alpha-\beta,\rho,\rho;\right.$$
$$\left.\times \frac{4}{(1+x)\,[2+t(x-1)]}, z, \frac{yt(x-1)}{2+(x-1)t}\right]. \tag{12}$$

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Proof. We start with the left-hand side

$$\sum_{n=0}^{\infty} \frac{(n+m)!}{n!} P_{n+m}^{(\alpha-m-n,\beta-m-n)}(x)\, F_1[\lambda, -n, \mu; \rho; y, z]\, t^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{r=0}^{n} \frac{(m+n)!\,(\lambda)_{r+k}(-n)_r(\mu)_k}{n!\,r!\,k!\,(\rho)_{r+k}} z^k y^r t^n P_{n+m}^{(\alpha-n-m,\beta-n-m)}(x)$$

$$= \sum_{r,k=0}^{\infty} \frac{(\lambda)_{r+k}(\mu)_k (m+r)!}{r!\,k!\,(\rho)_{r+k}} (-yt)^r z^k \sum_{n=0}^{\infty} \binom{m+n+r}{m+r} P_{m+n+r}^{(\alpha-n-m-r,\beta-m-n-r)}(x)\, t^n$$

$$= [1+\tfrac{1}{2}(x+1)t]^{\alpha-m}[1+\tfrac{1}{2}(x-1)t]^{\beta-m} \sum_{k=0}^{\infty} \frac{(\lambda)_k(\mu)_k}{k!\,(\rho)_k} z^k$$

$$\times \sum_{r=0}^{\infty} \frac{(m+r)!}{r!} \frac{(\lambda+k)_r}{(\rho+k)_r} P_{m+r}^{(\alpha-m-r,\beta-m-r)}\left[x+\frac{x^2-1}{2}t\right]$$

$$\times \left[\frac{-yt}{[1+\tfrac{1}{2}(x+1)t][1+\tfrac{1}{2}(x-1)t]}\right]^r \quad \text{by (7)}$$

$$= (-\alpha-\beta)_m [1+\tfrac{1}{2}(x-1)t]^{\alpha+\beta-2m} \left(\frac{x+1}{x-1}\right)^{\alpha-m},$$

$$\sum_{k=0}^{\infty} \frac{(\lambda)_k(\mu)_k}{k!\,(\rho)_k} z^k F_2\left[m-\alpha-\beta, -\alpha, \lambda+k; \ -\alpha-\beta, \rho+k;\right.$$

$$\left. \times \frac{4}{(1+x)[2+t(x-1)]}, \frac{yt(x-1)}{2-(1-x)t}\right] \quad \text{by (6)}$$

$$= (-\alpha-\beta)_m [1+\tfrac{1}{2}(x-1)t]^{\alpha+\beta-2m} \left(\frac{x+1}{x-1}\right)^{\alpha-m},$$

$$F_M\left[-\alpha, \lambda, \lambda, m-\alpha-\beta, \mu, m-\alpha-\beta, \ -\alpha-\beta, \rho, \rho;\right.$$

$$\left. \times \frac{4}{(1+x)[2+t(x-1)]}, z, \frac{yt(1-x)}{2+(x-1)t}\right] \quad \text{by (4).}$$

This completes the proof of (12). In case $m = 0$, and using the formula ((3), p. 83)

$$F_M[\alpha, \beta, \beta, \lambda, \mu, \lambda; \lambda, \delta, \delta; x, y, z] = (1-x)^{-\alpha},$$

$$F_D\left[\beta; \ \lambda-\alpha, \alpha, \mu; \ \delta; \ z, \frac{z}{1-x}, y\right] \tag{13}$$

in (12), it reduces to a known result (2). Taking $\lambda = -\delta, \rho = -\gamma-\delta, z = 0$ and using the formula ((5), p. 255)

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta)_{2n}}{n!\,(1+\alpha+\beta)_n} \left(\frac{x+1}{2}\right)^n {}_2F_1\left[\begin{matrix} -n, -\beta-n; \\ -\alpha-\beta-2n \end{matrix}; \frac{2}{1+x}\right] \tag{14}$$

in (12), we have

$$\sum_{n=0}^{\infty} \frac{(n+m)!}{(-\gamma-\delta)_n} P_n^{(\gamma-n,\delta-n)}(y)\, P_{n+m}^{(\alpha-m-n,\beta-m-n)}(x)\, t^n$$

$$= (-\alpha-\beta)_m [1-\tfrac{1}{4}t(x-1)(y+1)]^{\alpha+\beta-2m}$$

$$F_2\left[m-\alpha-\beta, -\alpha, -\delta; \ -\alpha-\beta, -\gamma-\delta;\right.$$

$$\left. \times \frac{8}{(x+1)[4-(x-1)(y+1)t]}, \frac{2t(1-x)}{4-(x-1)(y+1)t}\right]. \tag{15}$$

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Taking $y = z$ and using the formulae ((1), p. 239)

$$F_1(\alpha, \beta, \gamma: \delta; x, x) = {}_2F_1(\alpha, \beta+\gamma; \delta; x)$$

and (14) in (12) we get

$$\sum_{n=0}^{\infty} \frac{(n+m)!\,(n+r)!}{n!\,(-\gamma-\delta)_{n+r}} P_{n+m}^{(\alpha-m-n,\beta-m-n)}(x) P_{n+r}^{(\gamma-n-r,\delta-n-r)}(y)\, t^n$$

$$= (-\alpha-\beta)_m \left(\frac{x-1}{x+1}\right)^{m-\alpha} \left[1-(x-1)(y+1)\frac{t}{4}\right]^{\alpha+\beta-2m}$$

$$\times F_M\left[-\alpha, \delta, \delta, m-\alpha-\beta, -r, m-\alpha-\beta;\ -\alpha-\beta, -\gamma-\delta, -\gamma-\delta;\right.$$

$$\left.\times \frac{8}{(1+x)[4-(x-1)(y+1)t]}, \frac{2}{1+y}, \frac{2t(x-1)}{4-(x-1)(y+1)t}\right]. \tag{16}$$

Proceeding in the same way, we can obtain the following formulae by using the results (2), (5), (7) and (8)

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)\, F_2[\lambda, -n, \gamma; \mu, \delta; y, z]\, t^n = [1+\tfrac{1}{2}t(x+1)]^{\alpha}\,[1+\tfrac{1}{2}t(x-1)]^{\beta}$$

$$\times F_G\left[\lambda, \lambda, \lambda, \gamma, -\alpha, -\beta; \delta, \mu, \mu; z, \frac{(x+1)yt}{2+(x+1)t}, \frac{(x-1)yt}{2+(x+1)t}\right]. \tag{17}$$

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}(x)\, F_3[\lambda, -n, \gamma, \delta; \mu; y, z]\, t^n = [1+\tfrac{1}{2}t(x+1)]^{\alpha}\,[1+\tfrac{1}{2}t(x-1)]^{\beta}$$

$$\times F_S\left[\lambda, \delta, \delta, \gamma, -\beta, -\alpha; \mu, \mu, \mu; z, \frac{(x+1)yt}{2+(x+1)t}, \frac{(x-1)yt}{2+(x-1)t}\right]. \tag{18}$$

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