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COUNTEREXAMPLE TO CONJECTURES ON COMPLEMENTED ZERO-DIVISOR GRAPHS OF SEMIGROUPS

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Abstract

This paper is motivated by two conjectures proposed by Bender *et al.* ['Complemented zero-divisor graphs associated with finite commutative semigroups', *Comm. Algebra* **52**(7) (2024), 2852–2867], which have remained open questions. The first conjecture states that if the complemented zero-divisor graph $G(S)$ of a commutative semigroup S with a zero element has clique number three or greater, then the reduced graph $G_r(S)$ is isomorphic to the graph $G(\mathcal{P}(n))$. The second conjecture asserts that if $G(S)$ is a complemented zero-divisor graph with clique number three or greater, then $G(S)$ is uniquely complemented. We construct a commutative semigroup S with a zero element that serves as a counter-example to both conjectures.

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1. Introduction

The zero-divisor graph associated with a commutative ring was first introduced by Beck [2], with the goal of studying graph colouring. Later, Anderson and Livingston [1] modified the definition so that the vertex set consists only of the nonzero zero-divisors of the ring. This construction was further extended to the context of semigroups and posets in [4–6].

In a graph G, two distinct vertices a and b are called *orthogonal*, denoted $a \perp b$, if a and b are adjacent and there does not exist a vertex c adjacent to both a and b. Equivalently, adjacent vertices a and b are *orthogonal* if the edge $a - b$ is not an edge of any triangle (3-cycle). A graph G is called *complemented* if for every vertex a, there exists a vertex b such that $a \perp b$. A graph G is called *uniquely complemented* if G is complemented and has the property that if a, b, c are distinct vertices with $a \perp b$ and $a \perp c$, then $N(b) = N(c)$, where $N(x)$ denotes the neighbourhood of the vertex x.

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This graph-theoretic property has been studied in the context of the zero-divisor graph associated with a commutative ring (see [7]).

Throughout the paper, S denotes a finite commutative semigroup with 0. The set of zero-divisors of S, denoted by $Z(S)$, forms an ideal in S. The zero-divisor graph associated with S, denoted by $G(S)$, is the graph whose vertices are given by the nonzero zero-divisors, and two distinct vertices a and b are adjacent when $ab = 0$. A *clique* in a graph G is a sub-graph of G where every pair of distinct vertices in the sub-graph are adjacent. The *clique number* of a graph G is the number of vertices in a maximal clique in G and is denoted by $\omega(G)$.

Define an equivalence relation $\sim$ on the vertex set $V(G)$ of a graph G by $a \sim b$ if and only if $N(a) = N(b)$. Denote the equivalence class of $a \in V(G)$ by $[a]$. The *reduced graph* of G, denoted G_r, is the graph whose vertex set is the set of equivalence classes under the relation $\sim$, where two distinct vertices $[a]$, $[b]$ in the vertex set of G_r are adjacent if and only if vertices a and b are adjacent in the graph G (see [3]).

Let $\mathcal{P}(n)$ denote the semigroup consisting of the power set of an n-element set, equipped with the operation of intersection.

In [3], Bender *et al.* mentioned that every known example of a zero-divisor graph $G(S)$, for finite S, which is complemented and has the clique number $n \geq 3$, has reduced graph isomorphic to $G(\mathcal{P}(n))$ and is also a uniquely complemented graph. The case where the order of S is infinite is open. They posed the following conjectures.

CONJECTURE 1.1. If $G(S)$ is a complemented zero-divisor graph with the clique number $n \geq 3$, then $G(S)$ has the reduced graph G_r which is isomorphic to $G(\mathcal{P}(n))$.

CONJECTURE 1.2. If $G(S)$ is a complemented zero-divisor graph with the clique number 3 or greater, then $G(S)$ is uniquely complemented.

To settle these conjectures, we explored the underlying structures of complemented zero-divisor graphs in commutative semigroups with zero.

2. Counterexample

Consider the set $S = \mathcal{P}(3) \times \mathbb{Z}_4 = \{(X, \overline{a}) \mid X \in \mathcal{P}(3), \ \overline{a} \in \mathbb{Z}_4\}$. Clearly, S is a nonempty set, as $(\emptyset, \overline{0}) \in S$. We define a binary operation $*$ on the set S such that $(X, \overline{a}) * (Y, \overline{b}) = (X \cap Y, \ \overline{a} \times_4 \overline{b})$ for $(X, \overline{a}), (Y, \overline{b}) \in S$. For simplicity, we denote the binary operation $\times_4$ on $\mathbb{Z}_4$ by '$\cdot$'.

Since $(\mathcal{P}(3), \cap)$ and $(\mathbb{Z}_4, \cdot)$ are both commutative semigroups with zero elements, we can easily prove that $(S, *)$ is a commutative semigroup with zero element $(\emptyset, \overline{0})$.

Observe that

$$V(G(S)) = \{(X, \overline{a}) \in S \setminus (\emptyset, \overline{0}) \mid (X, \overline{a}) * (Y, \overline{b}) = (\emptyset, \overline{0}) \text{ for some } (Y, \overline{b}) \in S \setminus \{(\emptyset, \overline{0})\}\}.$$

Clearly, $V(G(S)) \neq \emptyset$, since $(\emptyset, \overline{1}) * (\{123\}, \overline{0}) = (\emptyset, \overline{0})$. Further, we can easily prove that $V(G(S)) = S \setminus \{(\emptyset, \overline{0}), (\{123\}, \overline{1}), (\{123\}, \overline{3})\}$.

First, we claim that the zero-divisor graph $G(S)$ is complemented.

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Let $(X,\overline{a}) \in V(G(S))$ be any element. Thus, $(X,\overline{a}) \neq (\emptyset,\overline{0})$ and there exists $(Y,\overline{b}) \neq (\emptyset,\overline{0})$ such that $(X,\overline{a}) * (Y,\overline{b}) = (\emptyset,\overline{0})$. Therefore, $X \cap Y = \emptyset$ and $\overline{a}\cdot\overline{b} = \overline{0}$. Since $(X,\overline{a}) \neq (\emptyset,\overline{0})$, we have $X \neq \emptyset$ or $\overline{a} \neq \overline{0}$. Thus, we have the following cases.

Case (I): $X = \emptyset$ and $\overline{a} \in \{\overline{1},\overline{2},\overline{3}\}$. Then, $(X,\overline{a}) \in \{(\emptyset,\overline{1}),(\emptyset,\overline{2}),(\emptyset,\overline{3})\}$. It is easy to observe that $(X,\overline{a}) \perp (\{1,2,3\},\overline{0})$.

Case (II): $X \neq \emptyset$ and $\overline{a} = \{\overline{0}\}$. Clearly, one can see that $(X,\overline{0}) \perp (Y,\overline{b})$, where $(Y,\overline{b}) \in \{(\{1,2,3\}\setminus X,\ \overline{1}),(\{1,2,3\}\setminus X,\ \overline{2}),(\{1,2,3\}\setminus X,\ \overline{3})\}$.

Case (III): $X \neq \emptyset$ and $\overline{a} \in \{\overline{1},\overline{2},\overline{3}\}$. Then, $(X,\overline{a}) \perp (\{1,2,3\}\setminus X,\overline{0})$ for any $X \subsetneq \{1,2,3\}$. Further, $(\{1,2,3\},\overline{2}) \perp (\emptyset,\overline{2})$.

Thus, in any case, any element $(X,\overline{a}) \in V(G(S))$ has a complement in $G(S)$. Therefore, $G(S)$ is a complemented graph. Clearly, $G(S)$ contains a cycle of length 3, namely, $(\{1\},\overline{0}) - (\{2\},\overline{0}) - (\{3\},\overline{0}) - (\{1\},\overline{0})$. This implies that $\omega(G(S)) \geq 3$.

2.1. Counterexample to Conjecture 1.1. For the reduced graph $G_r(S)$, first, we determine a neighbourhood $N((X,\overline{a}))$ for every vertex $(X,\overline{a}) \in V(G(S))$, as shown now.

(A) Let $X = \emptyset$ and $\overline{a} \in \{\overline{1},\overline{2},\overline{3}\}$.

(a) $N((\emptyset,\overline{1})) = \{(X,\overline{0}) \mid X(\neq\emptyset) \in \mathcal{P}(3)\}$
$= \{(\{1\},\overline{0}),(\{2\},\overline{0}),(\{3\},\overline{0}),(\{1,2\},\overline{0}),(\{1,3\},\overline{0})\},(\{2,3\},\overline{0}),(\{1,2,3\},\overline{0})\}$
$= N((\emptyset,\overline{3}))$.

(b) $N((\emptyset,\overline{2})) = \{(X,\overline{0}),(X,\overline{2}) \mid X(\neq\emptyset) \in \mathcal{P}(3)\}$
$= \{(\{1\},\overline{0}),(\{2\},\overline{0}),(\{3\},\overline{0}),(\{1,2\},\overline{0}),(\{1,3\},\overline{0})\},(\{2,3\},\overline{0}),(\{1,2,3\},\overline{0}),$
$(\{1\},\overline{2}),(\{2\},\overline{2}),(\{3\},\overline{2}),(\{1,2\},\overline{2}),(\{1,3\},\overline{2})\},(\{2,3\},\overline{2}),(\{1,2,3\},\overline{2})\}$.

(B) Let $X \neq \emptyset$ and $\overline{a} = \overline{0}$. Then,

$N((X,\overline{0}))$

$= \{(\emptyset,\overline{a}),(Y,\overline{b}) \mid \overline{a} \in \{\overline{1},\overline{2},\overline{3}\},\overline{b} \in \mathbb{Z}_4, Y(\neq\emptyset) \in \mathcal{P}(3) \text{ such that } X \cap Y = \emptyset\}$.

(a) $N((\{1\},\overline{0})) = \{(\emptyset,\overline{a}),(\{2\},\overline{b}),(\{3\},\overline{b}),(\{2,3\},\overline{b}) \mid \overline{a} \in \{\overline{1},\overline{2},\overline{3}\} \text{ and } \overline{b} \in \mathbb{Z}_4\}$.
(b) $N((\{2\},\overline{0})) = \{(\emptyset,\overline{a}),(\{1\},\overline{b}),(\{3\},\overline{b}),(\{1,3\},\overline{b}) \mid \overline{a} \in \{\overline{1},\overline{2},\overline{3}\} \text{ and } \overline{b} \in \mathbb{Z}_4\}$.
(c) $N((\{3\},\overline{0})) = \{(\emptyset,\overline{a}),(\{1\},\overline{b}),(\{2\},\overline{b}),(\{1,2\},\overline{b}) \mid \overline{a} \in \{\overline{1},\overline{2},\overline{3}\} \text{ and } \overline{b} \in \mathbb{Z}_4\}$.
(d) $N((\{1,2\},\overline{0})) = \{(\emptyset,\overline{a}),(\{3\},\overline{b}) \mid \overline{a} \in \{\overline{1},\overline{2},\overline{3}\} \text{ and } \overline{b} \in \mathbb{Z}_4\}$.
(e) $N((\{1,3\},\overline{0})) = \{(\emptyset,\overline{a}),(\{2\},\overline{b}) \mid \overline{a} \in \{\overline{1},\overline{2},\overline{3}\} \text{ and } \overline{b} \in \mathbb{Z}_4\}$.
(f) $N((\{2,3\},\overline{0})) = \{(\emptyset,\overline{a}),(\{1\},\overline{b}) \mid \overline{a} \in \{\overline{1},\overline{2},\overline{3}\} \text{ and } \overline{b} \in \mathbb{Z}_4\}$.
(g) $N((\{1,2,3\},\overline{0})) = \{(\emptyset,\overline{a}) \mid \overline{a} \in \{\overline{1},\overline{2},\overline{3}\}\}$.

Clearly, all the neighbourhoods $N((\{1\},\overline{0}))$, $N((\{2\},\overline{0}))$, $N((\{3\},\overline{0}))$, $N((\{1,2\},\overline{0}))$, $N((\{2,3\},\overline{0}))$ and $N((\{1,3\},\overline{0}))$ are distinct.

(C) Let $X \neq \emptyset$ and $\overline{a} \in \{\overline{1},\overline{3}\}$. Then,

$$N((X,\overline{a})) = \{(Y,\overline{0}) \mid Y(\neq\emptyset) \in \mathcal{P}(3) \text{ such that } X \cap Y = \emptyset\}.$$

(a) $N((\{1\},\overline{a})) = \{(\{2\},\overline{0}),(\{3\},\overline{0}),(\{2,3\},\overline{0})\}$.
(b) $N((\{2\},\overline{a})) = \{(\{1\},\overline{0}),(\{3\},\overline{0}),(\{1,3\},\overline{0})\}$.

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(c) $N((\{3\},\overline{a})) = \{(\{1\},\overline{0}),(\{2\},\overline{0}),(\{1,2\},\overline{0})\}$.
(d) $N((\{1,2\},\overline{a})) = \{(\{3\},\overline{0})\}$.
(e) $N((\{1,3\},\overline{a})) = \{(\{2\},\overline{0})\}$.
(f) $N((\{2,3\},\overline{a})) = \{(\{1\},\overline{0})\}$.

Clearly, all the neighbourhoods $N((\{1\},\overline{a}))$, $N((\{2\},\overline{a}))$, $N((\{3\},\overline{a}))$, $N((\{1,2\},\overline{a}))$, $N((\{1,3\},\overline{a}))$ and $N((\{2,3\},\overline{a}))$ are distinct.

(D) Let $X \neq \emptyset$ and $\overline{a} = \overline{2}$. Then,

$$N((X,\overline{2})) = \{(\emptyset,\overline{2}),(Y,\overline{a}) \mid Y(\neq \emptyset) \in \mathcal{P}(3) \text{ such that } X \cap Y = \emptyset \text{ and } \overline{a} \in \{\overline{0},\overline{2}\}\}.$$

(a) $N((\{1\},\overline{2})) = \{(\emptyset,\overline{2}),(\{2\},\overline{a}),(\{3\},\overline{a}),(\{2,3\},\overline{a}) \mid \overline{a} \in \{\overline{0},\overline{2}\}\}$.
(b) $N((\{2\},\overline{2})) = \{(\emptyset,\overline{2}),(\{1\},\overline{a}),(\{3\},\overline{a}),(\{1,3\},\overline{a}) \mid \overline{a} \in \{\overline{0},\overline{2}\}\}$.
(c) $N((\{3\},\overline{2})) = \{(\emptyset,\overline{2}),(\{1\},\overline{a}),(\{2\},\overline{a}),(\{1,2\},\overline{a}) \mid \overline{a} \in \{\overline{0},\overline{2}\}\}$.
(d) $N((\{1,2\},\overline{2})) = \{(\emptyset,\overline{2}),(\{3\},\overline{a}) \mid \overline{a} \in \{\overline{0},\overline{2}\}\}$.
(e) $N((\{1,3\},\overline{2})) = \{(\emptyset,\overline{2}),(\{2\},\overline{a}) \mid \overline{a} \in \{\overline{0},\overline{2}\}\}$.
(f) $N((\{2,3\},\overline{2})) = \{(\emptyset,\overline{2}),(\{1\},\overline{a}) \mid \overline{a} \in \{\overline{0},\overline{2}\}\}$.
(g) $N((\{1,2,3\},\overline{2})) = \{(\emptyset,\overline{2})\}$.

Clearly, all the neighbourhoods $N((\{1\},\overline{2}))$, $N((\{2\},\overline{2}))$, $N((\{3\},\overline{2}))$, $N((\{1,2\},\overline{2}))$, $N((\{1,3\},\overline{2}))$, $N((\{2,3\},\overline{2}))$ and $N((\{1,2,3\},\overline{2}))$ are distinct.

It follows that $[(X,\overline{1})] = [(X,\overline{3})]$ and

$$V(G_r) = \{[(\emptyset,\overline{2})],[(X,\overline{a})],[(Y,\overline{1})] \mid X(\neq \emptyset) \in \mathcal{P}(3), Y \in \mathcal{P}(3) \setminus \{1,2,3\} \text{ and } \overline{a} \in \{\overline{0},\overline{2}\}\}.$$

Therefore, $|V(G_r(S))| = 22$.

If we assume that the reduced graph $G_r(S)$ is isomorphic to the graph $G(\mathcal{P}(n))$, then we have $22 = |V(G_r(S))| = |V(G(\mathcal{P}(n)))| = 2^n - 2$. This implies that $2^n = 24$, which is a contradiction to the fact that $n \in \mathbb{N}$. Therefore, $G_r(S)$ is not isomorphic to the graph $G(\mathcal{P}(n))$ for any $n \in \mathbb{N}$.

Thus, Conjuncture 1.1 fails.

2.2. Counterexample to Conjecture 1.2. We now claim that $G(S)$ is not a uniquely complemented graph.

Let $u = (\emptyset,\overline{2})$, $v = (\{1,2,3\},\overline{0})$ and $w = (\{1,2,3\},\overline{2})$ be elements of $V(G(S))$. Observe that $u * v = (\emptyset,\overline{0})$ and hence u is adjacent to v in $G(S)$. Suppose there exists an element $t = (X,\overline{a}) \in V(G(S))$, with $t \neq u$ and $t \neq v$, such that t is adjacent to both u and v, that is, $t * u = t * v = (\emptyset,\overline{0})$. This implies $X = \emptyset$ and $\overline{a} \in \{\overline{0},\overline{2}\}$.

If $\overline{a} = \overline{0}$, then $t = (\emptyset,\overline{0})$, which is not a vertex of $G(S)$, since this element is the zero element of the semigroup and thus excluded from the vertex set.

If $\overline{a} = \overline{2}$, then $t = (\emptyset,\overline{2}) = u$, which contradicts the assumption $t \neq u$.

Therefore, no such t exists, and the edge u–v is not part of a 3-cycle. Thus, $u \perp v$ in $G(S)$. Similarly, it can be shown that $u \perp w$ in $G(S)$.

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Next, we examine the neighbourhoods of v and w. Clearly,

$$N(v) = \{(\emptyset, \overline{1}), (\emptyset, \overline{2}), (\emptyset, \overline{3})\}, \quad N(w) = \{(\emptyset, \overline{2})\}.$$

Since $N(v) \neq N(w)$, it follows that $G(S)$ is not a uniquely complemented graph.

Therefore, Conjecture 1.2 fails.

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