Bull. Aust. Math. Soc. (First published online 2026), page 1 of 11*
doi:10.1017/S0004972726102056
*Provisional—final page numbers to be inserted when paper edition is published

A NONNEGATIVITY CRITERION FOR ERDŐS MATRICES

FRÉDÉRIC MORNEAU-GUÉRIN

(Received 20 August 2026 UTC; accepted 27 August 2026 UTC)

Abstract

Let A be an $n \times n$ doubly stochastic matrix. The Marcus–Ree inequality asserts that $\|A\|_F^2 \leq \text{maxtrace}(A)$. Matrices attaining equality are called Erdős matrices. Recent work of Karmakar *et al.* ['Characterization of Erdős matrices by their zero entries', *Linear Algebra Appl.* **739** (2026), 154–169] shows that every Erdős matrix is a restricted common diagonal sum matrix. By the structural theory of Brualdi and Dahl ['Diagonal sums of doubly stochastic matrices', *Linear Multilinear Algebra* **70** (2022), 4946–4972], a restricted common diagonal sum matrix with fully indecomposable skeleton $S = (s_{ij})$ admits additive potentials satisfying $a_{ij} = (u_i + v_j)s_{ij}$. Karmakar *et al.* observed that $\min_i u_i + \min_j v_j \geq 0$ is sufficient for Erdős-ness. We prove that it is also necessary. Thus, recognition reduces to solving one linear system per fully indecomposable block and checking signs, with no residual maxtrace computation.

2020 *Mathematics subject classification*: primary 15B51; secondary 15A15, 90C27, 05C50.

Keywords and phrases: doubly stochastic matrix, Marcus–Ree inequality, Erdős matrix, RCDS matrix, assignment problem.

1. Introduction

Let Ω_n be the Birkhoff polytope of $n \times n$ doubly stochastic matrices and set

$$\text{maxtrace}(A) = \max_{\sigma \in S_n} \sum_{i=1}^{n} a_{i,\sigma(i)}.$$

Equivalently, maxtrace(A) is the optimal value of the assignment problem with A as its cost matrix; we discuss this interpretation in more detail in Section 3.

Marcus and Ree [8] proved that

$$\|A\|_F^2 \leq \text{maxtrace}(A), \tag{1.1}$$

where

$$\|A\|_F = \left(\sum_{i,j} a_{ij}^2 \right)^{1/2}$$

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denotes the Frobenius norm. Upon learning of this result, Erdős asked for a characterisation of the matrices for which equality holds in (1.1) (see [4, 8]). Thus, the problem is to determine all $A \in \Omega_n$ satisfying

$$\operatorname{maxtrace}(A) = \|A\|_{\mathrm{F}}^2. \tag{1.2}$$

Following terminology introduced by Tripathi [10], we call the matrices satisfying (1.2) *Erdős matrices*.

The question has received renewed attention. The 3×3 case was classified by Bouthat *et al.* [4]. Tripathi [10] subsequently proved that there are only finitely many Erdős matrices in each dimension, and that all of them are rational and have an algorithmic characterisation. The 4×4 case was later classified by Kushwaha and Tripathi [7].

Most relevant here is the recent work of Karmakar *et al.* [6]. They prove that every Erdős matrix belongs to the class of restricted common diagonal sum (RCDS) matrices studied by Brualdi and Dahl [5]. Questions concerning prescribed diagonal sums of doubly stochastic matrices go back several decades. Wang [11] studied maximal and minimal diagonal sums, while constant diagonal sums were considered by Sinkhorn [9], Achilles [1] and Balasubramanian [2, 3]. The RCDS matrices of Brualdi and Dahl [5] belong to this line of work. A permutation σ is *inner* for A if $a_{i,\sigma(i)} > 0$ for every i, and *outer* otherwise; A is RCDS when all inner permutation diagonals have the same sum.

For a fully indecomposable skeleton $S = (s_{ij})$, by which we mean the pattern of the matrix (see Section 2 for the precise definition), Brualdi and Dahl show that an RCDS matrix has the form

$$a_{ij} = (u_i + v_j)s_{ij}$$

for some real vectors $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$. We call u and v the *additive potentials* and the quantities $u_i + v_j$ the *cross-sums*. Although the potentials themselves are not unique, the cross-sums are uniquely determined. Karmakar *et al.* derive the criterion

$$\sum_{\substack{i \\ s_{i,\sigma(i)}=0}} (u_i + v_{\sigma(i)}) \geq 0$$

for every outer permutation σ and, in particular, record

$$\min_i u_i + \min_j v_j \geq 0 \tag{1.3}$$

as a sufficient condition for Erdős-ness [6, Lemma 1.6].

Our main result is that, for a fully indecomposable RCDS matrix, (1.3) is also necessary:

$$A \text{ is Erdős} \iff u_i + v_j \geq 0 \quad \text{for all } i, j \iff \min_i u_i + \min_j v_j \geq 0.$$

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The proof also yields a constructive interpretation: if $u_i + v_j < 0$ at a zero position, full indecomposability produces an outer permutation using that zero and no other one, whose diagonal sum exceeds $\|A\|_F^2$. Thus, the aggregate inequalities of [6] collapse to pointwise sign conditions. This gives a practical recognition procedure as well. After decomposing a doubly stochastic matrix into fully indecomposable blocks, one solves the linear system (2.3) on each block, verifies that the resulting cross-sums reproduce the block and checks the sign condition $\min_i u_i + \min_j v_j \geq 0$; no maxtrace computation is required. Moreover, when the sign condition fails, Proposition 3.4 produces an explicit permutation certifying the failure of Erdős-ness. Finally, applying the criterion blockwise gives a characterisation of Erdős matrices in arbitrary dimension.

2. RCDS matrices and additive potentials

For a nonnegative matrix $A = (a_{ij})$, write $\mathrm{skel}(A) = (s_{ij})$, where $s_{ij} = 1$ if $a_{ij} > 0$ and $s_{ij} = 0$ otherwise. Every doubly stochastic matrix has total support: each positive position belongs to an inner permutation by the Birkhoff–von Neumann theorem.

LEMMA 2.1. *Suppose* $A \in \Omega_n$ *and* $a_{ij} = (u_i + v_j)s_{ij}$, *where* $u = (u_1, \ldots, u_n)$ *and* $v = (v_1, \ldots, v_n)$ *are real vectors and* $S = \mathrm{skel}(A)$. *Then,*

$$\|A\|_F^2 = \sum_i u_i + \sum_j v_j \tag{2.1}$$

and, for every $\sigma \in S_n$,

$$\sum_i a_{i,\sigma(i)} = \|A\|_F^2 - \sum_{\substack{i \\ s_{i,\sigma(i)}=0}} (u_i + v_{\sigma(i)}). \tag{2.2}$$

In particular, A is RCDS.

PROOF. Since $a_{ij}^2 = (u_i + v_j)a_{ij}$ for all i, j, the row and column sums give (2.1). In addition,

$$\sum_i a_{i,\sigma(i)} = \sum_i (u_i + v_{\sigma(i)}) - \sum_{s_{i,\sigma(i)}=0} (u_i + v_{\sigma(i)})$$

and the first sum equals $\sum_i u_i + \sum_j v_j$. This proves (2.2); for an inner permutation, the second sum is empty. □

The following proposition is essentially the structural result of Brualdi and Dahl [5, Theorem 2.6]; see also the formulation of Karmakar *et al.* [6, Theorem 1.4].

PROPOSITION 2.2. *Let* $S = (s_{ij})$ *be fully indecomposable. Put* $r_i = \sum_j s_{ij}$, $c_j = \sum_i s_{ij}$, $D_R = \mathrm{diag}(r_i)$ *and* $D_C = \mathrm{diag}(c_j)$. *The system*

$$\begin{pmatrix} D_R & S \\ S^\intercal & D_C \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix} \tag{2.3}$$

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is solvable and any two solutions differ by $u \mapsto u + t\mathbf{1}$, $v \mapsto v - t\mathbf{1}$. *Moreover, S is the skeleton of an RCDS doubly stochastic matrix if and only if*

$$u_i + v_j > 0 \quad (s_{ij} = 1),$$

and then that matrix is uniquely determined by $a_{ij} = (u_i + v_j)s_{ij}$.

PROOF. The coefficient matrix in (2.3) is the signless Laplacian of the bipartite graph with biadjacency matrix S.

This graph is connected. Indeed, if it were disconnected, its vertex set would split into two nonempty parts with no edges between them; let I be the set of rows and J the set of columns lying in one of the parts. Then, $S[I, J^c] = 0$ and $S[I^c, J] = 0$. The dimension sums of these two zero submatrices add up to $2n$, so one of them is at least n; shrinking that submatrix if necessary yields an $r \times s$ all-zero submatrix with $r + s = n$, which contradicts full indecomposability. Since the graph is connected and bipartite, the kernel of the coefficient matrix is spanned by $\binom{\mathbf{1}}{-\mathbf{1}}$; since the right-hand side of (2.3) is orthogonal to this vector, the system is solvable and its solutions have exactly the stated ambiguity. Equation (2.3) is equivalent to the row and column sums of $(u_i + v_j)s_{ij}$ being 1. Positivity on the support therefore gives an RCDS matrix by Lemma 2.1, and the converse and uniqueness follow from the additive representation and invariance of the cross-sums. □

PROPOSITION 2.3 (Karmakar *et al.* [6, Proposition 2.7]). *Every Erdős matrix is RCDS.*

3. The potential criterion

3.1. Assignment duality. For a real $n \times n$ matrix $C = (c_{ij})$ we write, as in Section 1,

$$\operatorname{maxtrace}(C) = \max_{\sigma \in S_n} \sum_{i=1}^{n} c_{i,\sigma(i)}.$$

The maximum assignment problem for C is

$$\max \left\{ \sum_{i,j} c_{ij} x_{ij} : X = (x_{ij}) \in \Omega_n \right\} \tag{3.1}$$

and its dual is

$$\min \left\{ \sum_i \alpha_i + \sum_j \beta_j : \alpha_i + \beta_j \geq c_{ij} \text{ for all } i, j \right\}. \tag{3.2}$$

Since the extreme points of Ω_n are the permutation matrices, the optimum in (3.1) is $\operatorname{maxtrace}(C)$.

For an RCDS matrix A, the potentials u, v are dual feasible for the assignment problem restricted to the support of A: on the support, $u_i + v_j = a_{ij}$.

Passing to the unrestricted assignment problem adds the constraints

$$u_i + v_j \geq 0 \quad \text{whenever } s_{ij} = 0.$$

This gives the following criterion.

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THEOREM 3.1. *Let $A \in \Omega_n$ be RCDS with fully indecomposable skeleton $S = (s_{ij})$ and let u, v be its additive potentials, so that $a_{ij} = (u_i + v_j)s_{ij}$. The following conditions are equivalent:*

(i) *A is an Erdős matrix;*
(ii) *(u, v) is feasible for the assignment dual (3.2) with $C = A$;*
(iii) *$u_i + v_j \geq 0$ whenever $s_{ij} = 0$;*
(iv) *$u_i + v_j \geq 0$ for all i, j;*
(v) $\min_i u_i + \min_j v_j \geq 0$.

PROOF. We first prove that conditions (ii)–(v) are equivalent. On the support of A,

$$u_i + v_j = a_{ij} > 0,$$

so the dual constraints in (3.2) hold there with equality. At a zero position, they reduce to $u_i + v_j \geq 0$. Thus, conditions (ii) and (iii) are equivalent. Since the cross-sums are already positive on the support, condition (iii) is in turn equivalent to condition (iv). Finally,

$$\min_{i,j}(u_i + v_j) = \min_i u_i + \min_j v_j,$$

so conditions (iv) and (v) are equivalent.

Suppose now that condition (ii) holds. Weak duality for (3.1)–(3.2) with $C = A$ and Lemma 2.1 give

$$\operatorname{maxtrace}(A) \leq \sum_i u_i + \sum_j v_j = \|A\|_F^2.$$

The reverse inequality is precisely the Marcus–Ree inequality (1.1). Hence,

$$\operatorname{maxtrace}(A) = \|A\|_F^2$$

and condition (i) follows.

Conversely, suppose that condition (i) holds and let (α, β) be an optimal solution of the assignment dual (3.2) with $C = A$. Since A is doubly stochastic, every position (i, j) with $s_{ij} = 1$ belongs to an inner permutation (see Section 2). By Lemma 2.1, every such inner permutation has diagonal sum

$$\|A\|_F^2 = \operatorname{maxtrace}(A)$$

and is, therefore, optimal for the primal assignment problem. Complementary slackness then gives

$$\alpha_i + \beta_j = a_{ij} = u_i + v_j \quad \text{whenever } s_{ij} = 1. \tag{3.3}$$

The bipartite graph associated with S is connected, as shown in the proof of Proposition 2.2. Rewriting (3.3) as

$$(\alpha_i - u_i) + (\beta_j - v_j) = 0$$

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on every edge, connectedness implies that there exists $t \in \mathbb{R}$ such that

$$\alpha_i - u_i = t \quad \text{for every } i \quad \text{and} \quad \beta_j - v_j = -t \quad \text{for every } j.$$

Consequently,

$$\alpha_i + \beta_j = u_i + v_j \quad \text{for all } i, j.$$

At a zero position, dual feasibility therefore yields

$$u_i + v_j = \alpha_i + \beta_j \geq a_{ij} = 0.$$

Thus, condition (iii) holds, which completes the proof. □

REMARK 3.2. Theorem 3.1 says that the potentials for the support-restricted assignment problem remain dual feasible for the unrestricted problem exactly when A is Erdős. It also sharpens [6, Lemma 1.6]: for a fully indecomposable skeleton, the inequalities over the zero positions of outer permutations reduce to the pointwise conditions $u_i + v_j \geq 0$. The next proposition gives a constructive explanation.

3.2. A constructive cycle proof. The duality argument proves the criterion, but does not produce a permutation witnessing its failure. We now give such a construction.

Fix an inner permutation of A and permute the columns of A so that it becomes the identity, so that

$$s_{ii} = 1 \quad (1 \leq i \leq n).$$

This permutation of the columns permutes the entries of v and relabels the positions accordingly; it affects neither the hypotheses nor the conclusions below, so we may assume throughout this subsection that it has been carried out. Associate with S the directed graph D_S on $\{1, \ldots, n\}$, defined by

$$i \longrightarrow j \quad \Longleftrightarrow \quad s_{ij} = 1.$$

LEMMA 3.3. *If S is fully indecomposable, then D_S is strongly connected.*

PROOF. Suppose that D_S is not strongly connected. Then, we may choose a vertex k from which not every vertex is reachable. Let I be the set of vertices reachable from k. Then, I is nonempty and proper, and no edge can leave I, since the endpoint of such an edge would also be reachable from k. Hence,

$$s_{ij} = 0 \quad \text{whenever } i \in I \text{ and } j \notin I.$$

Thus, S contains an all-zero submatrix of size $|I| \times (n - |I|)$, which contradicts full indecomposability. □

The lemma allows us to isolate any single zero position.

PROPOSITION 3.4. *Under the hypotheses of Theorem 3.1, let (i, j) be a zero position of S. Then, there exists a permutation $\sigma \in S_n$ for which (i, j) is the only zero position among*

$$(k, \sigma(k)), \quad 1 \leq k \leq n.$$

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Consequently, if $u_i + v_j < 0$, *then*

$$\sum_k a_{k,\sigma(k)} = \|A\|_{\mathrm{F}}^2 - (u_i + v_j) > \|A\|_{\mathrm{F}}^2. \tag{3.4}$$

In particular, σ *certifies that* A *is not Erdős.*

PROOF. Since $s_{ii} = 1$, we have $i \neq j$. By Lemma 3.3, there is a directed path from j to i. Choose a simple such path,

$$j = i_0 \to i_1 \to \cdots \to i_m = i.$$

Define σ by

$$\sigma(i_\ell) = i_{\ell+1} \quad (0 \le \ell < m), \qquad \sigma(i) = j,$$

and let $\sigma(k) = k$ for all remaining indices.

Every position used by σ lies in the support of S, except for the closing position (i,j). Hence, (i,j) is the unique zero position used by σ. The defect identity (2.2) therefore gives

$$\sum_k a_{k,\sigma(k)} = \|A\|_{\mathrm{F}}^2 - (u_i + v_j).$$

If $u_i + v_j < 0$, the right-hand side is strictly larger than $\|A\|_{\mathrm{F}}^2$, as claimed. □

REMARK 3.5. Proposition 3.4 gives a constructive proof of (i) ⇒ (iii) in Theorem 3.1, independent of linear-programming duality.

4. Skeleton and blockwise characterisations

The potential criterion can be stated directly in terms of the skeleton.

THEOREM 4.1. *Let* $S = (s_{ij})$ *be a fully indecomposable zero–one matrix and let* u, v *solve* (2.3). *Then,* S *is the skeleton of an Erdős matrix if and only if*

$$u_i + v_j > 0 \quad (s_{ij} = 1), \tag{4.1}$$

$$u_i + v_j \ge 0 \quad (s_{ij} = 0). \tag{4.2}$$

Under (4.1), *condition* (4.2) *is equivalent to* $\min_i u_i + \min_j v_j \ge 0$. *When these conditions hold, the unique Erdős matrix with skeleton* S *is*

$$E(S)_{ij} = (u_i + v_j) s_{ij}.$$

PROOF. Proposition 2.2 gives the RCDS matrix with skeleton S exactly under (4.1); Theorem 3.1 then gives Erdős-ness exactly under (4.2). Conversely, every Erdős matrix is RCDS by Proposition 2.3, so the same two results give the necessity and uniqueness. Finally, under (4.1), (4.2) is equivalent to nonnegativity of all cross-sums and

$$\min_{i,j}(u_i + v_j) = \min_i u_i + \min_j v_j.$$

□

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REMARK 4.2. The uniqueness assertion in Theorem 4.1 is also a special case of [6, Theorem 1.8], which shows that a given skeleton carries at most one Erdős matrix; the present proof has the advantage of exhibiting $E(S)$ explicitly.

Every $A \in \Omega_n$ has total support; hence, after row and column permutations,

$$PAQ = A_1 \oplus \cdots \oplus A_m,$$

where the A_k are fully indecomposable doubly stochastic blocks [5].

LEMMA 4.3. *If $A = A_1 \oplus \cdots \oplus A_m$, then*

$$\operatorname{maxtrace}(A) = \sum_k \operatorname{maxtrace}(A_k), \quad \|A\|_{\mathrm{F}}^2 = \sum_k \|A_k\|_{\mathrm{F}}^2.$$

Hence, A is Erdős if and only if every A_k is Erdős.

PROOF. The norm identity is immediate. Maximising separately in each block gives one inequality. Conversely, any permutation of A selects a partial matching in each block; extending it to a permutation of the full row and column index sets of that block cannot decrease its weight, since the entries are nonnegative. Thus, its contribution from A_k is at most $\operatorname{maxtrace}(A_k)$. The last assertion follows from (1.1) applied blockwise. □

THEOREM 4.4. *Let $PAQ = A_1 \oplus \cdots \oplus A_m$ be the above decomposition and $S_k = \operatorname{skel}(A_k)$. For each k, let $u^{(k)}, v^{(k)}$ solve (2.3) for S_k. Then, A is Erdős if and only if, for every k,*

$$(A_k)_{ij} = (u_i^{(k)} + v_j^{(k)})(S_k)_{ij} \quad \textit{for all } i,j, \quad \textit{and} \quad \min_i u_i^{(k)} + \min_j v_j^{(k)} \geq 0.$$

PROOF. If A is Erdős, Lemma 4.3 makes every A_k Erdős and Proposition 2.3, together with Theorem 3.1, gives the stated conditions.

Conversely, the first condition gives the additive representation of each A_k, so Lemma 2.1 shows that A_k is RCDS. The second condition and Theorem 3.1 then make each A_k Erdős, and Lemma 4.3 completes the proof. □

5. Examples and consequences

Consider the matrix

$$E = \frac{1}{16}\begin{pmatrix} 0 & 11 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 & 11 \\ 10 & 0 & 6 & 0 & 0 \\ 3 & 0 & 0 & 8 & 5 \\ 3 & 5 & 0 & 8 & 0 \end{pmatrix},$$

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whose row and column sums are all equal to 1, and whose skeleton is

$$S = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

One checks directly that S has no all-zero submatrix of size $r \times s$ with $r + s = 5$, so S is fully indecomposable. A solution of the system (2.3) is given by

$$u = \left(0,\ 0,\ \frac{1}{16},\ -\frac{3}{8},\ -\frac{3}{8}\right), \quad v = \left(\frac{9}{16},\ \frac{11}{16},\ \frac{5}{16},\ \frac{7}{8},\ \frac{11}{16}\right).$$

Every cross-sum at a support position is positive and reproduces the corresponding entry,

$$e_{ij} = u_i + v_j > 0 \quad (s_{ij} = 1),$$

so E is RCDS by Lemma 2.1. At two of the zero positions, however,

$$u_4 + v_3 = u_5 + v_3 = -\frac{1}{16} < 0 \tag{5.1}$$

and Theorem 3.1 shows at once that E is not an Erdős matrix.

Proposition 3.4 converts either violation into an explicit permutation. Starting from the zero position $(4, 3)$ and the inner permutation $(2, 3, 1, 5, 4)$, the construction returns

$$\sigma = (2, 5, 1, 3, 4)$$

in one-line notation, which uses the position $(4, 3)$ and no other zero position. Its diagonal sum is

$$\sum_i e_{i,\sigma(i)} = \frac{11 + 11 + 10 + 0 + 8}{16} = \frac{5}{2},$$

whereas

$$\|E\|_{\mathrm{F}}^2 = \frac{39}{16}.$$

The excess is

$$\sum_i e_{i,\sigma(i)} - \|E\|_{\mathrm{F}}^2 = \frac{1}{16} = -(u_4 + v_3),$$

exactly as predicted by (3.4).

5.1. Adding one support entry can destroy the property. Call a zero–one matrix an *Erdős skeleton* if it is the skeleton of an Erdős matrix.

The class of Erdős skeletons is not upward closed.

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PROPOSITION 5.1. *There are fully indecomposable zero–one matrices $S < S'$ such that S' is obtained from S by changing a single* 0 *into a* 1*, S is an Erdős skeleton and S' is not.*

PROOF. Take

$$S = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad S' = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Both are fully indecomposable. For S, the potentials

$$u = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad v = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

solve (2.3); they are positive on the support and all cross-sums are nonnegative, so S is an Erdős skeleton by Theorem 4.1. Passing from S to S' increases only the first row and column degrees. Since $u_1 + v_1 = 0$, the same u, v still solve (2.3) for S'. However, now $(1, 1)$ belongs to the support and its cross-sum is 0, violating the strict positivity required in Theorem 4.1. Thus, S' is not an Erdős skeleton. □

6. Conclusion

The characterisation obtained above reduces Erdős-ness to a linear-algebraic sign test on the fully indecomposable blocks of the skeleton. In particular, it replaces the global maxtrace condition by the local requirement that all cross-sums $u_i + v_j$ are nonnegative, and a negative cross-sum yields an explicit permutation witnessing the failure of equality in the Marcus–Ree inequality.

Acknowledgements

The author is grateful to L. Bouthat, J. Mashreghi and S. Singh for previous collaborations.

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FRÉDÉRIC MORNEAU-GUÉRIN, Département Éducation,
Université TÉLUQ, Québec, Canada
e-mail: frederic.morneau-guerin@teluq.ca

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