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Extensive amenability and an application to interval exchanges

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Abstract. Extensive amenability is a property of group actions which has recently been used as a tool to prove amenability of groups. We study this property and prove that it is preserved under a very general construction of semidirect products. As an application, we establish the amenability of all subgroups of the group IET of interval exchange transformations that have angular components of rational rank less than or equal to two. In addition, we obtain a reformulation of extensive amenability in terms of inverted orbits and use it to present a purely probabilistic proof that recurrent actions are extensively amenable. Finally, we study the triviality of the Poisson boundary for random walks on IET and show that there are subgroups $G < \mathrm{IET}$ admitting no finitely supported measure with trivial boundary.

1. *Introduction*

Recall that an action of a group G on a set X is *amenable* if it admits a G-invariant mean, that is, a finitely additive probability measure defined on all subsets of X that is invariant under the action of G.

In this article, we study *extensive amenability* of an action $G \curvearrowright X$. As the name is intended to suggest, this is a much stronger form of amenability and it has an intimate

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connection with extensions of groups and of actions. This property was introduced (without a name) in [**JM13**] as a tool to prove amenability of groups, a role it continued to play in [**JNdlS13**]. In order to give its formal definition, we denote by $\mathscr{P}_{\mathrm{f}}(X)$ the set of all finite subsets of X.

Definition 1.1. The action of a group G on a set X is *extensively amenable* if there is a G-invariant mean m on $\mathscr{P}_{\mathrm{f}}(X)$ so that, for every $A \in \mathscr{P}_{\mathrm{f}}(X)$, the mean m gives full weight to the collection of subsets that contain A.

Remarks 1.2.

(i) Extensively amenable actions are amenable unless $X = \varnothing$, and every action of an amenable group is extensively amenable (see Lemma 2.1).

(ii) Both converses fail: there are amenable actions that are not extensively amenable (see [**JdlS13**] or §6 for a new example), as well as extensively amenable faithful actions of non-amenable groups [**JM13**].

(iii) A sufficient condition for extensive amenability was provided in [**JNdlS13**], namely, *recurrence* in the sense of random walks (see §4.2 for details and a new proof of this result).

(iv) We do not consider any topology on G or X. Moreover, it is not relevant to assume G or X countable, because an action $G \curvearrowright X$ is extensively amenable if and only if the action of every finitely generated subgroup $H \leq G$ on every H-orbit is so (Lemma 2.2).

An important advantage of extensive amenability over usual amenability of actions is that it is more robust. For instance, it is preserved under extensions of actions in a sense made precise in §2. Another stability property is fundamental for applications and concerns a wide class of group actions constructed functorially from $G \curvearrowright X$; here the group G itself will be replaced by a semidirect product extension of G. The first example is as follows.

Given $G \curvearrowright X$ one can form the *permutational wreath product*, or 'lamplighter group', $(\mathbf{Z}/2\mathbf{Z})^{(X)} \rtimes G$. Here $(\mathbf{Z}/2\mathbf{Z})^{(X)}$ denotes the group of finitely supported configurations $X \to \mathbf{Z}/2\mathbf{Z}$, on which G acts by translations; it can be identified with $\mathscr{P}_{\mathrm{f}}(X)$. The lamplighter group acts affinely on the 'lamp group' $(\mathbf{Z}/2\mathbf{Z})^{(X)}$, identified with the coset space $((\mathbf{Z}/2\mathbf{Z})^{(X)} \rtimes G)/G$. By simple Fourier analysis arguments, it was shown in [**JM13**, Lemma 3.1] that the amenability of this affine action is equivalent to the extensive amenability of $G \curvearrowright X$.

Our first result (Theorem 1.3 below) provides a vast non-commutative generalization of this equivalence, wherein the lamp groups $(\mathbf{Z}/2\mathbf{Z})^{(X)}$ will be replaced by rather arbitrary functors F applied to X. Specifically, we consider all 'finitary' functors to the category of amenable groups in the following precise sense.

Denote by $\mathbf{I}$ the category of finite sets whose morphisms are injective maps. Further, denote by **Amen** the category of amenable groups with group homomorphisms. Since **Amen** has direct limits, any functor $F : \mathbf{I} \to \mathbf{Amen}$ extends to the category of all sets with injective maps as morphisms, by setting $F(X)$ to be the direct limit of $F(Y)$ as Y runs over the directed set $\mathscr{P}_{\mathrm{f}}(X)$. We still denote the resulting functor by the same letter F. Several explicit examples are given in §3, but the reader can already have in mind $F(X) = A^{(X)}$

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for an amenable group A, or $F(X) = \mathrm{Sym}(X)$, the group of permutations of X with finite support.

Notice that any group action $G \curvearrowright X$ yields an action $G \curvearrowright F(X)$, by automorphisms, and hence a semidirect product group $F(X) \rtimes G$. This semidirect product acts on $F(X)$ identified with the coset space $(F(X) \rtimes G)/G$.

We say that the functor F is *tight* on a non-empty set X if, for all (equivalently, some) $x \in X$, the morphism $F(X \setminus \{x\}) \to F(X)$, induced by inclusion, is not onto. A particular case of our results from §3 is the following theorem.

THEOREM 1.3. *Let $F : \mathbf{I} \to \mathbf{Amen}$ be any functor, extended to arbitrary sets, as described above. Let G be a group acting on a set X.*

If the action $G \curvearrowright X$ is extensively amenable, then the action $F(X) \rtimes G \curvearrowright F(X)$ is amenable. Moreover, it is extensively amenable.

Conversely, assume that the action $F(X) \rtimes G \curvearrowright F(X)$ is amenable. Then $G \curvearrowright X$ is extensively amenable provided F is tight on X.

In particular, the amenability of the action $F(X) \rtimes G \curvearrowright F(X)$ does not depend on the choice of the functor F, provided that it is tight on X. Furthermore, for such 'affine' actions, amenability and extensive amenability coincide.

Beside the lamplighter case $F(X) = (\mathbf{Z}/2\mathbf{Z})^{(X)}$, another concrete example to which Theorem 1.3 applies is $F(X) = \mathrm{Sym}(X)$, the group of finitely supported permutations of a set X, on which G acts by conjugation. See §3 for more examples. A case in which the theorem *does not* apply, as it does not come from a functor on $\mathbf{I}$, is $F(X) = (\mathbf{Z}/2\mathbf{Z})^X$, the *unrestricted* direct product indexed by X (see §7 for a counterexample in this case).

Theorem 1.3 can be used to establish a criterion for the amenability of some subgroups of $F(X) \rtimes G$.

COROLLARY 1.4. *Let $G \curvearrowright X$ be an extensively amenable action and let $F : \mathbf{I} \to \mathbf{Amen}$ be any functor.*

A subgroup H of $F(X) \rtimes G$ is amenable as soon as the intersection $H \cap (\{1\} \times G)$ is so.

(The derivation of this result from Theorem 1.3 is given in §3.)

Remark 1.5. A particular case in which this criterion applies is when one is able to construct a *twisted embedding* $G \hookrightarrow F(X) \ltimes G$ of the form $g \mapsto (c_g, g)$, with the property that $\{g \in G : c_g = 1\}$ is an amenable subgroup of G. We then say that $c : G \to F(X)$, $g \mapsto c_g$ is a $F(X)$-*cocycle with amenable kernel*. The conclusion is then that G is amenable.

This method was used in the proof that the topological full group of any minimal Cantor system is amenable [**JM13**]; namely, the particular case of Corollary 1.4 for the 'lamp' functor $F(X) = (\mathbf{Z}/2\mathbf{Z})^{(X)}$ was used. In the next paragraph, we present a new application of this method, relying, this time, on the functor $F(X) = \mathrm{Sym}(X)$.

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1.1. *Application to groups of interval exchange transformations.* An interval exchange transformation is a permutation of a circle obtained by cutting this circle into a finite number of intervals (arcs), and reordering them. More precisely, an interval exchange transformation is a right-continuous permutation g of $\mathbf{R}/\mathbf{Z}$ such that the set $\{gx - x, x \in \mathbf{R}/\mathbf{Z}\}$, called the set of angles of g, is finite. Interval exchange transformations are traditionally defined as permutations of an interval $[a, b)$, but we get the same notion by identifying a and b.

The set of all interval exchange transformations is a group acting on $\mathbf{R}/\mathbf{Z}$, denoted by IET.

The interval exchange transformations have been a popular object of study in dynamical systems and ergodic theory; for example they model the dynamics on polygonal billiards with rational angles. The exchanges of three or more intervals were first considered by Katok and Stepin [**KS67**]. The systematic study started in the paper by Keane [**Kea75**], who introduced the terminology. For an introduction and account of the results, see the survey by Viana [**Via06**] or the book of Katok and Hasselblatt [**KH95**].

A basic question on IET was raised by Katok, namely whether or not IET contains a non-abelian free group. This problem attracted some attention recently; for instance, Dahmani–Fujiwara–Guirardel [**DFG13**] showed that free subgroups in IET are *rare* in the sense that a group generated by a generic pair of interval exchange transformations is not free. A related open question raised in [**dC13**, p. 4] is as follows.

Question 1.6. Is the group IET amenable?

It is sufficient for these questions to consider finitely generated subgroups of IET. The latter can be classified according to their *rational rank*, as follows. Given a subgroup $G \leq \mathrm{IET}$, we denote its *group of angles* by $\Lambda(G) \leq \mathbf{R}/\mathbf{Z}$, that is, the subgroup of $\mathbf{R}/\mathbf{Z}$ generated by all increments $gx - x$, where $g \in G$ and $x \in \mathbf{R}/\mathbf{Z}$.

Definition 1.7. The *rational rank* of G, denoted by $\mathrm{rk}_{\mathbf{Q}}(G) \in \mathbf{N} \cup \{\infty\}$, is the supremum of all d such that $\mathbf{Z}^d$ embeds in $\Lambda(G)$.

Remark 1.8. If G is finitely generated, then $\Lambda(G)$ is a finitely generated abelian group and thus $\Lambda(G) \simeq \mathbf{Z}^d \times H$ with H a finite abelian group. In this case $\mathrm{rk}_{\mathbf{Q}}(G) = d$.

THEOREM 1.9. *Let $G \leq \mathrm{IET}$. If $\mathrm{rk}_{\mathbf{Q}}(G) \leq 2$, then G is amenable.*

The proof is based on the method outlined in Remark 1.5 with $F(X) = \mathrm{Sym}(X)$, the group of finitely supported permutations of X. We first observe that there exists a $\mathrm{Sym}(\mathbf{R}/\mathbf{Z})$-cocycle $g \mapsto \tau_g$ with amenable kernel defined on the whole group IET. This reduces the problem of the amenability of a subgroup $G \leq \mathrm{IET}$ to proving the extensive amenability of the action $G \curvearrowright \mathbf{R}/\mathbf{Z}$. In the case $\mathrm{rk}_{\mathbf{Q}}(G) \leq 2$, this action is recurrent and this yields Theorem 1.9. The general case would follow from a positive answer to Question 1.11 below.

The possibility to define a twisted embedding $\mathrm{IET} \hookrightarrow \mathrm{Sym}(\mathbf{R}/\mathbf{Z}) \rtimes \mathrm{IET}$ also has an almost immediate application related to *Poisson–Furstenberg boundaries* for random walks on IET. A group endowed with a probability measure (G, μ) is said to have the *Liouville property* if its Poisson–Furstenberg boundary is trivial. If the support of μ

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generates G, the Liouville property for (G, μ) implies amenability of G [**KV83**, §4.2]. Conversely, any amenable group admits a symmetric measure supported on a (possibly infinite) generating set so that (G, μ) has the Liouville property [**KV83**, Theorem 4.3]. The next result shows that there is an obstruction to showing amenability of IET in this way, namely, that the Liouville property never holds for finitely supported measures generating a 'large enough' subgroup.

We say that a probability measure on a group is *non-degenerate* if its support generates the group.

THEOREM 1.10. *Let* $G < \mathrm{IET}$ *be finitely generated.*

(i) *If* $\mathrm{rk}_{\mathbf{Q}}(G) = 1$*, then every symmetric, finitely supported probability measure* μ *on* G *has the Liouville property.*

(ii) *Assume that* $\mathrm{rk}_{\mathbf{Q}}(G) \geq 3$ *and, viewing* $\Lambda(G)$ *as a subgroup of* IET *consisting of rotations, assume that* G *contains* $\Lambda(G)$ *strictly. Then every finitely supported, non-degenerate probability measure on* G *has non-trivial Poisson–Furstenberg boundary.*

Part (i) follows essentially from a combination of known facts, namely, a result from [**MB14**] combined with the observation [**dC13**] that a finitely generated $G < \mathrm{IET}$ can be realized as a subgroup of the *topological full group* of a certain minimal subshift on the group of angles $\Lambda(G)$. Part (ii) is proved by applying the twisted embedding $\mathrm{IET} \hookrightarrow \mathrm{Sym}(X) \rtimes \mathrm{IET}$ and then arguing in a similar way as for the classical example of the lamplighter group $(\mathbf{Z}/2\mathbf{Z}) \wr \mathbf{Z}^3$ from [**KV83**]. Details are given in §5.2.

There is a strong analogy between Theorems 1.9 and 1.10 and existing results on groups generated by finite *automata of polynomial activity*. These are a well-studied class of groups acting on rooted trees, which can be classified according to their *activity degree* $d \in \mathbf{N}$ (we refer to [**Sid00**] for details). Amenability of polynomial activity automata groups is an open problem, that has been answered affirmatively for $d \leq 2$ [**JNdlS13**]. Concerning the Liouville property, it was conjectured in [**AAV13**] that it holds up to $d = 2$; this is known for $d = 0$ [**BKN10**, **AAMBV13**] and for $d = 1$ in some important special cases [**AAV13**], while, for $d \geq 3$, the Liouville property does not hold [**AV14**].

1.2. *Criteria for extensive amenability.* It is an intriguing problem to find new criteria to establish extensive amenability of an action $G \curvearrowright X$. To this end, there is no loss of generality in assuming that G is finitely generated and acts transitively on X (Lemma 2.2). Assuming this, let S be a finite symmetric generating set of G. Recall that the *orbital Schreier graph* of the action is the oriented labelled graph $\Gamma(G, X, S)$ whose vertex set is X and edge set is $X \times S$, where the edge (x, s) connects x to sx. The corresponding edge is labelled by s.

We would like to find sufficient geometric conditions on the labelled Schreier graph $\Gamma(G, S, X)$ that imply extensive amenability of the action $G \curvearrowright X$. An obvious simplification for this problem is to seek sufficient conditions that only depend on the *unlabelled* Schreier graph. So far, the only available criterion [**JNdlS13**] of this kind is Theorem 4.2.

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Question 1.11. Assume that $\Gamma(G, X, S)$ grows polynomially. Does this imply that $G \curvearrowright X$ is extensively amenable?

A positive answer to Question 1.11 would imply that the whole group IET is amenable, as the Schreier graph of any finitely generated subgroup $G <$ IET admit an injective Lipschitz embedding into $\mathbf{Z}^d$ (see §5). Note that a counterexample is missing even for graphs with uniform sub-exponential growth. Relying on the main result from [**JNdlS13**], a positive answer in this case would also imply amenability of polynomial activity automata groups, for which the graphs grow uniformly sub-exponentially [**Bon12**].

1.3. *Organization of the paper.* In §2, we study general properties of extensive amenability. Section 3 deals with the functorial formalism, and contains proofs and generalizations of Theorem 1.3 and Corollary 1.4. In §4, we give a probabilistic reformulation of extensive amenability. The results on groups of interval exchange transformations are proved in §5. In §6, we describe an example of an action $G \curvearrowright X$ which is not extensively amenable, but where the action of every subgroup H of G on every H-orbit is amenable (see Corollary 2.3). Finally, in §7, we construct an example where the conclusion of Theorem 1.3 does not hold for the unrestricted wreath product action $(\mathbf{Z}/2\mathbf{Z})^X \rtimes G \curvearrowright (\mathbf{Z}/2\mathbf{Z})^X$.

2. *General properties of extensive amenability*

2.1. *Notation.* We will always identify a finitely additive probability measure m defined on all subsets of X with the corresponding positive unital linear form on $\ell^\infty(X)$, which we denote as $f \mapsto m(f)$ or $f \mapsto \int f\, dm$ or $\int_X f(x)\, dm(x)$.

The first lemma is easy and already known from [**JdlS13**]. We give a proof for convenience.

LEMMA 2.1. *Every action of an amenable group is extensively amenable, and every extensively amenable action on a non-empty set is amenable.*

Proof. Assume that G is amenable and acts on a set X. Then G acts on the set K of all means on $\mathscr{P}_{\mathrm{f}}(X)$ giving full weight to any given finite subset of X. Observe that K is a $\sigma(\ell^\infty(X)^*, \ell^\infty(X))$-closed (and hence compact) convex subset of $\ell^\infty(X)^*$. Moreover, K is non-empty because it contains any cluster point of the net $(\delta_A)_{A\in\mathscr{P}_{\mathrm{f}}(A)}$. Since G is amenable, G fixes a point in K, which is exactly extensive amenability.

If the action of a group G on a non-empty set X is extensively amenable, in particular, there is a G-invariant mean m on $\mathscr{P}_{\mathrm{f}}(X) \setminus \{\varnothing\}$. Then the mean $f \in \ell^\infty(X) \mapsto \int_{\mathscr{P}_{\mathrm{f}}(X)\setminus\{\varnothing\}} (1/|A|) \sum_{x\in A} f(x)\, dm(A)$ is G-invariant. □

In the next lemma, we study permanence properties of extensive amenability. The equivalence between (i) and (ii), in particular, shows that extensive amenability is preserved by direct limits.

LEMMA 2.2. *Let G be a group acting on a set X. The following are equivalent.*

(i) *The action of G on X is extensively amenable.*

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(ii) *For every finitely generated subgroup H of G and every H-orbit $Y \subset X$, the action of H on Y is extensively amenable.*

(iii) *For every finitely generated subgroup H of G and every $x_0 \in X$, there is an H-invariant mean on $\mathscr{P}_{\mathrm{f}}(Hx_0)$ that gives non-zero weight to $\{A \in \mathscr{P}_{\mathrm{f}}(Hx_0),\ x_0 \in A\}$.*

(iv) *There is a G-invariant mean on $\mathscr{P}_{\mathrm{f}}(X)$ that gives non-zero weight to $\{A \in \mathscr{P}_{\mathrm{f}}(X),\ x_0 \in A\}$, for all $x_0 \in X$.*

Proof. (i) $\Rightarrow$ (iv) holds, by definition.

(iv) $\Rightarrow$ (iii). Denote $Y = Hx_0$. The map $f \in \ell^\infty(\mathscr{P}_{\mathrm{f}}(Y)) \to f(\cdot \cap Y) \in \ell^\infty(\mathscr{P}_{\mathrm{f}}(X))$ is a positive unital H-map, so that the composition of the mean given by (iv) is a H-invariant mean giving positive weight to the sets containing x_0.

(iii) $\Rightarrow$ (ii) follows from [**JM13**, Lemma 3.1].

(ii) $\Rightarrow$ (i). For every finitely generated subgroup H of G and every finite union $Y = Y_1 \cup \cdots \cup Y_n$ of H-orbits, we have means m_i on $\mathscr{P}_{\mathrm{f}}(Y_i)$ given by (ii), and we construct a mean $m_{H,Y}$ on $\mathscr{P}_{\mathrm{f}}(X)$ as

$$\int f\, dm_{H,Y} = \int_{\mathscr{P}_{\mathrm{f}}(Y_1)} \cdots \int_{\mathscr{P}_{\mathrm{f}}(Y_n)} f(B_1 \cup \cdots \cup B_n)\, dm_1(B_1) \ldots dm_n(B_n).$$

The mean $m_{H,Y}$ is H-invariant and gives full weight to the sets containing any given finite subset of Y. If we order the pairs (H, Y) by inclusion, any cluster point of the net $(m_{H,Y})$ is G-invariant and gives full weight to the sets containing any given finite subset of X. □

We say that a group action $G \curvearrowright X$ is *hereditarily amenable* if, for every subgroup H of G, the action of H on every H-orbit is amenable. Thereby we have the following corollary.

COROLLARY 2.3. *Extensively amenable actions are hereditarily amenable.*

In §6, we shall see that the converse is not true.

We now prove that, in some sense, extensive amenability is preserved by extensions of actions. In the statement below, we denote the stabilizer of y in G by G_y.

PROPOSITION 2.4. *Let G be a group acting on two sets X, Y and let $q\colon X \to Y$ be a G-map. If $G \curvearrowright Y$ is extensively amenable and if $G_y \curvearrowright q^{-1}(y)$ is extensively amenable for every $y \in Y$, then $G \curvearrowright X$ is extensively amenable. The converse holds if q is surjective.*

The special case of transitive actions can be reformulated as the following particularly clean equivalence. We recall that the corresponding statement for amenability *does not hold* (see [**MP03**]).

COROLLARY 2.5. *Let $F \leq H \leq G$ be groups. Then the action of G on G/F is extensively amenable if and only if both the actions of H on H/F and G on G/H are extensively amenable.*

Proof of the corollary. Apply Proposition 2.4 to $X = G/F$ and $Y = G/H$ with the quotient map. □

Proof of Proposition 2.4. Assume that $G \curvearrowright Y$ and $G_y \curvearrowright q^{-1}(y)$ are extensively amenable for all $y \in Y$. Take a mean m_1 on $\mathscr{P}_{\mathrm{f}}(Y)$, as in Definition 1.1, for $G \curvearrowright Y$. We claim that,

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for all $y \in Y$, there is a mean m_y on $\mathscr{P}_{\mathrm{f}}(q^{-1}(y))$ giving full weight to the subsets of $q^{-1}(y)$ that contain any given finite subset of $q^{-1}(y)$ and with the property that m_{gy} is the push-forward by g of the mean m_y for all $y \in Y$ and $g \in G$. Indeed, by the assumption that $G_y \curvearrowright q^{-1}(Y)$ is extensively amenable, we can take such a G_y-invariant mean m_y on $\mathscr{P}_{\mathrm{f}}(q^{-1}(y))$ for each y in a fixed G-transversal of Y, and, for $y' = gy$ in the G-orbit of such y, define $m_{y'}$ as the push-forward of m_y by g; this definition does not depend on g because m_y is G_y-invariant, and this defines the requested mean on $\mathscr{P}_{\mathrm{f}}(q^{-1}(y'))$.

For any subset $A = \{y_1, \ldots, y_n\} \in \mathscr{P}_{\mathrm{f}}(Y)$, denote by m_A the mean on $\mathscr{P}_{\mathrm{f}}(X)$ by

$$m_A(f) = \frac{1}{n!} \sum_{\sigma\colon \{1,\ldots,n\} \to A} \int_{q^{-1}(\sigma(1))} \cdots \int_{q^{-1}(\sigma(n))} f\left(\bigcup_{i=1}^{n} B_i\right) dm_{\sigma(1)}(B_1) \ldots dm_{\sigma(n)}(B_n).$$

The average is taken over all bijections σ, in order to ensure that m_A does not depend on the chosen ordering of the elements of A. By the properties of the means m_y, we have the following two properties of the means m_A. Firstly, the push-forward of m_A by $g \in G$ is m_{gA}; secondly, m_A gives full weight to the subsets of X that contain any given subset of $q^{-1}(A)$. These properties ensure that the mean m on $\mathscr{P}_{\mathrm{f}}(X)$, defined by

$$m\colon f \in \ell^\infty(\mathscr{P}_{\mathrm{f}}(X)) = \int_{\mathscr{P}_{\mathrm{f}}(Y)} m_A(f)\, dm_1(A),$$

is G-invariant and gives full weight to the subsets of X that contain any given subset of X. This proves that $G \curvearrowright X$ is extensively amenable.

For the converse, assume, first, that $G \curvearrowright X$ is extensively amenable, and take m a mean on $\mathscr{P}_{\mathrm{f}}(X)$, as in Definition 1.1. Then, by Lemma 2.2, the action of the subgroup G_y on the subset $q^{-1}(y)$ of X is extensively amenable for all $y \in Y$. Consider $q'\colon A \in \mathscr{P}_{\mathrm{f}}(X) \mapsto \{q(a), a \in A\} \in \mathscr{P}_{\mathrm{f}}(Y)$. The map

$$f \in \ell^\infty(\mathscr{P}_{\mathrm{f}}(Y)) \mapsto m(f \circ q')$$

is a G-invariant mean that gives full weight to the sets containing every finite subset of $q(Y)$. This proves that $G \curvearrowright q(Y)$ is extensively amenable. □

COROLLARY 2.6. *Let G and H be two groups, and let $G \curvearrowright X$ and $H \curvearrowright Y$ be two extensively amenable actions. Then:*

(i) *the diagonal action of $G \times H$ on $X \times Y$ is extensively amenable; and*
(ii) *the action on $H^X \rtimes G$ on $X \times Y$ is extensively amenable.*

In the second statement, G acts diagonally on $X \times Y$ (with trivial action on Y) and H^X acts by $(h_x)_{x\in X} \cdot (x, y) = (x, h_x y)$; this gives rise to an action of $H^X \rtimes G$.

Proof. The first statement is Proposition 2.4 for the actions of $G \times H$ on $X \times Y$ and X (trivial action of H) and $q\colon X \times Y \to X$, the first coordinate projection.

The second statement is Proposition 2.4 for the actions on $H^X \rtimes G$ on $X \times Y$ and X (trivial action of H^X) for the same projection q. □

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3. *Functors from sets to amenable group actions*

This section deals with the proof (and generalizations) of Theorems 1.3 and 1.4. We start by giving examples of functors $\mathbf{I} \to \mathbf{Amen}$.

Example 3.1. Fix an amenable group A. Consider the functor F which maps any finite set Y to A^Y with the obvious extension map on inclusions. Then, for a general set X, we have $F(X) = A^{(X)}$, the restricted product.

Example 3.2. If $F(Y)$ is the symmetric group of the finite set Y, then $F(X)$ will be the finitely supported permutation group $\mathrm{Sym}(X)$ of a general set X.

In the following two examples, rings are always assumed to have a unit, but are not assumed to be commutative.

Example 3.3. Fix a finite ring R. Consider the functor $F(Y)$ given by the group of invertible matrices over R indexed by a finite set Y, with the 'corner' inclusions. For an infinite set X, the group $F(X)$ is a stable linear group, which we denote by $GL_{(X)}(R)$ (it is the usual one when X is countable). As a variation, we can define $F(Y)$ to be the group $EL_Y(R)$ generated by elementary matrices; this yields stable elementary groups $EL_X(R)$ for arbitrary sets X.

Example 3.4. Fix, again, a finite ring R. Given a finite set Y, consider the (unstable) Steinberg group of degree $|Y|$ over R. Then, for X general, we obtain a (stable) Steinberg group which we denote by $\mathrm{St}_{(X)}(R)$. Thus we have a natural transformation given by the morphisms $\mathrm{St}_{(X)}(R) \to EL_X(R)$ (recall that these are isomorphisms if R is a finite field).

Example 3.5. Finally, a trivial example is given by the constant functor $F(Y) = A$ for any given amenable group A. We also have the modified constant functor F_0 defined by $F_0(Y) = A$ for all non-empty sets Y and for $F_0(\varnothing)$ being the trivial group.

Recall the definition of a tight functor from the introduction.

Definition 3.6. A functor $F\colon \mathbf{I} \to \mathbf{Amen}$ is called *tight* on a (possibly infinite) set X if for all $x \in X$ the morphism $F(X \setminus \{x\}) \to F(X)$ is not onto.

It is straightforward to verify that all examples above are tight on every set X except some degenerate cases: see Example 3.5, Example 3.1 with A trivial and X non-empty, and Examples 3.2, 3.3 and 3.4 in which one should exclude $|X| = 1$.

We first treat, separately, a special case of Theorem 1.3, which is enough for the application to interval exchange transformations. The proof of the full statement of Theorem 1.3 will be given later in a more general setting (see Theorem 3.14).

PROPOSITION 3.7. *Let $F\colon \mathbf{I} \to \mathbf{Amen}$ be a functor, as in Theorem 1.3, and assume, moreover, that $F(A)$ is a finite group whenever A is a finite set.*

If the action of G on X is extensively amenable, then the action of $F(X) \rtimes G$ on $F(X)$ is extensively amenable.

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Proof. Assume that the action of G on X is extensively amenable. Let m be a G-invariant mean on $\mathscr{P}_{\mathrm{f}}(X)$ giving full weight to the collection of subsets that contain any given finite subset. Let F be a functor from $\mathbf{I}$ to the category of finite groups.

We first prove that the action of $F(X) \wr G$ on $F(X)$ is amenable. Then we will see how to adapt the proof to show that the action is extensively amenable.

For a finite set A, let m_A be the uniform probability measure on $F(A)$, which is a finite group, by assumption. We denote by m_A^X the mean on $F(X)$ obtained by push-forward through $F(A) \to F(X)$; this mean is $F(A)$-invariant by construction. Observe, also, that $gm_A^X = m_{gA}^X$ for every $g \in G$.

We obtain a mean $\widetilde{m}$ on $F(X)$ by integrating m_A^X over m; more precisely, given $f \in \ell^\infty(F(X))$ we define

$$\widetilde{m}(f) = \int_{\mathscr{P}_{\mathrm{f}}(X)} m_A^X(f)\, dm(A).$$

This mean is G-invariant, by construction, and we claim that it is also $F(X)$-invariant. It is enough to show that m is $F(A)$-invariant for every finite subset A of X. But this holds because m gives full weight to the set of finite subsets containing A and because m_B^X is $F(A)$-invariant whenever B contains A.

To prove that the (transitive) action of $F(X) \wr G$ on $F(X)$ is extensively amenable, by [**JM13**, Lemma 3.1], we have to prove that there is a mean on $(\mathbf{Z}/2\mathbf{Z})^{(F(X))}$ that is invariant under the action of $(\mathbf{Z}/2\mathbf{Z})^{(F(X))} \wr (F(X) \wr G)$. For every $A \in \mathscr{P}_{\mathrm{f}}(X)$, the map $F(\subset) : F(A) \to F(X)$ induces a group homomorphism $(\mathbf{Z}/2\mathbf{Z})^{F(A)} \to (Z/2\mathbf{Z})^{(F(X))}$ obtained by sending δ_s to $\delta_{F(\subset)(x)}$. If m_A is the uniform probability measure on the (finite) image of this group homomorphism, then the same argument as above shows that the mean obtained on $(\mathbf{Z}/2\mathbf{Z})^{(F(X))}$ by integrating m_A over m is invariant under $(\mathbf{Z}/2\mathbf{Z})^{(F(X))} \wr (F(X) \wr G)$. □

We now prove Corollary 1.4.

Proof of Corollary 1.4. To prove that H is amenable, we find an amenable action of H with amenable stabilizers. Let e be the unit element of $F(X)$, and consider the action of H on the H-orbit of e. By Theorem 1.3 and Lemma 2.2, this action is extensively amenable and, in particular, is amenable. Moreover, the stabilizer of e is $H \cap (\{1\} \times G)$ which, by assumption, is amenable. So is every other stabilizer that is a conjugate of the stabilizer of e. This implies that H is amenable. □

Remark 3.8. In the above proof, we appealed to Theorem 1.3 in full generality. However, it is sufficient to apply Proposition 3.7 if its assumptions are satisfied.

To prove the full statement of Theorem 1.3, it is convenient to pass to a slightly more general setting of functors from $\mathbf{I}$ to the category **AmenAct** of amenable actions. This generalization is needed in order to obtain that the action of $F(X) \ltimes G$ on $F(X)$ is not only amenable, but also extensively amenable.

The objects in **AmenAct** are given by a group acting amenably on a set, and a morphism from $G \curvearrowright X$ to $H \curvearrowright Y$ is given by a map $X \to Y$ and a group homomorphism $G \to H$ that intertwines the two actions. If $F : \mathbf{I} \to \mathbf{AmenAct}$ is a functor, we denote by $F_{\mathbf{Grp}}$ and $F_{\mathbf{Set}}$ the associated functors from $\mathbf{I}$ to the category of groups and sets, respectively. That is,

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$F(X)$ is the action of $F_{\mathbf{Grp}}(X)$ on $F_{\mathbf{Set}}(X)$. Since **AmenAct** has direct limits, any functor $F : \mathbf{I} \to \mathbf{AmenAct}$ extends to the category of all sets with injective maps as morphisms, in the same way as for functors to **Amen**. We still denote the resulting functor by F.

As before, notice that any group action $G \curvearrowright X$ yields an action $G \curvearrowright F_{\mathbf{Grp}}(X)$ by automorphisms and an action $G \curvearrowright F_{\mathbf{Set}}(X)$; and hence an action $F_{\mathbf{Grp}}(X) \rtimes G \curvearrowright F_{\mathbf{Set}}(X)$.

Example 3.9. Any functor $F : \mathbf{I} \to \mathbf{Amen}$ is, in particular, a functor $\mathbf{I} \to \mathbf{AmenAct}$ ($G \mapsto (G \curvearrowright G)$ realizes **Amen** as a subcategory of **AmenAct**).

Example 3.10. Take two functors $E, F : \mathbf{I} \to \mathbf{AmenAct}$. Assume, furthermore, that $F_{\mathbf{Set}}$ is a functor from $\mathbf{I}$ to the category of sets with injective maps as morphisms and that $F_{\mathbf{Grp}}$ is a functor $\mathbf{I} \to \mathbf{Amen}$. For a finite set Y, we have an action of an amenable group $F_{\mathbf{Grp}}(Y)$ on $F_{\mathbf{Set}}(Y)$, and hence, by the functoriality of E, an action of $E_{\mathbf{Grp}}(F_{\mathbf{Set}}(Y)) \rtimes F_{\mathbf{Grp}}(Y)$ on $E_{\mathbf{Set}}(F_{\mathbf{Set}}(Y))$. This action is amenable because $E_{\mathbf{Grp}}(F_{\mathbf{Set}}(Y)) \curvearrowright E_{\mathbf{Set}}(F_{\mathbf{Set}}(Y))$ is amenable and the group $F_{\mathbf{Grp}}(Y)$ is amenable. This defines a new functor, which we denote by $E \diamond F : \mathbf{I} \to \mathbf{AmenAct}$.

Example 3.11. The previous construction also makes sense if $F : \mathbf{I} \to \mathbf{AmenAct}$ is a functor such that $F_{\mathbf{Grp}} : \mathbf{I} \to \mathbf{Amen}$ and $E : \mathbf{Set} \to \mathbf{AmenAct}$.

Definition 3.12. A functor $F : \mathbf{I} \to \mathbf{AmenAct}$ is called *tight* on a (possibly infinite) set X if, for all $x \in X$, no invariant mean for $F(X)$ gives full weight to the image of $F_{\mathbf{Set}}(X \setminus \{x\}) \to F_{\mathbf{Set}}(X)$.

This definition extends Definition 3.6 to functors $F : \mathbf{I} \to \mathbf{AmenAct}$, because, if H is a subgroup of an amenable group G, then any G-invariant mean on G gives weight $|G : H|^{-1}$ to H, where $|G : H|$ is the index of H in G.

We now define the *support* maps associated to functors $F : \mathbf{I} \to \mathbf{Amen}$ and $F : \mathbf{I} \to \mathbf{AmenAct}$. Consider, first, the case of a functor $F : \mathbf{I} \to \mathbf{Amen}$. Given an arbitrary set X, define

$$\mathrm{supp}_X : F(X) \longrightarrow \mathscr{P}_{\mathrm{f}}(X)$$

as follows: for $a \in F(X)$, the set $\mathrm{supp}_X(a)$ is the intersection of all $Y \in \mathscr{P}_{\mathrm{f}}(X)$ such that a is in the image of the morphism $F(Y) \to F(X)$.

One checks that, for any injective map $i : X \to Z$,

$$\mathrm{supp}_Z(F(i)(a)) \subseteq i\ \mathrm{supp}_X(a).$$

Therefore, given any group action $G \curvearrowright X$, the map supp_X is equivariant with respect to the induced action on $F(X)$.

Remarks 3.13. The above inclusion can be strict. This happens, for instance, for the functor F_0 of Example 3.5 if A is non-trivial, $|X| = 1$ and $|Z| > 1$. Indeed, any non-trivial $a \in F(X) \cong A$ satisfies $\mathrm{supp}_X(a) = X$ and $\mathrm{supp}_Z(F(i)(a)) = \varnothing$.

We also observe that a non-trivial element can have empty support even if F is tight. For instance, let F be a tight functor and take the direct product F' of F with a constant functor associated to a non-trivial group A. Then F' is still tight but any element coming from A has empty support.

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In the case of a functor to **AmenAct**, the support map is defined in the same way, by replacing F by $F_{\mathbf{Set}}$, thereby yielding a G-equivariant map

$$\mathrm{supp}_X \colon F_{\mathbf{Set}}(X) \longrightarrow \mathscr{P}_{\mathrm{f}}(X).$$

We can now state and prove the results extending Theorem 1.3.

THEOREM 3.14. *Let $F \colon \mathbf{I} \to \mathbf{AmenAct}$ be any functor and let G be a group acting on a set X.*

If the action of G on X is extensively amenable, then the action of $F_{\mathbf{Grp}}(X) \rtimes G$ on $F_{\mathbf{Set}}(X)$ is amenable. Moreover, it is extensively amenable if $F_{\mathbf{Grp}}$ takes values in **Amen**.

Conversely, assume that the action of $F_{\mathbf{Grp}}(X) \rtimes G$ on $F_{\mathbf{Set}}(X)$ is amenable and that F is tight on X. Then the action of G on X is extensively amenable.

Proof. For each integer $n \geq 0$ we write $[n] = \{j \in \mathbf{N} : 1 \leq j \leq n\}$ and choose a mean m_n on $F_{\mathbf{Set}}([n])$ invariant under the (amenable) action of $F_{\mathbf{Grp}}([n]) \rtimes \mathrm{Sym}([n])$. Given any finite set A, we obtain a mean m_A on $F_{\mathbf{Set}}(A)$ by transporting m_n for $n = |A|$ through some bijection $[n] \to A$. This mean m_A does not depend on the chosen bijection since m_n is $\mathrm{Sym}([n])$-invariant, and any bijection between two finite sets A, A' maps m_A to $m_{A'}$. We denote by m_A^X the mean on $F_{\mathbf{Set}}(X)$ obtained by push-forward through $F_{\mathbf{Set}}(A) \to F_{\mathbf{Set}}(X)$; this mean is $F_{\mathbf{Grp}}(A)$-invariant, and $gm_A^X = m_{gA}^X$ for every $g \in G$.

If m is a G-invariant mean on $\mathscr{P}_{\mathrm{f}}(X)$ giving full weight to the collection of subsets that contain any given finite subset, we get a mean $\widetilde{m}$, as in the proof of Proposition 3.7, by setting, for every $f \in \ell^\infty(F_{\mathbf{Set}}(X))$,

$$\widetilde{m}(f) = \int_{\mathscr{P}_{\mathrm{f}}(X)} m_A^X(f)\, dm(A).$$

This mean is G-invariant and $F_{\mathbf{Grp}}(X)$-invariant for the same reason as in the proof of Proposition 3.7. This proves that the action of $F_{\mathbf{Grp}}(X) \rtimes G$ on $F_{\mathbf{Set}}(X)$ is amenable. Before proving that it is extensively amenable if $F_{\mathbf{Grp}}$ takes values in **Amen**, let us prove the converse part of the statement.

Assume that $F_{\mathbf{Grp}}(X) \rtimes G \curvearrowright F_{\mathbf{Set}}(X)$ is amenable and that F is tight on X. Given a G-invariant mean m on $F_{\mathbf{Set}}(X)$, the G-map supp_X provides us, by push-forward, with a G-invariant mean $\overline{m}$ on $\mathscr{P}_{\mathrm{f}}(X)$. Let $x_0 \in X$. By definition, the value of $\overline{m}$ on the collection of finite sets containing x_0 is $m(B)$, where $B = \left\{b \in F_{\mathbf{Set}}(X) : x_0 \in \mathrm{supp}_X(b)\right\}$. This can be rewritten as

$$B = \{b \in F_{\mathbf{Set}}(X) : \forall\, A \in \mathscr{P}_{\mathrm{f}}(X) \text{ with } x_0 \notin A : b \notin \mathrm{Im}(F_{\mathbf{Set}}(A) \to F_{\mathbf{Set}}(X))\}.$$

In plain words, B is the complement in $F_{\mathbf{Set}}(X)$ of the image of the direct limit of $F_{\mathbf{Set}}(A)$ over all finite A not containing x_0. It is thus the complement of the image of $F_{\mathbf{Set}}(X \setminus \{x_0\})$. Since F is tight, we deduce that $m(B) > 0$ if m is $F_{\mathbf{Grp}}(X)$-invariant. By Lemma 2.2, we conclude that $G \curvearrowright X$ is extensively amenable.

It remains to be proved that the action $F_{\mathbf{Grp}}(X) \rtimes G \curvearrowright F_{\mathbf{Set}}(X)$ is extensively amenable whenever $F_{\mathbf{Grp}}$ takes values in **Amen**. By the above, it is enough to find a tight functor $F_1 \colon \mathbf{I} \to \mathbf{Amen}$ such that the action of $F_1(F_{\mathbf{Set}}(X)) \rtimes (F_{\mathbf{Grp}}(X) \rtimes G)$ on $F_1(F_{\mathbf{Set}}(X))$ is amenable. We consider the functor $F_1(X) = (\mathbf{Z}/2\mathbf{Z})^{(X)}$ and we observe

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that we can see F_1 as a functor **Set** $\to$ **Amen** (for a map $f\colon X \to Y$ between sets, the associated group homomorphism $(\mathbf{Z}/2\mathbf{Z})^{(X)} \to (\mathbf{Z}/2\mathbf{Z})^{(Y)}$ maps δ_x to $\delta_{f(x)}$). By the assumption that $F_{\mathbf{Grp}}(Y)$ is amenable for all Y, we can consider the functor $F_1 \diamond F$, as defined in Example 3.11. Since F_1 commutes with direct limits, we see that $F_1 \diamond F(X)$ is the action of $F_1(F_{\mathbf{Set}}(X)) \rtimes F_{\mathbf{Grp}}(X)$ on $F_1(F_{\mathbf{Set}}(X))$, not only for finite sets X, but also infinite sets. The first part of the statement applied to the functor $F_1 \diamond F$ implies that the action of $F_1(F_{\mathbf{Set}}(X)) \rtimes (F_{\mathbf{Grp}}(X) \rtimes G)$ on $F_1(F_{\mathbf{Set}}(X))$ is amenable and concludes the proof. □

Remark π. It is possible to slightly relax the assumption in Theorem 3.14 that the functor F is obtained as a direct limit from a functor defined on finite sets. This allows us to recover, in this language, some known amenability results, for example the result from [**JNdlS13**] that states that groups generated by *finite state automata* of *activity degree up to two* are amenable. For this application, it is enough to show that Theorem 3.14 holds for the functor $F(X) = \mathrm{Fun}_K(X, H) \curvearrowright (H/K)^{(X)}$, where H is any group, $K < H$ is a co-amenable subgroup, $\mathrm{Fun}_K(X, H)$ denotes the group of all functions from X to H that take values in K outside a finite set and the action $\mathrm{Fun}_K(X, H) \curvearrowright (H/K)^{(X)}$ is the diagonal action. It is possible to give general conditions on a functor F in order to include this case in Theorem 3.14, without essentially affecting the proof. It is also possible to give more complicated statements in this functorial language, that recover the main result from [**JNdlS13**] and unify it with Theorem 1.9. However, we omit further details, as we are not aware of any applications to show amenability of new groups.

4. *Probabilistic reformulation and recurrent actions*

4.1. *The inverted orbit.* We now give a more probabilistic reformulation of extensive amenability. To simplify the statements, we make the assumption that $G = \langle S \rangle$ is finitely generated and acts transitively on X (this is very inessential, by Lemma 2.2). We fix a symmetric probability measure μ on G with generating support and a basepoint $x_0 \in X$. We can consider the (left) random walk $(g_n)_{n\geq 0}$ on G defined by $g_0 = e$ and $g_n = h_n g_{n-1}$ for $n \geq 1$, where $(h_i)_{i\geq 1}$ are independent with law μ.

The inverted orbit $\mathbf{O}_n$ is then the (random) subset of X

$$\mathbf{O}_n = \{x_0, g_1^{-1}x_0, \ldots, g_n^{-1}x_0\}. \tag{1}$$

If G is not abelian, the inverted orbit need not have the same distribution as the directed orbit of the random walk on X. The inverted orbit is a well-known object, central to the study of growth and random walks on permutational wreath products (see [**BE12**, **BE11**, **AV12**]).

It is sometimes convenient to consider the following variation of the inverted orbit

$$\mathbf{O}'_n = \{x_0,\ h_n x_0,\ h_n h_{n-1} x_0, \ldots,\ h_n \cdots h_1 x_0\}. \tag{2}$$

For a fixed n, $\mathbf{O}_n$ and $\mathbf{O}'_n$ have the same distribution, although the joint distributions of the processes $(\mathbf{O}_n)_n$ and $(\mathbf{O}'_n)_n$ differ.

The next proposition shows that proving extensive amenability of an action $G \curvearrowright X$ boils down to a fine understanding of the asymptotic behaviour of the distribution of $|\mathbf{O}_n|$.

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The third reformulation was suggested to us by Gidi Amir, to whom we are grateful for letting us include it here.

PROPOSITION 4.1. *Fix $G \curvearrowright X$, $x_0 \in X$ and μ, as before. The following properties are all equivalent to extensive amenability.*

(i) $\lim_{n\to\infty} -(1/n)\log \mathbb{E}(2^{-|\mathbf{O}_n|}) = 0$.

(ii) *For every $\varepsilon > 0$, $\mathbb{P}(|\mathbf{O}_n| < \varepsilon n) > e^{-\varepsilon n}$ for an infinite number of n.*

(iii) *There exists a sequence of events $A_n \in \sigma(g_1, \ldots, g_n)$ verifying $-(1/n)\log \mathbb{P}(A_n) \to 0$, conditioned to which $(1/n)\mathbb{E}(|\mathbf{O}_n| : A_n) \to 0$.*

In particular, these conditions do not depend on μ and x_0.

In the last part, $\mathbb{E}(|\mathbf{O}_n| : A_n) = \mathbb{E}(|\mathbf{O}_n| 1_{A_n})/\mathbb{P}(A_n)$ denotes the expectation of $|\mathbf{O}_n|$, conditioned to the event A_n.

Proof. In the proof, we will freely use the result from [**JM13**] that $G \curvearrowright X$ is extensively amenable if and only if $(\mathbf{Z}/2\mathbf{Z})^{(X)} \rtimes G \curvearrowright (\mathbf{Z}/2\mathbf{Z})^{(X)}$ is amenable (as follows from Theorem 3.14). Since we assume that G is finitely generated and acts transitively on X, the wreath product $(\mathbf{Z}/2\mathbf{Z})^{(X)} \rtimes G$ is finitely generated and acts transitively on $(\mathbf{Z}/2\mathbf{Z})^{(X)}$. Thereby, if we fix any non-degenerate symmetric, finitely supported probability measure ν on $(\mathbf{Z}/2\mathbf{Z})^{(X)} \rtimes G$, we can consider the Schreier graph Γ associated to the action $(\mathbf{Z}/2\mathbf{Z})^{(X)} \rtimes G \curvearrowright (\mathbf{Z}/2\mathbf{Z})^{(X)}$ with generating set $\mathrm{supp}(\nu)$ and basepoint the trivial configuration $f_0 = 0_{(\mathbf{Z}/2\mathbf{Z})^{(X)}}$. The left random walk on $(\mathbf{Z}/2\mathbf{Z})^{(X)} \rtimes G$ with step measure ν then induces a nearest neighbour random walk (f_n) on Γ. By Kesten's amenability criterion for a graph (see [**Woe00**, Theorem 10.6]), amenability of the action $(\mathbf{Z}/2\mathbf{Z})^{(X)} \rtimes G \curvearrowright (\mathbf{Z}/2\mathbf{Z})^{(X)}$ is thereby equivalent to

$$\lim_{n\to\infty} -\frac{1}{n}\log \mathbb{P}(f_n = f_0) = 0. \tag{3}$$

Moreover, it is sufficient to show this for any choice of ν with the above properties. We can therefore choose ν to be the *switch-walk-switch* measure $\widetilde{\mu}$ on $(\mathbf{Z}/2\mathbf{Z})^{(X)} \rtimes G$ associated to the measure μ on G and to the basepoint $x_0 \in X$. Recall that if λ is the uniform probability measure on $\{0_{(\mathbf{Z}/2\mathbf{Z})^{(X)}}, \delta_{x_0}\} \subset (\mathbf{Z}/2\mathbf{Z})^{(X)}$, then, by definition, $\widetilde{\mu} = \lambda * \mu * \lambda$, where λ and μ are naturally seen as probability measures on $(\mathbf{Z}/2\mathbf{Z})^{(X)} \rtimes G$. With this choice of ν, it is then easy to see that f_{n+1} is obtained from f_n through the following steps: first change the value of $f_n(x_0)$ to a uniform random value in $\mathbf{Z}/2\mathbf{Z}$ to obtain a new f_n', then translate f_n' to $f_n'' = h_{n+1} \cdot f_n'$, where $h_{n+1} \in G$ has distribution μ and, finally, randomize again $f_n''(x_0)$ to obtain f_{n+1}. It follows, from this description, that $\mathrm{supp} f_n \subset \mathbf{O}_n'$ for every n, where $\mathbf{O}_n'$ is as in (2). Moreover, f_n lights each point in $\mathbf{O}_n'$ independently with probability 1/2. From this, we immediately get that $\mathbb{P}(f_n = f_0) = \mathbb{E}(2^{-|\mathbf{O}_n'|}) = \mathbb{E}(2^{-|\mathbf{O}_n|})$. The equivalence between (i) and extensive amenability follows.

The equivalence between (i) and (ii) is essentially Markov's inequality.

The second condition implies the third by a diagonal extraction argument.

To see that the third condition implies the first, observe that $2^{-|\mathbf{O}_n|} \geq 2^{-|\mathbf{O}_n|} 1_{A_n}$ and, thereby, by Jensen's inequality,

$$-\frac{1}{n}\log \mathbb{E}(2^{-|\mathbf{O}_n|}) \leq -\frac{1}{n}\log \mathbb{E}(2^{-|\mathbf{O}_n|} 1_{A_n}) \leq -\frac{1}{n}\log \mathbb{P}(A_n) - \frac{1}{n}\mathbb{E}(|\mathbf{O}_n| : A_n)\log 2 \to 0.$$

□

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4.2. *Application to recurrent actions.* As an application, let us give a new proof of a criterion for extensive amenability in [**JNdlS13**]. Here a group action $G \curvearrowright X$ is called *recurrent* if, for every symmetric, finitely supported probability measure μ on G and every $x_0 \in X$, the random walk $(g_n x_0)$ on the orbit of x_0 induced by the left random walk on G is recurrent. If G is finitely generated, it is sufficient to check this for a symmetric, finitely supported probability measure with generating support. Equivalently, it is sufficient to check that, for every $x \in X$ and for a symmetric, finite generating set S of G, the Schreier graph $\Gamma(G, x_0, S)$ is recurrent for a simple random walk.

THEOREM 4.2. ([**JNdlS13**, Theorem 1.2]) *Recurrent actions are extensively amenable.*

Let us give a direct proof based on the inverted orbit. The main ingredient is the following lemma.

LEMMA 4.3. [**BE11**, **AV12**] *Assume that G is finitely generated and that the action $G \curvearrowright X$ is transitive. Then $G \curvearrowright X$ is recurrent if and only if $(1/n)\mathbb{E}|\mathbf{O}_n| \to 0$ for some (equivalently, for any) non-degenerate symmetric, finitely supported probability measure μ on G.*

Although this is exactly [**BE11**, Lemma 3.1], we report a proof for the convenience of the reader.

Proof. Let $T = \min\{n \geq 1 \,:\, g_n x_0 = x_0\} \in \mathbf{N} \cup \{\infty\}$.

$$\begin{aligned}
&\mathbb{E}|\mathbf{O}_{n+1}| - \mathbb{E}|\mathbf{O}_n| \\
&\quad = \mathbb{P}(g_{n+1}^{-1} x_0 \notin \mathbf{O}_n) \\
&\quad = \mathbb{P}(h_{n+1}^{-1} x_0 \neq x_0,\ h_n^{-1} h_{n+1}^{-1} x_0 \neq x_0, \ldots,\ h_1^{-1} \cdots h_{n+1}^{-1} x_0 \neq x_0) = \mathbb{P}(T > n+1),
\end{aligned}$$

since, by symmetry, $(h_{n+1}^{-1},\ h_n^{-1} h_{n+1}^{-1}, \ldots,\ h_1^{-1} \cdots h_{n+1}^{-1})$ has the same law as $(g_1, \ldots, g_{n+1})$. This computation shows that $(1/n)\mathbb{E}|\mathbf{O}_n| \to \mathbb{P}(T = \infty)$, which vanishes if and only if $G \curvearrowright X$ in recurrent. □

Proof of Theorem 4.2. By Lemma 2.2, we may assume that G is finitely generated and acts transitively on X. Apply part (1) of Proposition 4.1. By convexity,

$$-\frac{1}{n} \log \mathbb{E}(2^{-|\mathbf{O}_n|}) \leq \frac{1}{n} \mathbb{E}(|\mathbf{O}_n|) \log 2 \to 0,$$

where we used Lemma 4.3. Part (i) of Proposition 4.1 gives the conclusion. □

5. *Interval exchange transformations*

5.1. *Amenability of subgroups of low rational rank.* Let $\Lambda < \mathbf{R}/\mathbf{Z}$ be a finitely generated subgroup of the circle. Thus, Λ is isomorphic to $\mathbf{Z}^d \times F$ for some finite abelian group F and some integer d, the *rational rank* of Λ, denoted by $\mathrm{rk}_{\mathbf{Q}}(\Lambda)$. With this notation, the rational rank (as defined in Definition 1.7) of a finitely generated subgroup $G \leq \mathrm{IET}$ is an abbreviation for $\mathrm{rk}_{\mathbf{Q}}(\Lambda(G))$. We denote by $\mathrm{IET}(\Lambda)$ the subgroup of all $g \in \mathrm{IET}$ so that the angle $gx - x$ is in Λ for every $x \in \mathbf{R}/\mathbf{Z}$.

The following result is a reformulation of Theorem 1.9.

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THEOREM 5.1. *Let* $\Lambda < \mathbf{R}/\mathbf{Z}$ *be finitely generated. If* $\mathrm{rk}_{\mathbf{Q}}(\Lambda) \leq 2$, *then* $\mathrm{IET}(\Lambda)$ *is amenable.*

Our goal is to apply Corollary 1.4 for the functor $F\colon \mathrm{Sym}\colon \mathbf{I} \to \mathbf{Amen}$, given by $F(A) = \mathrm{Sym}(A)$, the symmetric group. This extends to an infinite set as $F(X) = \mathrm{Sym}(X)$, the group of permutations of X with finite support (see Example 3.2). If $G \curvearrowright X$, we recall that we have an action of G on $\mathrm{Sym}(X)$, by conjugation. For $\tau \in \mathrm{Sym}(X)$ and $g \in G$, we denote ${}^g\tau = g\tau g^{-1}$. The first ingredient is the following lemma (see §4.2 for the definition of a recurrent action).

LEMMA 5.2. *If* $\mathrm{rk}_{\mathbf{Q}}(\Lambda) \leq 2$, *the action of* $\mathrm{IET}(\Lambda)$ *on* $\mathbf{R}/\mathbf{Z}$ *is recurrent. In particular, it is extensively amenable.*

Proof. Equip Λ with a finite symmetric generating set and the corresponding Cayley graph structure. Then Λ is a recurrent graph. Let $H < \mathrm{IET}(\Lambda)$ be a finitely generated subgroup, equipped with a finite symmetric generating set S. For $x \in \mathbf{R}/\mathbf{Z}$, let $\Gamma(x, H, S)$ be the orbital Schreier graph for the action of H on H_x. We have an injective map

$$\begin{aligned} \Gamma(x, H, S) &\to \Lambda \\ y &\mapsto y - x, \end{aligned}$$

where the difference is taken in $\mathbf{R}/\mathbf{Z}$. This maps takes values in Λ, since $y = hx$ for some $h \in H < \mathrm{IET}(\Lambda)$. It is not hard to check that this map is Lipschitz (for some constant depending on the generating set S of H only). The Schreier graph $\Gamma(x, H, S)$ is thus Lipschitzly embedded in a recurrent graph and is therefore recurrent (see [**LP16**, Theorem 2.17]). The action is then extensively amenable, by Theorem 4.2. □

The other ingredient in the proof of Theorem 5.1 is the following result, which concludes the proof.

PROPOSITION 5.3. *A subgroup* $G \leq \mathrm{IET}$ *is amenable if and only if the action* $G \curvearrowright \mathbf{R}/\mathbf{Z}$ *is extensively amenable.*

Proof. The 'only if part' is obvious (Lemma 2.1).

The 'if part' is based on the method explained in Remark 1.5. If we replace the convention that interval exchanges are right-continuous by the condition that they are left continuous, we get a group of permutations of $\mathbf{R}/\mathbf{Z}$, which we denote by $\widetilde{\mathrm{IET}}$. The map $g \in \mathrm{IET} \mapsto \widetilde{g} \in \widetilde{\mathrm{IET}}$, where $\widetilde{g}$ is the unique left-continuous map that coincides with g except on the points of discontinuity of g, is a group isomorphism. Then

$$\tau_g = \widetilde{g} g^{-1} \tag{4}$$

is a permutation of $\mathbf{R}/\mathbf{Z}$ with finite support equal to the points of discontinuity of g^{-1}. Moreover,

$$\tau_{gh} = \widetilde{g}\widetilde{h}h^{-1}g^{-1} = \tau_g g \tau_h g^{-1} = \tau_g({}^g\tau_h), \tag{5}$$

so that the map

$$\begin{aligned} \iota\colon \mathrm{IET} &\to \mathrm{Sym}(\mathbf{R}/\mathbf{Z}) \rtimes \mathrm{IET} \\ g &\mapsto (\tau_g, g) \end{aligned}$$

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is an injective group homomorphism. Observe that $\tau_g = 1$ if and only if g is continuous, that is, a rotation. If $G \leq \mathrm{IET}$ is a subgroup, the restriction of ι to G takes values in $\mathrm{Sym}(\mathbf{R}/\mathbf{Z}) \rtimes G$. Moreover, $\iota(G) \cap \{1\} \times G$ consists of rotations and thereby is amenable. In the terminology of Remark 1.5, $g \mapsto \tau_g$ is a $\mathrm{Sym}(\mathbf{R}/\mathbf{Z})$-cocycle with amenable kernel. The conclusion follows from Corollary 1.4. □

5.2. *Non-trivial boundary for subgroups of high rank.* We use notation from the previous section. Again, let Λ be a finitely generated subgroup of $\mathbf{R}/\mathbf{Z}$. Note that Λ identifies with an abelian subgroup of $\mathrm{IET}(\Lambda)$, acting on $\mathbf{R}/\mathbf{Z}$ by rotations.

THEOREM 5.4. *Let $\Lambda < \mathbf{R}/\mathbf{Z}$ be a finitely generated group.*

(1) *If $\mathrm{rk}_{\mathbf{Q}}(\Lambda) = 1$, then $\mathrm{IET}(\Lambda)$ has the Liouville property for every symmetric, finitely supported measure.*

(2) *Assume that $\mathrm{rk}_{\mathbf{Q}}(\Lambda) \geq 3$. Let $G \leq \mathrm{IET}(\Lambda)$ be a non-abelian group that contains Λ. Then any finitely supported, non-degenerate probability measure μ on G has non-trivial Poisson–Furstenberg boundary.*

We do not know whether (1) can be extended to the case $\mathrm{rk}_{\mathbf{Q}}(\Lambda) = 2$.

The proof of (1) is postponed to §5.3. To prove (2), we will use the following fact that allows us to restrict to symmetric measures. Given a probability measure μ on a countable group G, we denote the measure $\check{\mu}(g) = \mu(g^{-1})$ by $\check{\mu}$.

LEMMA 5.5. (Baldi *et al* [**BLP77**]) *Let G be a countable group acting transitively on a set X and let μ be a probability measure on G. Consider the symmetric measure $\nu = \frac{1}{2}(\mu + \check{\mu})$. Assume that the Markov chain on X induced by ν is transient. Then so is the Markov chain induced by μ.*

Proof. For $G = X$ acting on itself, this is exactly [**BLP77**, Proposition 1], and the proof extends with no changes to group actions. □

Let (G, μ) be as in (2) of Theorem 5.4. For the proof, it will be convenient to consider the *right* random walk $g_n = h_1 \cdots h_n$ with step measure μ. When speaking about recurrence or transience of left group actions, we refer, however, to the left random walk. Thereby, with the right random walk notation, saying that the action of G on the orbit of $x_0 \in \mathbf{R}/\mathbf{Z}$ is transient means that the Markov chain $(g_n^{-1} x_0)$ is transient. Observe that this is the case under the assumptions of (2) of Theorem 5.4.

For $g \in \mathrm{IET}$, let $\tau_g \in \mathrm{Sym}(\mathbf{R}/\mathbf{Z})$ be the permutation introduced in the proof of Proposition 5.1.

LEMMA 5.6. *For every $x \in \mathbf{R}/\mathbf{Z}$, the value $\tau_{g_n}(x) \in \mathbf{R}/\mathbf{Z}$ is almost surely constant for large enough n. We denote its eventual value by $\tau_\infty(x)$. The map $\tau_\infty \colon \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ is an injective map that preserves every Λ-coset.*

Proof. Let $\Sigma \subset \mathbf{R}/\mathbf{Z}$ be the union of $\mathrm{supp}\tau_s$ for $s \in \mathrm{supp}\,\mu$. By transience, for every $x \in \mathbf{R}/\mathbf{Z}$ there exists a random N so that $x \notin g_n \Sigma$ for $n \geq N$. Now observe that, by the cocycle relation (5),

$$\tau_{g_n} = \tau_{h_1} \circ {}^{g_1}\tau_{h_2} \cdots \circ {}^{g_{n-1}}\tau_{h_n}$$

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and that the support of ${}^{g_{i-1}}\tau_{h_i}$ is contained in $g_{i-1}\Sigma$. Hence $\tau_{g_n}(x) = \tau_{g_N}(x)$ for every $n \geq N$. The fact that τ_∞ is injective follows immediately from the fact that τ_{g_n} is injective for every n, and it preserves every Λ-coset in $\mathbf{R}/\mathbf{Z}$, since so does G. □

Denote by $\mathrm{Inj}_\Lambda(\mathbf{R}/\mathbf{Z}, \mathbf{R}/\mathbf{Z})$ the set of all injective maps of the circle to itself that preserve every Λ coset. The group $\mathrm{IET}(\Lambda)$ acts on $\mathrm{Inj}_\Lambda(\mathbf{R}/\mathbf{Z}, \mathbf{R}/\mathbf{Z})$ by

$$g \cdot f = \tau_g \circ {}^g f = \tau_g \circ g \circ f \circ g^{-1},$$

where $g \in \mathrm{IET}(\Lambda)$ and $f \in \mathrm{Inj}_\Lambda(\mathbf{R}/\mathbf{Z}, \mathbf{R}/\mathbf{Z})$. Endow with the topology induced by the product of the *discrete* topology on $\mathbf{R}/\mathbf{Z}$. Then the above construction shows that τ_{g_n} converges almost surely to a $\tau_\infty \in \mathrm{Inj}_\Lambda(\mathbf{R}/\mathbf{Z}, \mathbf{R}/\mathbf{Z})$. If ν is the distribution of τ_∞, then $(\mathrm{Inj}(\mathbf{R}/\mathbf{Z}, \mathbf{R}/\mathbf{Z}), \nu)$ is a quotient of the Poisson–Furstenberg boundary. To conclude the proof, we only need to check that this quotient is not the trivial one, namely, that ν is not concentrated on a single point.

LEMMA 5.7. *Let $f \in \mathrm{Inj}_\Lambda(\mathbf{R}/\mathbf{Z}, \mathbf{R}/\mathbf{Z})$ and G be as in (2) of Theorem 5.4. Then there exists $g \in G$ so that $g \cdot f \neq f$.*

Proof. Assume that G stabilizes f. A rotation $r \in \Lambda < G$ acts on f, by conjugation. Hence the increment $x \mapsto f(x) - x$ is Λ-invariant. It follows that f coincides on every Λ-coset with a translation by a fixed element in Λ. Observe that this implies the following: if $g \in G$ is arbitrary, then $g \circ f \circ g^{-1}$ is either equal to f or it differs from f on a infinite number of points. Indeed, observe that the restriction of $g \circ f \circ g^{-1}$ on every Λ-coset coincides with the restriction of an interval exchange transformation, and two interval exchange transformations are either equal or differ on an infinite number of points in every Λ-coset (since these are dense). Now consider $g \in G$ which is not a rotation. For such a g, $\tau_g \neq \mathrm{Id}$. Hence $g \cdot f = \tau_g \circ g \circ f \circ g^{-1}$ cannot be equal to f, in the first case, since τ_g is nontrivial, and, in the second case, since it is finitely supported. □

Proof of Theorem 5.4 (part (2)). Assume that τ_∞ is deterministic. Then the Dirac mass on τ_∞ is a μ-stationary measure. This implies that the support of μ stabilizes τ_∞, and hence so does G since μ is non-degenerate. But this is impossible, by Lemma 5.7 □

Question 5.8. With the above notation, is $(\mathrm{Inj}_\Lambda(\mathbf{R}/\mathbf{Z}, \mathbf{R}/\mathbf{Z}), \nu)$ isomorphic to the Poisson–Furstenberg boundary of (G, μ)?

Similar ideas yield the following alternative for a subgroup containing an irrational rotation, which could also be proved more directly.

PROPOSITION 5.9. *Let G be a finitely generated subgroup of* IET *that contains an irrational rotation. Then either G is an abelian group of rotations or G has exponential growth.*

Proof. Suppose that G is nonabelian. Let $r \in G$ be an irrational rotation and $h \in G$ be an element which is not a rotation, so that $\tau_h \neq 0$. Consider the non-symmetric measure $\mu = \frac{1}{3}\delta_r + \frac{1}{3}\delta_{h^{-1}} + \frac{1}{3}\delta_h$. The random walk on the Schreier graphs of the set of angles is drifted in the direction of r (perhaps after replacing r with a big enough power). The same idea as in the previous proof shows that μ has non-trivial Poisson–Furstenberg boundary.

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Existence of a finitely supported measure with non-trivial boundary implies that G has exponential growth (see [**KV83**, Proposition 1.4]). □

5.3. *Connection with topological full groups and Liouville property in rank one.* In this section, we show (1) of Theorem 5.4. First, we explain, in some detail, a construction that has already appeared implicitly in [**dC13**], showing that the study of finitely generated subgroups of IET is closely related to the study of the *topological full groups* of a special family of minimal actions of finitely generated abelian groups on the Cantor set.

Let Γ be a group acting by homeomorphisms on a topological space X. The *topological full group* of the action, $[[\Gamma]]$, is the group of all homeomorphisms h of X such that every point of X admits a neighbourhood where h agrees with an element of Γ.

The dynamical system (Γ, X) is *minimal* if there are no non-trivial closed Γ-invariant subsets in X.

A *Cantor* Γ*-system* is a dynamical system (Γ, X), where X is the Cantor set. Let A be a finite alphabet. The Γ*-shift* over A is the Cantor system (Γ, A^Γ), where Γ acts on A^Γ by translations. A Γ*-subshift* is a Cantor Γ-system (Γ, X), where $X \subset A^\Gamma$ is a closed Γ-invariant subset.

In what follows, we use the notation $\vee_{\alpha\in I}\mathcal{P}_\alpha$ for the join of a family of partitions $(\mathcal{P}_\alpha)_{\alpha\in I}$ of a set. Recall the following elementary criterion to establish whether a Cantor Γ-system is conjugate to a subshift. For a proof, see, for example, [**dC13**, Fait 2.2].

LEMMA 5.10. *Let (Γ, X) be a Cantor Γ-system. Then (Γ, X) is conjugate to a Γ-subshift over a finite alphabet if and only if there exists a finite partition $\mathcal{P}$ of X into clopen sets so that the partition $\vee_{\gamma\in\Gamma}\gamma\mathcal{P}$ is the point partition of X (that is, for any $x \neq y \in X$, there exists $\gamma \in \Gamma$ so that the partition $\gamma\mathcal{P}$ separates x and y).*

As before, let $\Lambda < \mathbf{R}/\mathbf{Z}$ be a finitely generated and infinite subgroup, and let $\Sigma = \{x_1, \ldots, x_r\}$ be a finite subset of $\mathbf{R}/\mathbf{Z}$. We denote by $\mathrm{IET}(\Lambda; \Sigma)$ the subgroup of $\mathrm{IET}(\Lambda)$ consisting of interval exchange transformations so that all extrema of the defining intervals lie in cosets $x_i + \Lambda$ of points in Σ. Every finitely generated $G < \mathrm{IET}$ is contained in $\mathrm{IET}(\Lambda, \Sigma)$, where $\Lambda = \Lambda(G)$ and Σ is the set of extrema of the defining intervals of elements in some generating set of G.

A construction going back essentially to [**Kea75**] realizes $\mathrm{IET}(\Lambda, \Sigma)$ as a group of homeomorphisms of a Cantor set. Using Lemma 5.10, we will see, through this construction, that $\mathrm{IET}(\Lambda; \Sigma)$ is isomorphic to the topological full group of a minimal Λ-subshift, and even that the actions are semiconjugate.

PROPOSITION 5.11. *Let $\Lambda < \mathbf{R}/\mathbf{Z}$ be infinite and finitely generated and Σ be as above. There is a minimal Λ-subshift (Λ, X), an isomorphism $\pi : [[\Lambda]] \to \mathrm{IET}(\Lambda; \Sigma)$ and a continuous surjective map $h : X \to \mathbf{R}/\mathbf{Z}$ such that $h(g \cdot x) = \pi(g) \cdot x$ for all $x \in X$ and $g \in [[\Lambda]]$.*

Remark 5.12. The Cantor minimal Λ-systems arising in this way have a very special form. Recall that the topological full group of a Cantor minimal $\mathbf{Z}^d$-system may fail to be amenable even for $d = 2$ [**EM13**], while this does not happen for IET, as shown by Theorem 5.1.

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Proof. Let $\mathcal{C}$ be the space constructed out of the circle in the following way. Take a circle and double all points in $\Sigma + \Lambda$. Namely, we replace each point $x \in \Sigma + \Lambda$ by two points: left and right, which we denote by x_- and x_+, respectively. There is a natural topology on $\mathcal{C}$ that makes it homeomorphic to the Cantor set. The action of Λ on $\mathbf{R}/\mathbf{Z}$ by rotations induces an action on $\mathcal{C}$ by homeomorphisms. This action is minimal as soon as Λ is infinite. It is then easy to see that the map $x_\pm \in X \mapsto x \in \mathbf{R}/\mathbf{Z}$ is a continuous surjection that implements a semiconjugation between the action of $[[\Lambda]]$ and $\mathrm{IET}(\Lambda;\ \Sigma)$.

To see that the Λ-Cantor system is in fact a subshift, we apply Lemma 5.10 to the following partition. Let S be a generating set of Λ. For each $\lambda \in S$ and $x \in \Sigma$, consider the partition $\mathcal{P}_{\lambda,x}$ of $\mathcal{C}$ in the two clopen sets $[x_+, (x+\lambda)_-]$ and $[(x+\lambda)_+, x_-]$. Set $\mathcal{P} = \vee_{\lambda,x}\mathcal{P}_{\lambda,x}$. Then the Λ-translates of $\mathcal{P}$ separate points. □

We now explain how this implies part 1 of Theorem 5.4. We first recall the notion of *complexity* of a subshift. Let Γ be a finitely generated group with generating set S and let X be a Γ-subshift. Let $\mathcal{P}$ be a partition of X satisfying Lemma 5.10. The *one-dimensional complexity* of the subshift (Γ, X) with respect to S and $\mathcal{P}$ is the function $\rho_{S,\mathcal{P}}\colon \mathbf{N} \to \mathbf{N}$ which counts the number of elements of the partition $\vee_{|\gamma|_s \le n}\gamma\mathcal{P}$, where $|\cdot|_S$ is the word metric associated to S. For a $\mathbf{Z}$-subshift $X \subset A^{\mathbf{Z}}$, this coincides with the more standard definition of complexity of sequences if one takes the standard generating set of $\mathbf{Z}$ and the partition indexed by A into cylinder sets corresponding to the letter at position zero. The proof of the following lemma is elementary and not difficult.

LEMMA 5.13. *Let $(\Lambda, \mathcal{C})$ the Cantor minimal system constructed in the previous proposition, and assume that $\mathrm{rk}_{\mathbf{Q}}(\Lambda) = d$. Then, for every generating set S of Λ and every partition $\mathcal{P}$ satisfying Lemma 5.10, there exists a constant $C > 0$ so that $\rho_{S,\mathcal{P}}(n) \le Cn^d$.*

Proof of part (1) of Theorem 5.4. It is proved, in [**MB14**, Theorem 1.2], that if $(\Lambda, \mathcal{C})$ is a subshift over a virtually cyclic group whose complexity satisfies $\rho_{S,\mathcal{P}}(n) = o(n^2/\log^2 n)$ (with respect to some generating set and some partition $\mathcal{P}$, as above), then $[[\Lambda]]$ has the Liouville property. (The statement of [**MB14**, Theorem 1.2] assumes that $\Lambda = \mathbf{Z}$, but the proof extends with no changes if Λ is virtually cyclic). The conclusion follows by Proposition 5.11 and Lemma 5.13. □

6. *A hereditarily amenable action which is not extensively amenable*

Corollary 2.3 shows that an obstruction for an amenable action to be extensively amenable is to fail to be *hereditarily amenable*. In this section, we show that this is not the only obstruction.

We consider a group H of piecewise projective orientation-preserving homeomorphisms of $\mathbf{R}$, as introduced in [**Mon13**]. A self-homeomorphism f of $\mathbf{R}$ belongs to this group if there exist intervals $I_1, \dots, I_n$ covering $\mathbf{R}$ and $g_1, \dots, g_n \in \mathrm{PSL}(2, \mathbf{R})$ such that f coincides with g_i on I_i for all $i \in \{1, \dots, n\}$. Here we consider the standard projective action of $\mathrm{PSL}(2, \mathbf{R})$ on $P^1(\mathbf{R}) = \mathbf{R} \cup \infty$: $g \cdot x = (ax+b)/(cx+d)$ if $g = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The goal of this section is to show the following theorem.

THEOREM 6.1. *The action $H \curvearrowright \mathbf{R}$ is hereditarily amenable, but it is not extensively amenable.*

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Remark 6.2. By Lemma 2.2, this implies that there exists a finitely generated subgroup $L \leq H$ and an L-orbit $X \subset \mathbf{R}$ so that the action of L on X is hereditarily amenable but not extensively amenable.

The first ingredient of the proof is the following lemma.

LEMMA 6.3. *$H \curvearrowright \mathbf{R}$ is hereditarily amenable.*

Proof. Let $H^{(0)} \leq H$ be any subgroup and let $x \in \mathbf{R}$. Denote by $H^{(1)}$ the commutator subgroup of $H^{(0)}$ and by $H^{(2)}$ the commutator subgroup of $H^{(1)}$. Denote $M = \sup(H^{(0)}x) \in \mathbf{R} \cup \{\infty\}$. It is a fixed point of $H^{(0)}$. Let $x_n \in H^{(0)}x$ be a sequence converging to M. Since the double commutator of the stabilizer of M in $\mathrm{PSL}(2, \mathbf{R})$ is trivial, every element h of $H^{(2)}$ is trivial on a neighbourhood of M, and hence satisfies $hx_n = x_n$ for all n large enough, depending on h. Therefore any w-* cluster point of the sequence δ_{x_n} is an $H^{(2)}$-invariant mean on $H^{(0)}x$.

This shows that the compact convex subset $K \subset \ell^\infty(H^{(0)}x)^*$ of all $H^{(2)}$-invariant means on X is non-empty. The action of $H^{(0)}$ on K factors to an action of the amenable group $H^{(0)}/H^{(2)}$, and hence has a fixed point. Such a fixed point is an $H^{(0)}$-invariant mean on X. □

The fact that $H \curvearrowright \mathbf{R}$ is not extensively amenable follows from the following Theorem because it was proved in [**Mon13**] that H is not amenable.

THEOREM 6.4. *A subgroup H_1 of H is amenable if and only if $H_1 \curvearrowright \mathbf{R}$ is extensively amenable.*

The proof relies on the main result in [**JNdlS13**]. Recall that given a topological space X and a groupoid of germs of homeomorphims $\mathcal{G}$ acting on X, the *topological full group* $[[\mathcal{G}]]$ of $\mathcal{G}$ is the group of all self-homeomorphisms of X whose germs belong to the groupoid X.

THEOREM 6.5. [**JNdlS13**] *Let G be a group acting on a topological space X with groupoid of germs $\mathcal{G}$. Assume that there is a groupoid of germs of homeomorphisms $\mathcal{H}$ acting on X so that the following hold.*

(i) *For every $g \in G$, the germ of g at x belongs to $\mathcal{H}$ for all but a finite number of $x \in X$.*
(ii) *For every $x \in X$, the isotropy group $\mathcal{G}_x$ is amenable.*
(iii) *The action $G \curvearrowright X$ is extensively amenable.*
(iv) *The group $[[\mathcal{H}]]$ is amenable.*

Then G is amenable.

In fact, condition (iii) in the original statement in [**JNdlS13**] was that the action is recurrent, but this is used in the proof only through Theorem 4.2.

Proof of Theorem 6.4. The only if part holds in view of Lemma 2.1. Assume that $H_1 \curvearrowright \mathbf{R}$ is extensively amenable. Apply Theorem 6.5 to $G = H_1$ with $\mathcal{H}$ the groupoid of germs of the partial action of $\mathrm{PSL}(2, \mathbf{R})$ on the real line. Condition (ii) is satisfied, since $\mathcal{G}_x \simeq \mathrm{Aff}(\mathbf{R}) \times \mathrm{Aff}(\mathbf{R})$. To check condition (iv), let $h \in [[\mathcal{H}]]$. Since projective homeomorphisms are analytic, it is easy to see that the germ of h in at any two points $x, y \in \mathbf{R}$ is represented by the same element of $\mathrm{PSL}_2(\mathbf{R})$. Therefore $h \in \mathrm{PSL}(2, \mathbf{R})$ and h stabilizes $\mathbf{R}$ globally. This shows that $[[\mathcal{H}]] = \mathrm{Aff}(\mathbf{R})$ is amenable. Thereby, H_1 is amenable. □

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7. *Unrestricted wreath product actions*

This section is a complement to §3. In the results therein, we restricted ourselves to functors that can be written as direct limits of functors defined on finite sets. The next proposition shows that Theorem 1.3 fails if this assumption is removed.

PROPOSITION 7.1. *There exists an extensively amenable, transitive action $G \curvearrowright X$ of a finitely generated group so that the action of the unrestricted wreath product $(\mathbf{Z}/2\mathbf{Z})^X \rtimes G$ on $(\mathbf{Z}/2\mathbf{Z})^X$ is not amenable.*

Here $(\mathbf{Z}/2\mathbf{Z})^X$ denotes the abelian group of all configurations $f\colon X \to \mathbf{Z}/2\mathbf{Z}$.

We will use the following simple fact.

LEMMA 7.2. *Let G be a finitely generated group acting on a set Y, and let μ be a non-degenerate, symmetric finitely supported probability measure on G. Then the action of G on Y is non-amenable if and only if the spectral radius of μ on every G-orbit is uniformly bounded sway from one: that is, if there exists $\rho < 1$ so that, for any $y_0 \in Y$, $\lim_n \mathbb{P}(g_{2n} y_0 = y_0)^{1/2n} < \rho$, where G_n is the left random walk on G with step measure μ.*

Proof. The action is non-amenable if and only if the spectral radius (that is, the norm of the convolution operator $f \mapsto \mu * f$ on $\ell^2(Y)$) is smaller than one. This is equivalent to the fact that the spectral radius on every G-orbit is uniformly bounded away from one. It is well known that the spectral radius on the orbit of y_0 equals $\lim_n \mathbb{P}(g_{2n} y_0 = y_0)^{1/2n}$ (see [**Woe00**]). □

To show Proposition 7.1, we construct a finitely generated group $G = \langle S \rangle$ acting faithfully and transitively on a set X so that:

(1) the Schreier graph of the action is isomorphic to $\mathbf{Z}$; and
(2) the action of $(\mathbf{Z}/2\mathbf{Z})^X \rtimes G$ on $(\mathbf{Z}/2\mathbf{Z})^X$ is not amenable.

Note that the first condition guarantees that the action is extensively amenable, by Theorem 4.2.

Set $G = \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$, freely generated by three involutions b, r, y and set $X = \mathbf{Z}$. Let B, R, Y denote three colours (blue, red and yellow). Consider the space of *proper colourings* of edges of the line $\mathbf{Z}$, that is, colourings where two adjacent edges never have the same colour. Every such colouring defines an action of G on X, where the generators b, r, y act as involutions by switching two adjacent integers whenever the edge between them has the corresponding colour B, R, Y. Consider any proper colouring so that every finite proper word in B, R, Y appears an infinite number of times (for example, endow the space of proper colourings with the natural Markov measure and pick a random one). This colouring gives the action $G \curvearrowright X$ that we consider (this is, essentially, the same action constructed in [**vD90**]). We will abuse notation and identify a proper word in the colours B, R, Y with an element of the group G.

Let $f \in (\mathbf{Z}/2\mathbf{Z})^X$, to be determined later, and consider the group $H \leq (\mathbf{Z}/2\mathbf{Z})^X \rtimes G$ generated by G and f. We will construct an element f such that the action of H on any H-orbit in $(\mathbf{Z}/2\mathbf{Z})^X$ has spectral radius uniformly bounded away from one. This concludes, by Lemma 7.2.

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To construct f, arrange all reduced words in B, R, Y in a list (that is, consider a numeration of elements of G; but we avoid introducing the numeration explicitly). Consider the first word in the list w, and let $l = |w|$ be its length. Pick an auxiliary word $\tilde{w}$ so that $|\tilde{w}| \geq 5l$. Find $l+1$ copies of the word $\tilde{w}w\tilde{w}$ in the coloured line X that do not overlap. For $i = 0, \ldots, l$, let $x_i \in X$ be the vertex between the first $\tilde{w}$ and w in the ith copy of $\tilde{w}w\tilde{w}$.

Consider a big interval I_w that contains all these words and define f on I_w to be $f(x_i + i) = 1$, for every $i = 0, \ldots, l$, and $f(y) = 0$, for all other $y \in I_w$. Here $x_i + i$ denotes the point lying i positions to the right of x_i.

Consider the second word w' in the list and repeat the same construction, taking care that everything happens outside of I_w, then define a bigger interval $I_{w'}$ containing I_w, and so on. This defines f everywhere. In fact, a random $f \in (\mathbf{Z}/2\mathbf{Z})^X$ distributed according to the uniform Bernoulli measure will also have the features that we need.

For every word $w \in G$, we denote by $x_0(w), x_1(w), \ldots x_{|w|}(w) \in X$ the points x_i appearing in the construction.

Now let ν be the equidistributed measure on the standard generating set of G, set $\eta = \frac{1}{2}\delta_f + \frac{1}{2}\delta e$, and consider the 'switch-walk-switch like' measure on $H \leq (\mathbf{Z}/2\mathbf{Z})^X \rtimes G$, given by $\mu = \eta * \nu * \eta$. Let $g_n = (f_n, w_n)$ be the right random walk on H with step measure μ, where $f_n \in (\mathbf{Z}/2\mathbf{Z})^X$ and $w_n \in G$. Also, let $h_1, \ldots h_n$ be the increments of (w_j), so that $w_n = h_n \cdots h_1$.

Let $t \in (\mathbf{Z}/2\mathbf{Z})^X$ be arbitrary, and set $t_n = g_n \cdot t$. The following lemma shows that the action of H on $(\mathbf{Z}/2\mathbf{Z})^X$ is non-amenable, by Lemma 7.2, thereby concluding the proof.

LEMMA 7.3. *There exists a constant $c > 0$ that does not depend on t such that*

$$\mathbb{P}(t_n = t) \leq e^{-cn}. \tag{6}$$

To prove the lemma, first recall that the length $|w_n|$ is, essentially, a drifted random walk on $\mathbf{N}$ with drift $1/3$ and its behaviour is completely understood. By classical large deviations, there is a constant $c_1 > 0$ such that

$$\mathbb{P}(|w_n| \leq n/4) \leq e^{-c_1 n}.$$

Hence

$$\begin{aligned}\mathbb{P}(t_n = t) &= \mathbb{P}(t_n = t, |w_n| \leq n/4) + \mathbb{P}(t_n = t, |w_n| \geq n/4)\\ &\leq e^{-c_1 n} + \mathbb{P}(t_n = t, |w_n| \geq n/4),\end{aligned}$$

so it is sufficient to study the probability $\mathbb{P}(t_n = t)$ conditionally to the event $|w_n| \geq n/4$.

LEMMA 7.4. *Conditionally to w_n, and assuming that $|w_n| \geq n/4$, the values of t_n on the points from the construction $t_n(x_0(w_n))$, $t_n(x_1(w_n))$, $\ldots$ $t_n(x_{|w_n|}(w_n))$ are independent uniform Bernoulli in $\mathbf{Z}/2\mathbf{Z}$.*

This concludes the proof of Lemma 7.3, since then (conditionally to w_n) the probability that these values are equal to the corresponding values of t is bounded above by $2^{-|w_n|} \leq 2^{-n/4}$ and, by integrating over possible values of w_n, we get

$$\mathbb{P}(t_n = t, |w_n| \geq n/4) \leq \sum_{|w| \geq n/4} \mathbb{P}(w_n = w) 2^{-n/4} \leq 2^{-n/4}.$$

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Proof of Lemma 7.4. Fix i and consider the sequence of vertices $(h_r \cdots h_n x_i(w_n))_{r \le n}$. For $i \neq j$, these sequences are obtained one from each other by a suitable translation (they stay in a region where the colouring looks the same, using that $n \le 4|w_n|$ and that the auxiliary words $\tilde{w}$ from the construction have length $\ge 5|w_n|$). In particular, it is not possible that, for $i \neq j$ and for the same $0 \le r \le n$, $h_r \cdots h_n x_i(w_n) = x_i(w_n) + i$ and $h_r \cdots h_n x_j(w_n) = x_j(w_n) + j$. For each $i = 0, \ldots, |w_n|$, let Σ_i be the set of times $r \le n$ so that $h_r \cdots h_n x_i(w_n) = x_i(w_n) + i$. Then the following two properties are satisfied.

(1) $\Sigma_i \cap \Sigma_j = \varnothing$ whenever $i \neq j$ (because of the above observation).

(2) $\Sigma_i \neq \varnothing$ for every i (since, by construction, $w_n = h_n \cdots h_1$ is the word read on the right of $x_i(w_n)$ and hence $h_1 \cdots h_n x_i(w_n) = x_i(w_n) + |w_n|$).

Let $\xi_0, \ldots, \xi_n$ be independent Bernoulli random variables taking values in $\mathbf{Z}/2\mathbf{Z}$. Then it is readily checked that, for every i, the value of $t_n(x_i(w_n))$ has the same distribution as $t(h_1 \cdots h_n x_i(w_n)) + \sum_{r \in \Sigma_i} \xi_r$. These are all Bernoulli random variables because of the second property above, and they are independent because of the first property. □

We conclude with a question; a negative answer to it would trivialize Proposition 7.1.

Question 7.5. Does there exist a faithful action of a non-amenable group G on a set X such that the action of $(\mathbf{Z}/2\mathbf{Z})^X \rtimes G$ on $(\mathbf{Z}/2\mathbf{Z})^X$ is amenable?

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