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On Kelly's congruence theorem for trees

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1. *Introduction.* A *graph* G is a pair $(V(G), E(G))$; $V(G)$ is the *set of vertices* of G, and $E(G)$ the *set of edges* of G; $E(G)$ is a subset of the Cartesian product $V(G) \times V(G)$ (all pairs (v_i, v_j) where $v_i \in V(G)$, $v_j \in V(G)$). If v is a vertex of G we shall write $v \in G$. The graph-theoretic terminology used in this paper is that of Berge (1). We shall consider only graphs which are finite, undirected, and without loops.

If G_1 is a subgraph of G, $G - G_1$ denotes that subgraph of G which is obtained from G by deleting G_1 and all edges of G that are joined to vertices of G_1. If $G_1, G_2, \ldots, G_n$ are subgraphs of G, $\bigcup_{i=1}^{n} G_i = G_1 \cup G_2 \cup \ldots \cup G_n$ is the minimal subgraph of G containing $G_1, G_2, \ldots, G_n$. The symbol $\mathbf{U}$ will be reserved for *set-theoretic* unions. Two graphs, G and H, are *isomorphic* ($G \simeq H$) if there is a 1–1 correspondence between the vertices $\{u_i\}$ of G and the vertices $\{v_i\}$ of H such that, when u_i corresponds to v_i and u_j corresponds to v_j, $(u_i, u_j) \in E(G) \Leftrightarrow (v_i, v_j) \in E(H)$. The notation $\{a_i\}$, where the suffix i can range over some set I indicated by the context, denotes the set $\{a_i : i \in I\}$; the set with sole element a will be denoted by $[a]$.

We shall denote the shortest distance between vertices u, v by $d(u, v)$.

$$\rho = \min_{u \in G} \max_{v \in G} d(u, v)$$

is the *radius* of the graph G. Any vertex c for which $\max_{v \in G} d(c, v) = \rho$ is called a *centre* of G. A *peripheral vertex* is a vertex p for which $\max_{v \in G} d(p, v) = \max_{u \in G} \max_{v \in G} d(u, v)$. A *pendant vertex* is a vertex of degree one.

In (5) Ulam suggests the following problem.

Problem 1. Suppose two graphs G, H, with the same number of vertices (i.e. $|V(G)| = |V(H)| = N$) have the following property: the vertices $\{u_i\}_1^N$ of G and $\{v_i\}_1^N$ of H may be paired in such a way that $G - u_i \simeq H - v_i$, $1 \leqslant i \leqslant N$. Then is it true that $G \simeq H$?

Kelly (3) has proved that this is so in the case when G and H are trees. Closely associated with Problem 1 is,

Problem 2. Suppose two graphs G, H, with the same number of edges and with

$$|E(G)| = |E(H)| = M$$

have the following property: the edges $\{e_i\}_1^M$ of G and $\{f_i\}_1^M$ of H may be paired in such a way that $G - e_i \simeq H - f_i$, $1 \leqslant i \leqslant M$. Then is it true that $G \simeq H$?

In (2) Harary and Palmer strengthened Kelly's theorem by showing that every tree

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is determined by its maximal sub*trees*. The purpose of this paper is to prove the following theorem.

THEOREM 1. *Let G, H, be two trees, each containing N peripheral vertices. If the peripheral vertices $\{u_i\}_1^N$ of G and $\{v_i\}_1^N$ of H may be paired so that $G-u_i \simeq H-v_i$, $1 \leqslant i \leqslant N$, then $G \simeq H$.*

Theorem 1 has, as a corollary, the theorem of Harary and Palmer, which in turn yields the solution to Problem 2 in the case when G and H are trees. For convenience, we shall abbreviate peripheral vertex to *p-vertex*.

2. *Preliminary results.*

THEOREM 2. *The centres of a tree are mutually adjacent. A tree therefore has at most two centres.*

A proof may be found in (4). In view of Theorem 2 we call a tree *central* or *bicentral* according as it has one or two centres. Throughout the paper pairs (u_i, v_i), (u_i', v_i'), etc., with the same index, will denote corresponding p-vertices in the sense of Theorem 1; u_i is not a particular p-vertex but the label for a general p-vertex. References to u_i in different sections are independent. We shall now assume that G and H satisfy the hypotheses of Theorem 1.

THEOREM 3. *G and H have the same radius and are either both central or both bicentral.*

Proof. If T is a central tree of radius r, longest chains in T have length $2r$; if T is a bicentral tree of radius r, longest chains in T have length $2r-1$. Hence it suffices to show that longest chains in G and H have the same length. Now let d_G, d_H be the lengths of longest chains in G, H respectively and let C_G, C_H be two such chains. We assume that G and H each have N p-vertices.

(*a*) $N = 2$. If u_i is a p-vertex in G, longest chains in $G-u_i$ have length d_G-1 and longest chains in $H-v_i$ (where v_i corresponds to u_i) have length d_H-1. Since $G-u_i \simeq H-v_i$ we must have $d_G-1 = d_H-1$. Therefore $d_G = d_H$.

(*b*) $N > 2$. There exists a p-vertex u_i in G such that $u_i \notin C_G$. Then

$$C_G \subset G-u_i \simeq H-v_i \subset H$$

and therefore $d_G \leqslant d_H$. Similarly there exists a p-vertex v_j in H such that $v_j \notin C_H$. Then $C_H \subset H-v_j \simeq G-u_j \subset G$ and so $d_H \leqslant d_G$. Hence $d_G = d_H$.

We shall now assume that G and H are both central trees of radius r with centres g and h respectively and that each tree has N p-vertices, $\{u_i\}_1^N$ in G, $\{v_i\}_1^N$ in H. Proofs for the case when G and H are bicentral trees will follow later.

LEMMA. *Let T and U be central trees of radius r. Then, whenever T is isomorphic to a subgraph of U, p-vertices in T correspond under the isomorphism to p-vertices in U.*

Proof. Suppose $T \simeq T' \subset U$. We must show that every p-vertex of T' is a p-vertex of U. Let u be the centre of U. Then

$$d(u, v) \leqslant r \quad (\forall\, v \in U).$$

A fortiori

$$d(u, v) \leqslant r \quad (\forall\, v \in T')$$

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and therefore u is the centre of T'. If v' is a p-vertex of T' then $d(u, v') = r$ and hence v' is a p-vertex of U.

THEOREM 4. *Let T be any central tree of radius r with n pendant vertices ($n < N$), all of which are p-vertices. Suppose there are α distinct subgraphs of G which are isomorphic to T, and that the p-vertex u_i of G is in α_i of these subgraphs; that there are β distinct subgraphs of H which are isomorphic to T and that the p-vertex v_i of H is in β_i of these subgraphs. Then*

$$\alpha = \beta \quad \text{and} \quad \alpha_i = \beta_i \quad (1 \leqslant i \leqslant N).$$

Proof. There are $\alpha - \alpha_i$ subgraphs of $G - u_i$, and $\beta - \beta_i$ subgraphs of $H - v_i$, which are isomorphic to T. Since $G - u_i \simeq H - v_i$ we have

$$\alpha - \alpha_i = \beta - \beta_i \quad (1 \leqslant i \leqslant N). \tag{1}$$

By the Lemma, when T is isomorphic to a subgraph of G, every p-vertex in T corresponds to a p-vertex in G. Therefore

$$\sum_1^N \alpha_i = n\alpha. \tag{2}$$

Similarly

$$\sum_1^N \beta_i = n\beta. \tag{3}$$

Subtracting (3) from (2) and using (1)

$$N(\alpha - \beta) = n(\alpha - \beta).$$

But $n < N$. Hence $\alpha = \beta$ and, from (1), $\alpha_i = \beta_i (1 \leqslant i \leqslant N)$.

A *limb* of a central tree T is a subtree which contains the centre of T as a pendant vertex and is joined to the rest of T at the centre. A limb which contains a p-vertex of T is called a *radial limb*. We shall denote the sets of radial limbs $\{G_i\}$, $\{H_i\}$ of G, H by $\mathbf{G}$, $\mathbf{H}$ respectively; the sets of non-radial limbs $\{K_i\}$, $\{L_i\}$ of G, H by $\mathbf{K}$, $\mathbf{L}$ respectively. In addition we write $K = \bigcup_i K_i$, $L = \bigcup_i L_i$.

THEOREM 5. *Let $N > 2$. Then if the p-vertex u_i is on a limb containing γ_i p-vertices, v_i is also on a limb containing γ_i p-vertices.*

Proof. Since a tree is connected and contains no cycles, there is one and only one chain between any pair u, v of its vertices. We shall denote this chain by $C(u, v)$.

If u_1 is on a limb containing γ_i p-vertices then u_i lies on $N - \gamma_i$ chains of length $2r$ in G, namely $\{C(u_i, g) \cup C(g, u_j)\}_{u_j \in I}$ where I is the set of p-vertices of G not on the limb containing u_i, and g is the centre of G. Now a chain of length $2r$ is a central tree with two pendant vertices, both of which are p-vertices, and therefore, by Theorem 4, the p-vertex v_i of H lies on $N - \gamma_i$ chains of length $2r$ in H. Hence v_i is on a limb containing γ_i p-vertices.

COROLLARY. *If G has M radial limbs $\{G_i\}_1^M$, where G_i contains γ_i p-vertices, then H also has M radial limbs $\{H_i\}_1^M$ and these may be ordered so that H_i contains γ_i p-vertices.*

Proof. If $N = 2$ then both G and H have just two radial limbs, with one p-vertex on each limb.

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If $N > 2$ denote by u_{ij}, $1 \leqslant j \leqslant \gamma_i$, the p-vertices on G_i. Then u_{ij} lies on a limb containing γ_i p-vertices. Therefore, by Theorem 5, v_{ij} lies on a limb containing γ_i p-vertices, $1 \leqslant j \leqslant \gamma_i$, $1 \leqslant i \leqslant M$, and this implies the truth of the Corollary.

3. *Proof of main theorem: central case.* By the Corollary to Theorem 5, G and H have the same number of radial limbs, say M. For $1 \leqslant i \leqslant M$ we may suppose that G_i and H_i each contain γ_i p-vertices, where $\gamma_1 \leqslant \gamma_2 \leqslant \ldots \leqslant \gamma_M$.

θ, ϕ, ϕ' respectively will denote isomorphisms of $G-u_1$ onto $H-v_1$, of $G-u_2$ onto $H-v_2$, and of $G-u_2'$ onto $H-v_2'$, where $v_2' = \theta(u_2)$. As for the p-vertices u_i, references to θ, ϕ, ϕ' in different sections are to be considered independent.

The notation $(U,u) \simeq (V,v)$ will mean that the graphs U, V are isomorphic under an isomorphism mapping vertex $u \in U$ into vertex $v \in V$. If α is such an isomorphism we shall also write $\alpha(U,u) = (V,v)$. Two sets $\mathbf{S} = \{S_i\}$, $\mathbf{T} = \{T_i\}$ of graphs are isomorphic ($\mathbf{S} \simeq \mathbf{T}$) if there is a 1–1 function from $\mathbf{S}$ onto $\mathbf{T}$ which maps each graph in $\mathbf{S}$ onto an isomorphic graph in $\mathbf{T}$. $(\mathbf{S},u) \simeq (\mathbf{T},v)$ (or $\alpha(\mathbf{S},u) = (\mathbf{T},v)$) will mean that $\mathbf{S} \simeq \mathbf{T}$, that $u \in S_i$, $v \in T_i$ for all i, and that for each pair of isomorphic graphs there is an isomorphism (α) mapping vertex u into vertex v.

The proof will be achieved by showing that $(\mathbf{G},g) \simeq (\mathbf{H},h)$ and $(\mathbf{K},g) \simeq (\mathbf{L},h)$.

3.1. $M > 2$. If u_i is a p-vertex in G then, since $M > 2$, $G-u_i$ and $H-v_i$ are central trees of radius r. In any isomorphism of $G-u_i$ onto $H-v_i$ the centres must correspond. Hence

$$\theta(g) = h, \quad \phi(g) = h, \quad \phi'(g) = h. \tag{4}$$

It is also clear that these isomorphisms map radial limbs into radial limbs. Let u_1, u_2 be p-vertices in G_1, G_2 respectively. We may assume that $v_1 \in H_1$. Since $G_1 - u_1$, $H_1 - v_1$ each contain $\gamma_1 - 1$ p-vertices, $\theta(G-u_1) = H-v_1 \Rightarrow \theta(\{G_i\}_2^M, g) = (\{H_i\}_2^M, h)$ and there is no loss of generality in assuming that

$$\theta(G_i, g) = (H_i, h) \quad (2 \leqslant i \leqslant M). \tag{5}$$

We now consider three subcases.

(*a*) $\gamma_2 > \gamma_1 + 1$. Then $\phi(G-u_2) = H-v_2 \Rightarrow \phi(G_1) = H_1$.

(*b*) $\gamma_2 = \gamma_1 + 1$. From (5) $\theta(u_2) = v_2' \in H_2$, and hence $\theta(G_2 - u_2) = H_2 - v_2'$,

$$\phi(G-u_2) = H-v_2 \Rightarrow \quad \text{either} \quad \phi(G_1) = H_1 \quad \text{or} \quad \phi(G_2-u_2) = H_1,$$
$$\phi'(G-u_2') = H-v_2' \Rightarrow \quad \text{either} \quad \phi'(G_1) = H_1 \quad \text{or} \quad \phi'(G_1) = H_2-v_2'.$$

Therefore at least one of ϕ, ϕ', $\phi\theta^{-1}\phi'$ (product of isomorphisms) maps G_1 onto H_1.

(*c*) $\gamma_2 = \gamma_1$. Suppose $v_2 \in H_k$. Then

$$\phi(G-u_2) = H-v_2 \Rightarrow \phi(\{G_i\}_{i \neq 2}, g) = (\{H_i\}_{i \neq k}, h).$$

Put $\phi_r = \phi(\theta^{-1}\phi)^r$. We shall show that, for some r, either

$$\phi_r(G_1) = H_1 \quad \text{or} \quad \phi_r(G_1) = H_2. \tag{6}$$

For assume otherwise. Then $\phi_r(G_1)$ is well defined for all r and we may put

$$\phi_r(G_1) = H_{j_r}, \quad \text{where} \quad j_r \neq 1, 2.$$

Thus

$$\theta^{-1}\phi_r(G_1) = \theta^{-1}(H_{j_r}) = G_{j_r}. \tag{7}$$

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Since M is finite there are integers m, n, $(m < n)$ such that $H_{j_m} = H_{j_n}$. Then

$$\phi_m(G_1) = \phi_n(G_1); \quad \text{i.e.} \quad \theta^{-1}\phi_{n-m-1}(G_1) = G_1,$$

contradicting (7). Hence (6) follows. From (5) $v_2' \in H_2$ and by an analogous argument to the above, $\phi'(G-u_2') = H - v_2' \Rightarrow$

$$\text{either} \quad \phi_s'(G_1) = H_1 \quad \text{or} \quad \phi_s'(G_2) = H_1 \tag{8}$$

for some s, where $\phi_s' = \phi'(\theta^{-1}\phi')^s$. Therefore for some r, s, at least one of $\phi_r, \phi_s', \phi_s'\theta^{-1}\phi_r$ maps G_1 onto H_1.

(a), (b), (c) and (4) together imply that an isomorphism θ_1 exists such that

$$\theta_1(G_1, g) = (H_1, h).$$

We now show that $(\mathbf{K}, g) \simeq (\mathbf{L}, h)$.

If $\gamma_1 > 1$ then $G - u_1$ has the same non-radial limbs as G and $H - v_1$ has the same non-radial limbs as H. Hence

$$\theta(G-u_1) = H - v_1 \Rightarrow \theta(\mathbf{K}, g) = (\mathbf{L}, h).$$

If $\gamma_1 = 1$ then $\theta_1(G_1, g) = (H_1, h) \Rightarrow \theta_1(u_1) = v_1$ and hence

$$\theta_1(G_1 - u_1, g) = (H_1 - v_1, h). \tag{9}$$

Now $G-u_1$ has non-radial limbs $[G_1-u_1] \cup \mathbf{K}$ and $H-v_1$ has non-radial limbs $[H_1-v_1] \cup \mathbf{L}$. Therefore $\theta([G_1-u_1)] \cup \mathbf{K}, g) = ([H_1-v_1] \cup \mathbf{L}, h)$. If

$$\theta(G_1-u_1, g) = (H_1-v_1, h) \quad \text{then} \quad \theta(\mathbf{K}, g) = (\mathbf{L}, h).$$

Otherwise $\theta(G_1-u_1, g) = (L_j, h)$ and $\theta(K_k, g) = (H_1-v_1, h)$ for some j, k. Then

$$\theta(\{K_i\}_{i \neq k}, g) = (\{L_i\}_{i \neq j}, h) \quad \text{and from (9)} \quad \theta\theta_1^{-1}\theta(K_k, g) = (L_j, h).$$

Hence $(\mathbf{K}, g) \simeq (\mathbf{L}, h)$. Therefore $G \simeq H$.

3.2. $M = 2$. For $\gamma_1 > 1$ the proof of 3.1. suffices. We shall therefore assume that $\gamma_1 = 1$. For $i = 1, 2$, denote by g_i the vertex adjacent to g on G_i and by h_i the vertex adjacent to h on H_i. Let u_1 be the p-vertex in G_1. We may assume that $v_1 \in H_1$. Then $G-u_1$ and $H-v_1$ are bicentral trees of radius r. In the isomorphism $\theta: G-u_1 \to H-v_1$ the bicentres g, g_2 of $G-u_1$ and h, h_2 of $H-v_1$ must correspond in some order. Hence either (α_1): $\theta(g) = h$, $\theta(g_2) = h_2$ or (α_2): $\theta(g) = h_2$, $\theta(g_2) = h$.

$$(\alpha_1) \Rightarrow \theta([G_1-u_1] \cup \mathbf{K}, g) = ([H_1-v_1] \cup \mathbf{L}, h), \quad \theta(G_2, g) = (H_2, h). \tag{10}$$

$$(\alpha_2) \Rightarrow \theta((G_1-u_1) \cup K, g) = (H_2-h, h_2), \quad \theta(G_2-g, g_2) = ((H_1-v_1) \cup L, h). \tag{11}$$

We first make two points.

$$(G_1, g) \simeq (H_1, h) \Rightarrow (G_1-u_1, g) \simeq (H_1-v_1, h) \tag{12}$$

since u_1 and v_1 must correspond under any such isomorphism.

$$(G_i-g, g_i) \simeq (H_i-h, h_i) \Rightarrow (G_i, g) \simeq (H_i, h) \quad (i = 1, 2) \tag{13}$$

since g is joined to g_i and h is joined to h_i.

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(*a*) $\gamma_2 > 2$. Let u_2 be a p-vertex in G_2. Then $v_2 \in H_2$ since v_1 is the p-vertex in H_1. $G-u_2, H-v_2$ are central trees of radius r and $\phi(G-u_2) = H-v_2 \Rightarrow$

$$\phi(G_1, g) = (H_1, h), \quad \phi(\mathbf{K}, g) = (\mathbf{L}, h). \tag{14}$$

(α_1) From (10), $\theta(G_2, g) = (H_2, h)$.

(α_2) From (12) and (14),

$$\phi((G_1, -u_1) \cup K, g) = ((H_1 - v_1) \cup L, h).$$

Hence, by (11), $\theta\phi^{-1}\theta$ maps $G_2 - g$ onto $H_2 - h$. Thus

$$(G_2 - g, g_2) \simeq (H_2 - h, h_2) \quad \text{and, by (13),} \quad (G_2, g) \simeq (H_2, h).$$

Hence in both cases $(G_2, g) \simeq (H_2, h)$ and this, with (14), shows that $G \simeq H$.

(*b*) $\gamma_2 = 2$. Let u_2 be a p-vertex in G_2. Then $\phi(G-u_2) = H-v_2 \Rightarrow$

$$\phi(\mathbf{K}, g) = (\mathbf{L}, h) \tag{15}$$

and either (β_1): $\phi(g_1) = h_1, \phi(g_2) = h_2$ or (β_2): $\phi(g_1) = h_2, \phi(g_2) = h_1$.

$$(\beta_1) \Rightarrow \phi(G_1, g) = (H_1, h), \quad \phi(G_2 - u_2, g) = (H_2 - v_2, h). \tag{16}$$

$$(\beta_2) \Rightarrow \phi(G_1, g) = (H_2 - v_2, h), \quad \phi(G_2 - u_2, g) = (H_1, h). \tag{17}$$

(α_1) From (10), $\theta(G_2, g) = (H_2, h)$. Since $\theta(u_2) = v_2'$,

$$\theta(G_2 - u_2, g) = (H_2 - v_2', h). \tag{18}$$

Now $$\phi'(G - u_2') = H - v_2' \Rightarrow$$

either (γ_1): $\phi'(g_1) = h_1, \phi'(g_2) = h_2$ or (γ_2): $\phi'(g_1) = h_2, \phi'(g_2) = h_1$.

$$(\gamma_1) \Rightarrow \phi'(G_1, g) = (H_1, h), \quad \phi'(G_2 - u_2', g) = (H_2 - v_2', h), \tag{19}$$

$$(\gamma_2) \Rightarrow \phi'(G_1, g) = (H_2 - v_2', h), \quad \phi'(G_2 - u_2', g) = (H_1, h). \tag{20}$$

(16)–(20) imply that at least one of ϕ, ϕ', $\phi\theta^{-1}\phi'$ maps G_1 onto H_1. Hence

$$(G_1, g) \simeq (H_1, h),$$

and with (10) and (15) this shows that $G \simeq H$.

(α_2) If (β_1) holds then, from (16), $\phi(G_1, g) = (H_1, h)$, and, by (11), (12), (15) and (16), $\theta\phi^{-1}\theta$ maps $G_2 - g$ onto $H_2 - h$. Therefore $(G_2, -g, g_2) \simeq (H_2 - h, h_2)$ and, by (13), $(G_2, g) \simeq (H_2, h)$. Hence $G \simeq H$.

If (β_2) holds then

from (11) $$|E(H_2)| = |E(G_1)| + |E(K)|,$$

from (17) $$|E(H_2)| = |E(G_1)| + 1.$$

Therefore K has just one edge. By means of (11), (17), one can construct G_1 and H_2 and show that they must be the following graphs:

G_1 is a chain of length r with an edge attached to each vertex except two at one end and the one at the other (central) end. H_2 is a chain of length r with an edge attached to each vertex except the ones at either end.

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By a similar argument one may prove that L has just one edge, that H_1 is a graph isomorphic to G_1 and that G_2 is a graph isomorphic to H_2 (indeed that $(G_1, g) \simeq (H_1, h)$ and $(G_2, g) \simeq (H_2, h)$). Hence $G \simeq H$.

(*c*) $\gamma_2 = 1$. Under the isomorphism $\phi: G - u_2 \to H - v_2$ either

$$(\boldsymbol{\beta}_1): \phi(g) = h, \phi(g_1) = h_1 \quad \text{or} \quad (\boldsymbol{\beta}_2): \phi(g) = h_1, \phi(g_1) = h.$$

$$(\boldsymbol{\beta}_1) \Rightarrow \phi(G_1, g) = (H_1, h), \quad \phi([G_2 - u_2] \cup \mathbf{K}, g) = ([H_2 - v_2] \cup \mathbf{L}, h). \tag{21}$$

$$(\boldsymbol{\beta}_2) \Rightarrow \phi(G_1 - g, g_1) = ((H_2 - v_2) \cup L, h), \quad \phi((G_2 - u_2) \cup K, g) = (H_1 - h, h_1). \tag{22}$$

We have four cases to consider: $(\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1), (\boldsymbol{\alpha}_1, \boldsymbol{\beta}_2), (\boldsymbol{\alpha}_2, \boldsymbol{\beta}_1), (\boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$. As $(\boldsymbol{\alpha}_1, \boldsymbol{\beta}_2)$ and $(\boldsymbol{\alpha}_2 \boldsymbol{\beta}_1)$ are symmetric cases it suffices to consider just one of the two.

$(\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ From (10), $\theta(G_2, g) = (H_2, h)$, from (21) $\phi(G_1, g) = (H_1, h)$ and, with (12), (10) and (21) also imply (by an argument similar to that used in 3.1 (*c*)) that $(\mathbf{K}, g) \simeq (\mathbf{L}, h)$. Therefore $G \simeq H$.

$(\boldsymbol{\alpha}_1, \boldsymbol{\beta}_2)$ From (10), $\theta(G_2, g) = (H_2, h)$ and hence, since $\gamma_2 = 1$,

$$\theta(G_2 - u_2, g) = (H_2 - v_2, h).$$

If $(\mathbf{K}, g) \simeq (\mathbf{L}, h)$ then, from (22), $(G_1 - g, g_1) \simeq (H_1 - h, h_1)$ and so $(G_1, g) \simeq (H_1, h)$. Then $G \simeq H$.

If $(\mathbf{K}, g) \not\simeq (\mathbf{L}, h)$ then, by (10), $(G_1 - u_1, g) \not\simeq (H_1 - v_1, h)$. Therefore $G_1 - u_1$ is isomorphic to a non-radial limb in H, say $(G_1 - u_1, g) \simeq (\mathbf{L}_1, h)$. Similarly we may put

$$(H_1 - v_1, h) \simeq (K_1, g). \tag{23}$$

From (22) $$|E(H_1)| = |E(K)| + |E(G_2)|$$

and from (23) $$|E(H_1)| = |E(K_1)| + 1 \leqslant |E(K)| + 1.$$

Therefore $|E(G_2)| \leqslant 1$. Hence $\mathbf{K}$ and $\mathbf{L}$ are both empty, contradicting the supposition $(\mathbf{K}, g) \not\simeq (\mathbf{L}, h)$.

$(\boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$ From (11) $$|E(G_1)| + |E(K)| = |E(H_2)|$$

and from (22) $$|E(H_2)| + |E(L)| = |E(G_1)|.$$

Therefore $|E(K)| = |E(L)| = 0$ and so $\mathbf{K}$ and $\mathbf{L}$ are both empty. Hence, by (11), (22), $(G_1 - u_1, g) \simeq (H_2 - h, h_2)$ and $(G_1 - g, g_1) \simeq (H_2 - v_2, h)$. Using these two equations one can construct G_1 and H_2 and show that they must both be chains of length r. Similarly G_2 and H_1 are chains of length r. Therefore G and H are both chains of length $2r$ and hence isomorphic.

This completes the proof of the central case of Theorem 1.

4. *Proof of main theorem : bicentral case.* 4.1. The following is the analogue of Theorem 4 for bicentral trees, and may be proved in a similar way.

THEOREM 4A. *Let T be any bicentral tree of radius r with n pendant vertices $(n < N)$, all of which are p-vertices. Suppose there are α distinct subgraphs of G which are isomorphic to T, and that the p-vertex u_i of G is in α_i of these subgraphs; that there are β distinct sub-*

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graphs of H which are isomorphic to T, and that the p-vertex v_i of H is in β_i of these subgraphs. Then $\alpha = \beta, \quad \alpha_i = \beta_i \quad (1 \leqslant i \leqslant N).$

The *central edge* of a bicentral tree is the edge joining its bicentres. A *limb* of a bicentral tree T is a subtree which contains the central edge and both bicentres of T, one as a pendant vertex and the other as a vertex of degree two, and which is joined to the rest of T at the bicentres. Let G have bicentres $g, \bar{g}$ and radius r; let H have bicentres h, $\bar{h}$ and radius r. Denote by $\mathbf{G} = \{G_i\}$, $\bar{\mathbf{G}} = \{\overline{G_i}\}$ respectively, the sets of radial limbs of G having $\bar{g}$, g as pendant vertices, and by $\mathbf{H} = \{H_i\}$, $\bar{\mathbf{H}} = \{\overline{H_i}\}$ respectively, the sets of radial limbs of H having $\bar{h}$, h as pendant vertices; by $\mathbf{K} = \{K_i\}$, $\bar{\mathbf{K}} = \{\overline{K_i}\}$ respectively, the sets of non-radial limbs of G having $\bar{g}$, g as pendant vertices, and by $\mathbf{L} = \{L_i\}$, $\bar{\mathbf{L}} = \{\overline{L_i}\}$ respectively, the sets of non-radial limbs of H having $\bar{h}$, h as pendant vertices. As before we write $K = \bigcup_i K_i$, $\bar{K} = \bigcup_i \bar{K}_i$, $L = \bigcup_i L_i$, $\bar{L} = \bigcup_i \bar{L}_i$.

θ, ϕ, ψ respectively will denote isomorphisms of $G - u_1$ onto $H - v_1$, of $G - u_2$ onto $H - v_2$, and of $G - u_3$ onto $H - v_3$. Again, references to θ, ϕ, ψ in different sections are independent. We extend the notation of Section 3 by writing $(\mathbf{S}, u, u') \simeq (\mathbf{T}, v, v')$ (or $\alpha(\mathbf{S}, u, u') = (\mathbf{T}, v, v')$) if there is an isomorphism α such that $\alpha(\mathbf{S}, u) = (\mathbf{T}, v)$, $\alpha(\mathbf{S}, u') = (\mathbf{T}, v')$.

THEOREM 5A. *Suppose there are $\gamma_G, \bar{\gamma}_G$ p-vertices of G in $\{G_i\}$, $\{\bar{G}_i\}$ respectively, and $\gamma_H, \bar{\gamma}_H$ p-vertices of H in $\{H_i\}$, $\{\bar{H}_i\}$, respectively. We may assume that $g, \bar{g}$ and $h, \bar{h}$ are ordered so that $\gamma_G \leqslant \bar{\gamma}_G$, $\gamma_H \leqslant \bar{\gamma}_H$. Then $\gamma_G = \gamma_H$, $\bar{\gamma}_G = \bar{\gamma}_H$.*

Proof. G and H each have N p-vertices.

If $N = 2$, then $\gamma_G = \bar{\gamma}_G = \gamma_H = \bar{\gamma}_H = 1$.

If $N > 2$, let u_i be a p-vertex in $\{G_i\}$ Then u_i lies on $\bar{\gamma}_G$ chains of length $2r-1$ in G. Hence, by Theorem 4A, v_i lies on $\bar{\gamma}_G$ chains of length $2r-1$ in H. If $v_i \in \{H_i\}$ then $\bar{\gamma}_G = \bar{\gamma}_H$ and $\gamma_G = N - \bar{\gamma}_G = N - \bar{\gamma}_H = \gamma_H$. If $v_i \in \{\bar{H}_i\}$ then $\bar{\gamma}_G = \gamma_H$ and $\bar{\gamma}_H = \gamma_G$. Therefore $\bar{\gamma}_G = \gamma_H \leqslant \bar{\gamma}_H = \gamma_G$, and since $\gamma_G \leqslant \bar{\gamma}_G$ by assumption, we have

$$\gamma_G = \gamma_H = \bar{\gamma}_G = \bar{\gamma}_H.$$

In view of Theorem 5A we write

$$\gamma_G = \gamma_H = \gamma, \quad \bar{\gamma}_G = \bar{\gamma}_H = \bar{\gamma} \quad \text{where} \quad \gamma \leqslant \bar{\gamma}.$$

4.2. $\gamma > 1$. Then for any p-vertex u_i in G, $G - u_i$ and $H - v_i$ are bicentral trees of radius r. If X is a bicentral tree, denote by X' the central tree obtained from X by adding a vertex on its central edge.

$$\begin{aligned} &G - u_i \simeq H - v_i \quad (1 \leqslant i \leqslant N) \\ &\Rightarrow (G - u_i)' \simeq (H - v_i)' \quad (1 \leqslant i \leqslant N) \\ &\Rightarrow G' - u_i \simeq H' - v_i \quad (1 \leqslant i \leqslant N), \end{aligned}$$

since $(G - u_i)' = G' - u_i$ and $(H - v_i)' = H' - v_i$

$$\Rightarrow G' \simeq H',$$

since Theorem 1 has been proved for central trees

$$\Rightarrow G \simeq H.$$

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4.3. $\gamma = 1$. The proof will be based on showing that $(\mathbf{G}, \bar{g}) \simeq (\mathbf{H}, \bar{h})$, $(\bar{\mathbf{G}}, g) \simeq (\bar{\mathbf{H}}, h)$, $(\mathbf{K}, g, \bar{g}) \simeq (\mathbf{L}, h, \bar{h})$, $(\bar{\mathbf{K}}, g, \bar{g}) \simeq (\bar{\mathbf{L}}, h, \bar{h})$.

Let u_1 be the p-vertex in G_1, and let u_2 be a p-vertex in $\bar{G}_1$. We may assume that $v_1 \in H_1$; then v_2 is in a limb of $\bar{\mathbf{H}}$, which we may take to be $\bar{H}_1$. Since $G - u_1$, $H - v_1$ are central trees with centres $\bar{g}$, $\bar{h}$ respectively, $\theta(G - u_1) = H - v_1 \Rightarrow$

$$\theta(\bar{\mathbf{K}}, g, \bar{g}) = (\bar{\mathbf{L}}, h, \bar{h}), \quad \theta([(G_1 - u_1) \cup K] \mathbf{U} \{\bar{G}_i - g\}, \bar{g}) = ([(H_1 - v_1) \cup L] \mathbf{U} \{\bar{H}_i - h\}, \bar{h}). \tag{24}$$

(*a*) $\bar{\gamma} > 2$. $G - u_2$ and $H - v_2$ are bicentral trees and therefore

$$\phi(G_1, \bar{g}) = (H_1, \bar{h}), \quad \phi(\mathbf{K}, g, \bar{g}) = (\mathbf{L}, h, \bar{h}). \tag{25}$$

Hence $\phi((G_1 - u_1) \cup K, \bar{g}) = ((H_1 - v_1) \cup L, \bar{h})$ and so, from (24),

$$(\{\bar{G}_i - g\}, \bar{g}) \simeq (\{\bar{H}_i - h\}, \bar{h}).$$

Therefore $(\bar{\mathbf{G}}, g) \simeq (\bar{\mathbf{H}}, h)$ (by an argument analogous to that used in (13)) and, with (24) and (25), this shows that $G \simeq H$.

(*b*) $\bar{\gamma} = 2$. Let u_3 be the other p-vertex in $\bar{\mathbf{G}}$, say $u_3 \in \overline{G_j}$; then v_3 is in a limb $\bar{H}_k$ of $\bar{\mathbf{H}}$. $\phi(G - u_2) = H - v_2 \Rightarrow$ either $(\boldsymbol{\alpha}_1)$: $\phi(g) = h$, $\phi(\bar{g}) = \bar{h}$ or $(\boldsymbol{\alpha}_2)$: $\phi(g) = \bar{h}$, $\phi(\bar{g}) = h$.

$$(\boldsymbol{\alpha}_1) \Rightarrow \phi(G_1, \bar{g}) = (H_1, \bar{h}), \quad \phi(\mathbf{K}, g, \bar{g}) = (\mathbf{L}, h, \bar{h}), \tag{26}$$

$$(\boldsymbol{\alpha}_2) \Rightarrow \left.\begin{aligned} \phi([G_1] \mathbf{U} \mathbf{K}, g, \bar{g}) &= (\{\bar{H}_i\}_{i \neq 1} \mathbf{U} [\bar{H}_1 - v_2] \mathbf{U} \bar{\mathbf{L}}, \bar{h}, h), \\ \phi(\{\bar{G}_i\}_{i \neq 1} \mathbf{U} [\bar{G}_1 - u_2] \mathbf{U} \bar{\mathbf{K}}, g, \bar{g}) &= ([H_1] \mathbf{U} \mathbf{L}, \bar{h}, h). \end{aligned}\right\} \tag{27}$$

$\psi(G - u_3) = H - v_3 \Rightarrow$ either $(\boldsymbol{\beta}_1)$: $\psi(g) = h, \psi(\bar{g}) = \bar{h}$ or $(\boldsymbol{\beta}_2)$: $\psi(g) = \bar{h}, \psi(\bar{g}) = h$.

$$(\boldsymbol{\beta}_1) \Rightarrow \psi(G_1, \bar{g}) = (H_1, \bar{h}) \quad \psi(\mathbf{K}, g, \bar{g}) = (\mathbf{L}, h, \bar{h}), \tag{28}$$

$$(\boldsymbol{\beta}_2) \Rightarrow \left.\begin{aligned} \psi([G_1] \mathbf{U} \mathbf{K}, g, \bar{g}) &= (\{\bar{H}_i\}_{i \neq k} \mathbf{U} [\bar{H}_k - v_3] \mathbf{U} \bar{\mathbf{L}}, \bar{h}, h), \\ \psi(\{\bar{G}_i\}_{i \neq j} \mathbf{U} [\bar{G}_j - u_3] \mathbf{U} \bar{\mathbf{K}}, g, \bar{g}) &= ([H_1] \mathbf{U} \mathbf{L}, \bar{h}, h). \end{aligned}\right\} \tag{29}$$

If $(\boldsymbol{\alpha}_1)$ holds, then, by (26), $((G_1 - u_1) \cup K, \bar{g}) \simeq ((H_1 - v_1) \cup L, \bar{h})$. Hence, as in 4.3.(*a*), (24) and (26) combine to show that $G \simeq H$. Similarly $G \simeq H$ if $(\boldsymbol{\beta}_1)$ holds.

Otherwise both $(\boldsymbol{\alpha}_2)$ and $(\boldsymbol{\beta}_2)$ hold. From (24) $\bar{\mathbf{G}}$, $\bar{\mathbf{H}}$ have the same number of radial limbs. Since $\bar{\gamma} = 2$ either

$$\bar{\mathbf{G}} = \{\bar{G}_1, \bar{G}_2\}, \quad \bar{\mathbf{H}} = \{\bar{H}_1, \bar{H}_2\} \quad \text{or} \quad \bar{\mathbf{G}} = \{\bar{G}_1\}, \quad \bar{\mathbf{H}} = \{\bar{H}_1\}.$$

If $\bar{\mathbf{G}} = \{\bar{G}_1, \bar{G}_2\}$, $\bar{\mathbf{H}} = \{\bar{H}_1, \bar{H}_2\}$ then from (27)

$$\phi(G_1, \bar{g}) = (\bar{H}_2, h), \quad \phi(\mathbf{K}, g, \bar{g}) = ([\bar{H}_1 - v_2] \mathbf{U} \bar{\mathbf{L}}, \bar{h}, h),$$

$$\phi(\bar{G}_2, g) = (H_1, \bar{h}), \quad \phi([\bar{G}_1 - u_2] \mathbf{U} \bar{\mathbf{K}}, g, \bar{g}) = (\mathbf{L}, \bar{h}, h)$$

and from (29) $$\psi(G_1, \bar{g}) = (\bar{H}_1, h), \quad \psi(\bar{G}_1, g) = (H_1, \bar{h}).$$

Hence $$\phi\psi^{-1}(\bar{H}_1, h) = \phi(G_1, \bar{g}) = (\bar{H}_2, h) \tag{30}$$

and $$\phi^{-1}\psi(\bar{G}_1, g) = \phi^{-1}(H_1, \bar{h}) = (\bar{G}_2, g). \tag{31}$$

By (30) $(\bar{H}_1 - h, \bar{h}) \simeq (G_1 - \bar{g}, g) \simeq (\bar{H}_2 - h, \bar{h})$ and therefore, from (24), either

$$(\bar{G}_1 - g, \bar{g}) \simeq (G_1 - \bar{g}, g) \quad \text{or} \quad (\bar{G}_2 - g, \bar{g}) \simeq (G_1 - \bar{g}, g),$$

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i.e. either $(\bar{G}_1, g) \simeq (G_1, \bar{g})$ or $(\bar{G}_2, g) \simeq (G_1, \bar{g})$. (32)

(30)–(32) show that

$$(G_1, \bar{g}) \simeq (H_1, \bar{h}) \simeq (\bar{G}_1, g) \simeq (\bar{H}_1, h) \simeq (\bar{G}_2, g) \simeq (\bar{H}_2, h)$$

(which imply that $(\bar{G}_1 - u_2, g) \simeq (\bar{H}_1 - v_2), h)$.

From (24) $(\bar{\mathbf{K}}, g\bar{g}) \simeq (\bar{\mathbf{L}}, h, \bar{h})$ and (27) yields $(\mathbf{K}, g, \bar{g}) \simeq (\mathbf{L}, h, \bar{h})$. (We have here assumed that $(\bar{\mathbf{G}}, g) \simeq (\bar{\mathbf{H}}, h) \Rightarrow (\bar{\mathbf{G}}, g, \bar{g}) \simeq (\bar{\mathbf{H}}, h, \bar{h})$, a fact which follows from the definitions. Similarly, we shall use the fact that $(\mathbf{G}, \bar{g}) \simeq (\mathbf{H}, \bar{h}) \Rightarrow (\mathbf{G}, g, \bar{g}) \simeq (\mathbf{H}, h, \bar{h})$.)
Hence $G \simeq H$.

If $\bar{\mathbf{G}} = \{G_1\}, \bar{\mathbf{H}} = \{H_1\}$ then, from (27) and (29),

$$\phi(\mathbf{K}, g, \bar{g}) = (\bar{\mathbf{L}}, \bar{h}, h), \quad \phi(\bar{\mathbf{K}}, g, \bar{g}) = (\mathbf{L}, \bar{h}, h) \tag{33}$$

and

$$\left.\begin{aligned} \phi^{-1}(\bar{H}_1 - v_2, \bar{h}, h) = (G_1, g, \bar{g}) = \psi^{-1}(\bar{H}_1 - v_3, \bar{h}, h), \\ \phi(\bar{G}_1 - u_2, g, \bar{g}) = (H_1, \bar{h}, h) = \psi(\bar{G}_1 - u_3, g, \bar{g}). \end{aligned}\right\} \tag{34}$$

By (24), $(\bar{\mathbf{K}}, g, \bar{g}) \simeq (\bar{\mathbf{L}}, h, \bar{h})$ and hence, using (33),

$$(\bar{\mathbf{K}}, g, \bar{g}) \simeq (\bar{\mathbf{L}}, h, \bar{h}) \simeq (\mathbf{K}, \bar{g}, g) \simeq (\mathbf{L}, \bar{h}, h). \tag{35}$$

Also from (24) either $(\bar{G}_1 - g, \bar{g}) \simeq (\bar{H}_1 - h, \bar{h})$ or $(\bar{G}_1 - g, \bar{g}) \simeq ((H_1 - v_1) \cup L, \bar{h})$. If $(\bar{G}_1 - g, \bar{g}) \simeq (\bar{H}_1 - h, \bar{h})$ then $(\bar{G}_1, g) \simeq (\bar{H}_1, h)$ and under this isomorphism u_2 corresponds to v_2 or to v_3. Therefore either

$$(\bar{G}_1 - u_2, g) \simeq (\bar{H}_1 - v_2, h) \quad \text{or} \quad (\bar{G}_1 - u_2, g) \simeq (\bar{H}_1 - v_3, h).$$

In both cases (34) shows that $(G_1, \bar{g}) \simeq (H_1, \bar{h})$ and so $G \simeq H$. If

$$(\bar{G}_1 - g, \bar{g}) \simeq ((H_1 - v_1) \cup L, \bar{h})$$

then

$$|E(\bar{G}_1)| = |E(H_1)| + |E(L)| - 1.$$

But from (34)

$$|E(\bar{G}_1)| = |E(H_1)| + 1.$$

Therefore $|E(L)| = 2$ and so L has just two edges, one being the central edge. By (35), the same is true of K, $\bar{K}$ and $\bar{L}$. As in 3.2.(*b*) we may construct the graphs $\bar{G}_1, H_1$, and $\bar{H}_1, G_1$, and show that $G \simeq H$. In fact G and H are chains of length $2r-1$ with an edge attached to each vertex except two at one end and one at the other.

(*c*) $\bar{\gamma} = 1$. From (24), $\theta(\bar{\mathbf{K}}, g, \bar{g}) = (\bar{\mathbf{L}}, h, \bar{h})$ and

$$\theta([(G_1 - u_1) \cup K] \cup [\bar{G}_1 - g], \bar{g}) = ([(H_1 - v_1) \cup L] \cup [\bar{H}_1 - h], \bar{h}). \tag{36}$$

$\phi(G - u_2) = H - v_2 \Rightarrow \phi(\mathbf{K}, g, \bar{g}) = (\mathbf{L}, h, \bar{h})$ and

$$\phi([(\bar{G}_1 - u_2) \cup \bar{K}] \cup [G_1 - \bar{g}], g) = ([(\bar{H}_1 - v_2) \cup \bar{L}] \cup [H_1 - \bar{h}], h). \tag{37}$$

If $(G_1, \bar{g}) \simeq (H_1, \bar{h})$ then $((G_1 - u_1) \cup K, \bar{g}) \simeq ((H_1 - v_1) \cup L, \bar{h})$ and so, by (36),

$$(\bar{G}_1 - g, \bar{g}) \simeq (\bar{H}_1 - h, \bar{h}).$$

Hence $(\bar{G}_1, g) \simeq (\bar{H}_1, h)$ and $G \simeq H$. Similarly if $(\bar{G}_1, g) \simeq (\bar{H}_1, h)$ then (37) implies that $(G_1, \bar{g}) \simeq (H_1, \bar{h})$ and $G \simeq H$.

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Otherwise, from (36) and (37),

$$((H_1-v_1)\cup L,\bar{h})\simeq(\bar{G}_1-g,\bar{g}),\quad((\bar{G}_1-u_2)\cup\bar{K},g)\simeq(H_1-\bar{h},h),\tag{38}$$

$$((G_1-u_1)\cup K,\bar{g})\simeq(\bar{H}_1-h,\bar{h}),\quad((\bar{H}_1-v_2)\cup\bar{L},h)\simeq(G_1-\bar{g},g).\tag{39}$$

From (38) $$|E(H_1)|+|E(L)|=|E(\bar{G}_1)|+1$$

and $$|E(\bar{G}_1)|+|E(\bar{K})|=|E(H_1)|+1.$$

Therefore $|E(L)|=|E(\bar{K})|=1$ and $L,\bar{K}$ each consist solely of the central edge, which is common to all limbs. Hence $\mathbf{L},\bar{\mathbf{K}}$ and $\mathbf{K},\bar{\mathbf{L}}$ are all empty. Then equations (38) imply that $\bar{G}_1$, H_1 are chains of length r. Similarly, by (39), G_1, $\bar{H}_1$ are chains of length r.

Therefore G and H are both chains of length $2r-1$.

This completes the proof of the bicentral case of Theorem 1.

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