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On Moser's boundedness problem of dual Billiards

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Abstract. In this paper, the boundedness of the dual billiard orbits of trapezoids is obtained. This gives an example showing that there are also polygons, which are neither rational nor quasi-rational, with all of their dual billiard orbits bounded. Moreover, most of the orbits are Poisson-stable, which means that most points will come sufficiently near to their original positions under iterations of the dual billiard map.

1. *Introduction*

For a general polygonal dual billiard problem, we consider a simple convex polygon P on the plane. Its dual billiard map is defined as follows. For any point p outside P, we draw two supporting lines of P and choose one whose direction agrees with the orientation of ∂P. In general this supporting line meets ∂P at a single point q and when this is the case, we reflect p about q, and let $\Phi(p)$ denote the so-obtained point (Figure 1). The map Φ is called the dual billiard map of P.

Here Φ is defined not on the whole set $\mathbf{R}^2 \setminus P$, but a full-measured subset of it. Clearly Φ is one-to-one and Φ^{-1} is the reflection about the supporting point of the other supporting line (if it meets ∂P at a single point). Let Φ^n denote the n times combination of Φ and let $\mathbf{F}_\infty(P)$ denote the set consisting of all points whose positive Φ-orbits with respect to P are well defined, e.g. $p \in \mathbf{F}_\infty(P)$ if and only if $\Phi^n(p)$ is well defined for every $n \in \mathbf{N}$. Here $\mathbf{F}_\infty(P)$ is also a full-measured subset of $\mathbf{R}^2 \setminus P$. It can be easily seen that affine equivalent polygons have affine equivalent dual billiard orbits.

It is natural to ask whether for any P and for any point $p \in \mathbf{F}_\infty(P)$, the dual billiard orbits of p are bounded. This question was proposed by Moser [**1**] and he claimed that for some special polygons, indeed all triangles and parallelograms, the answer is affirmative. Later, in the 1980s, Vivaldi and Shaidenko [**2**] proved that the dual billiard orbits of all rational or quasi-rational polygons are bounded. The same result can also be found in [**3**]. A different point of view of this problem was given by Gutkin and Simanyi in [**4**]. For

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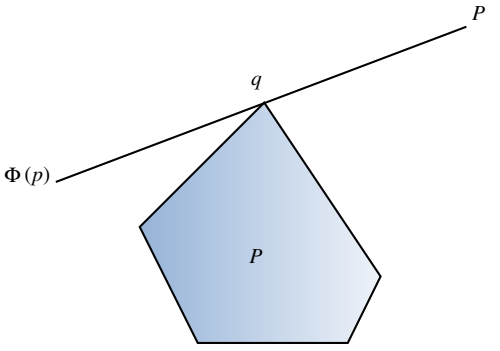


FIGURE 1. Polygonal dual billiard map.

more classical results on polygonal dual billiards, we refer the interested reader to [**5–8**] and references therein.

In this paper, we mainly treat a kind of so-called irrational polygon, namely trapezoids, and obtain the following results.

THEOREM 1. *All dual billiard orbits of all trapezoids are bounded.*

Definition 1.1. Let X be a topology space, and $\{\phi^n, n \in \mathbf{N}\}$ a cascade defined on $U \subset X$, then a point $p \in U$ is called *Poisson-stable* if for arbitrarily small vicinity V_0 of p, there exists $n_0 \in \mathbf{N}$ such that $\phi^{n_0}(p) \in V_0$.

This implies that a Poisson-stable point is an accumulation point of its own orbit. Also, every periodic point is Poisson-stable.

THEOREM 2. *Let P be a trapezoid, then all periodic points of its dual billiard map form a set with positive measure. In addition, if there exists at least one non-periodic orbit, then all non-periodic points that are Poisson-stable form a set with positive measure as well. Indeed, for any infinite Φ-orbit which is sufficiently far from P, all points on it are Poisson-stable.*

It should be remarked that the same result as Theorem 1 here was already obtained by Genin in his PhD thesis [**9**] in 2005. We give a different proof here. A new and exciting result concerning the boundedness of polygonal dual billiards was given by Schwartz [**10**] who found a class of quadrilaterals, namely the so-called irrational kites, having certain orbits escaping to infinity.

In §2, the notion of half-strip region is introduced. Noting that $T(p) := \Phi^2(p)$ is actually a translation of p, although the translation vector varies with the position of p, one can divide the plane into regions in terms of different translation vectors. The point p can move from one region to another under T. Considering merely the zones in which p arrives while it leaps into one region from another, it can be found that these zones coincide with some half-strips far from P. So we obtain several half-strip regions. Under Φ^2, the point p repeats the same translation within one half-strip region until it enters another region and then starts repeating a different translation, and so on. In this way, the dual billiard orbits can be described quite simply.

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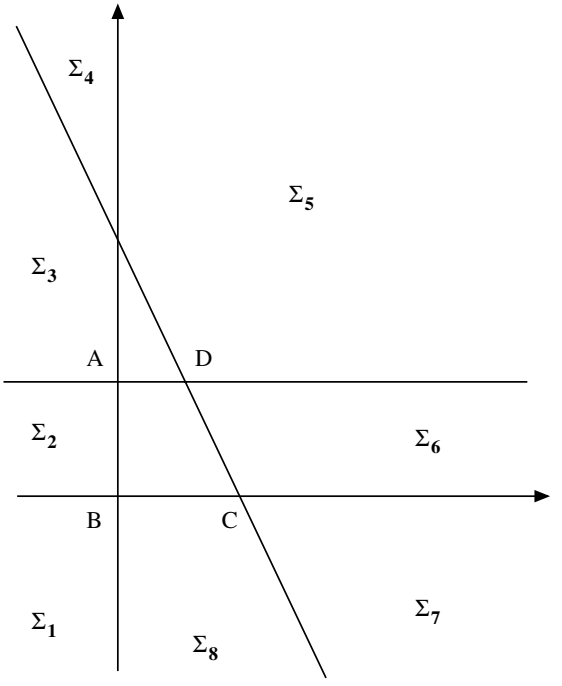


FIGURE 2. Eight zones, in each zone points translate by the same vector under Φ (inverse of Φ).

In §3 we introduce the so-called return map and obtain its concrete expression. Taking a certain half-strip region, we consider the displacement of points in this region when they go back into the same region at the next step and call it a return map. Its expression varies with the initial position of p.

In §4, the return map is reconsidered on a circle, where it turns into the motion of three objects, a point and two intervals. Every step of the motion depends on the relative position among these three objects. In other words, whether p goes further or comes closer to P under the return map is determined completely by the relative position mentioned above. It is sufficient to show that the case of going further and that of coming closer occur at roughly the same rate. Therefore, any point cannot go to infinity under iterations of Φ^2, and that is what we need.

In §§5 and 6 we prove Theorems 1 and 2. Actually, Theorem 2 can be regarded as a corollary of Theorem 1.

2. *Half-strip region*

As any trapezoid can be transformed into the trapezoid P whose vertices have coordinates $A(0, 1)$, $B(0, 0)$, $C(1, 0)$, $D(\alpha, 1)$ with $0 < \alpha < 1$ (Figure 2), it is sufficient to prove that all Φ-orbits of P are bounded. If α is rational, all orbits are periodic (see [**2**]).

The four lines AB, BC, CD, DA divide the plane into several regions. Denote by $\Sigma_1, \Sigma_2, \ldots, \Sigma_8$, the eight unbounded regions (Figure 2). Obviously, for any point $p \in \Sigma_i$ $(i = 1, 2, \ldots, 8)$ and far away from P, $\Phi(p)$ and $\Phi^{-1}(p)$ are both well defined.

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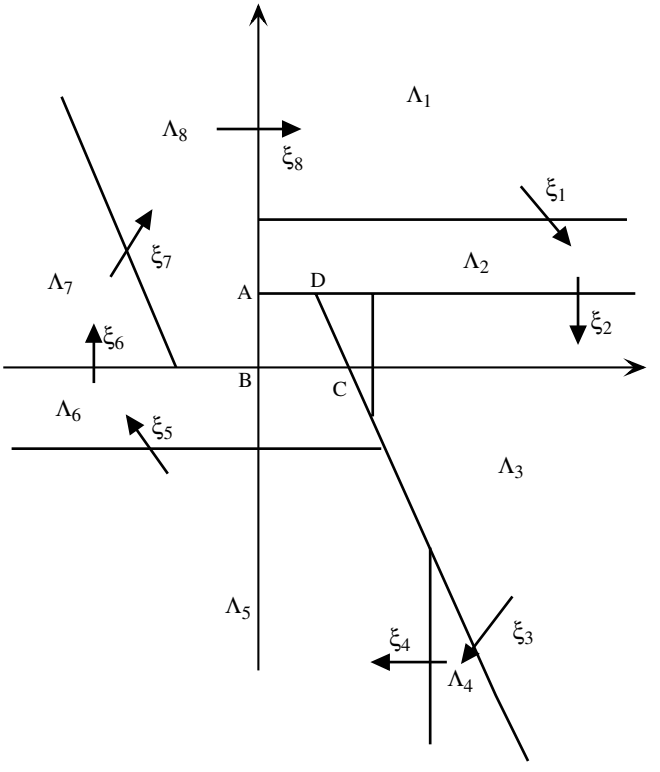


FIGURE 3. Eight regions and translation vectors between them.

Moreover, in this case, the two reflection points are determined completely. For example, if p is in Σ_5, then the reflection points must be A (from p to $\Phi(p)$) and C (from p to $\Phi^{-1}(p)$).

Let $T = \Phi^2$. Then the conclusion of Theorem 1 is equivalent to the boundedness of orbits of T. So it suffices to prove that all orbits of T are bounded.

Let $\Lambda_i = \Phi^{-1}(\Sigma_i)$ (Figure 3). Then for any point $p \in \Lambda_i$ which is far away from P, the points $\Phi(p)$ and $\Phi^2(p)$ are both well defined and the two reflection points from p to $\Phi(p)$ and from $\Phi(p)$ to $\Phi^2(p)$ are determined uniquely. So the map T^2 is well defined for $p \in \Lambda_i$ far away from P.

Let

$$\xi_1 = (2, -2), \quad \xi_2 = (0, -2), \quad \xi_3 = (-2\alpha, -2), \quad \xi_4 = (-2, 0),$$
$$\xi_5 = (-2, 2), \quad \xi_6 = (-2(1-\alpha), 2), \quad \xi_7 = (2\alpha, 2), \quad \xi_8 = (2, 0).$$

Then for $p \in \Lambda_i$ far away from the trapezoid P, we have (see Figure 3)

$$T(p) = p + \xi_i.$$

Note that under T, p might move from Λ_1 directly into Λ_3 without entering Λ_2, or from Λ_5 directly into Λ_7 without entering Λ_6. Except for these two cases, p can only move within one region or from Λ_i into Λ_{i+1} ($i = 1, \ldots, 8$, $\Lambda_9 = \Lambda_1$).

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Let

$$\tilde{\Gamma}_i = \Lambda_i \cap T(\Lambda_{i-1}) \quad (i = 1, 2, 4, 5, 6, 8),$$
$$\tilde{\Gamma}_i = \Lambda_i \cap (T(\Lambda_{i-1}) \cup T(\Lambda_{i-2})) \quad (i = 3, 7),$$

and

$$\Gamma_1 = \{(x, y) \mid 0 < x < 2, y > 0\},$$
$$\Gamma_2 = \{(x, y) \mid x > 0, 1 < y < 2\},$$
$$\Gamma_3 = \{(x, y) \mid x > 0, -1 < y < 1\},$$
$$\Gamma_4 = \{(x, y) \mid -1 < x + (1 - \alpha)y < 1, x > 0\},$$
$$\Gamma_5 = \{(x, y) \mid 0 < x < 2, y < 0\},$$
$$\Gamma_6 = \{(x, y) \mid x < 0, -1 < y < 0\},$$
$$\Gamma_7 = \{(x, y) \mid x < 0, 0 < y < 2\},$$
$$\Gamma_8 = \{(x, y) \mid -1 < x + (1 - \alpha)y < 1, x < 0\}.$$

Then Γ_i coincides with $\tilde{\Gamma}_i$ when far from P (for $i = 3, 7$, the difference between Γ_i and $\tilde{\Gamma}_i$ when far from P is just a ray on which the points have finite Φ-orbits). We call these Γ_i half-strip regions [**3**] (Figure 4).

Let $p \in \Gamma_1$ be a point far from P, then under T it translates by vector a ξ_1 repeatedly until entering Γ_2 or Γ_3. In the latter case it subsequently translates by ξ_3 repeatedly until entering Γ_4, while in the former case it first translates by ξ_2 once and then translates by ξ_3 repeatedly until entering Γ_4. In both cases it then translates by ξ_4 repeatedly to enter Γ_5, and then by ξ_5 repeatedly to enter Γ_6 or Γ_7. Similarly, in the latter case, it directly translates by ξ_7 repeatedly until entering Γ_8, while in the former case, it first translates by ξ_6 once and then by ξ_7 repeatedly until entering Γ_8. Then for both cases, it translates by ξ_8 repeatedly until going back into Γ_1 and the process is repeated as long as the orbit remains far from P. So we can define a map Ψ as follows. For $p \in \Gamma_1$ and far from P, let $\Psi(p)$ be the position when p first comes back to Γ_1 under iterations of T. (Not Φ!) Here Ψ is called the return map of P.

Remark 2.1. If $\Phi^k(p)$ lies on the boundary of some Γ_i for some k, then Φ is undefined at the next step. In this case, the Φ-orbit of p consists of finite points, so is certainly bounded. If p lies on the ray $\{(x, y) \mid x = 2, y > 0\}$, then $\{\Phi^n(p), n \in \mathbf{N}\}$ might be infinite, but we can consider $\Psi(p)$ instead of p, $\Psi(p)$ also lies in Γ_1 but definitely not on the ray $\{(x, y) \mid x = 2, y > 0\}$. Therefore it is sufficient to take the initial point in Γ_1.

3. *Return map*

Now we come to derive the expression of the return map Ψ. Let $S_{2n} = \{(x, y) \mid 0 < x < 2, 2n < y < 2n + 2\}$ $(n \in \mathbf{Z})$ be a sequence of squares (Figure 5), and $p \in \Gamma_1$ be a point far from P, then p belongs to S_{2n} or lies on some line segment $\{(x, y) \mid 0 < x < 2, y = 2n\}$ for some positive integer n. If the latter happens, then $T^{n-1}(p)$ lies on the boundary of Γ_2 so the Φ-orbit of p is finite. Therefore, we assume that $p \in S_{2n}$ from now on.

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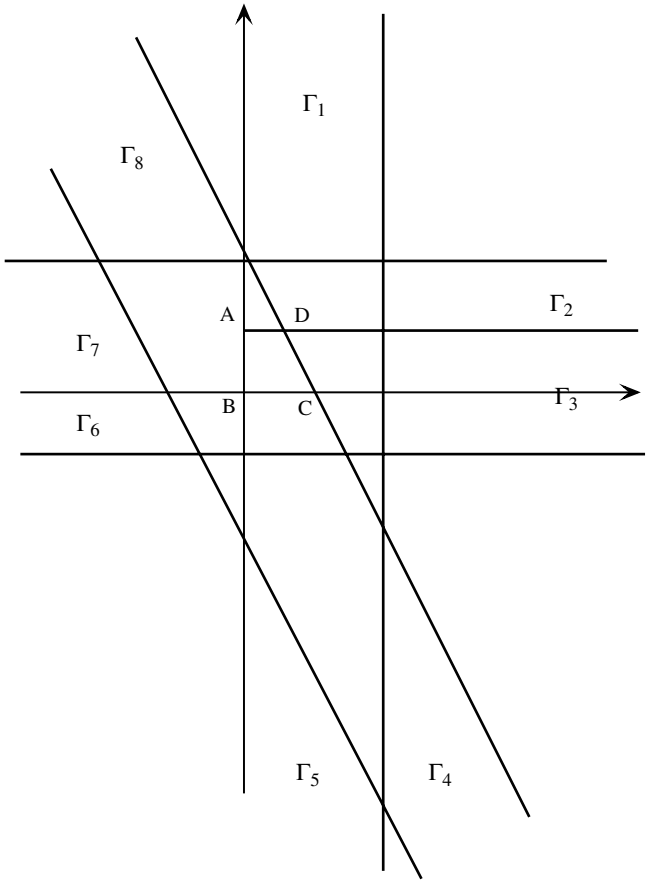


FIGURE 4. Half-strip regions.

According to the previous discussions, we know that under T, p first translates by $\xi_1 = (2, -2)$ until entering Γ_2 or Γ_3. More precisely, if $p \in S_{2n}^+ = \{(x, y) \mid 0 < x < 2, 2n + 1 < y < 2n + 2\}$, then $T^n(p) \in \{(x, y) \mid 2n < x < 2n + 2, 1 < y < 2\} \subset \Gamma_2$; if $p \in S_{2n}^- = \{(x, y) \mid 0 < x < 2, 2n < y < 2n + 1\}$, then $T^n(p) \in \{(x, y) \mid 2n < x < 2n + 2, 0 < y < 1\} \subset \Gamma_3$. The case that $p \in \{(x, y) \mid 0 < x < 2, y = 2n + 1\}$ is omitted for the reason given in Remark 2.1. We now treat these two cases separately.

3.1. *The case of* $p \in S_{2n}^-$. In this case, we have $T^n(p) \in \{(x, y) \mid 2n < x < 2n + 2, 0 < y < 1\} \subset \Gamma_3$. Hence, it translates repeatedly by $\xi_3 = (-2\alpha, -2)$ until entering $\Gamma_4 = \{(x, y) \mid -1 < x + (1 - \alpha)y < 1, x > 0\}$. Note that each translation by ξ_3 makes the value of $x + (1 - \alpha)y$ reduce by 2, so for any $p_0 = (x_0, y_0) \in \Gamma_3$, it enters Γ_4 after exactly $[(x_0 + (1 - \alpha)y_0 + 1)/2]$ of ξ_3-translations. Here $[x]$ denotes the maximum integer not larger than x for any $x \in \mathbf{R}$. To calculate this number we divide S_{2n}^- into two components, namely

$$P_{2n}^- = \{(x, y) \mid 2n < y < 2n + 1, 0 < x < 1 - (1 - \alpha)(y - 2n)\}$$

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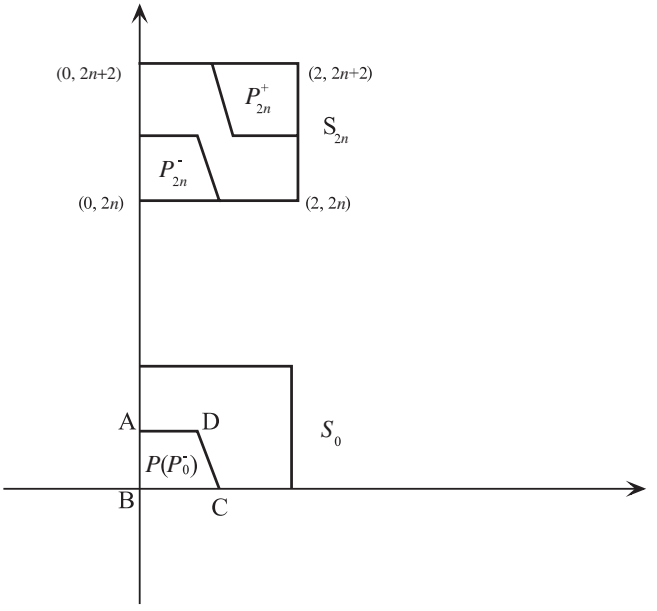


FIGURE 5. Divide every square into four trapezoids.

and

$$\tilde{P}^{-}_{2n} = \{(x, y) \mid 2n < y < 2n + 1,\ 1 - (1 - \alpha)(y - 2n) < x < 2\}.$$

If $p = (x, y) \in P^{-}_{2n}$, then

$$0 < x + (1 - \alpha)(y - 2n) < 1,$$

so for $T^n(p) = (x + 2n, y - 2n)$, after exactly

$$\left[\frac{x + 2n + (1 - \alpha)(y - 2n) + 1}{2}\right] = \left[\frac{x + (1 - \alpha)(y - 2n) + 1}{2}\right] + n = n$$

ξ_3-translations, it enters Γ_4.

If $p = (x, y) \in \tilde{P}^{-}_{2n}$, then

$$1 < x + (1 - \alpha)(y - 2n) < 2 + (1 - \alpha)(y - 2n) < 3,$$

so for $T^n(p) = (x + 2n, y - 2n)$, after exactly

$$\left[\frac{x + 2n + (1 - \alpha)(y - 2n) + 1}{2}\right] = \left[\frac{x + (1 - \alpha)(y - 2n) + 1}{2}\right] + n = n + 1$$

ξ_3-translations, it enters Γ_4.

Once the point enters Γ_4, it begins translating by $\xi_4 = (-2, 0)$ repeatedly until entering $\Gamma_5 = \{(x, y) \mid 0 < x < 2,\ y < 0\}$. We do not calculate the exact number of ξ_4-translations, but introduce a map to represent this procedure.

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Define $\Omega : \mathbf{R}^2 \to \mathbf{R}^2$ as

$$\Omega((x, y)) = \left(x - 2\left[\frac{x}{2}\right], y\right) \quad \text{for all } (x, y) \in \mathbf{R}^2.$$

Thus, no matter how many ξ_4-translations are performed, Ω gives the map mapping any point in Γ_4 into its arrival position in Γ_5.

If $p = (x, y) \in P_{2n}^-$, then before entering Γ_5, it experiences n ξ_1-translations, n ξ_3-translations and, finally, the action of Ω. So the resulting point in Γ_5 is

$$\Omega(T^{2n}(p)) = \Omega((x + 2n - 2n\alpha, y - 4n)) = \Omega((x - 2n\alpha, y - 4n)).$$

(Obviously $\Omega((x + 2, y)) = \Omega((x, y))$ for any (x, y).) Since $p = (x, y)$ satisfies $2n < y < 2n + 1$ and $-2n < y - 4n < -2n + 1$, $\Omega(T^{2n}(p))$ lies in S_{-2n}^-.

If $p = (x, y) \in \tilde{P}_{2n}^-$, then before entering Γ_5, it experiences n ξ_1-translations, $n + 1$ ξ_3-translations and, finally, the action of Ω. So the resulting point in Γ_5 is

$$\begin{aligned}\Omega(T^{2n+1}(p)) &= \Omega((x + 2n - (2n + 2)\alpha, y - 4n - 2)) \\ &= \Omega((x - (2n + 2)\alpha, y - 4n - 2)).\end{aligned}$$

As $p = (x, y)$ satisfies $2n < y < 2n + 1$ and $-2n - 2 < y - 4n - 2 < -2n - 1$, so $\Omega(T^{2n+1}(p))$ lies in S_{-2n-2}^-.

On the other hand, consider an arbitrary $p' = (x', y') \in S_{-2n'}^- = \{(x, y) \mid 0 < x < 2, -2n' < y < -2n' + 1\}$ $(n' \in \mathbf{N})$ which is far from P. Under iterations of T, p' translates by $\xi_5 = (-2, 2)$ repeatedly until entering $\Gamma_6 = \{(x, y) \mid x < 0, -1 < y < 0\}$ or $\Gamma_7 = \{(x, y) \mid x < 0, 0 < y < 2\}$. Since $-2n' < y' < -2n' + 1$, it must directly enter Γ_7 after exactly n' ξ_5-translations, without entering Γ_6. Then it translates by $\xi_7 = (2\alpha, 2)$ repeatedly until entering $\Gamma_8 = \{(x, y) \mid -1 < x + (1 - \alpha)y < 1, x < 0\}$. Each ξ_7-translation makes the value of $x + (1 - \alpha)y$ increase by 2, so for any $(x_0, y_0) \in \Gamma_7$, after exactly $[(-x_0 - (1 - \alpha)y_0 + 1)/2]$ ξ_7-translations, it enters Γ_8. To calculate this number we also need to consider $P_{-2n'}^-$ and $\tilde{P}_{-2n'}^-$ separately. Recall that

$$\begin{aligned}P_{-2n'}^- &= \{(x, y) \mid -2n' < y < -2n' + 1, 0 < x < 1 - (1 - \alpha)(y + 2n')\}, \\ \tilde{P}_{-2n'}^- &= \{(x, y) \mid -2n' < y < -2n' + 1, 1 - (1 - \alpha)(y + 2n') < x < 2\}.\end{aligned}$$

If $p' = (x', y') \in P_{-2n'}^-$, then $0 < x' + (1 - \alpha)(y' + 2n') < 1$, so for $T^{n'}(p') = (x' - 2n', y' + 2n')$, after exactly

$$\left[\frac{-(x' - 2n') - (1 - \alpha)(y' + 2n') + 1}{2}\right] = \left[\frac{-x' - (1 - \alpha)(y' + 2n') + 1}{2}\right] + n' = n'$$

ξ_7-translations, it enters Γ_8.

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If $p' = (x', y') \in \tilde{P}^-_{-2n'}$, then $1 < x' + (1-\alpha)(y' + 2n') < 2 + (1-\alpha)(y' + 2n') < 3$, so for $T^{n'}(p') = (x' - 2n', y' + 2n')$, after exactly

$$\left[\frac{-(x' - 2n') - (1-\alpha)(y' + 2n') + 1}{2}\right] = \left[\frac{-x' - (1-\alpha)(y' + 2n') + 1}{2}\right] + n' = n' - 1$$

ξ_7-translations, it enters Γ_8.

Obviously Ω just defined gives the map mapping of any point in Γ_8 into its arrival point in Γ_1.

Therefore, if $p' = (x', y') \in P^-_{-2n'}$, then before entering Γ_1, it experiences n' ξ_5-translations, n' ξ_7-translations and the action of Ω. So the resulting point in Γ_1 is

$$\Omega(T^{2n'}(p')) = \Omega((x' - 2n' + 2n'\alpha, y' + 4n')) = \Omega((x' + 2n'\alpha, y' + 4n')).$$

Since $-2n' < y < -2n' + 1$, $2n' < y' + 4n' < 2n' + 1$, so $\Omega(T^{2n'}(p'))$ must lie in $S^-_{2n'}$.

If $p' = (x', y') \in \tilde{P}^-_{-2n'}$, then before entering Γ_1, it experiences n' ξ_5-translations, $n' - 1$ ξ_7-translations and, finally, the action of Ω. So the resulting point in Γ_1 is

$$\begin{aligned}\Omega(T^{2n'-1}(p')) &= \Omega((x' - 2n' + (2n' - 2)\alpha, y' + 4n' - 2)) \\ &= \Omega((x' + (2n' - 2)\alpha, y' + 4n' - 2)).\end{aligned}$$

Since $-2n' < y' < -2n' + 1$, $2n' - 2 < y' + 4n' - 2 < 2n' - 1$, so $\Omega(T^{2n'-1}(p))$ must lie in $S^-_{2n'-2}$.

Now combine the above two processes, namely, point $p = (x, y) \in S^-_{2n}$ ($n \in \mathbf{N}$) moving to point $p' = (x', y') \in \Gamma_5$ (indeed, S^-_{-2n} or S^-_{-2n-2}), and p' moving back into Γ_1. Therefore, it is necessary to examine whether p' lies in P^-_{-2n} (P^-_{-2n-2}) or $\tilde{P}^-_{-2n}$ ($\tilde{P}^-_{-2n-2}$) if $p' \in S^-_{-2n}$ (S^-_{-2n-2}). To do this, let

$$\begin{aligned}R^-_{2n} &= \{(x, y) \in S^-_{2n} \mid \Omega((x - 2n\alpha, y)) \in P^-_{2n}\}, \\ \tilde{R}^-_{2n} &= \{(x, y) \in S^-_{2n} \mid \Omega((x - 2n\alpha, y)) \in \tilde{P}^-_{2n}\}, \\ Q^-_{2n} &= \{(x, y) \in S^-_{2n} \mid \Omega((x - (2n + 2)\alpha, y)) \in P^-_{2n}\}, \\ \tilde{Q}^-_{2n} &= \{(x, y) \in S^-_{2n} \mid \Omega((x - (2n + 2)\alpha, y)) \in \tilde{P}^-_{2n}\}.\end{aligned}$$

If $p = (x, y) \in P^-_{2n} \cap R^-_{2n}$, then $(x', y') = \Omega((x - 2n\alpha, y - 4n))$ is just the point obtained by translating $\Omega((x - 2n\alpha, y))$ by vector $(0, -4n)$. Noting that after translating by vector $(0, -4n)$, P^-_{2n} ($\tilde{P}^-_{2n}$) coincides with P^-_{-2n} ($\tilde{P}^-_{-2n}$), we have $(x', y') \in P^-_{-2n}$, because $\Omega((x - 2n\alpha, y)) \in P^-_{2n}$. Similarly, if $p \in P^-_{2n} \cap \tilde{R}^-_{2n}$, then $p' \in \tilde{P}^-_{-2n}$; if $p \in \tilde{P}^-_{2n} \cap Q^-_{2n}$, then $p' \in P^-_{-2n-2}$; if $p \in \tilde{P}^-_{2n} \cap \tilde{Q}^-_{2n}$, then $p' \in \tilde{P}^-_{-2n-2}$ (the cases when the orbits are finite are omitted).

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Now we give the expression of the return map Ψ in these four cases separately.

(1) We have $p=(x,\,y)\in P_{2n}^{-}\cap R_{2n}^{-}$, then $(x',\,y')=\Omega((x-2n\alpha,\,y-4n))\in P_{-2n}^{-}$, so

$$\Psi(p)=\Omega((x'+2n\alpha,\,y'+4n))=(x,\,y),$$

which yields that all points in $P_{2n}^{-}\cap R_{2n}^{-}$ are invariant under Ψ.

(2) We have $p=(x,\,y)\in P_{2n}^{-}\cap \tilde{R}_{2n}^{-}$, then $(x',\,y')=\Omega((x-2n\alpha,\,y-4n))\in \tilde{P}_{-2n}^{-}$, so

$$\Psi(p)=\Omega((x'+(2n-2)\alpha,\,y'+4n-2))=\Omega((x-2\alpha,\,y-2)).$$

(3) We have $p=(x,\,y)\in\tilde{P}_{2n}^{-}\cap Q_{2n}^{-}$, then $(x',\,y')=\Omega((x-(2n+2)\alpha,\,y-4n-2))\in P_{-2n-2}^{-}$, so

$$\Psi(p)=\Omega((x'+2(n+1)\alpha,\,y'+4(n+1)))=(x,\,y+2).$$

(4) We have $p=(x,\,y)\in\tilde{P}_{2n}^{-}\cap \tilde{Q}_{2n}^{-}$, then $(x',\,y')=\Omega((x-(2n+2)\alpha,\,y-4n-2))\in \tilde{P}_{-2n-2}^{-}$, so

$$\Psi(p)=\Omega((x'+(2(n+1)-2)\alpha,\,y'+4(n+1)-2))=\Omega((x-2\alpha,\,y)).$$

In the conclusion, S_{2n}^{-} is divided into four components, $P_{2n}^{-}\cap R_{2n}^{-}$, $P_{2n}^{-}\cap \tilde{R}_{2n}^{-}$, $\tilde{P}_{2n}^{-}\cap Q_{2n}^{-}$ and $\tilde{P}_{2n}^{-}\cap \tilde{Q}_{2n}^{-}$, for points in these four regions, the return map Ψ is just a translation by vector $(0,\,0)$, $(-2\alpha,\,-2)$, $(0,\,2)$, $(-2\alpha,\,0)$, respectively, combined with the action of Ω.

3.2. *The case of* $p\in S_{2n}^{+}$. In this case, we have $T^{n}(p)\in\{(x,\,y)\mid 2n<x<2n+2,\ 1<y<2\}\subset\Gamma_2$, hence it translates by $\xi_2=(0,\,-2)$ once to enter Γ_3, so $T^{n+1}(p)\in\{(x,\,y)\mid 2n<x<2n+2,\ -1<y<0\}$. It then translates by $\Gamma_3=(-2\alpha,\,-2)$ repeatedly until entering $\Gamma_4=\{(x,\,y)\mid -1<x+(1-\alpha)y<1,\ x>0\}$. Similarly we need to calculate the number $[(x_0+(1-\alpha)y_0+1)/2]$ for any point $p_0=(x_0,\,y_0)\in\{(x,\,y)\mid 2n<x<2n+2,\ -1<y<0\}$. Divide S_{2n}^{+} into two components, namely

$$P_{2n}^{+}=\{(x,\,y)\mid 2n+1<y<2n+2,\ 3-2\alpha-(1-\alpha)(y-2n)<x<2\},$$
$$\tilde{P}_{2n}^{+}=\{(x,\,y)\mid 2n+1<y<2n+2,\ 0<x<3-2\alpha-(1-\alpha)(y-2n)\}.$$

If $p=(x,\,y)\in P_{2n}^{+}$, then

$$1<x+(1-\alpha)(y-2n-2)<2+(1-\alpha)(y-2n-2)<2,$$

so for $T^{n+1}(p)=(x+2n,\,y-2n-2)$, after exactly

$$\left[\frac{x+2n+(1-\alpha)(y-2n-2)+1}{2}\right]=\left[\frac{x+(1-\alpha)(y-2n-2)+1}{2}\right]+n=n+1$$

ξ_3-translations, it enters Γ_4.

If $p=(x,\,y)\in\tilde{P}_{2n}^{+}$, then

$$-1<(1-\alpha)(y-2n-2)<x+(1-\alpha)(y-2n-2)<1,$$

so for $T^{n+1}(p)=(x+2n,\,y-2n-2)$, after exactly

$$\left[\frac{x+2n+(1-\alpha)(y-2n-2)+1}{2}\right]=\left[\frac{x+(1-\alpha)(y-2n-2)+1}{2}\right]+n=n$$

ξ_3-translations, it enters Γ_4.

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Therefore, if $p = (x, y) \in P^+_{2n}$, then before entering Γ_5, it experiences n ξ_1-translations, one ξ_2-translation, $n+1$ ξ_3-translations and, finally, the action of Ω. So the resulting point is

$$\begin{aligned}\Omega(T^{2n+2}(p)) &= \Omega((x + 2n - 2(n+1)\alpha,\ y - 2n - 2 - 2(n+1))) \\ &= \Omega((x - (2n+2)\alpha,\ y - 4n - 4)).\end{aligned}$$

Since $p = (x, y)$ satisfies $2n + 1 < y < 2n + 2$, $-2n - 3 < y - 4n - 4 < -2n - 2$, so $\Omega(T^{2n+2}(p))$ must lie in S^+_{-2n-4}.

If $p = (x, y) \in \tilde{P}^+_{2n}$, then before entering Γ_5, it experiences n ξ_1-translations, one ξ_2-translation, n ξ_3-translations and, finally, the action of Ω. So the resulting point is

$$\Omega(T^{2n+1}(p)) = \Omega((x + 2n - 2n\alpha,\ y - 2n - 2 - 2n)) = \Omega((x - 2n\alpha,\ y - 4n - 2)).$$

Since $p = (x, y)$ satisfies $2n + 1 < y < 2n + 2$, $-2n - 1 < y - 4n - 2 < -2n$, so $\Omega(T^{2n+1}(p))$ must lie in S^+_{-2n-2}.

On the other hand, consider an arbitrary $p' = (x', y') \in S^+_{-2n'} = \{(x, y) \mid 0 < x < 2, -2n' + 1 < y < -2n' + 2\}$ $(n' \in \mathbf{N})$ which is far from P. Under iterations of T, p' translates by $\xi_5 = (-2, 2)$ repeatedly until entering $\Gamma_6 = \{(x, y) \mid x < 0, -1 < y < 0\}$ or $\Gamma_7 = \{(x, y) \mid x < 0, 0 < y < 2\}$. Since $-2n' + 1 < y' < -2n' + 2$, it must enter Γ_6 after exactly $n' - 1$ ξ_5-translations. After this it translates by $\xi_6 = (-2(1-\alpha), 2)$ once to enter Γ_7 and then translates by $\xi_7 = (2\alpha, 2)$ repeatedly until entering $\Gamma_8 = \{(x, y) \mid -1 < x + (1-\alpha)y < 1, x < 0\}$. As we have stated, for any $(x_0, y_0) \in \Gamma_7$, after exactly $[(-x_0 - (1-\alpha)y_0 + 1)/2]$ ξ_6-translations, it enters Γ_8. Similarly, to calculate this number we need to consider $P^+_{-2n'}$ and $\tilde{P}^+_{-2n'}$ separately. Recall that

$$\begin{aligned}P^+_{-2n'} &= \{(x, y) \mid -2n' + 1 < y < -2n' + 2,\ 3 - 2\alpha - (1-\alpha)(y + 2n') < x < 2\}, \\ \tilde{P}^+_{-2n'} &= \{(x, y) \mid -2n' + 1 < y < -2n' + 2,\ 0 < x < 3 - 2\alpha - (1-\alpha)(y + 2n')\}.\end{aligned}$$

If $p' = (x', y') \in P^+_{-2n'}$, then $3 < x' + 2\alpha + (1-\alpha)(y' + 2n') < 2 + 2\alpha + (1-\alpha)(y' + 2n') < 4$, so for $T^{n'}(p') = (x' - 2(n'-1) - 2(1-\alpha),\ y' + 2(n'-1) + 2)$, after exactly

$$\begin{aligned}&\left[\frac{-(x' - 2(n'-1) - 2(1-\alpha)) - (1-\alpha)(y' + 2(n'-1) + 2) + 1}{2}\right] \\ &= \left[\frac{-x' - 2\alpha - (1-\alpha)(y' + 2n') + 1}{2}\right] + n' \\ &= n' - 2\end{aligned}$$

ξ_7-translations, it enters Γ_8.

If $p' = (x', y') \in \tilde{P}^+_{-2n'}$, then $1 < 1 + \alpha < 2\alpha + (1-\alpha)(y' + 2n') < x' + 2\alpha + (1-\alpha)(y' + 2n') < 3$, so for $T^{n'}(p') = (x' - 2(n'-1) - 2(1-\alpha),\ y' + 2(n'-1) + 2)$, after exactly

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$$\left[\frac{-(x'-2(n'-1)-2(1-\alpha))-(1-\alpha)(y'+2(n'-1)+2)+1}{2}\right]$$
$$=\left[\frac{-x'-2\alpha-(1-\alpha)(y'+2n')+1}{2}\right]+n'$$
$$=n'-1$$

ξ_7-translations, it enters Γ_8.

Therefore, if $p'=(x',\,y')\in P^+_{-2n'}$, then before entering Γ_1, it experiences $n'-1$ ξ_5-translations, one ξ_6-translation, $n'-2$ ξ_7-translations and, finally, the action of Ω. So the resulting point in Γ_1 is

$$\begin{aligned}\Omega(T^{2n'-1}(p')) &= \Omega((x'-2(n'-1)-2(1-\alpha)+2(n'-2)\alpha,\, y'+2(n'-1)\\ &\quad +2+2(n'-2)))\\ &= \Omega((x'-2n'+(2n'-2)\alpha,\, y'+4n'-4))\\ &= \Omega((x'+(2n'-2)\alpha,\, y'+4n'-4)).\end{aligned}$$

Since $-2n'+1<y'<-2n'+2,\ 2n'-3<y'+4n'-4<2n'-2,$ so $\Omega(T^{2n'-1}(p'))$ must lie in $S^+_{2n'-4}$.

If $p'=(x',\,y')\in\tilde P^+_{-2n'}$, then before entering Γ_1, it experiences $n'-1$ ξ_5-translations, one ξ_6-translation, $n'-1$ ξ_7-translations and, finally, the action of Ω. So the resulting point in Γ_1 is

$$\begin{aligned}\Omega(T^{2n'}(p')) &= \Omega((x'-2(n'-1)-2(1-\alpha)+2(n'-1)\alpha,\, y'+2(n'-1)\\ &\quad +2+2(n'-1)))\\ &= \Omega((x'-2n'+2n'\alpha,\, y'+4n'-2))\\ &= \Omega((x'+2n'\alpha,\, y'+4n'-2)).\end{aligned}$$

Since $-2n'+1<y'<-2n'+2,\ 2n'-1<y'+4n'-2<2n'$, so $\Omega(T^{2n'}(p))$ must lie in $S^+_{2n'-2}$.

Now combine the above two processes, namely, point $p=(x,\,y)\in S^+_{2n}$ $(n\in\mathbf{N})$ moving to point $p'=(x',\,y')\in\Gamma_5$ (indeed, S^+_{-2n-4} or S^+_{-2n-2}), and p' moving back into Γ_1. Therefore, it is necessary to examine whether p' lies in P^+_{-2n-4} (P^+_{-2n-4}) or $\tilde P^+_{-2n-2}$ $(\tilde P^+_{-2n-2})$ if $p'\in S^+_{-2n-4}$ (S^+_{-2n-2}). To do this, let

$$\begin{aligned}R^+_{2n} &= \{(x,\,y)\in S^+_{2n}\mid \Omega((x-(2n+2)\alpha,\, y))\in P^+_{2n}\},\\ \tilde R^+_{2n} &= \{(x,\,y)\in S^+_{2n}\mid \Omega((x-(2n+2)\alpha,\, y))\in \tilde P^+_{2n}\},\\ Q^+_{2n} &= \{(x,\,y)\in S^+_{2n}\mid \Omega((x-2n\alpha,\, y))\in P^+_{2n}\},\\ \tilde Q^+_{2n} &= \{(x,\,y)\in S^+_{2n}\mid \Omega((x-2n\alpha,\, y))\in \tilde P^+_{2n}\}.\end{aligned}$$

If $p=(x,\,y)\in P^+_{2n}\cap R^+_{2n}$, then $(x',\,y')=\Omega((x-(2n+2)\alpha,\, y-4n-4))$ is just the point obtained by translating $\Omega((x-(2n+2)\alpha,\, y))$ by vector $(0,\,-4n-4)$. Noting that after translating by vector $(0,\,-4n-4)$, P^+_{2n} $(\tilde P^+_{2n})$ coincides with P^+_{-2n-4} $(\tilde P^+_{-2n-4})$, we have $(x',\,y')\in P^+_{-2n-4}$, because $\Omega((x-(2n+2)\alpha,\, y))\in P^+_{2n}$. Similarly, if $p\in P^+_{2n}\cap \tilde R^+_{2n}$, then $p'\in\tilde P^+_{-2n-4}$; if $p\in\tilde P^+_{2n}\cap Q^+_{2n}$, then $p'\in P^+_{-2n-2}$; if $p\in\tilde P^+_{2n}\cap\tilde Q^+_{2n}$, then $p'\in\tilde P^+_{-2n-2}$ (the cases when the orbits are finite are omitted).

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Now we give the expression of the return map Ψ for these four cases separately.

(1) We have $p=(x, y)\in P_{2n}^{+}\cap R_{2n}^{+}$, then $(x', y')=\Omega((x-(2n+2)\alpha, y-4n-4))\in P_{-2n-4}^{+}$, so

$$\Psi(p)=\Omega((x'+(2(n+2)-2)\alpha, y'+4(n+2)-4))=\left(x-2\left[\frac{x}{2}\right], y\right)=(x, y).$$

So all points in $P_{2n}^{+}\cap R_{2n}^{+}$ are invariant under Ψ.

(2) We have $p=(x, y)\in P_{2n}^{+}\cap \tilde{R}_{2n}^{+}$, then $(x', y')=\Omega((x-(2n+2)\alpha, y-4n-4))\in \tilde{P}_{-2n-4}^{+}$, so

$$\Psi(p)=\Omega((x'+2(n+2)\alpha, y'+4(n+2)-2))=\left(x+2\alpha-2\left[\frac{x+2\alpha}{2}\right], y+2\right)$$
$$=\Omega((x+2\alpha, y+2)).$$

(3) We have $p=(x, y)\in \tilde{P}_{2n}^{+}\cap Q_{2n}^{+}$, then $(x', y')=\Omega((x-2n\alpha, y-4n-2))\in P_{-2n-2}^{+}$, so

$$\Psi(p)=\Omega((x'+(2(n+1)-2)\alpha, y'+4(n+1)-4))$$
$$=\left(x-2\left[\frac{x}{2}\right], y-2\right)=(x, y-2).$$

(4) We have $p=(x, y)\in \tilde{P}_{2n}^{+}\cap \tilde{Q}_{2n}^{+}$, then $(x', y')=\Omega((x-2n\alpha, y-4n-2))\in \tilde{P}_{-2n-2}^{+}$, so

$$\Psi(p)=\Omega((x'+2(n+1)\alpha, y'+4(n+1)-2))=\left(x+2\alpha-2\left[\frac{x+2\alpha}{2}\right], y\right)$$
$$=\Omega((x+2\alpha, y)).$$

In other words, if we divide S_{2n}^{+} into four components, $P_{2n}^{+}\cap R_{2n}^{+}$, $P_{2n}^{+}\cap \tilde{R}_{2n}^{+}$, $\tilde{P}_{2n}^{+}\cap Q_{2n}^{+}$ and $\tilde{P}_{2n}^{+}\cap \tilde{Q}_{2n}^{+}$, then for points in these four regions, the return map Ψ is just a translation by a vector $(0, 0)$, $(2\alpha, 2)$, $(0, -2)$, $(2\alpha, 0)$, respectively, combined with the action of Ω. Note that these four vectors have nothing to do with n either, and are the opposite of those corresponding to S_{2n}^{-}.

Concluding the above discussion we have the following.

PROPOSITION 3.1. *There exists $N_0\in\mathbf{N}$ such that for any integer $n>N_0$, the return map Ψ of P has the following expression,*

$$\Psi((x, y))=\begin{cases}(x, y), & (x, y)\in P_{2n}^{-}\cap R_{2n}^{-},\\ \Omega((x-2\alpha, y-2)), & (x, y)\in P_{2n}^{-}\cap \tilde{R}_{2n}^{-},\\ (x, y+2), & (x, y)\in \tilde{P}_{2n}^{-}\cap Q_{2n}^{-},\\ \Omega((x-2\alpha, y)), & (x, y)\in \tilde{P}_{2n}^{-}\cap \tilde{Q}_{2n}^{-},\\ (x, y), & (x, y)\in P_{2n}^{+}\cap R_{2n}^{+},\\ \Omega((x+2\alpha, y+2)), & (x, y)\in P_{2n}^{+}\cap \tilde{R}_{2n}^{+},\\ (x, y-2), & (x, y)\in \tilde{P}_{2n}^{+}\cap Q_{2n}^{+},\\ \Omega((x+2\alpha, y)), & (x, y)\in \tilde{P}_{2n}^{+}\cap \tilde{Q}_{2n}^{+}.\end{cases}$$

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4. *Reconsideration of the problem*

In this section we consider the orbit of Ψ instead of that of T, because if all Ψ-orbits of P are bounded then so are all T-orbits of P. Throughout this section, let N_0 be the integer defined in Proposition 3.1, and let $p_0 = (x_0, y_0) \in S_{2n_0}$ $(n_0 > N_0)$. Then Proposition 3.1 gives the expression of $\Psi(p_0)$ if it is well defined. Note that this expression depends on the domain in which the point p_0 belongs.

In the following, we only study the case of $p_0 \in S^-_{2n_0}$. The case of $p_0 \in S^+_{2n_0}$ can be studied similarly.

For $p_0 \in S^-_{2n_0}$, we have $\Psi^i(p_0) \in S^-_{2n_i}$ for every i.

If $p_0 \in P^-_{2n_0} \cap R^-_{2n_0}$, then p_0 is a fixed point of Ψ, so the boundedness of orbit of p_0 is obvious. Hence it suffices to study the other three possibilities, that is, $p_0 \in P^-_{2n_0} \cap \tilde{R}^-_{2n_0}$, $p_0 \in \tilde{P}^-_{2n_0} \cap Q^-_{2n_0}$ and $p_0 \in \tilde{P}^-_{2n_0} \cap \tilde{Q}^-_{2n_0}$. We may distinguish these three cases by $p_0 \in P^-_{2n_0}$ or $p_0 \in Q^-_{2n_0}$.

Iterations of Ψ obey the rules stated in Proposition 3.1, whenever Ψ is well defined. It is however possible, after some iterations, for Ψ to become undefined or for $\Psi^i(p_0)$ to become close to P for some i. The former case can be easily eliminated by assuming that $p_0 \in \mathbf{F}_\infty(P)$, while the latter case is dealt with in the proof of Theorem 1. Actually, we have the following.

PROPOSITION 4.1. *Let $p_0 \in S^-_{2n_0}$ $(n_0 > N_0)$ be a point belonging to $\mathbf{F}_\infty(P)$. Then there exists $M > 0$, such that for any $N \in \mathbf{N} \cup \{+\infty\}$ satisfying $n_0, n_1, \ldots, n_N > N_0$, we have $y_k < M$ $(k = 0, 1, \ldots, N)$. Here y_k denotes the y-coordinate of $\Psi^k(p_0)$.*

Before proving this proposition, we reinvestigate the return map Ψ first.

(a) If $\Psi^i(p_0) = (x_i, y_i) \in P^-_{2n_i}$, then $\Psi^{i+1}(p_0) = \Omega((x_i - 2\alpha, y_i - 2)) \in S^-_{2n_i-2}$ (the case of fixed point has been excluded). It is not difficult to see that for any n,

$$P^-_{2n-2} = \{(x, y-2) \mid (x, y) \in P^-_{2n}\},$$
$$Q^-_{2n-2} = \{\Omega((x-2\alpha, y-2)) \mid (x, y) \in Q^-_{2n}\}.$$

For simplification, we regard $(x+2, y)$ and (x, y) as the same point for every $(x, y) \in \mathbf{R}^2$. Then P^-_{2n} and Q^-_{2n} are both trapezoids for every n under this consideration. Imagine that a point p at the position of $\Psi^i(p_0)$ translates by $(-2\alpha, -2)$ to the position of $\Psi^{i+1}(p_0)$. Also imagine that a trapezoid Z_1 at the position of $P^-_{2n_i}$ translates by $(0, -2)$ to the position of $P^-_{2n_i-2}$, and another trapezoid Z_2 at the position of $Q^-_{2n_i}$ translates by $(-2\alpha, -2)$ to the position of $Q^-_{2n_i-2}$. Then under Ψ, p moves by $(-2\alpha, 0)$ relative to Z_1 and still remains relative to Z_2.

(b) If $\Psi^i(p_0) = (x_i, y_i) \notin P^-_{2n_i}$ and $\Psi^i(p_0) \in Q^-_{2n_i}$, then $\Psi^{i+1}(p_0) = (x_i, y_i + 2) \in S^-_{2n+2}$ (excluding the case when $\Psi^{i+1}(p_0)$ is undefined). Similarly to (a), we regard $(x+2, y)$ and (x, y) as the same point for every $(x, y) \in \mathbf{R}^2$. In addition, imagine that point p translates by $(0, 2)$ from $\Psi^i(p_0)$ to $\Psi^{i+1}(p_0)$, trapezoid Z_1 translates by $(0, 2)$ from $P^-_{2n_i}$ to $P^-_{2n_i+2}$ and trapezoid Z_2 translates by $(2\alpha, 2)$ from $Q^-_{2n_i}$ to $Q^-_{2n_i+2}$. So under Ψ, p still remains relative to Z_1 and moves by $(-2\alpha, 0)$ relative to Z_2.

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(c) If $\Psi^i(p_0) = (x_i, y_i) \notin P^-_{2n_i}$ and $\Psi^i(p_0) \notin Q^-_{2n_i}$, then $\Psi^{i+1}(p_0) = \Omega((x_i - 2\alpha, y_i)) \in S^-_{2n_i}$ (omitting the case when $\Psi^{i+1}(p_0)$ is undefined). Similarly to (a) and (b), we regard $(x + 2, y)$ and (x, y) as the same point for every $(x, y) \in \mathbf{R}^2$. In addition, let point p translate by $(-2\alpha, 0)$ from $\Psi^i(p_0)$ to $\Psi^{i+1}(p_0)$, with Z_1 and Z_2 not moving since p still lies in $S^-_{2n_i}$. So under Ψ, p moves by $(-2\alpha, 0)$ relative to both Z_1 and Z_2.

Clearly, if initially we put a point p, a trapezoid Z_1 and another trapezoid Z_2 at the position of p_0, $P^-_{2n_0}$ and $Q^-_{2n_0}$, respectively, and let them move as above, e.g. at each step, p translates from $\Psi^i(p_0)$ to $\Psi^{i+1}(p_0)$, Z_1 translates from $P^-_{2n_i}$ to $P^-_{2n_{i+1}}$ and Z_2 translates from $Q^-_{2n_i}$ to $Q^-_{2n_{i+1}}$, for any i, then the next move always depends on the relative position of p, Z_1 and Z_2 at that moment. Therefore, it is sufficient to investigate the relative motion among p, Z_1 and Z_2. In each case, the relative motion of the three objects is along the horizontal direction. Thus, we can use line segments I and II to take the place of Z_1 and Z_2. Here I (II) is the intersection of Z_1 (Z_2) with the horizontal line passing through p, and translates by the same vector to that of Z_1 (Z_2) at each step. All of these considerations are available while regarding $(x + 2, y)$ and (x, y) as the same point for every $(x, y) \in \mathbf{R}^2$.

Consider a circle with perimeter 2, endowed with a one-dimensional coordinate system, counterclockwise direction being positive. For convenience, we let the positive direction on the circle correspond to the negative horizontal direction in $\mathbf{R}^2$. Place p, I and II onto the circle according to their initial horizontal relative position, and set the rest point of the circle at the initial position of p. Then the relative motion among these three objects on the circle is as follows (rotating by a positive number means rotating counterclockwise, while rotating by a negative number means rotating clockwise).

(a′) If $p \in I$, then at the next step p rotates by 2α relative to I, while the relative position between p and II remains invariant. Instead of this we use the equivalent relative motion that I rotates by -2α with p and II fixed in the following.

(b′) If $p \notin I$ and $p \in II$, then at the next step p rotates by 2α relative to II, while the relative position between p and I remains invariant. Instead of this we use the equivalent relative motion that II rotates by -2α with p and I fixed in the following.

(c′) If $p \notin I$ and $p \notin II$, then at the next step p rotates by 2α relative to both I and II. So in this case we let p rotate by 2α with I and II fixed in the following.

Notice that case (a) implies that the y-coordinate of $p \in \mathbf{R}^2$ (not the point on the circle) decreases by 2 at the next move, while case (b) implies that the y-coordinate of p increases by 2 at the next move and case (c) implies that the y-coordinate of p remains unchanged at the next move. Moreover, cases (a′), (b′) and (c′) correspond to the cases (a), (b) and (c), respectively, stated previously. Therefore, in order to prove Proposition 4.1, it is sufficient to prove that for the prescribed motion on the circle, the differences between the times (a′) occurs and the times (b′) occurs are bounded forever. In fact, we have the following.

LEMMA 4.1. *For the previously described motion of p, I and II on the circle (e.g. the motion obeying* (a′)*,* (b′) *and* (c′) *while the initial relative position of p, I and II agrees with their initial horizontal relative position in* $\mathbf{R}^2$*), there exists* $M > 0$ *such that, for any* $N \in \mathbf{N}$*, within the N steps at the beginning, the times* (a′) *occurs (denoted by* $k_1(N)$*) and the times* (b′) *occurs (denoted by* $k_2(N)$*) satisfy* $|k_1(N) - k_2(N)| < M$.

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Obviously Lemma 4.1 implies Proposition 4.1. The proof of Lemma 4.1 is given at the end of this section.

Without loss of generality, we may assume that $0 < \alpha < \frac{1}{2}$. Indeed, if $\frac{1}{2} < \alpha < 1$, we can use $1 - \alpha$ to take the place of α and reverse the orientation on the circle. Take the initial position of p on the circle to be the point of zero, and let $0 < d < 1$ denote the length of I (equal to that of II). Use intervals $(a, a + d)$ and $(b, b + d)$ $(0 < a, b < 2)$ to represent the initial positions of I and II, respectively. First suppose that p does not lie in I or II initially. We use a trick changing the rotation rules slightly which does not influence the correctness of the result as follows.

At any time when p does not lie in $I \cup II$, p will then rotate by 2α repeatedly until entering $I \cup II$. When it first enters $I \cup II$, if it enters $I \setminus II$, then I rotates by -2α repeatedly until p leaps out of I; if $p \in II \setminus I$, then II rotates by -2α repeatedly until p leaps out of II; these are agreeable with the original rules. If when first entering $I \cup II$ from outside, p is in $I \cap II$, then consider the previous step when p is still outside $I \cup II$, and compare the positive directional distance from p to I with that from p to II at that time (here the positive directional distance from p to I (II) means the distance travelled when we walk from p counterclockwise until meeting I (II) for the first time). If the positive directional distance from p to I (II) is smaller, then after p entering $I \cup II$ let I (II) first rotate by -2α repeatedly until p leaves I (II) and then let II (I) rotate by -2α repeatedly until p leaves II (I). This makes a little difference, because owing to the original rules, as long as p enters $I \cup II$, I must rotate before II. However, since this new rule just changes the order of rotations, it will not have any influence on the boundedness of $|k_1(N) - k_2(N)|$; in fact, it changes each of $k_1(N)$ and $k_2(N)$ by at most $[d/2\alpha] + 1$, for any N.

Under the new rules, every time p is outside $I \cup II$, either I or II, whichever has the smaller positive directional distance from p, will rotate soon after p enters $I \cup II$, no matter whether p jumps into one or both of them. With this we have the following result.

LEMMA 4.2. *Let p, I and II move under the new rules, then there exists $M' > 0$ such that for any $N \in \mathbf{N}$, within the N steps at the beginning, the times that p passes through I (denoted by $w_1(N)$) and the times that p passes through II (denoted by $w_2(N)$) satisfy $|w_1(N) - w_2(N)| < M'$.*

Proof. Since initially p is at the origin and belongs to neither I nor II owing to our assumption, before p passes through I for the first time, I must have rotated by -2α exactly $[(a + d)/2\alpha] - [a/2\alpha]$ times, which is the number of integral multiples of 2α between a and $a + d$ (p cannot coincide with either end point of I at any time, otherwise the Ψ-orbit of p will be finite; therefore, neither a nor $a + d$ is an integral multiple of 2α). It should be noted that this number is not influenced by the movement of II, because when II moves, p and I are both fixed.

Now I is at the position of $(a - 2\alpha([(a + d)/2\alpha] - [a/2\alpha]), a + d - 2\alpha([(a + d)/2\alpha] - [a/2\alpha]))$ and does not contain p, so until the next time that p passes through I, I will

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additionally rotate by -2α exactly

$$\left[\frac{a+d-2\alpha([(a+d)/2\alpha]-[a/2\alpha])+2}{2\alpha}\right]-\left[\frac{a-2\alpha([(a+d)/2\alpha]-[a/2\alpha])+2}{2\alpha}\right]$$
$$=\left[\frac{a+d+2}{2\alpha}\right]-\left[\frac{a+2}{2\alpha}\right]$$

times. Generally, for the ith time that p passes through I, I rotates by -2α exactly $[(a+d+2(i-1))/2\alpha]-[(a+2(i-1))/2\alpha]$ times. Similarly, for the ith time that p passes through II, II rotates by -2α exactly $[(b+d+2(i-1))/2\alpha]-[(b+2(i-1))/2\alpha]$ times.

Let $a_0=a$, $b_0=b$ and let

$$a_i=a_{i-1}+2-2\alpha\left(\left[\frac{a_{i-1}+d}{2\alpha}\right]-\left[\frac{a_{i-1}}{2\alpha}\right]\right),$$
$$b_i=b_{i-1}+2-2\alpha\left(\left[\frac{b_{i-1}+d}{2\alpha}\right]-\left[\frac{b_{i-1}}{2\alpha}\right]\right),$$

for all $i\in\mathbf{N}$, then (a_i, a_i+d) $((b_i, b_i+d))$ is the position of I (II) after p passes through I (II) exactly i times. Moreover,

$$2>a_i-a_{i-1}\geq 2-2\alpha\left(\left[\frac{d}{2\alpha}\right]+1\right)\geq 2-d-2\alpha>0,$$
$$2>b_i-b_{i-1}\geq 2-2\alpha\left(\left[\frac{d}{2\alpha}\right]+1\right)\geq 2-d-2\alpha>0,$$

for all $i\in\mathbf{N}$. so the two series $\{a_i\}$ and $\{b_i\}$ also give the order among all passings under the new rules, as described in the following.

Initially, p is at the origin, I is at (a_0, a_0+d) and II is at (b_0, b_0+d), so p will first pass through I (II) if $a_0<b_0$ $(b_0<a_0)$. For example, take $a_0<b_0$, then p first passes through I and I moves to (a_1, a_1+d). Since $a_0<a_1<a_0+2$, so at the next step whether p will pass through I or II depends on the relationship between a_1 and b_0, if $a_1<b_0$ $(a_1>b_0)$ then I (II) will be passed through first. Then let the selected interval move to the new place and compare with the other, and so on.

In other words, p will pass through the intervals in $\{(a_i, a_i+d)\}\cup\{(b_i, b_i+d)\}$ according to the order among elements of the set $\{a_i\}\cup\{b_i\}$. Also, we have $0<a_i-a_{i-1}<2$, $0<b_i-b_{i-1}<2$, for all $i\in\mathbf{N}$ together with $|a_0-b_0|<2$. This means that, during the N steps at the beginning, $w_1(N)$ (the times that p passes through I) and $w_2(N)$ (the times that p passes through II) must satisfy

$$|a_{w_1(N)}-b_{w_2(N)}|<2.$$

On the other hand, for any $i\in\mathbf{N}$, we have

$$a_i=a_0+\sum_{k=1}^{i}\left(2-2\alpha\left(\left[\frac{a_0+d+2(k-1)}{2\alpha}\right]-\left[\frac{a_0+2(k-1)}{2\alpha}\right]\right)\right).$$

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In fact, this is obviously true for $i = 1$. Suppose that it holds for $i = j - 1$, then

$$\begin{aligned}\frac{a_{j-1}+d}{2\alpha} &= \frac{a_0+d+\sum_{k=1}^{j-1}(2-2\alpha([(a_0+d+2(k-1))/2\alpha]-[(a_0+2(k-1))/2\alpha]))}{2\alpha}\\ &= \frac{a_0+d+2(j-1)}{2\alpha}-\sum_{k=1}^{j-1}\left(\left[\frac{a_0+d+2(k-1)}{2\alpha}\right]-\left[\frac{a_0+2(k-1)}{2\alpha}\right]\right),\\ \frac{a_{j-1}}{2\alpha} &= \frac{a_0+\sum_{k=1}^{j-1}(2-2\alpha([(a_0+d+2(k-1))/2\alpha]-[(a_0+2(k-1))/2\alpha]))}{2\alpha}\\ &= \frac{a_0+2(j-1)}{2\alpha}-\sum_{k=1}^{j-1}\left(\left[\frac{a_0+d+2(k-1)}{2\alpha}\right]-\left[\frac{a_0+2(k-1)}{2\alpha}\right]\right),\end{aligned}$$

so

$$\left[\frac{a_{j-1}+d}{2\alpha}\right]-\left[\frac{a_{j-1}}{2\alpha}\right]=\left[\frac{a_0+d+2(j-1)}{2\alpha}\right]-\left[\frac{a_0+2(j-1)}{2\alpha}\right],$$

and, therefore,

$$\begin{aligned}a_j &= a_{j-1}+2-2\alpha\left(\left[\frac{a_{j-1}+d}{2\alpha}\right]-\left[\frac{a_{j-1}}{2\alpha}\right]\right)\\ &= a_0+\sum_{k=1}^{j}\left(2-2\alpha\left(\left[\frac{a_0+d+2(k-1)}{2\alpha}\right]-\left[\frac{a_0+2(k-1)}{2\alpha}\right]\right)\right).\end{aligned}$$

Similarly, for any $i \in \mathbf{N}$,

$$b_i = b_0+\sum_{k=1}^{i}\left(2-2\alpha\left(\left[\frac{b_0+d+2(k-1)}{2\alpha}\right]-\left[\frac{b_0+2(k-1)}{2\alpha}\right]\right)\right),$$

hence

$$\begin{aligned}b_i-a_i &= 2\alpha\sum_{k=1}^{i}\left(\left[\frac{a_0+d+2(k-1)}{2\alpha}\right]-\left[\frac{a_0+2(k-1)}{2\alpha}\right]\right)\\ &\quad -2\alpha\sum_{k=1}^{i}\left(\left[\frac{b_0+d+2(k-1)}{2\alpha}\right]-\left[\frac{b_0+2(k-1)}{2\alpha}\right]\right).\end{aligned}$$

Recall that initially I and II are placed according to the horizontal relative position between $P_{2n}^- \cap \{y = y_0\}$ and $Q_{2n}^- \cap \{y = y_0\}$ in $\mathbf{R}^2$. Since

$$Q_{2n}^- = \{(x, y) \in S_{2n}^- \mid \Omega((x-(2n+2)\alpha, y)) \in P_{2n}^-\},$$

there exists $m \in \mathbf{Z}$ such that

$$a_0 = b_0+(2n+2)\alpha-2m,$$

therefore, for all $k > m$,

$$\begin{aligned}&\left[\frac{a_0+d+2(k-1)}{2\alpha}\right]-\left[\frac{a_0+2(k-1)}{2\alpha}\right]\\ &\quad=\left[\frac{b_0+d+2(k-m-1)}{2\alpha}\right]-\left[\frac{b_0+2(k-m-1)}{2\alpha}\right],\end{aligned}$$

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so, for every $i > m$,

$$\begin{aligned}|a_i - b_i| &= \left|2\alpha \sum_{k=1}^{m}\left(\left[\frac{a_0 + d + 2(k-1)}{2\alpha}\right] - \left[\frac{a_0 + 2(k-1)}{2\alpha}\right]\right)\right. \\ &\quad \left. - 2\alpha \sum_{k=i-m+1}^{i}\left(\left[\frac{b_0 + d + 2(k-1)}{2\alpha}\right] - \left[\frac{b_0 + 2(k-1)}{2\alpha}\right]\right)\right| \\ &\leq 2m\alpha.\end{aligned}$$

The inequality is due to the fact that for any $x \in \mathbf{R}$ the value of $[(x+d)/2\alpha] - [x/2\alpha]$ can only be $[d/2\alpha]$ or $[d/2\alpha] + 1$. Obviously the inequality $|a_i - b_i| \leq 2m\alpha$ also holds for $i \leq m$.

Let $M' = (2 + 2m\alpha)/(1 - 2\alpha)$, thus for any $N \in \mathbf{N}$, if $|w_1(N) - w_2(N)| \geq M'$, then

$$\begin{aligned}|a_{w_1(N)} - b_{w_2(N)}| &\geq |a_{w_1(N)} - a_{w_2(N)}| - |a_{w_2(N)} - b_{w_2(N)}| \\ &\geq |w_1(N) - w_2(N)|\left(2 - 2\alpha\left(\left[\frac{d}{2\alpha}\right] + 1\right)\right) - 2m\alpha \\ &\geq M'(2 - d - 2\alpha) - 2m\alpha \\ &> M'(1 - 2\alpha) - 2m\alpha \\ &= 2,\end{aligned}$$

and leads to a contradiction. Therefore, $|w_1(N) - w_2(N)| < M'$ for every $N \in \mathbf{N}$. □

Now we come to prove Lemma 4.1.

Proof of Lemma 4.1. First suppose that p lies outside $I \cup II$ initially and after the Nth step. This means p just passes through I for $w_1(N)$ times and through II for $w_2(N)$ times. Also, whether the original or new rules are used does not matter, because p is outside $I \cup II$ at the end. Thus,

$$\begin{aligned}k_1(N) &= \sum_{i=1}^{w_1(N)}\left(\left[\frac{a_{i-1} + d}{2\alpha}\right] - \left[\frac{a_{i-1}}{2\alpha}\right]\right) \\ &= \sum_{i=1}^{w_1(N)}\left(\left[\frac{a_0 + d + 2(i-1)}{2\alpha}\right] - \left[\frac{a_0 + 2(i-1)}{2\alpha}\right]\right), \\ k_2(N) &= \sum_{i=1}^{w_2(N)}\left(\left[\frac{b_{i-1} + d}{2\alpha}\right] - \left[\frac{b_{i-1}}{2\alpha}\right]\right) \\ &= \sum_{i=1}^{w_2(N)}\left(\left[\frac{b_0 + d + 2(i-1)}{2\alpha}\right] - \left[\frac{b_0 + 2(i-1)}{2\alpha}\right]\right)\end{aligned}$$

so, for N large enough such that $w_1(N) > m$ (m is the integer satisfying $a_0 = b_0 + (2n+2)\alpha - 2m$),

$$\begin{aligned}|k_1(N) - k_2(N)| \leq &\left|\sum_{i=1}^{w_1(N)}\left(\left[\frac{a_0 + d + 2(i-1)}{2\alpha}\right] - \left[\frac{a_0 + 2(i-1)}{2\alpha}\right]\right)\right. \\ &\left. - \sum_{i=1}^{w_1(N)}\left(\left[\frac{b_0 + d + 2(i-1)}{2\alpha}\right] - \left[\frac{b_0 + 2(i-1)}{2\alpha}\right]\right)\right|\end{aligned}$$

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$$
\begin{aligned}
&\quad + |w_1(N) - w_2(N)|\left(\left[\frac{d}{2\alpha}\right] + 1\right) \\
&< \left|\sum_{i=1}^{m}\left(\left[\frac{a_0 + d + 2(i-1)}{2\alpha}\right] - \left[\frac{a_0 + 2(i-1)}{2\alpha}\right]\right)\right. \\
&\quad \left. - \sum_{i=w_1(N)-m+1}^{w_1(N)}\left(\left[\frac{b_0 + d + 2(i-1)}{2\alpha}\right] - \left[\frac{b_0 + 2(i-1)}{2\alpha}\right]\right)\right| \\
&\quad + M'\left(\left[\frac{1}{2\alpha}\right] + 1\right) \\
&\le m + M'\left(\left[\frac{1}{2\alpha}\right] + 1\right).
\end{aligned}
$$

It is obvious that for $w_1(N) < m$ the inequality $|k_1(N) - k_2(N)| < m + M'([1/2\alpha] + 1)$ also holds.

Generally, p may not be outside $I \cup II$ initially or after the Nth step. If p lies in I initially, then we may take $-d < a_0 < 0$, thus in the expression of $k_1(N)$, the first term $[(a_0 + d)/2\alpha] - [a_0/2\alpha]$ is replaced by $[(a_0 + d)/2\alpha] + 1$ with the other terms remaining the same, and this reduces $k_1(N)$ by at most $-[-d/2\alpha] - 1 \le -[-1/2\alpha] - 1$. Similarly, if p lies in II initially, then we may take $-d < b_0 < 0$, thus in the expression of $k_2(N)$, the first term $[(b_0 + d)/2\alpha] - [b_0/2\alpha]$ is replaced by $[(b_0 + d)/2\alpha] + 1$ with the other terms remaining the same, and this reduces $k_2(N)$ to at most $-[-1/2\alpha] - 1$ for the same reason. On the other hand, if p lies in $I \cup II$ after the Nth step, suppose that the last time that p lay outside $I \cup II$ was the end of the $\tilde{N}$th step ($\tilde{N} < N$). Then we have

$$
\begin{aligned}
0 < k_1(N) - k_1(\tilde{N}) &\le \left[\frac{d}{2\alpha}\right] + 1 \le \left[\frac{1}{2\alpha}\right] + 1, \\
0 < k_2(N) - k_2(\tilde{N}) &\le \left[\frac{d}{2\alpha}\right] + 1 \le \left[\frac{1}{2\alpha}\right] + 1,
\end{aligned}
$$

because it needs at most $[d/2\alpha] + 1$ steps for p to pass through I (or II).

Henceforth, for the general case, we have for every $N \in \mathbf{N}$

$$
\begin{aligned}
|k_1(N) - k_2(N)| &< m + M'\left(\left[\frac{1}{2\alpha}\right] + 1\right) + \left(-\left[\frac{-1}{2\alpha}\right] - 1\right) + \left(\left[\frac{1}{2\alpha}\right] + 1\right) \\
&= m + M'\left(\left[\frac{1}{2\alpha}\right] + 1\right) + \left[\frac{1}{2\alpha}\right] - \left[\frac{-1}{2\alpha}\right].
\end{aligned}
$$

Therefore, if we take

$$
M = m + M'\left(\left[\frac{1}{2\alpha}\right] + 1\right) + \left[\frac{1}{2\alpha}\right] - \left[\frac{-1}{2\alpha}\right],
$$

then, for any $N \in \mathbf{N}$, $|k_1(N) - k_2(N)| < M$. □

Proposition 4.1 can be derived directly from Lemma 4.1, although the upper bound M may be different to that in Lemma 4.1. Moreover, from the proof of the lemma, we can see that M' and M can be taken only depending on n_0 (the integer satisfying $p_0 \in S^-_{2n_0}$). Therefore, the upper bound M in Proposition 4.1 can also be taken only depending on n_0.

Similarly, for $p \in S^+_{2n}$, we have the following.

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PROPOSITION 4.2. *Let N_0 be as described in Proposition 3.1, and $p_0 \in S^+_{2n_0}$ $(n_0 > N_0)$ be a point belonging to $\mathbf{F}_\infty(P)$. Then there exists $M > 0$, such that for any $N \in \mathbf{N} \cup \{+\infty\}$ satisfying $n_0, n_1, \ldots, n_N > N_0$, we have $y_k < M$ $(k = 0, 1, \ldots, N)$. Here y_k denotes the y-coordinate of $\Psi^k(p_0)$.*

The proof is analogous to that of Proposition 4.1 and is omitted. From the above discussions, we have the following.

PROPOSITION 4.3. *Let N_0 be as described in Proposition 3.1, and $p_0 \in S_{2n_0}$ $(n_0 > N_0)$ be a point belonging to $\mathbf{F}_\infty(P)$. Then there exists $M > 0$, such that for any $N \in \mathbf{N} \cup \{+\infty\}$ satisfying $n_0, n_1, \ldots, n_N > N_0$, we have $y_k < M$ $(k = 0, 1, \ldots, N)$. Here y_k denotes the y-coordinate of $\Psi^k(p_0)$. Moreover, M can be taken only depending on n_0.*

5. *Proof of Theorem 1*

With no loss of generality, we suppose that P is the trapezoid with vertices $A(0, 1)$, $B(0, 0)$, $C(1, 0)$, $D(\alpha, 1)$ $(0 < \alpha < 1)$, since any trapezoid can be transformed into this form through affine transformations. Consider the dual billiard orbit of a point $p_0 \in \mathbf{F}_\infty(P)$ outside P. It is sufficient to prove this for the case when $p_0 \in \Gamma_1$, because every point in $\mathbf{F}_\infty(P)$ will enter Γ_1 after some iterations of T (therefore, Φ). Moreover, we can assume $p_0 \in S_{2n_0}$ with $n_0 > N_0$, with N_0 as described in Proposition 3.1, because every orbit which might possibly tend to infinity must have at least one point in $\bigcup_{n=N_0+1}^{\infty} S_{2n}$. Now take such p_0, and let M be as obtained in Proposition 4.3.

We claim that for all $i \in \mathbf{N}$, the y-coordinate of $\Psi^i(p_0)$ is less than M. Otherwise, there exists $i_0 \in \mathbf{N}$ such that the y-coordinate of $\Psi^{i_0}(p_0)$ is not less than M. Let n_i be the integer such that $\Psi^i(p_0) \in S_{2n_i}$ for each i, then $n_1, n_2, \ldots, n_{i_0}$ may not all be larger than N_0, however, there is a maximal integer $i_{\max} \le i_0$ such that $n_{i_{\max}} = n_0$. Thus $n_{i_{\max}}, n_{i_{\max}+1}, \ldots, n_{i_0}$ must be all larger than N_0, since from below S_{2n_0} to $S_{2n_{i_0}}$, the point must enter S_{2n_0} at some time, and this contradicts Proposition 4.3. Therefore, the Ψ-orbit of p_0 is bounded, which implies that there exists $N \in \mathbf{N}$ such that $\{\Psi^i(p_0), i \in \mathbf{N} \cup \{0\}\} \subset \bigcup_{n=0}^{N} S_{2n}$. Henceforth, every point in $\{T^i(p_0)\}$ can be expressed as the image of at most finite many times (depending on N only) of iterations of T acting on some point in $\bigcup_{n=0}^{N} S_{2n}$. So the T-orbit of p_0 is bounded for every p_0 under consideration.

Since $T = \Phi^2$, every point in $\{\Phi^i(p_0)\}$ belongs to either $\{T^i(p_0)\}$ or $\{\Phi \circ T^i(p_0)\}$. Note that Φ is just a reflection with its reflection point being one of the vertices of P. Therefore, the Φ-orbit of p_0 is bounded.

6. *Proof of Theorem 2*

We first give a corollary of Theorem 1 and then prove Theorem 2.

COROLLARY 1. *There exists $N_1 \in \mathbf{N}$ such that for any $n > N_1$, $n \in \mathbf{N}$, and for any $p \in S_{2n}$,*

$$\{\Psi^i(p), i \in \mathbf{N}\} \subset \{(x, y) \mid 0 < x < 2, y > 2N_0 + 2\},$$

where N_0 is as described in Proposition 3.1.

Proof. The proof is elementary. From Theorem 1 we know that for any $n \le N_0$, $n \in \mathbf{N} \cup \{0\}$ and for any $p \in S_{2n}$, $\{\Psi^{-i}(p), i \in \mathbf{N}\}$ is bounded (regarding Φ^{-1} as a dual

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billiard map). Moreover, the bound can be taken depending on n only (see the proof of Proposition 4.1), thus there exists $N_1 \in \mathbf{N}$ such that for any $p \in \bigcup_{n=0}^{N_0} S_{2n}$,

$$\{\Psi^{-i}(p),\, i \in \mathbf{N}\} \subset \{(x,\, y) \mid 0 < x < 2,\, 0 < y < 2N_1 + 2\}.$$

Therefore, for any $p \in \bigcup_{n=N_1+1}^{\infty} S_{2n}$, we have

$$\{\Psi^{i}(p),\, i \in \mathbf{N}\} \subset \{(x,\, y) \mid 0 < x < 2,\, y > 2N_0 + 2\}. \qquad \square$$

Corollary 1 implies that for p sufficiently far from P, its Ψ-orbit is bounded away from P, so that Proposition 3.1 applies at all steps.

Now we prove Theorem 2.

Proof of Theorem 2. The first half of the theorem is obvious owing to Proposition 3.1.

For the second half, we first take α to be irrational (if α is rational, then all orbits are periodic). We claim that on every infinite Ψ-orbit whose points all belong to $\{(x,\, y) \mid 0 < x < 2,\, y > 2N_0 + 2\}$, there is at least one point which is Poisson-stable. Let N_1 be as described in Corollary 1, and p be any point belonging to $\mathbf{F}_\infty(P) \cap S_{2n}$ $(n > N_1)$, then $\{\Psi^i(p),\, i \in \mathbf{N}\} \subset \{(x,\, y) \mid 0 < x < 2,\, y > 2N_0 + 2\}$. This implies that the Ψ-orbit of p obeys Proposition 3.1 at every step, thus in the horizontal direction each iteration of Ψ either does nothing or is a rotation by 2α on the circle with perimeter 2. Moreover, the latter case occurs for infinite times for otherwise the orbit will be unbounded. Therefore, the x-coordinates of points in $\{\Psi^i(p),\, i \in \mathbf{N}\}$ must be dense in the interval $(0,\, 2)$.

Suppose that there exists an infinite Ψ-orbit $\{\Psi^i(p_0),\, i \in \mathbf{N}\}$, whose points all lie in $\{(x,\, y) \mid 0 < x < 2,\, y > 2N_0 + 2\}$, such that all points on it are not Poisson-stable. That is, any point $\Psi^j(p_0)$ $(j \in \mathbf{N})$ on this orbit has a neighborhood such that $\{\Psi^i(p_0),\, i > j\}$ has no point in it. With no loss of generality, let such a neighborhood of $\Psi^j(p_0) = (x_j,\, y_j)$ be $\{(x,\, y) : |x - x_j| < \varepsilon(j),\, |y - y_j| < 1\}$ (strictly speaking, the vertical width should be taken small enough to be contained in the given neighborhood, but it makes no difference in fact, for the vertical directional displacement of a point under Ψ can only be 0, 2 or -2), with $\varepsilon(j)$ taken less than $\min\{x_j,\, 2 - x_j\}$ for every $j \in \mathbf{N}$. Let $\varepsilon_0 = \varepsilon(0)$. The density mentioned above implies that there exists infinite many points in $\{\Psi^i(p_0),\, i \in \mathbf{N}\}$ whose x-coordinate lies in $(x_0 - \varepsilon_0/2,\, x_0 + \varepsilon_0/2)$. Let $\Psi^{i_1}(p_0)$ be one of these points, then $y_{i_1} \neq y_0$ from the definition of $\varepsilon(0)$. Let $\varepsilon_1 = \min\{\varepsilon(i_1),\, \varepsilon_0/2\}$, then there exists infinite points in $\{\Psi^i(p_0),\, i > i_1,\, i \in \mathbf{N}\}$ whose x-coordinate lies in $(x_{i_1} - \varepsilon_1/2,\, x_{i_1} + \varepsilon_1/2)$. Let $\Psi^{i_2}(p_0)$ be one of these points, then $y_{i_2} \neq y_0,\, y_{i_1}$. Let $\varepsilon_2 = \min\{\varepsilon(i_2),\, \varepsilon_1/2\}$ and so on. In this way one can construct an infinite series of integers $\{i_k,\, k \in \mathbf{N}\}$, satisfying that y_{i_k} are pairwise different and all different from y_0, but this leads to a contradiction with the boundedness of $\{\Psi^i(p_0),\, i \in \mathbf{N}\}$.

We have now proved that on every infinite Ψ-orbit whose points all belong to $\{(x,\, y) \mid 0 < x < 2,\, y > 2N_0 + 2\}$, there is at least one point which is Poisson-stable. As $\mathbf{F}_\infty(P) \cap \{(x,\, y) \mid 0 < x < 2,\, y > 2N_0 + 2\}$ is of positive measure, points which are Poisson-stable must be of positive measure. Note that if $\Phi^i(p_0)$ is Poisson-stable, then so is $\Phi^{i+1}(p_0)$ and $\Phi^{i-1}(p_0)$ (if $i \geq 1$), for there exists sufficiently small neighborhood of $\Phi^i(p_0)$ (a half-neighborhood may be taken if $\Phi^i(p_0)$ or $\Phi^{i-1}(p_0)$ lies on the boundary of some Γ_j) such that every point in it has the same translation vector with $\Phi^i(p_0)$ under Φ and Φ^{-1}. Consequently, every point on this orbit is Poisson-stable. $\square$

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