Math. Struct. in Comp. Science (2015), *vol.* 25, *pp.* 46–82. © Cambridge University Press 2014
doi:10.1017/S0960129514000036

Inclusions for partiality

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Received 15 May 2013; revised 31 October 2013

In this paper we introduce *stable systems of inclusions*, which feature chosen arrows $A \hookrightarrow B$ to capture the notion that A is a subobject of B, and proposes them as an alternative context to stable systems of monics to discuss partiality. A category $\mathbf{C}$ equipped with such a system $\mathscr{I}$, called an *i-category*, is shown to give rise to an associated category $\partial(\mathbf{C},\mathscr{I})$ of partial maps, which is a split restriction category whose restriction monics are inclusions. This association is the object part of a 2-equivalence between such *inclusively* split restriction categories and i-categories. $\mathscr{I}$ determines a stable system of monics $\mathscr{I}^+$ on $\mathbf{C}$, and, conversely, a stable system of monics $\mathscr{M}$ on $\mathbf{C}$ yields an i-category $\left(\mathbf{C}[\mathscr{M}],\mathscr{M}^+\right)$, giving a 2-adjunction between i-categories and m-categories. The category of partial maps $\mathsf{Par}(\mathbf{C},\mathscr{M})$ is isomorphic to the full subcategory of $\partial\left(\mathbf{C}[\mathscr{M}],\mathscr{M}^+\right)$ comprising the objects of $\mathbf{C}$, and $\partial(\mathbf{C},\mathscr{I}) \cong \mathsf{Par}\left(\mathbf{C},\mathscr{I}^+\right)$.

1. Introduction

It is in the nature of computation that functions are often only partially defined. Typically, the algorithm that implements a function also determines which data inputs the function is defined on. This state of affairs is reflected in such notions as the μ-recursive functions of computability theory, and abstract data types with partial operations in software engineering. In view of their ubiquitous role in computing science, substantial work has been done in recent decades on a category theoretic treatment of partial functions.

Partial functions $A \to B$ amount to pairs consisting of a subset X of A – the domain of definition – and a function $X \to B$. General categories, lacking actual 'subobjects', do not admit an analogously straightforward description of a 'partial map'. A partial map $A \to B$ as a pair (m, f) consisting of a monic $m : X \to A$ and an arrow $f : X \to B$ will not do: in Set, these do not give the partial functions because there is no 1–1 correspondence between injective functions into a set A and subsets of A. The prevailing approach is to take an *equivalence class* of such pairs, with $(m, f) \sim (n, g)$ if there exists an isomorphism r such that $m = rn$ and $f = rg$. To arrive at a theory of partiality, these monics are drawn from what is called a *stable system of monics* in Cockett and Lack (2002), and a *dominion* in Rosolini (1986), where the concept appears to originate. Thus, a category $\mathbf{C}$ with a stable system of monics $\mathscr{M}$, called an *m-category*[†], determines a category $\mathsf{Par}(\mathbf{C},\mathscr{M})$ of partial maps.

[†] The concept is called an '$\mathscr{M}$-category' in Cockett and Lack (2002).

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Conversely, we can *specify* categories of partial maps using operations and equations. According to Rosolini (1986), the first to do so were Robert A. Di Paola and Alex Heller in the form of *dominical categories* (Heller 1987). Rosolini, working alone and then with E. Robinson, introduced the related but simplified notion of *p-categories* Rosolini (1986) and Robinson and Rosolini (1988). The most recent and most general refinement of these ideas are the *restriction categories* pioneered by J.R.B. Cockett and Stephen Lack, in which every arrow has an associated idempotent on its domain to represent its degree of partiality, subject to four simple axioms – see Cockett and Lack (2002), Cockett and Lack (2003), Cockett and Lack (2007) and Cockett and Manes (2009), which also discuss the history of the field. Restriction categories form a 2-category rCat, an important full sub 2-category of which is defined by the *split* restriction categories: those in which every idempotent splits into a *restriction monic* and *restriction retraction*. A basic result states that the map $(\mathbf{C},\mathscr{M}) \mapsto \mathsf{Par}\,(\mathbf{C},\mathscr{M})$ constitutes a 2-equivalence $\mathsf{mCat} \simeq \mathsf{rCat}_s$ between m-categories and split restriction categories (Cockett and Lack 2002).

In the current paper, we propose another alternative, in which we legislate for subobjects directly. In fact, this has been done before, though not in the context of partiality. We get from Căzănescu and Roşu (2000) that a category has *inclusions* if it has a broad subcategory[†], which amounts to a partial order on its objects, the *subobject* relation. Inclusions originated as part of *inclusion systems*, which were introduced in Diaconescu *et al.* (1993), which was a general study of the properties required of logical systems to support modularisation. They have since been used to give semantics to the module system of the algebraic specification language CafeOBJ (Diaconescu and Futatsugi 1998), and they play an important role in abstract model theory (Diaconescu 2008). Inclusion systems have been much clarified and developed by Grigore Roşu and Virgil Emil Căzănescu (Căzănescu and Roşu 1997; Căzănescu and Roşu 2000).

The *i-categories* introduced in Section 2 of the current paper are categories $\mathbf{C}$ with a *stable system of inclusions* $\mathscr{I}$ – inclusions subject to axioms reminiscent of m-categories. Like the latter, they form a 2-category, iCat. Examples include subsets, subcategories, subalgebras and topological subspaces. We will describe several methods for building i-categories. We will also discuss the dual notion, in the form of *stable surclusions*, in Section 3, giving constructions for the categories of topological spaces, well-founded sets and frames (which imply stable inclusions for locales).

In the context of i-categories (Section 4) a partial map $A \to B$ is a pair (X, f) consisting of a subobject X of A and an arrow $f : X \to B$ of $\mathbf{C}$. To avoid confusion with the established notion, we call them *partial arrows*. They form a split restriction category $\partial\,(\mathbf{C},\mathscr{I})$. In fact, $\partial\,(\mathbf{C},\mathscr{I})$ is *inclusively* split, meaning it has right cancellable inclusions[‡] and its restriction monics are inclusions. This notion is introduced and explored in Section 5, culminating in a 2-equivalence $\mathsf{iCat} \simeq \mathsf{rCat}_{is}$ between i-categories and inclusively split restriction categories, similar to $\mathsf{mCat} \simeq \mathsf{rCat}_s$ (Cockett and Lack 2002).

[†] A subcategory X of Y is broad if $|X| = |Y|$.

[‡] Any arrow f whose composition $f\,;i$ with an inclusion i is an inclusion is itself an inclusion.

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Cockett and Lack (2002) proved that every restriction category $\mathbf{X}$ is isomorphic to a full subcategory of a split restriction category $K_r(\mathbf{X})$, the values of a 2-functor $K_r : \mathsf{rCat} \to \mathsf{rCat}_s$ that is left biadjoint to the inclusion $\mathsf{rCat}_s < \mathsf{rCat}$. It was also shown by general algebraic means that this inclusion also has a left 2-adjoint L, to which K_r is therefore naturally equivalent. A key result of the current paper (Theorem 6.1) is that $K_r(\mathbf{X})$ comes naturally equipped with inclusions that render it *inclusively* split, and that the map $\mathbf{X} \mapsto K_r(\mathbf{X})$ constitutes a left 2-adjoint to the forgetful 2-functor $U : \mathsf{rCat}_{is} \to \mathsf{rCat}$.

In Section 7, we apply this result to undertake a comparison between the two approaches: monics versus inclusions. First, it implies that there exists a 2-adjunction between i-categories and m-categories (Theorem 7.1). If a category $\mathbf{C}$ has a stable system of inclusions $\mathscr{I}$, then it also has a stable system of monics $\mathscr{I}^+$, which is obtained by composing inclusions and isomorphisms. Conversely, every m-category $(\mathbf{C},\mathscr{M})$ gives rise to a free i-category $\left(\mathbf{C}[\mathscr{M}],\mathscr{M}^+\right)$, with the property that $\mathbf{C}$ is (up to isomorphism) a full subcategory of $\mathbf{C}[\mathscr{M}]$. Theorem 7.2 states that $\partial(\mathbf{C},\mathscr{I})$ and $\mathsf{Par}\left(\mathbf{C},\mathscr{I}^+\right)$ are isomorphic restriction categories. In this case, there is no need for a wider class of monics, and partiality can be expressed directly rather than through equivalence classes. But even when $\mathbf{C}$ has no inclusions, we can work with partiality based on inclusions: Theorem 7.3 says that there is an isomorphism of restriction categories between $\mathsf{Par}(\mathbf{C},\mathscr{M})$ and a full subcategory of $\partial\left(\mathbf{C}[\mathscr{M}],\mathscr{M}^+\right)$ corresponding to the objects of $\mathbf{C}$.

The upshot of Theorems 7.2 and 7.3 is that no generality is lost in basing partiality on inclusions rather than monics. One might say that there is little benefit to using inclusions as no generality is gained either, but that would be to miss the point that inclusions bring important advantages. With regard to theoretical discussions, partial maps based on inclusions are technically much simpler. Furthermore, they directly express the intuition that a partial map has a domain of definition that is a subobject of its domain. Partiality based on stable inclusions is also easier to apply. In any specific category with chosen stable inclusions, the partial maps are immediately known. In contrast, the monics-based partial maps involve a tedious and possibly difficult process of identification. To take Cat as an example, we must *demonstrate* that the partial maps based on all monic functors coincide with the partial functors.

Finally, given that we have restriction categories, we might ask why we should bother with i-categories at all. After all, they are more general, simply specified by equations, and rCat has nice model-theoretic properties thanks to these equations. The answer, essentially, is that they are complementary. In contrast to restriction categories, i-categories are not, *a priori*, about partiality. They are about subobjects, with partiality *arising* in a specific, concrete process. In that sense, they serve as a source of examples of restriction categories. The fact that this process is so intuitive also means that i-categories can be a useful point of departure for generalising partiality to realms other than Cat: that is, we adapt the subobject notion, and partiality follows. For example, in a paper soon to be submitted for publication, we consider partiality in a 2-category. Whereas the step to 2i-categories is easy and obvious, 2-restriction categories are a much more difficult proposition; partiality based on the former is essential to guide and validate the latter. Finally, as we mentioned earlier, Theorem 6.1 tells us that inclusions and i-categories arise naturally in the context of restriction categories.

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Notation

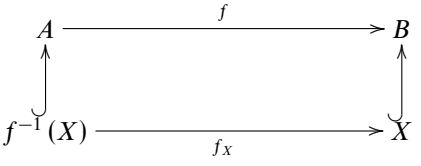
Composition is always written in diagrammatic order and denoted by juxtaposition, though we will sometimes write $f;g$ where fg might be confusing. Functorial and functional application are written in similar style, except in rare cases where this might be confusing, when we write aF instead of $F(a)$.

2. Inclusions for partiality

It is a basic fact that preordered sets amount to categories that have at most one arrow between any pair of objects, with $A \leqslant B$ if and only if $A \to B$, and reflexivity and transitivity corresponding to identities and composition. The notion of a *category of inclusions*, which was first introduced in Căzănescu and Roşu (2000, Definition 5), corresponds to a preorder that is also anti-symmetric, in other words a partial order: if $A \to B$ and $B \to A$, then $A = B$. Furthermore, a category $\mathbf{C}$ *has inclusions* (Căzănescu and Roşu 2000, Definition 7) if it has a broad subcategory $\mathscr{I}$ of inclusions. In order to facilitate partiality in $\mathbf{C}$, some extra properties are required of the pair $(\mathbf{C},\mathscr{I})$, which are listed in the following definition. We will follow Căzănescu and Roşu (2000) by referring to the arrows of $\mathscr{I}$ as inclusions, denoting them by $A \hookrightarrow B$ and calling such an A a *subobject* of B. We also say that a set of objects is *bounded* if there exists an object, called a *bound*, of which each member of the set is a subobject.

Definition 2.1. A category $\mathbf{C}$ has a *stable system of inclusions* $\mathscr{I}$ if it has a broad subcategory $\mathscr{I}$ such that:

(I.0) For any A, B there is at most one inclusion $A \hookrightarrow B$. If $B \hookrightarrow A$ also, then $A = B$.
(I.1) Any $f : A \to B$ is an inclusion if its composition with some $B \hookrightarrow C$ is an inclusion.
(I.2) For any $f : A \to B$ and $X \hookrightarrow B$, there exists a subobject $f^{-1}(X)$ of A and an arrow $f_X : f^{-1}(X) \to X$ such that



is a pullback diagram.

We call the pair $(\mathbf{C},\mathscr{I})$ an *i-category*.

Following Căzănescu and Roşu (1997), we refer to axiom (I.1) as *right cancellability*. We call the f_X of (I.2) the *restriction of f into X* and we call $f^{-1}(X)$[†] the *inverse image of X under f*. If $f^{-1}(X) = A$, then f is said to *restrict into X*. Căzănescu and Roşu (1997, Proposition 25) implies that restrictions into subobjects exist in $\mathbf{C}$ if $\mathbf{C}$ has a weak inclusion system and pullbacks. We call the restriction of an inclusion $A \hookrightarrow B$ into X the *bounded*

[†] The notation $f^{-1}(X)$ is from Căzănescu and Roşu (2000).

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intersection of A and X in **C**. More generally, we call the limit of the diagram $X \hookrightarrow A$ as X ranges over a set $\mathcal{X}$ the *bounded intersection of* $\mathcal{X}$.

The compositions of an arrow $f : A \to B$ with inclusions $X \hookrightarrow A$ and $B \hookrightarrow Y$ are written $f|_X$ and $f|^Y$, respectively, and called the *restriction of f to X* and the *extension of f to Y*, respectively.

Lemma 2.2. Suppose **C** has a stable system of inclusions $\mathscr{I}$. Then

(1) Restrictions into subobjects are unique.
(2) If $f : A \to B$ equals $f'|^B$ for some $f' : A \to X$, then

$$f^{-1}(X) = A$$

and

$$f' = f_X.$$

(3) The inclusions are monic in **C**.
(4) Every right-inverse inclusion is an identity.
(5) The restriction of an inclusion into a subobject is a product in $\mathscr{I}$.
(6) **C** is closed under finite bounded intersections.

Proof.

(1) Let $g : Y \to X$ be another restriction of $f : A \to B$ into X. Then it follows from the pullback property of $f^{-1}(X)$ and (I.1) that

$$Y \hookrightarrow f^{-1}(X).$$

Similarly,

$$f^{-1}(X) \hookrightarrow Y,$$

so $Y = f^{-1}(X)$ and $g = f_X$.

(2) Since $1_A f = f'|^C$, it follows from (I.1) that $A \hookrightarrow f^{-1}(X)$. But $f^{-1}(X) \hookrightarrow A$ too, so the result follows.

(3) Suppose $f, g : A \to B$, $B \hookrightarrow C$ such that $f|^C = g|^C$. Then

$$\begin{aligned} f &= \left(f|^C\right)_B \\ &= \left(g|^C\right)_B \\ &= g \end{aligned}$$

by (2).

(4) Let $A \hookrightarrow B$ have a left inverse r. Then r is an inclusion by (I.1), so $A = B$ and $r = 1$.

(5) This is immediate from (I.1).

(6) Let $A_1, \cdots, A_n$ be bounded by A. We will prove by induction on n that their product (in $\mathscr{I}$) L_n is the limit of the diagram $A_i \hookrightarrow A$ where i varies from 1 to n.

The result is trivially true when $n = 1$.

For $n \geqslant 1$, let

$$L_{n+1} = L_n \cap A_{n+1}.$$

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The inductive step then follows easily from the induction hypothesis and the fact that



is a pullback diagram. □

Products in $\mathscr{I}$ are called *intersections* in Căzănescu and Roşu (2000), so bounded intersections are intersections, and thus independent of the bound. But the converse is not true: for example, subgroups constitute stable inclusions in the category of small groups (see Examples 2.10, 2.11), but not every pair of groups has an intersection.

The following definition corresponds to the definition of an *m-functor* in Cockett and Lack (2002).

Definition 2.3. An *i-functor* $\mathbf{F} : (\mathbf{C},\mathscr{I}) \to (\mathbf{D},\mathscr{J})$ is a functor $\mathbf{F} : \mathbf{C} \to \mathbf{D}$ that:

(1) maps $\mathscr{I}$-inclusions to $\mathscr{J}$-inclusions;
(2) preserves restrictions into subobjects.

Lemma 2.2 implies i-functors preserve bounded intersections. The *i-cartesian* notion in the following definition is the *m-cartesian* notion in Cockett and Lack (2002) applied to inclusions.

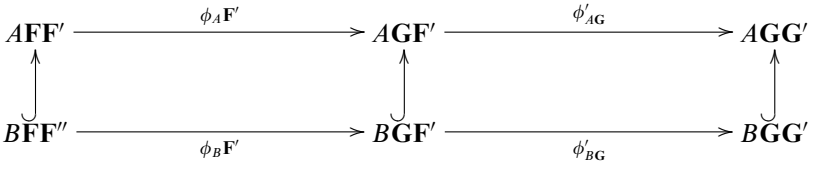
Definition 2.4. Let $\mathbf{F},\mathbf{G} : (\mathbf{C},\mathscr{I}) \to (\mathbf{D},\mathscr{J})$ be i-functors. Then an *i-cartesian natural transformation* $\phi : \mathbf{F} \to \mathbf{G}$ is a natural transformation that is *i-cartesian*: whenever $B \hookrightarrow A$, the component ϕ_B is the restriction of ϕ_A into $B\mathbf{G}$.

Proposition 2.5. i-Categories, i-functors and i-cartesian natural transformations form a 2-category iCat.

Proof. It follows easily from the definitions that i-functors are closed under composition and include the identity functors, and that i-cartesian natural transformations are closed under vertical composition and include the vertical identities. For closure under horizontal composition, let

$$\phi : \mathbf{F} \to \mathbf{G} : (\mathbf{C},\mathscr{I}) \to (\mathbf{D},\mathscr{J})$$
$$\phi' : \mathbf{F}' \to \mathbf{G}' : (\mathbf{D},\mathscr{J}) \to (\mathbf{E},\mathscr{K})$$

be i-cartesian, and $B \hookrightarrow A$. Then for any objects A, B of $\mathbf{C}$, the composite



is a pullback because ϕ and ϕ' are i-cartesian and $\mathbf{F}'$ preserves restrictions into subobjects. The conclusion then follows because the horizontal composites are $(\phi \fatsemi \phi')_A$ and $(\phi \fatsemi \phi')_B$,

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together with associativity, the identity laws and the interchange law are inherited from 2Cat. □

Lemma 2.6. An i-functor $\mathbf{F} : (\mathbf{C},\mathscr{I}) \to (\mathbf{D},\mathscr{J})$ is an isomorphism if $\mathbf{C} \cong \mathbf{D}$ and $\mathscr{I} \cong \mathscr{J}$ under $\mathbf{F}$ (in Cat).

Proof. The functor $\mathbf{F}$ has an inverse $\mathbf{G}$ that preserves inclusions because $\mathbf{F}$ only maps inclusions to inclusions. Therefore, it just remains to show that $\mathbf{G}$ preserves restrictions into subobjects (Definition 2.3). So let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \uparrow & (\dagger) & \uparrow \\ X & \xrightarrow[g]{} & C \end{array}$$

be a pullback in $\mathbf{D}$, let $f'\mathbf{F} = f$ and $C'\mathbf{F} = C$ for some f' and C', and let

$$\begin{array}{ccc} A' & \xrightarrow{f'} & B' \\ \uparrow & (\ddagger) & \uparrow \\ X' & \xrightarrow[g']{} & C' \end{array}$$

be the pullback in $\mathbf{C}$. Since $\mathbf{F}$ preserves restrictions into subobjects, it follows from Lemma 2.2 that (†) is the image of (‡) under $\mathbf{F}$. Hence (‡) is the image of (†) under $\mathbf{G}$. □

Example 2.7 (identities). The identity arrows constitute stable inclusions in any category.

Example 2.8 (sets). It is clear that the subset relation yields right cancellable inclusions for Set. For (I.2), the restriction of a function $F : A \to B$ into a subset $B' \subseteq B$ is the inverse image $F^{-1}(B')$.

Example 2.9 (topological spaces). For topological spaces X, Y, recall that X is a subspace of Y if $X \subseteq Y$ and X has the subspace topology. The inclusion of X into Y is a continuous map, and such maps constitute right cancellable inclusions for Top, the category of topological spaces and continuous maps. For (I.2), suppose $Y' \hookrightarrow Y$ and $f : X \to Y$ is continuous. Then their pullback in Set is the set

$$X' = \{x \in X | xf \in Y'\}$$

with projections the set inclusion $X' \hookrightarrow X$ and the function $g : X' \to Y'$ that maps any x to xf. With the subspace topology on X', the former becomes a topological inclusion and the latter continuous, since

$$\begin{aligned} g^{-1}(O \cap Y') &= g^{-1}(O) \cap g^{-1}(Y') \\ &= g^{-1}(O) \cap X', \end{aligned}$$

which is open in X' whenever O is open in Y. So $X' \hookrightarrow X$, f' equalise f, $Y' \hookrightarrow Y$ in Top. It is easy to verify that they constitute a pullback.

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Example 2.10 (many-sorted algebras). In the category $\mathsf{Alg}(S,\Sigma)$, we take for the inclusions those homomorphisms $A \to B$ whose underlying maps $A_s \to B_s$ are set inclusions. In other words, those families of set inclusions $A_s \hookrightarrow B_s$ such that the square

$$\begin{array}{ccc} A_{s_1} \times \cdots \times A_{s_n} & \xrightarrow{\subseteq} & B_{s_1} \times \cdots \times B_{s_n} \\ \downarrow{\scriptstyle \rho_A} & & \downarrow{\scriptstyle \rho_B} \\ A_s & \xrightarrow[\subseteq]{} & B_s \end{array}$$

commutes for every operation symbol $\rho : s_1 \cdots s_n \to s$ of Σ. It is clear that they constitute right cancellable inclusions. For (I.2), suppose $B' \hookrightarrow B$ and let $h : A \to B$ be a Σ-homomorphism. Define A'_s, h'_s by the pullback diagrams

$$\begin{array}{ccc} A_s & \xrightarrow{h_s} & B_s \\ \uparrow & & \uparrow \\ A'_s & \xrightarrow[h'_s]{} & B'_s \end{array}$$

for each $s \in S$. Then it follows that there exist unique arrows

$$\rho_{A'} : A'_{s_1} \times \cdots \times A'_{s_n} \to A'_s$$

that render the sets A'_s a subalgebra of A and the functions h'_s a Σ-homomorphism. Hence the square

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \uparrow & & \uparrow \\ A' & \xrightarrow[h']{} & B' \end{array}$$

commutes, and it can also be easily shown to be a pullback.

Example 2.11 (many-sorted algebras satisfying equations). $\mathsf{Alg}(S,\Sigma,E)$ is the full subcategory of $\mathsf{Alg}(S,\Sigma)$ consisting of those algebras that satisfy E, and its inclusions are those inherited from the latter. They are stable because the inverse image A' of $B' \hookrightarrow B$ under $h : A \to B$ satisfies E because A satisfies E and $A' \hookrightarrow A$.

Example 2.12 (F-coalgebras). The category of F-coalgebras of an i-functor $F : A \to A$ has stable inclusions if A does. It is easy to see that those $r : (a,f) \to (b,g)$ such that $r : a \to b$ is an inclusion constitute right cancellable inclusions. For (I.2), suppose

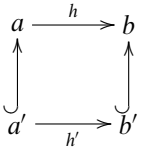
$$(b',g') \hookrightarrow (b,g)$$

and

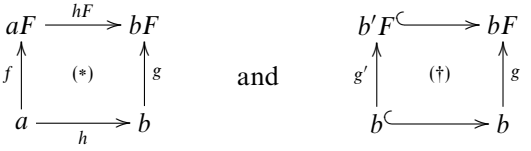
$$h : (a,f) \to (b,g).$$

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Let a' be a pullback, as in



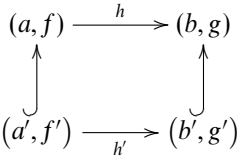
Since F preserves restrictions into subobjects and



commute, it follows that there exists a unique $f' : a' \to a'F$ such that

$$f'|^{aF} = f|_{a'}$$
$$f'; h'F = h'; g'F.$$

Hence, $(a', f') \hookrightarrow (a, f)$, h' is a morphism of F-coalgebras and



commutes. Suppose next that

$$r : (c,k) \to (a,f)$$
$$s : (c,k) \to (b',g)$$

such that

$$rh = s|^{(b,g)}.$$

Then there exists a unique $t : c \to a'$ such that

$$t|^{a} = r$$
$$th' = s$$

because a' is a pullback. Finally, note that t constitutes a morphism of F-algebras

$$(c,k) \to (a',f')$$

because the relation

$$k; tF = tf'$$

follows from the fact that $a'F \hookrightarrow aF$ is monic and r is a morphism of F-algebras.

Example 2.13 (F-algebras). In this case we have the same result, with a similar proof.

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Example 2.14 (Categories). For inclusions in Cat, we take the subcategories. These are stable, with the inverse image of B' under a functor $F : A \to B$ having for its objects, those a such that aF lies in B', and for its arrows $a \to b$, those f such that fF lies in $B'(aF, bF)$.

We will now describe several ways of building inclusions for partiality. We say an object in a category is *empty* if it is both *initial* (the colimit of the empty graph) and *anti-final* (an object x in a category A is anti-final if, whenever $f : a \to x$, we have $a = x$ and $f = 1$). There is at most one empty object, and we denote it by $\emptyset$. In Set and Cat, $\emptyset$ is the empty set and empty category, respectively.

Lemma 2.15. Suppose $\mathbf{C}$ and $\mathbf{C}'$ have stable systems of inclusions $\mathscr{I}$ and $\mathscr{I}'$, respectively. Then:

(1) Any broad subcategory $\mathscr{J}$ of $\mathscr{I}$ is a stable system of inclusions in $\mathbf{C}$ if:
   (a) Whenever $B \hookrightarrow C$ and the composite $A \hookrightarrow B \hookrightarrow C$ lie in $\mathscr{J}$, so does $A \hookrightarrow B$.
   (b) For any $f : A \to B$ of $\mathbf{C}$ and $B' \hookrightarrow B$ of $\mathscr{J}$, we have $f^{-1}(B') \hookrightarrow A$ lies in $\mathscr{J}$.

(2) If $\emptyset$ exists in $\mathbf{C}$, the category $\mathscr{I}_\emptyset$ resulting from the addition to $\mathscr{I}$ of every $\emptyset \to A$ that does not already belong to $\mathscr{I}$ is a stable system of inclusions in $\mathbf{C}$.

(3) $\mathscr{I} \times \mathscr{I}'$ is a stable system of inclusions in $\mathbf{C} \times \mathbf{C}'$.

Proof.

(1) The fact that $\mathscr{J}$ is a category of inclusions follows immediately from the fact that it is a broad subcategory of $\mathscr{I}$. Suppose the composition of $f : A \to B$ with an inclusion $B \hookrightarrow C$ from $\mathscr{J}$ is also an inclusion from $\mathscr{J}$. Then f lies in $\mathscr{I}$ by (I.1), and therefore in $\mathscr{J}$ by assumption (a), which proves (I.1).
(I.2) follows from assumption (b).

(2) It is easy to verify that $\mathscr{I}_\emptyset$ is a category of inclusions. For (I.1), any $f : A \to \emptyset$ is necessarily $1_\emptyset$ and thus a member of $\mathscr{I}_\emptyset$, and for (I.2), it suffices to note that the pullback of any $f : A \to B$ and $\emptyset \to B$ is $(\emptyset \to A, 1_\emptyset)$.

(3) $\mathscr{I} \times \mathscr{I}'$ is a category of inclusions for $\mathbf{C} \times \mathbf{C}'$ because it is a subcategory, and it is a partial order because $\mathscr{I}$ and $\mathscr{I}'$ are both partial orders. Both (I.1) and (I.2) hold pointwise. □

The following lemma is an adaptation of Căzănescu and Roşu (1997, Lemma 16).

Lemma 2.16. Suppose $\mathbf{D}$ has stable inclusions $\mathscr{I}$ and $\mathbf{F} : \mathbf{C} \to \mathbf{D}$ is a faithful functor that creates those pullbacks of the form $f : A \to B$, $X \hookrightarrow B$. Then the family

$$\mathscr{I}_\mathbf{F} = \{f \in \mathbf{C} | f\mathbf{F} \in \mathscr{I}\}$$

is a stable system of inclusions in $\mathbf{C}$, and $\mathbf{F}$ constitutes an i-functor $(\mathbf{C}, \mathscr{I}_\mathbf{F}) \to (\mathbf{D}, \mathscr{I})$.

Proof. By Căzănescu and Roşu (1997, Lemma 16), $\mathscr{I}_\mathbf{F}$ is a category of right cancellable inclusions for $\mathbf{C}$ provided $\mathbf{F}$ has the property that for any isomorphism $f : A \cong B$, we have $f\mathbf{F} = 1$ implies $f = 1$. In fact, pullback creation implies this property since the image under $\mathbf{F}$ of $(f, 1_B)$ is the pair $(1_{A\mathbf{F}}, 1_{A\mathbf{F}})$ whose pullback may be given as $A\mathbf{F}$ with projections equal to the identity. Since both $(1_A, f)$ and $(f^{-1}, 1_B)$ are cones to $(f, 1_B)$, and

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both map under $\mathbf{F}$ to $(1_{A\mathbf{F}}, 1_{A\mathbf{F}})$, it follows that $A = B$ and $f = 1$. For (I.2), any inverse image $f^{-1}(X)$ of an arrow $f : A \to B$ is created by $\mathbf{F}$. The projection $f^{-1}(X) \to A$ is therefore an inclusion because its image under $\mathbf{F}$ is an inclusion. □

Lemma 2.17. For categories $\mathbf{X}$ and $\mathbf{A}$, if $\mathscr{I}$ is a stable system of inclusions in $\mathbf{A}$, then the collection $\mathscr{I}^{\mathbf{X}}$ of natural transformations of the form $\mathbf{F} \to \mathbf{G} : \mathbf{X} \to \mathbf{A}$ whose components are all inclusions constitute stable inclusions in $[\mathbf{X}, \mathbf{A}]$.

Proof. The proof is entirely straightforward. □

Each of the above examples, apart from sets and categories, could have been deduced using Lemma 2.16 applied to, respectively:

— the forgetful functor $U : \mathsf{Top} \to \mathsf{Set}$;
— the forgetful functor $U : \mathsf{Alg}(S, \Sigma) \to [S, \mathsf{Set}]$;
— the inclusion $\mathsf{Alg}(S, \Sigma, E) \leqslant \mathsf{Alg}(S, \Sigma)$;
— and the functor $\mathsf{Coalg}(A) \to A$ that maps (a, f) to a and $h : (a, f) \to (b, g)$ to g.

For partiality in Set, we have the following complete result.

Proposition 2.18. The following constitute stable inclusions in Set:
(1) all subsets;
(2) all identities;
(3) all identities plus the empty subset.
For any other stable system of inclusions in Set, there is, for all A, a 1–1 correspondence between inclusions $i : A' \hookrightarrow A$ and subsets X of A through the relation $X = i(A')$.

Proof. Categories (2) and (3) constitute stable inclusions in Set by Example 2.7 and Lemma 2.15.

Now suppose Set has a stable system of inclusions $\mathscr{I}$ that contains at least one inclusion $j : B' \hookrightarrow B$ that is neither the identity nor empty. Let A be any set and X be any subset of A. We now pick an element $y \in B'$ and $z \in B \setminus j(B')$, and define $f : A \to B$ by setting xf equal to yj if $x \in X$, and equal to z otherwise. By (I.2), the inverse image A' of B' under f exists, with $i : A' \hookrightarrow A$. From the construction of pullbacks in Set, it follows that

$$\begin{aligned} A' &\cong \{a \in A | af \in j(B')\} \\ &= \{a \in A | af = yj\} \\ &= X, \end{aligned}$$

and

$$i = A' \cong X \subseteq A.$$

This property is unique to i since, if $i' : A'' \hookrightarrow A$ with

$$i' = A'' \cong X \subseteq A,$$

then $i = gi'$ for some isomorphism g, which implies $A' \hookrightarrow A''$ and $A'' \hookrightarrow A'$ by (I.1), so $A' = A''$ and $i = i'$.

Conversely, given $i : A' \hookrightarrow A$, we set $X = i(A')$. This property is again unique to $i(A')$, which completes the proof. □

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The final clause of Proposition 2.18 is not vacuous. Let $\mathscr{I}$ consist of the functions:

(1) $(0 \mapsto 1) : \{0\} \to B$ whenever $|B| \geqslant 2$ and $1 \in B$.
(2) $(1 \mapsto 0) : \{1\} \to B$ whenever $|B| \geqslant 2$ and $0 \in B$.
(3) Any inclusion $A \subseteq B$ where (A, B) is distinct from $\big(\{0\}, B\big)$, $\big(\{1\}, B\big)$ above.

It is straightforward to verify that $\mathscr{I}$ is closed under composition, contains all identities and contains at most one function $A \to B$ for any A, B. Suppose

$$i : A \to B$$
$$j : B \to A.$$

The composite ij must be 1, which is only possible if both are subset inclusions, so $A = B$ and $i = j = 1$. Hence, $\mathscr{I}$ is a category of inclusions of $\mathbf{C}$.

Next, let $f : A \to B$ be a function, and

$$j : A \to C$$
$$i : B \to C$$

be arrows of $\mathscr{I}$ such that $j = fi$. We will now prove that f must lie in $\mathscr{I}$ for all possible choices of i and j. Suppose j is an instance of (1). Then i cannot be an instance of (2). If i were an instance of (1), f would have to be $1 : \{0\} \to \{0\}$, which lies in $\mathscr{I}$. Otherwise, i would have to be a subset inclusion in $\mathscr{I}$ (an instance of (3)), with $B \neq \{1\}$, $1 \in B$ and $f : 0 \mapsto 1$. As this implies $|B| \geqslant 2$, we would have f lies in $\mathscr{I}$. By symmetry, f must lie in $\mathscr{I}$ if j is an instance of (2). So we now suppose j is a subset inclusion. If A is empty, then f is a subset inclusion and lies in $\mathscr{I}$. Otherwise, A is non-empty, and

$$\{0\} \neq A \neq \{1\}.$$

It then follows that B is non-empty and

$$\{0\} \neq B \neq \{1\}.$$

Hence, i must be a subset inclusion in $\mathscr{I}$, so f must be too, and we have proved (I.1).

For (I.2), let $f : A \to B$ be a function and $j : B' \to B$ an arrow of $\mathscr{I}$. We are looking to construct a pullback

$$i : A' \to A$$
$$f' : A' \to B,$$

with $i \in \mathscr{I}$. Knowing that

$$f^{-1}\big(j\,(B')\big) \subseteq A,$$

and

$$g : f^{-1}\big(j\,(B')\big) \to B',$$

where $g : a \mapsto j^{-1}(af)$, is a pullback, we proceed as follows:

— If $\{0\} \neq f^{-1}\big(j\,(B')\big) \neq \{1\}$, we set $A' = f^{-1}\big(j\,(B')\big)$ with i an instance of (3) and $f' = g$.
— If $\{0\} = f^{-1}\big(j\,(B')\big) = A$, we set $A' = \{0\}$, $i = 1$ and $f' = g$.

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— If $\{0\} = f^{-1}\left(j\left(B'\right)\right) \neq A$, we set $A' = \{1\}$ with i an instance of (2) because $|A| \geqslant 2$, and $f'(1) = g(0)$.

The remaining case, that is, $f^{-1}\left(j\left(B'\right)\right) = \{1\}$, is similar.

In view of our Example 2.9 on topological spaces, it is natural to ask whether the subspace topology can be generalised to *locales*. Recall that a locale is a complete lattice, which is also called a *frame*. Frames and their morphisms (order-preserving functions that also preserve all joins and finite meets) form a category Frm. The category of locales and *localic maps* is then defined as $\mathsf{Frm}^{\bullet}$. If (X, τ) is a topological space, then τ is a frame, and if $f : (X, \tau) \to (Y, \mu)$ is continuous, the contravariant powerset functor $\mathscr{P}(f)$ is a frame morphism $\mu \to \tau$. To ask, then, for a stable system of inclusions in Loc is to ask for the dual concept in Frm.

3. Duality

By analogy with injections and surjections, we say a category $\mathbf{C}$ has *stable surclusions* $\mathscr{S}$ if $(\mathbf{C}^{\bullet}, \mathscr{S}^{\bullet})$ is an i-category. This means:

(S.0) For any A, B, there is at most one surclusion $A \twoheadrightarrow B$. If $B \twoheadrightarrow A$ also, then $A = B$.
(S.1) Any $f : B \to C$ is a surclusion if its composition with some $A \twoheadrightarrow B$ is a surclusion.
(S.2) For any $f : A \to B$ and $A \twoheadrightarrow X$, there exists a *surobject* $f \dagger X$ of A and an arrow $f^X : X \to f \dagger X$ such that

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow[\quad f^X \quad]{} & f \dagger X \end{array}$$

is a pushout diagram.

Example 3.1 (topological spaces). The identity $1 : X \to X$ constitutes a continuous map $(X, \tau) \to (X, \mu)$ if and only if $\mu \subseteq \tau$, in other words if and only if τ is a finer topology than μ. We will now show that such maps constitute stable surclusions for Top. Let $f : (X, \tau) \to (Y, \mu)$ be continuous, and let $s : (X, \tau) \twoheadrightarrow \left(X, \tau'\right)$ be a surclusion. Then the pullback μ' of $\mathscr{P}(f) : \mu \to \tau$ to τ', is a topology on Y. In fact, it is the largest sub topology of μ such that $f : X \to Y$ is continuous with respect to τ'. With f' denoting $f : \left(X, \tau'\right) \to \left(Y, \mu'\right)$, the square

$$\begin{array}{ccc} (X, \tau) & \xrightarrow{\quad f \quad} & (Y, \mu) \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} \\ \left(X, \tau'\right) & \xrightarrow[\quad f' \quad]{} & \left(Y, \mu'\right) \end{array}$$

commutes. We now suppose there exist continuous

$$h : \left(X, \tau'\right) \to (Z, \nu)$$
$$k : (Y, \mu) \to (Z, \nu)$$

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that coequalise s and f. Let $U \in v$. Then $h^{-1}(U) \in \tau'$, so

$$f^{-1}\left(k^{-1}(U)\right) \in \tau'$$

because $h = fk$. It follows that $k^{-1}(U) \in \mu'$ by the definition of μ', so k also constitutes a continuous map

$$l : \left(Y, \mu'\right) \to (Z, v).$$

It is obviously unique with the property that its composition with t equals k. Furthermore,

$$fl = fk = h$$

by the definition of l and the hypothesis on h and k. Hence, the square is a pushout and Top has stable surclusions.

In order to treat Frm, we need the following proposition.

Proposition 3.2. Let **C** be a category with chosen coequalisers $c : A \to C$ comprising, for every A, one representative from each equivalence class. Suppose all identities are chosen, the composition of two chosen coequalisers is also chosen and $c = c'$ whenever $C = C'$. Then **C** has stable surclusions.

Proof. The surclusions are those arrows $f : C \to C'$ such that $cf = c'$, where (C, c) and $\left(C', c'\right)$ are chosen coequalisers. Since c might be 1, such f include the chosen coequalisers themselves, and they clearly constitute a partial order on the objects of **C**.

Suppose $sf = s'$ with s and s' surclusions. As s bears the relation $cs = c'$ where c and c' are chosen coequalisers, it follows that $c'f = cs'$, so f is again a surclusion, which proves (S.1).

Next note that for any epi $e : A \to B$, a pushout of arrows (f, g) from B is the same thing as a pushout of (ef, eg). So to prove (S.2), it suffices to consider a chosen coequaliser $c : A \to C$ of the pair $l, r : K \to A$, and an arrow $f : A \to B$. Let $c' : B \to C'$ be the chosen coequaliser of (lf, rf). Since fc' coequalises (l, r), it follows that there exists a unique $f' : C \to C'$ such that $cf' = fc'$. Now suppose

$$h : B \to D$$
$$k : C \to D$$

such that $fh = ck$. Then there exists a unique $d : C' \to D$ such that $c'd = h$ because

$$lfh = lck = rck = rfh.$$

Since we also have $f'd = k$ because c is epic, we can conclude that $\left(c', f'\right)$ is the pushout of (c, f). □

Example 3.3 (frames). A nucleus on a frame L is a mapping $v : L \to L$ such that

$$a \leqslant av$$
$$avv = av$$
$$(a \wedge b)v = av \wedge bv$$

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for all elements a, b of L. Nuclei on L are in a 1–1 correspondence with congruences C on L through the mappings

— $C \mapsto v_C$ defined by

$$av_C = \bigvee aC,$$

— $v \mapsto C_v$ defined by

$$(a, b) \in C_v$$

if and only if

$$av = bv.$$

Furthermore, every pair of arrows $l, r : K \to L$ has a coequaliser of the form

$$v_C : L \to L/C,$$

where C is the least congruence containing the relation on L defined by (l, r), and L/C is the image of L under v_C. To obtain stable surclusions for Frm, we will verify that these coequalisers satisfy the hypotheses of Proposition 3.2. If v_C and $v_{C'}$ are in the same equivalence class, it follows that they determine the same congruence, so $C = C'$. Identities correspond to the diagonal congruence. Closure under composition follows from the fact that given coequalisers

$$\begin{aligned} v_C &: L \to L/C \\ v_{C'} &: L/C \to (L/C)/C', \end{aligned}$$

the composite

$$v_C v_{C'} : L \to L$$

is a nucleus:

$$\begin{aligned} av_C v_{C'} v_C v_{C'} &= av_C v_{C'} v_{C'} \\ &= av_C v_{C'}, \end{aligned}$$

where the second equality follows because $av_C v_{C'}$ is invariant under v_C because it lies in L/C, and the other two properties follow easily. Finally, suppose $L/C = L/C'$. Then, for any $a \in L$, there exists a $b \in L$ such that

$$av_C = bv_{C'},$$

which implies

$$\begin{aligned} av_C v_{C'} &= bv_{C'} v_{C'} \\ &= bv_{C'} \\ &= av_C. \end{aligned}$$

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It then follows that

$$\begin{aligned} av_C &= av_C v_{C'} \\ &\geqslant av_{C'} \\ &= av_{C'} v_C \\ &\geqslant av_C, \end{aligned}$$

and thus

$$av_C = av_{C'}.$$

In the case of a topological space (Y, τ) and $X \subseteq Y$ with the subspace topology τ_X, let C be the congruence defined by the frame morphism

$$(U \mapsto U \cap X) : \ \tau \to \tau_X.$$

Then τ/C consists of the open sets

$$\bigcup\{V \mid V \cap X = U \cap X\},$$

which the unique frame morphism $\tau/C \to \tau_X$, whose composition with v_C equals $\tau \to \tau_X$, sends to $U \cap X$. As this mapping is clearly 1–1, it follows that $\tau/C \cong \tau_X$.

Constructing stable surclusions for sets is trickier. Essentially, the sursets of a set S are its partitions, but the difficulty arises when S is itself a partition. Recall that a *partition* of a set A is a set P of pairwise disjoint non-empty subsets of A whose union equals A. We write $p : \ A \to P$ to denote the map that sends elements a to the member of P that contains a. P is *discrete* if each of its elements is a singleton set.

Lemma 3.4. Assuming the axiom of foundation, every partition P originates in a set O_P that is not a partition in the sense that there exists a finite composition of partitions $O_P \to P$. Furthermore, there is an injective mapping from partitions Q of P to partitions R of O_P through a commutative square

$$\begin{array}{ccc} O_P & \longrightarrow & P \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ R & \xrightarrow[\cong]{} & Q \end{array}$$

Proof. Foundation implies that every set S has an associated ordinal $\mathsf{rank}(S)$ called its *rank*, given by the formula

$$\mathsf{rank}(S) = \bigcup\{\mathsf{rank}(a)^+ \mid a \in S\}.$$

For a partition P, we define $\alpha(P)$ to be the least ordinal of the set

$$\{\mathsf{rank}(X) \mid X \in P\},$$

and write P' to denote the set

$$\{X \in P \mid \mathsf{rank}(X) = \alpha(P)\}.$$

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So we have

$$\mathsf{rank}\left(\bigcup P'\right) = \alpha(P) \in \mathsf{rank}\left(P'\right) \leqslant \mathsf{rank}(P).$$

We now suppose there exists an infinite sequence of partitions of the form

$$\cdots \xrightarrow{p} P_2 \xrightarrow{p} P_1 \xrightarrow{p} P_0$$

and recursively define a sequence of non-empty $Q_i \subseteq P_i$ by setting

$$Q_0 = P_0$$
$$Q_{i+1} = \bigcup Q_i',$$

and noting that

$$\varnothing \neq Q_{i+1} \subseteq P_{i+1}$$

if

$$\varnothing \neq Q_i \subseteq P_i$$

because

$$\varnothing \neq Q_i' \subseteq P_i$$

and

$$\bigcup Q_i' \subseteq \bigcup P_i = P_{i+1}.$$

For all i, we define

$$\beta_i = \mathsf{rank}(Q_i).$$

Thus,

$$\begin{aligned}\beta_{i+1} &= \mathsf{rank}\left(\bigcup Q_i'\right)\\ &= \alpha(Q_i) \in \mathsf{rank}(Q_i)\\ &= \beta_i\end{aligned}$$

for all i, contradicting the axiom of foundation. So no such sequence of partitions can exist, which proves the first statement.

To prove the second statement, we can assume the composition is trivial and $O_P \to P$ is a partition since the general case then follows by repetition. Set

$$R = \{\bigcup X \mid X \in Q\}.$$

If $a \in O_P$, there exist $Y \in P$ and $X \in Q$ such that $a \in Y \in X$, so $a \in \bigcup X$. Hence,

$$\bigcup R = O_P.$$

Now suppose $X, Y \in Q$ with $X \neq Y$. Then $X \cap Y = \varnothing$, which means that for all $X' \in X$ and $Y' \in Y$, we have $X' \neq Y'$ and thus $X' \cap Y' = \varnothing$ because P is a partition. Hence,

$$\bigcup X \cap \bigcup Y = \varnothing,$$

and R is a partition of O_P. A corollary is that the map $X \mapsto \bigcup X$ defines a bijection from Q to R and the square commutes. Finally, since they are partitions, Q and R determine each other uniquely. □

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Example 3.5 (well-founded sets). For the chosen coequalisers, we select, along with the identities, all composites $p : P \to Q \cong R$ where p is a non-discrete partition and R is the partition of O_P that corresponds to Q by Lemma 3.4, with the convention that if P is not a partition, O_P is just P, so R must be Q and $\cong$ must be the identity. To see that they are closed under composition, suppose

$$c : P \to R$$
$$d : R \to S$$

are chosen coequalisers. Then both c and d must be partitions of O_P. In Set, the composition of two coequalisers is a coequaliser, which means there exists a partition $p : P \to T$ and an isomorphism $f : T \cong S$ such that $pf = cd$. In other words, S corresponds to T by Lemma 3.4, and cd is chosen. The remaining conditions of Proposition 3.2 follow easily from Lemma 3.4.

4. Partiality based on inclusions

A *partial arrow*

$$(X, f) : A \to B$$

consists of an $\mathscr{I}$-subobject X of A and an arrow $f : X \to B$ of $\mathbf{C}$. We say X is its *domain of definition* and f is its *totality*. It is *total* if $X = A$. Partial arrrows

$$(X, f) : A \to B$$
$$(X', f') : B \to C$$

may be composed to give

$$\left(f^{-1}(X'), f_{X'} f'\right) : A \to C.$$

It is easy to see that this composition is associative because the composition of pullbacks is a pullback, and the partial arrows $(A, 1_A) : A \to A$ are clearly identities for it. Observe also that the partial arrows

$$(A, A \hookrightarrow B) : A \to B$$

form a broad subcategory of $\partial(\mathbf{C}, \mathscr{I})$ that partially orders its objects, and they thus constitute inclusions. They are also right cancellable since for given

$$(X, f) : A \to B, B \hookrightarrow C$$

such that $(X, f)|^C$ equals $(A, A \hookrightarrow C)$, it follows that $X = A$ and $f|^C$ is an inclusion of $\mathbf{C}$, so f is an inclusion by (I.1). They are also monic since given

$$(X, f), (Y, g) : A \to B, B \hookrightarrow C$$

such that

$$(X, f)|^C = (Y, g)|^C,$$

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it follows that

$$X = Y$$
$$f|^C = g|^C,$$

so $f = g$ because the inclusions of $\mathbf{C}$ are monic.

We have now proved the following proposition.

Proposition 4.1. The partial arrows form a category $\partial(\mathbf{C}, \mathscr{I})$ with right cancellable, monic inclusions.

However, we cannot, in general, iterate to form $\partial(\partial\mathbf{C})$, as the following counterexample shows.

Counterexample 4.2. The inclusions of $\partial\mathbf{C}$ are not stable.

Proof. We will show that axiom (2) of Definition 2.1 does not hold in $\partial\,\mathsf{Cat}$. Let A be freely generated by arrows

$$f : a \to b$$
$$g : b \to c$$
$$h : a \to c$$

subject to the relation

$$h = fg.$$

Let B be generated by the same data subject to the further constraint that $a = c$, and let B' be the subcategory of B containing just g. Let F map a to c, be undefined on f, and be the identity elsewhere.

Suppose the pullback A' exists. It is easy to verify that

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ \uparrow & & \uparrow \\ B' & \xrightarrow[1]{} & B' \end{array}$$

commutes, so it follows that $g \in A'$. Next let A'' be the subcategory of A containing only the arrow $f : a \to b$, and let $F'' : b \mapsto b$, $a \mapsto c$, and be undefined elsewhere. Then

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ \uparrow & & \uparrow \\ A'' & \xrightarrow[F'']{} & B' \end{array}$$

also commutes, so $f \in A'$. But then $h = fg \in A'$, which cannot be the case because F maps $h : a \to b$ to $h : c \to c$, and the latter does not lie in B'. □

As in the case of total arrows, we call the composition of a partial arrow $f : A \to B$ and inclusion $X \hookrightarrow A$ the *restriction of f to X*, and denote it by $f|_X : A \to B$. Its totality evaluates to $f^{\mathsf{t}}|_{A_f \cap X}$.

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At this point, we will recall the definition of split restriction category from Cockett and Lack (2002).

Definition 4.3. A *restriction category* is a category together with an assignment of a *restriction idempotent* $\overline{f} : A \to A$ to each arrow $f : A \to B$, subject to the axioms:
(R.1) $\overline{f} f = f$.
(R.2) $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ whenever $g : A \to C$.
(R.3) $\overline{f}\,\overline{g} = \overline{f g}$ whenever $g : A \to C$.
(R.4) $f\overline{g} = \overline{f g}\, f$ whenever $g : B \to C$.
A restriction category is *split* if all its restriction idempotents $\overline{f}$ factor as rm for some m and r such that $mr = 1$, which are called the *restriction monic* and *restriction retraction*, respectively.

Proposition 4.4. $\partial(\mathbf{C},\mathscr{I})$ is a split restriction category.

Proof. For $(X, f) : A \to B$, we set

$$\overline{f} = (X, X \hookrightarrow A) : A \to A$$

and verify that the axioms (R.1)–(R.4) hold in $\partial(\mathbf{C},\mathscr{I})$:
(R.1) We have

$$(X, X \hookrightarrow A)\ ;(X, f) = (X, f).$$

(R.2) We have

$$\begin{aligned}(X, X \hookrightarrow A)\ ;(Y, Y \hookrightarrow A) &= (X \cap Y, X \cap Y \hookrightarrow A)\\ &= (Y, Y \hookrightarrow A)\ ;(X, X \hookrightarrow A).\end{aligned}$$

(R.3) This follows because both

$$\begin{aligned}&(X, X \hookrightarrow A)\ ;(Y, g : Y \to C)\\ &(X, X \hookrightarrow A)\ ;(Y, Y \hookrightarrow A)\end{aligned}$$

have domain of definition $X \cap Y$.
(R.4) Given

$$(X, f) : A \to B, \left(X', f'\right) : B \to C,$$

the left-hand side of (R.4)

$$(X, f)\ ;\left(X', X' \hookrightarrow B\right)$$

equals

$$\left(f^{-1}(X), f_{X'}|^{B}\right).$$

For the right-hand side, note that

$$(X, f)\ ;\left(X', f'\right) = \left(f^{-1}\left(X'\right), f_{X'} f'\right),$$

for which the restriction idempotent is

$$\left(f^{-1}\left(X'\right), f^{-1}\left(X'\right) \hookrightarrow A\right),$$

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which when composed with (X, f) equals

$$\left(f^{-1}\left(X'\right), f|_{f^{-1}(X')}\right),$$

which equals the left-hand side of (R.4) by the definition of $f_{X'}$.

Now every

$$(X, X \hookrightarrow A) \; : \; A \to A$$

equals

$$(X, X \hookrightarrow A) \; ; (X, X \hookrightarrow A) \; : \; A \to X \to A,$$

with

$$(X, X \hookrightarrow A) \; : \; A \to X$$

having the property that its restriction to X is the identity, so $\partial(\mathbf{C}, \mathscr{I})$ is split. □

5. Inclusively split restriction categories

The previous result motivates the following definition.

Definition 5.1. A restriction category $\mathbf{X}$ is *inclusively split* if:

(1) $\mathbf{X}$ has right cancellable inclusions $\mathscr{I}$.
(2) Every restriction idempotent $f : A \to A$ equals $r : A \to X \hookrightarrow A$ for some X and r such that $r|_X = 1$.

The inclusion of (2) is called a *restriction inclusion*, r is called a *restriction retraction*, the pair (X, r) is called an *inclusive splitting* of f and we say f *splits at* X.

Lemma 5.2. In any inclusively split restriction category $\mathbf{X}$:

(1) Restriction inclusions are monic.
(2) The restriction idempotent of an inclusion is the identity.
(3) If (X, r) splits a restriction idempotent $f : A \to A$, then $f = \overline{r}$.
(4) Restriction inclusions constitute right cancellable inclusions for $\mathbf{X}$.
(5) If $r, s : A \to X$ are restriction retractions then $r = s$.
(6) Restriction retractions constitute inclusions for $\mathbf{X}$.
(7) Every split restriction idempotent has a unique inclusive splitting.
(8) Inclusion preserving restriction functors preserve inclusive splittings.
(9) Every restriction monic is of the form $A \cong X \hookrightarrow B$.

Proof.

(1) This part follows because restriction inclusions have a right inverse.
(2) Let f be the restriction idempotent of inclusion $X \hookrightarrow A$. Then $f|^A$ is an inclusion by (R.1), so f is an inclusion by right cancellability, and thus f must be the identity on X.
(3) We have

$$f = \overline{f} = \overline{r|^A} = \overline{r}$$

by Cockett and Lack (2002, Lemma 2.1 (iv) and (iii)) and part (1) of the lemma.

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(4) We will first prove that restriction inclusions are closed under composition, and include all identities.
Consider the restriction idempotent $\overline{1} : A \to A$. Since **X** is a split restriction category, it factorises as

$$r : A \to A' \hookrightarrow A$$

for some A' and r. Now $\overline{1} = 1$ because $1 : A \to A$ is monic, which implies that $r|^A = 1$, and thus is an inclusion. It follows that r is an inclusion, and hence that $A' = A$ and $r = 1$. So $1 : A \to A$ is a restrictive inclusion. We now let $B \hookrightarrow A$ and $C \hookrightarrow B$ be restriction inclusions, and let $r : A \to B$ and $s : B \to C$ be the restriction retractions associated with them. Then

$$\begin{aligned} \overline{rs} &= \overline{\overline{r}rs} \\ &= \overline{r}\,\overline{rs} \\ &= \overline{rs}\,\overline{r} \end{aligned}$$

by (R.1), (R.3) and (R.2), which equals

$$\begin{aligned} (\overline{rs}\,r)|^A &= (r\overline{s})|^A \\ &= (rs)|^A \end{aligned}$$

by the definition of r, (R.4) and the definition of s. Hence, $C \hookrightarrow A$ is also a restriction inclusion (of $\overline{rs}$).
We now need to verify (I.1). Suppose the composite $f : C \to B \hookrightarrow A$ is an inclusion and let

$$\begin{aligned} s &: A \to C \\ r &: A \to B \end{aligned}$$

be the associated restriction retractions. Now f must be an inclusion, so $C \hookrightarrow B$. Then

$$\begin{aligned} (s|_B)|^A &= \overline{s}|_B \\ &= \left(\overline{s|_B}\right)|^A \end{aligned}$$

by the definition of s and (R.4), so

$$s|_B = \overline{s|_B}$$

because $B \hookrightarrow A$ is monic. As we also have

$$\begin{aligned} f\,;s|_B &= s|_C \\ &= 1 \end{aligned}$$

by the definition of s, it follows that f is a restriction inclusion (of $\overline{s|_B}$).

(5) We have

$$\begin{aligned} r &= r|^A\,;\,s|^A\,;r \\ &= s|^A\,;r|^A\,;r \\ &= s \end{aligned}$$

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by (R.2) and because

$$r|_X = s|_X = 1.$$

(6) The proof of part (4) also proves that all identities are restriction retractions, and that restriction retractions are closed under composition. Also, given restriction retractions

$$r : A \to B$$
$$s : B \to A,$$

we have $B \hookrightarrow A$ and $A \hookrightarrow B$, so $A = B$ and $r = s = 1$, and the statement of part (6) now follows from part (5).

(7) Suppose (X, r) and(Y, s) both split the same restriction idempotent. Since

$$r|_X = 1,$$

we have $(r|^A)|_X$ is an inclusion, so $(s|^A)|_X$ is an inclusion too because $r|^A = s|^A$. This implies $s|_X$ is an inclusion, that is, $X \hookrightarrow Y$. But then $Y \hookrightarrow X$ too, so $X = Y$, and $r = s$ by part (5).

(8) Statement (8) follows immediately from definitions.

(9) Suppose $\overline{f} : A \to A$ has splitting

$$s\,; m : A \to Y \to A$$

and inclusive splitting (X, r). Then $s|_X : X \to Y$ and $mr : Y \to X$ constitute an isomorphism $X \cong Y$. □

We call the pair $(\mathbf{X}, \mathscr{I})$ an *inclusively split restriction category* if $\mathscr{I}$ only contains restriction inclusions. Observe that all inclusions of $\partial(\mathbf{C}, \mathscr{I})$ are restriction inclusions: any $A \hookrightarrow B$ is the restriction inclusion of the restriction idempotent

$$(A, A \hookrightarrow B) : B \to B.$$

Note also that if $(\mathbf{X}, \mathscr{J})$ is also inclusively split, then every element j of $\mathscr{J}$ is of the form $r\,; i$ for some isomorphism r and $i \in \mathscr{I}$. The remarks following Proposition 2.18 show that $\mathscr{I}$ and $\mathscr{J}$ need not be equal.

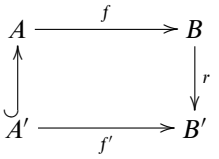
Recall from Cockett and Lack (2002) that an arrow f is *total* if $\overline{f} = 1$, and that they constitute a broad subcategory $\mathsf{Total}(\mathbf{X})$ of $\mathbf{X}$. The following result mirrors Cockett and Lack (2002, Proposition 3.3).

Lemma 5.3. If $(\mathbf{X}, \mathscr{I})$ is inclusively split, then $\mathscr{I}$ is stable in $\mathsf{Total}(\mathbf{X})$.

Proof. Lemma 5.2 (2) and (4) imply that $\mathscr{I}$ constitutes right cancellable inclusions for $\mathsf{Total}(\mathbf{X})$. For the second axiom, let $f : A \to B$ be total and $B' \hookrightarrow B$. Then there exists a $r : B \to B'$ such that $r|^B = \overline{r}$, and, because $\mathbf{X}$ is split, the restriction idempotent $\overline{fr}$ equals

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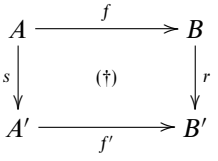
$s|^A$ for some $s : A \to A'$. We now define f' by



Then

$$\begin{aligned} sf' &= s|^A fr \\ &= \overline{fr} fr \\ &= fr \end{aligned}$$

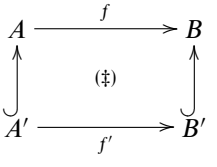
by the definitions of f' and s, and (R.1). So



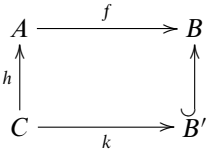
commutes. It follows that

$$\begin{aligned} f'|^B &= s|_{A'} ; f'|^B \\ &= f|_{A'} ; r|^B = (f\bar{r})|_{A'} \\ &= \left(\overline{fr} f\right)|_{A'} \\ &= f|_{A'}, \end{aligned}$$

by appeal to the relation $s|_{A'} = 1$, (†), the definition of r, (R.4) and the fact that $\overline{fr}|_{A'}$ is the inclusion $A' \hookrightarrow A$. Hence,

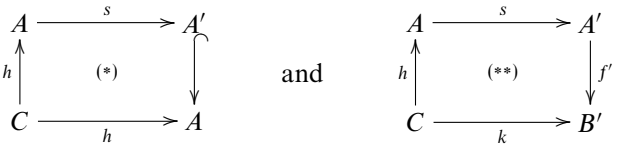


commutes, which in turn implies that f' is total by Cockett and Lack (2002, Lemma 2.2). To complete the proof, we need to show that (‡) is a pullback. So let h and k be total such that



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commutes. Since $A' \hookrightarrow A$ is monic, we just need to prove that



commute. We have

$$\begin{aligned} h\,;s|^A &= h\overline{fr} \\ &= \overline{hfr}h \\ &= \overline{k|^B r}h \\ &= \overline{k|^B}h \\ &= h \end{aligned}$$

by, respectively, the definition of s, (R.4), the definition of h and k, the fact that $\overline{r}|_{B'}$ is the inclusion $B' \hookrightarrow B$, and the fact that k and $B' \hookrightarrow B$ are total. Hence $(*)$ commutes. We also have

$$\begin{aligned} hs\,;f'|^B &= hf\,;r|^B \\ &= hf\overline{r} \\ &= k|^B\overline{r} \\ &= k|^B \end{aligned}$$

by $(\dagger)$, the definition of r, the definition of h and k, and the fact that $\overline{r}|_{B'}$ is the inclusion $B' \hookrightarrow B$. Since $B' \hookrightarrow B$ is monic, the commutativity of $(**)$ then follows and $(\ddagger)$ is a pullback. □

Note that whenever $(\dagger)$ and $(\ddagger)$ in the above proof commute for total f and f', and restriction retractions r and s, we have that $(\ddagger)$ must be a pullback since then

$$\overline{fr} = \overline{sf'} = \overline{s}.$$

Lemma 5.4. In any inclusively split restriction category $(\mathbf{X},\mathscr{I})$:

(1) Every $f : A \to B$ is a unique composite of a restriction retraction and a total arrow, namely

$$r_f f^{\mathrm{t}} : A \to A_f \to B$$

where $f^{\mathrm{t}} = f|_{A_f}$ and (A_f, r_f) is the splitting of $\overline{f}$.

(2) For any $f, g : A \to B$,

$$\begin{aligned} A_f &= A_g \\ f^{\mathrm{t}} &= g^{\mathrm{t}} \end{aligned}$$

implies

$$f = g.$$

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(3) Given $f : A \to B$ and $A_f \hookrightarrow X \hookrightarrow A$ with $X \hookrightarrow A$ monic, $\overline{f|_X}$ has splitting $\left(A_f, r_f|_X\right)$.
(4) For restriction idempotents $f, g : A \to A$, the domain of definition of fg is $A_f \cap A_g$.
(5) We have

$$(ff' : A \to B \to C)^{\mathsf{t}} = f^{\mathsf{t}}_{f'} f'^{\mathsf{t}} : A_{ff'} \to B_{f'} \to C,$$

where $f^{\mathsf{t}}_{f'}$ is the restriction, in $\mathsf{Total}\,(\mathbf{X})$, of f^{t} into $B_{f'}$.

Proof.

(1) We have

$$f = \overline{f} f = r_f f^{\mathsf{t}}.$$

f^{t} is total because

$$\begin{aligned}\overline{f|_{A_f}} &= \overline{\overline{f}|_{A_f}} \\ &= \overline{(r_f|^A)|_{A_f}} \\ &= \overline{A_f \hookrightarrow A} \\ &= 1\end{aligned}$$

using Cockett and Lack (2002, Lemma 2.1 (iii)) Lemma 5.2 (2). Suppose also that $f = sg$ for some restriction retraction $s : A \to X$ and total g. Then

$$\begin{aligned}\overline{r_f} &= \overline{f} \\ &= \overline{sg} \\ &= \overline{s}\end{aligned}$$

by Lemma 5.2 (3) and because g is total. Hence $s = r_f$ by Lemma 5.2 (7), so $g = f^{\mathsf{t}}$.
(2) $A_f = A_g$ implies $r_f = r_g$ by Lemma 5.2 (5), so $f = g$ by part (1).
(3) We have

$$\begin{aligned}\overline{f|_X} &= \overline{\overline{f}|_X} \\ &= \overline{(r_f|^A)|_X} \\ &= \overline{r_f|_X}\end{aligned}$$

by Cockett and Lack (2002, Lemma 2.1 (iii)), the definition of r_f and the fact that $A_f \hookrightarrow A$ is total. Now

$$\begin{aligned}\overline{r_f|_X}|^A &= \overline{r_f}|_X \\ &= (r_f|_X)|^A\end{aligned}$$

by (R.4) and the fact that r_f is the restriction retraction of $\overline{r_f}$, so

$$\overline{r_f|_X} = (r_f|_X)|^X$$

because $X \hookrightarrow A$ is assumed monic. However, $A_f \hookrightarrow X$ is a left inverse for $r_f|_X$, so $\left(A_f, r_f|_X\right)$ is the splitting for $\overline{r_f|_X}$, and $\overline{f|_X}$.
(4) Since fg is a restriction idempotent, it has a splitting (Z, r). It then follows that $(fg)|_Z$ is an inclusion, so $Z \hookrightarrow A_g$ by right cancellability because g has a splitting $\left(A_g, r_g\right)$.

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But

$$fg = gf$$

by (R.2), so $Z \hookrightarrow X$ too. Next note that $(fg)|_{A_f \cap A_g}$ evaluates to an inclusion, so $r|_{A_f \cap A_g}$ must be an inclusion, that is,

$$A_f \cap A_g \hookrightarrow Z.$$

Hence,

$$Z = A_f \cap A_g.$$

(5) We have

$$(ff')^{\mathsf{t}} = (ff')|_{A_{ff'}} = f^{\mathsf{t}}|_{A_{ff'}} r_{f'} f'^{\mathsf{t}}$$

by Lemma 5.4 (1). But

$$\overline{f^{\mathsf{t}} r_{f'}} = \overline{(ff')|_{A_f}}$$

because f'^{t} is total, and the restriction inclusion $A_f \hookrightarrow A$ is monic by Lemma 5.2. It then follows from part (1) that $\overline{f^{\mathsf{t}} r_{f'}}$ has splitting $\left(A_{ff'}, r_{ff'}|_{A_f}\right)$, and this implies that $f^{\mathsf{t}}|_{A_{ff'}} r_{f'}$ is the restriction, in $\mathsf{Total}(\mathbf{X})$, of f^{t} into $B_{f'}$ (see the proof of Lemma 5.3). □

An *inclusive restriction functor* $(\mathbf{X}, \mathscr{I}) \to \left(\mathbf{Y}, \mathscr{J}\right)$ is a restriction functor $\mathbf{X} \to \mathbf{Y}$ that maps inclusions to inclusions. We write rCat_{is} to denote the category consisting of inclusively split restriction categories, inclusive restriction functors and all restriction transformations. Note there is an obvious 2-functor $U : \mathsf{rCat}_{is} \to \mathsf{rCat}_s$ that forgets the inclusions.

Lemma 5.5. A restriction functor is an isomorphism if it is an isomorphism in Cat.

Proof. Let $\mathbf{F} : \mathbf{X} \to \mathbf{Y}$ be a restriction functor with functorial inverse $\mathbf{G}$, let f be a restriction idempotent in $\mathbf{Y}$ and let $f' = f\mathbf{G}$. Then

$$\begin{aligned} f'\mathbf{F} &= f \\ &= \overline{f} \\ &= \overline{f'\mathbf{F}} \\ &= \overline{f'}\mathbf{F}, \end{aligned}$$

which implies $f' = \overline{f'}$. In other words, $f\mathbf{G}$ is a restriction idempotent. □

Lemma 5.6. Suppose $\mathbf{F} : (\mathbf{X}, \mathscr{I}) \to \left(\mathbf{Y}, \mathscr{J}\right)$ is an inclusive restriction functor. Then:

(1) $f^{\mathsf{t}}\mathbf{F} = (f\mathbf{F})^{\mathsf{t}}$ for any $f \in \mathbf{X}$.

(2) $\mathbf{F}$ is an isomorphism in rCat_{is} if it is an isomorphism in Cat.

Proof.

(1) We have

$$\begin{aligned} f\mathbf{F} &= (r_f f^{\mathsf{t}})\mathbf{F} \\ &= r_f\mathbf{F} \,;\, f^{\mathsf{t}}\mathbf{F} \end{aligned}$$

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where $r_f\mathbf{F}$ is a restriction retraction by Lemma 5.2(8) and $f^{\mathsf{t}}\mathbf{F}$ is total because f^{t} is. Hence,

$$f^{\mathsf{t}}\mathbf{F} = (f\mathbf{F})^{\mathsf{t}}$$

by Lemma 5.4(1).

(2) By Lemma 5.5, $\mathbf{F}$ has an inverse $\mathbf{G}$ in rCat. We must show $\mathbf{G}$ maps inclusions to inclusions. So suppose $X \hookrightarrow A$ and restriction idempotent $f : A \to A$ splits at X. Let $f\mathbf{G}$ split at $\mathbf{Y}$. Applying $\mathbf{F}$, it follows that f inclusively splits at $Y\mathbf{F}$, so $Y\mathbf{F} = \mathbf{X}$, and thus $X\mathbf{G} = Y$ and $X\mathbf{G} \hookrightarrow A\mathbf{G}$. □

We write $\mathsf{Total}(\mathbf{X},\mathscr{I})$ to denote the i-category consisting of $\mathsf{Total}(\mathbf{X})$ and inclusions $\mathscr{I}$. It extends to a 2-functor

$$\mathsf{Total} : \mathsf{rCat}_{is} \to \mathsf{iCat}$$

as follows. Any inclusive restriction functor

$$\mathbf{F} : (\mathbf{X},\mathscr{I}) \to (\mathbf{Y},\mathscr{J})$$

restricts to a functor

$$\mathsf{Total}(\mathbf{F}) : \mathsf{Total}(\mathbf{X}) \to \mathsf{Total}(\mathbf{Y})$$

because $\mathbf{F}$ preserves totality. To see that $\mathsf{Total}(\mathbf{F})$ preserves restrictions into subobjects, recall how they are constructed from Lemma 5.3, and let $f' : A' \to B'$ be the restriction of $f : A \to B$ into B'. Then

$$\overline{fr} = \overline{s}$$

where r and s are the restriction retractions associated with $B' \hookrightarrow B$ and $A' \hookrightarrow A$. It follows that

$$\begin{aligned}\overline{f\mathbf{F}\,;r\mathbf{F}} &= \overline{(fr)\mathbf{F}}\\ &= \overline{fr}\mathbf{F}\\ &= \overline{s}\mathbf{F}\\ &= \overline{s\mathbf{F}}\end{aligned}$$

because $\mathbf{F}$ preserves restriction idempotents, where $r\mathbf{F}$ and $s\mathbf{F}$ are the restriction retractions associated with

$$\mathbf{F}(B') \hookrightarrow \mathbf{F}(B)$$

and

$$\mathbf{F}(A') \hookrightarrow \mathbf{F}(A)$$

by Lemma 5.2(8). Hence, $f'\mathbf{F}$ is the restriction of $f\mathbf{F}$. As $\mathsf{Total}(\mathbf{F})$ also preserves inclusions, we deduce that it constitutes an i-functor

$$\mathsf{Total}(\mathbf{X},\mathscr{I}) \to \mathsf{Total}(\mathbf{Y},\mathscr{J}).$$

And given inclusive

$$\mathbf{F},\mathbf{G} : (\mathbf{X},\mathscr{I}) \to (\mathbf{Y},\mathscr{J}),$$

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a restriction transformation $\phi : \mathbf{F} \to \mathbf{G}$ is a natural transformation

$$\mathsf{Total}(\phi) : \mathsf{Total}(\mathbf{F}) \to \mathsf{Total}(\mathbf{G})$$

because its components are total. Furthermore, it is an i-cartesian natural transformation

$$\mathsf{Total}(\phi) : \mathsf{Total}(\mathbf{F}) \to \mathsf{Total}(\mathbf{G})$$

because

$$\phi_A ; f\mathbf{G} = f\mathbf{F} ; \phi_B$$

whenever f is an inclusion or restriction retraction since ϕ is natural. The 2-functoriality of Total follows immediately from the definitions.

We will now show that the definition of the inclusively split restriction category $\partial(\mathbf{C},\mathscr{I})$ extends to a 2-functor $\partial : \mathsf{iCat} \to \mathsf{rCat}_{is}$. An i-functor

$$\mathbf{F} : (\mathbf{C},\mathscr{I}) \to \left(\mathbf{D}, \mathscr{J}\right)$$

gives rise to an inclusive restriction functor

$$\partial\mathbf{F} : \partial(\mathbf{C},\mathscr{I}) \to \partial\left(\mathbf{D}, \mathscr{J}\right)$$

defined by the assignments

$$\partial\mathbf{F}(A) = A$$
$$\partial\mathbf{F}(X, f) = (X\mathbf{F}, f\mathbf{F})$$

for any object A of $\mathbf{C}$ and partial arrow $(X, f) : A \to B$ of $\mathbf{C}$. The functoriality of $\partial\mathbf{F}$ is easily shown to follow from the fact that $\mathbf{F}$ preserves restrictions into subobjects, and it obviously preserves restriction idempotents and inclusions. Next, a partiality inclusion transformation $\phi : \mathbf{F} \to \mathbf{G}$ gives rise to a restriction 2-natural transformation $\partial\phi : \partial\mathbf{F} \to \partial\mathbf{G}$ defined by

$$\partial\phi_A = (A\mathbf{F}, \phi_A).$$

The naturality follows from that of ϕ because the commutativity of

$$\begin{array}{ccc} A\mathbf{F} & \xrightarrow{(A\mathbf{F},\phi_A)} & A\mathbf{G} \\ {\scriptstyle (X\mathbf{F},f\mathbf{F})}\downarrow & & \downarrow{\scriptstyle (X\mathbf{G},f\mathbf{G})} \\ B\mathbf{F} & \xrightarrow[(B\mathbf{F},\phi_B)]{} & B\mathbf{G} \end{array}$$

for any partial arrow $(X, f) : A \to B$ of $\mathbf{C}$, amounts to the commutativity of

$$\begin{array}{ccc} X\mathbf{F} & \xrightarrow{\phi_X} & X\mathbf{G} \\ {\scriptstyle f\mathbf{F}}\downarrow & & \downarrow{\scriptstyle f\mathbf{G}} \\ B\mathbf{F} & \xrightarrow[\phi_B]{} & B\mathbf{G} \end{array}$$

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since ϕ_B is total and ϕ is i-cartesian. Finally, the 2-functoriality of ∂ follows easily from the definitions.

Lemma 5.7. $(\mathbf{C},\mathscr{I}) \cong \mathsf{Total}\,(\partial\,(\mathbf{C},\mathscr{I}))$ in iCat. The isomorphism is 2-natural in $(\mathbf{C},\mathscr{I})$.

Proof. A partial arrow $(X, f) : A \to B$ of $\mathbf{C}$ is total if and only if it is total as an arrow of a restriction category because

$$(X, X \hookrightarrow A) = 1$$

if and only if

$$X = A.$$

Since $\mathbf{C}$ and $\mathsf{Total}\,(\partial\,(\mathbf{C},\mathscr{I}))$ have the same objects, the 1–1 correspondence

$$f : A \to B \leftrightarrow (A, f : A \to B)$$

yields an isomorphism. It is clear that it is bijective on inclusions, and it is easy to verify that it preserves pullbacks along inclusions, so it is an isomorphism by Definition 2.3 and Lemma 2.6. The 2-naturality in $(\mathbf{C},\mathscr{I})$ follows immediately from the definitions. □

Lemma 5.8. $(\mathbf{X},\mathscr{I}) \cong \partial\,\mathsf{Total}\,(\mathbf{X},\mathscr{I})$ in rCat_{is}. The isomorphism is 2-natural in $(\mathbf{X},\mathscr{I})$.

Proof. Our desired isomorphism is the identity on objects and maps arrows $f : A \to B$ to partial arrows $\left(A_f, f^{\mathsf{t}}\right) : A \to B$. It is an inclusive restriction functor because:

(1) We have

$$\left(A_f, f^{\mathsf{t}}\right) ; \left(B_{f'}, f'^{\mathsf{t}}\right) = \left(A_{ff'}, (ff')^{\mathsf{t}}\right)$$

by Lemma 5.4 and Proposition 4.1.

(2) $1 : A \to A$ maps to $(A, 1_A)$ by Lemma 5.4 (1).

(3) Restriction idempotents $f : A \to A$ map to

$$\left(A_f, A_f \hookrightarrow A\right) : A \to A$$

by Lemma 5.4 (1).

(4) $A \hookrightarrow B$ maps to

$$(A, A \hookrightarrow B) : A \to B,$$

which is an inclusion.

Since the structural morphisms are bijections by Lemma 5.4 (1) and (2), we obtain an isomorphism by Lemma 5.6. The 2-naturality in $(\mathbf{X},\mathscr{I})$ follows easily. □

Note that the two notions of totality are equivalent. The totality of a partial arrow of an i-category $(\mathbf{C},\mathscr{I})$ corresponds, under the isomorphism of Lemma 5.7, to its totality in the sense of Lemma 5.4. And the totality of an arrow of an inclusively split restriction category $(\mathbf{X},\mathscr{I})$ is the same as the totality of its image under the isomorphism of Lemma 5.8. Similar remarks apply to the 'domain of definition'.

We have therefore proved the following theorem.

Theorem 5.9. The 2-functors Total and ∂ constitute a 2-equivalence $\mathsf{iCat} \simeq \mathsf{rCat}_{is}$.

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6. Formally split restriction idempotents

Every restriction category $\mathbf{X}$ is isomorphic to a full subcategory of a split restriction category $K_r(\mathbf{X})$, which is obtained from $\mathbf{X}$ by 'formally splitting' all restriction idempotents. In this section, we show that the $K_r(\mathbf{X})$ split *inclusively*. Furthermore, we will also show that the embeddings $\mathbf{X} \to K_r(\mathbf{X})$ constitute a left 2-adjoint to

$$U : \mathsf{rCat}_{is} \to \mathsf{rCat}.$$

Now $K_r(\mathbf{X})$ has for its objects pairs (X, e) where $e : X \to X$ is a restriction idempotent of $\mathbf{X}$. Its arrows $(X, e) \to (X', e')$ are arrows $f : X \to X'$ such that

$$efe' = f.$$

It follows immediately that for such f, the relations

$$ef = f = fe'$$

hold, so

$$e\overline{f} = \overline{ef} = \overline{f}$$

by (R.3), and

$$\overline{e}f = \overline{ef}e = \overline{f}e$$

by (R.4). Composition is composition in $\mathbf{X}$, with identities the arrows $e : e \to e$. The restriction idempotent of $f : e \to e'$ is just $\overline{f} : e \to e$, which is a legitimate arrow of $K_r(\mathbf{X})$ because

$$e\overline{f}e = \overline{f}e = \overline{f}.$$

The axioms (R.1)–(R.4) hold in $K_r(\mathbf{X})$ precisely because they hold in $\mathbf{X}$, so the former is a restriction category.

For the inclusions, we take those $e : (X, e) \to (X, e')$ such that

$$ee' = e,$$

which it is easy to see form a partial order on the objects of $K_r(\mathbf{X})$. Suppose the composite

$$fe' : (X, e) \to (X', e') \to (X', e'')$$

is an inclusion. It follows that

$$X = X'$$
$$fe' = e.$$

Hence,

$$f = efe' = ee = e,$$

so f is an inclusion, and the inclusions are right cancellable.

Every restriction idempotent $\overline{f} : e \to e$ equals the composite

$$\overline{f}\,\overline{f} : e \to \overline{f} \to e,$$

where $\overline{f} : e \to \overline{f}$ is legitimate because

$$e\overline{f}\,\overline{f} = \overline{f},$$

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and

$$\overline{f}\,\overline{f} : \overline{f} \to e \to \overline{f}$$

obviously equals the identity. We write $\mathscr{K}(\mathbf{X})$ to denote these inclusions, and since they are all restriction inclusions, the pair $(K_r(\mathbf{X}), \mathscr{K}(\mathbf{X}))$ is an inclusively split restriction category.

The unit of our 2-adjunction has for its components $\eta : \mathbf{X} \to K_r(\mathbf{X})$ restriction functors that map objects A to the pair $(A, 1_A)$, and arrows $f : A \to B$ to the morphism

$$f : (A, 1_A) \to (B, 1_B).$$

Suppose $(\mathbf{Y}, \mathscr{I})$ is an inclusively split restriction category and $\mathbf{F} : \mathbf{X} \to \mathbf{Y}$ is a restriction functor. We will now prove that there exists a unique inclusive restriction functor

$$\mathbf{F}^\flat : (K_r(\mathbf{X}), \mathscr{K}(\mathbf{X})) \to (\mathbf{Y}, \mathscr{I})$$

such that

$$\eta ; \mathbf{F}^\flat = \mathbf{F}.$$

In fact, such an $\mathbf{F}^\flat$ necessarily maps objects 1_X to $X\mathbf{F}$ and arrows $f : 1_X \to 1_{X'}$ to $f\mathbf{F}$. In particular, $\mathbf{F}^\flat$ maps restriction idempotents of the form $e : 1_X \to 1_X$ to $e\mathbf{F}$. Since the former splits at (X, e), it follows from Lemma 5.2(8) that the restriction idempotent $e\mathbf{F}$ necessarily splits at $\mathbf{F}^\flat(X, e)$, fixing $\mathbf{F}^\flat$ on objects. Now an arbitrary $f : (X, e) \to (X', e')$ factorises as

$$ef e' : e \to 1_X \to 1_{X'} \to e',$$

where we recall that the first and third factors are an inclusion and restriction retraction, respectively. It thus follows that $\mathbf{F}^\flat$ maps $f : e \to e'$ to the composite

$$(X, e)\,\mathbf{F}^\flat \hookrightarrow X\mathbf{F} \xrightarrow{f\mathbf{F}} X'\mathbf{F} \xrightarrow{r} (X', e')\,\mathbf{F}^\flat$$

where r is the restriction retraction of

$$(X', e')\,\mathbf{F}^\flat \hookrightarrow X'\mathbf{F}.$$

We have thus proved uniqueness.

Conversely, given the above assignments for $\mathbf{F}^\flat$, we easily find that

$$\begin{array}{ccc} (X, e)\,\mathbf{F}^\flat & \xrightarrow{f\mathbf{F}^\flat} & (X', e')\,\mathbf{F}^\flat \\ \downarrow & (*) & \downarrow \\ X\mathbf{F} & \xrightarrow[f\mathbf{F}]{} & X'\mathbf{F} \end{array}$$

commutes. In fact, it defines $f\mathbf{F}^\flat$ because inclusions are monic. The functoriality of $\mathbf{F}^\flat$ now follows immediately from that of $\mathbf{F}$. For the preservation of restriction idempotents,

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let $f : (X,e) \to (X',e')$. Then

$$\begin{aligned}\overline{f\mathbf{F}^\flat}|^{X\mathbf{F}} &= \overline{f\mathbf{F}^\flat|^{X'\mathbf{F}}}|^{X\mathbf{F}} \\ &= \overline{f\mathbf{F}|_{(X,e)\mathbf{F}^\flat}}|^{X\mathbf{F}} \\ &= \overline{f\mathbf{F}}|_{(X,e)\mathbf{F}^\flat} \\ &= \overline{f}\mathbf{F}|_{(X,e)\mathbf{F}^\flat}\end{aligned}$$

because inclusions are total by (*) and (R.4), and because $\mathbf{F}$ preserves restriction idempotents. Hence, $\overline{f\mathbf{F}^\flat}$ must be $\overline{f}\mathbf{F}^\flat$ by (*). As for the preservation of inclusions, note that with f equal to $e : (X,e) \to (X,e')$, the left leg of (*) is an inclusion because $e\mathbf{F}$ splits at $(X,e)\,\mathbf{F}^\flat$, so $e\mathbf{F}^\flat$ is an inclusion by right cancellability.

Finally, suppose $\mathbf{G} : \mathbf{X} \to \mathbf{Y}$ is another restriction functor, and $\phi : \mathbf{F} \to \mathbf{G}$ is a restriction transformation. Then we define a restriction transformation $\phi^\flat : \mathbf{F}^\flat \to \mathbf{G}^\flat$ by setting each component $\phi^\flat_e$ equal to the composite

$$(X,e)\,\mathbf{F}^\flat \rightarrowtail X\mathbf{F} \xrightarrow{\phi_X} X\mathbf{G} \xrightarrow{r} (X,e)\,\mathbf{G}^\flat$$

These constitute a natural family: given $f : (X,e) \to (X',e')$, we compose the three squares

$$\begin{array}{ccc} (X,e)\,\mathbf{F}^\flat & \xrightarrow{\phi^\flat_e} & (X,e)\,\mathbf{G}^\flat \\ \downarrow & (\dagger) & \downarrow \\ X\mathbf{F} & \xrightarrow[\phi_X]{} & X\mathbf{G} \end{array} \qquad \begin{array}{ccc} X\mathbf{F} & \xrightarrow{\phi_X} & X\mathbf{G} \\ {\scriptstyle f\mathbf{F}}\downarrow & & \downarrow{\scriptstyle f\mathbf{G}} \\ X'\mathbf{F} & \xrightarrow[\phi_{X'}]{} & X'\mathbf{G} \end{array} \qquad \begin{array}{ccc} X'\mathbf{F} & \xrightarrow{\phi_{X'}} & X'\mathbf{G} \\ {\scriptstyle r}\downarrow & (\ddagger) & \downarrow{\scriptstyle r} \\ (X,e')\,\mathbf{F}^\flat & \xrightarrow[\phi^\flat_{e'}]{} & (X,e')\,\mathbf{G}^\flat \end{array}$$

the second of which commutes because ϕ is natural, while (†) and (‡) follow easily from the definition of $\phi^\flat$. It is clear that $\eta \mathbin{;} \phi^\flat = \phi$. For uniqueness, note that naturality at $e : e \to 1_X$ is exactly (†), which defines $\phi^\flat_e$ because inclusions are monic. We can thus conclude with the following theorem.

Theorem 6.1. K_r is left 2-adjoint to $U : \mathsf{rCat}_{is} \to \mathsf{rCat}$.

7. Stable systems of monics

We turn now to the question of how partiality based on stable inclusions relates to partiality based on stable monics. Since U maps into rCat_s, which is a full sub 2-category of rCat, it follows that K_r restricted to rCat_s is left 2-adjoint to $U : \mathsf{rCat}_{is} \to \mathsf{rCat}_s$. Composing this adjunction with those of Theorem 5.9 and Cockett and Lack (2002, Theorem 3.4) immediately yields the following theorem.

Theorem 7.1. There is a 2-adjunction between i-categories and m-categories.

Identifying the right adjoint requires a bit of work. From the definitions, we see that it maps i-category $(\mathbf{C},\mathscr{I})$ to the m-category

$$\left(\mathsf{Total}\left(\partial\left(\mathbf{C},\mathscr{I}\right)\right), \mathscr{M}_{\partial(\mathbf{C},\mathscr{I})}\right),$$

https://doi.org/10.1017/S0960129514000036 Published online by Cambridge University Press

where $\mathscr{M}_{\partial(\mathbf{C},\mathscr{I})}$ consists of all the restriction monics in $\partial(\mathbf{C},\mathscr{I})$. In fact, the latter are all composites (in $\partial(\mathbf{C},\mathscr{I})$) of the form

$$B \cong X \hookrightarrow A.$$

These composites are certainly all restriction monics, and supposing the restriction idempotent

$$(X, X \hookrightarrow A) \ : \ A \to A$$

factors as

$$(Y, g) \ ; (Z, h) \ : \ A \to B \to A,$$

with

$$(Z, h) \ ; (Y, g) = 1_B,$$

it follows that

$$g^{-1}(Z) = X$$

and $g_Z h : \ X \to Z \to A$ is an inclusion, and

$$h^{-1}(Y) = B$$

and $h_Y g : \ B \to Y \to B$ is the identity. Hence,

$$\begin{aligned} Z &= B \\ g^{-1}(Z) = Y &= X \\ g_Z &= g \\ h_Y &= h_X. \end{aligned}$$

Therefore,

$$\begin{aligned} h_X g &= 1_B \\ h &= h_X|^A. \end{aligned}$$

Hence, gh_X must be an inclusion because gh is an inclusion, so

$$gh_X = 1_X,$$

and thus (B, h_X) is an isomorphism and (Z, h) factors as

$$(B, h_X) \, ; (X, X \hookrightarrow A).$$

It is now easy to see that the isomorphism of Lemma 5.7 constitutes an isomorphism

$$\left(\mathsf{Total}\,(\partial(\mathbf{C},\mathscr{I})), \mathscr{M}_{\partial(\mathbf{C},\mathscr{I})}\right) \cong \left(\mathbf{C},\mathscr{I}^+\right),$$

which is 2-natural in $(\mathbf{C},\mathscr{I})$, where $\mathscr{I}^+$ comprises the composites $B \cong X \hookrightarrow A$ (in $(\mathbf{C},\mathscr{I})$). So the right adjoint $\mathsf{iCat} \to \mathsf{mCat}$ may be taken to map $(\mathbf{C},\mathscr{I})$ to $\left(\mathbf{C},\mathscr{I}^+\right)$, and is the identity on 1-cells and 2-cells.

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Next, the left 2-adjoint maps an m-category $(\mathbf{C},\mathscr{M})$ to the i-category, which we denote by $\big(\mathbf{C}[\mathscr{M}],\mathscr{M}^+\big)$, consisting of

$$\mathsf{Total}\,(K_r\,(\mathsf{Par}\,(\mathbf{C},\mathscr{M})))$$

and inclusions

$$\mathscr{K}\,(\mathsf{Par}\,(\mathbf{C},\mathscr{M}))\,.$$

The objects of $\mathbf{C}[\mathscr{M}]$ are pairs consisting of an object X of $\mathbf{C}$ and a restriction idempotent

$$[(m,m)]\,:\,X\to X$$

of $\mathsf{Par}\,(\mathbf{C},\mathscr{M})$, where m lies in $\mathscr{M}$ with codomain X. Its arrows

$$(X,[(m,m)])\to[(Y,(n,n)])$$

are arrows

$$[(k,f)]\,:\,X\to Y$$

of $\mathsf{Par}\,(\mathbf{C},\mathscr{M})$ such that

$$[(m,m)]\,;[(k,f)]\,;[(n,n)]=[(k,f)]$$

and

$$\overline{[(k,f)]}=1,$$

that is,

$$[(k,k)]=[(m,m)].$$

The first relation amounts to k and f factoring through m and n, respectively, while the second means k equals rm for some isomorphism r. Inclusions of $\mathscr{M}^+$ are of the form

$$[(m,m)]\,:\;\;(X,[(m,m)])\to(X,[(n,n)])$$

whenever

$$[(m,m)]\,;[(n,n)]=[(m,m)].$$

All told, the arrows of $\mathbf{C}[\mathscr{M}]$ are of the form

$$[(m,fn)]\,:\;\;(X,[(m,m)])\to(Y,[(n,n)])\,,$$

with:

— composition given by the formula

$$[(m,fn)]\,;[(n,gk)]=[(m,fgk)];$$

— $[(m,m)]$ the identity on $(X,[(m,m)])$; and
— inclusions

$$[(nm,nm)]\,:\;\;(X,[(nm,nm)])\to(X,[(m,m)])\,.$$

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The unit of the adjunction is the usual composite of the units of the constituent adjunctions, and in view of our chosen right adjoint, it is easy to see that its component at $(\mathbf{C},\mathscr{M})$ maps objects A to $(A,[(1_A,1_A)])$ and arrows $f : A \to B$ to $[(1_A,f)]$.

Theorem 7.2. $U(\partial(\mathbf{C},\mathscr{I})) \cong \mathsf{Par}(\mathbf{C},\mathscr{I}^+)$. The isomorphism is 2-natural in $(\mathbf{C},\mathscr{I})$.

Proof. The 2-isomorphism is the composite $\partial U \fatsemi \varepsilon$ where ε is the counit of $\mathsf{Par} \dashv \mathsf{Total}$ □

Theorem 7.3. $\mathsf{Par}(\mathbf{C},\mathscr{M})$ is isomorphic to the full subcategory of $\partial(\mathbf{C}[\mathscr{M}],\mathscr{M}^+)$ given by the objects $(A,[(1_A,1_A)])$. The isomorphism is 2-natural in $(\mathbf{C},\mathscr{M})$.

Proof. The isomorphisms are the components at $(\mathbf{C},\mathscr{M})$ of the composite

$$\mathsf{Par} \xrightarrow{\ \eta\ } \mathsf{Par}\,K_r U \xrightarrow{\ (\mathsf{Par}\,K_r)\eta U\ } \mathsf{Par}\,K_r\,\mathsf{Total}\,\partial U$$ □

Note that under the isomorphism of Theorem 7.3, a partial map

$$[(m,f)] : A \to B$$

corresponds to a partial arrow

$$((A,[(m,m)]),[(m,f)]) : (A,[(1_A,1_A)]) \to (B,[(1_B,1_B)]).$$

Finally, we should say a few words about formal subobjects as discussed in Cockett and Lack (2002). A category $\mathbf{C}$ for which each of its objects has an associated meet-semilattice comprising its subobjects may be modelled as a contravariant functor from $\mathbf{C}^\bullet$ to the category of meet-semilattices, which we denote by sLat. Such fibred semilattices live in a category $\mathsf{sLatFib}$. Every m-category $(\mathbf{C},\mathscr{M})$ induces a fibred meet-semilattice $(\mathbf{C},Sub_{\mathscr{M}})$ where

$$Sub_{\mathscr{M}} : \mathbf{C}^\bullet \to \mathsf{sLat}$$

sends objects A to the semilattice of all $\mathscr{M}$-subobjects of A. This mapping extends to a fully faithful 2-functor

$$\mathcal{I} : \mathsf{mCat} \to \mathsf{sLatFib}$$

that has a 2-left adjoint (Cockett and Lack 2002, Theorem 4.3, page 257). The latter amounts to a universal way of realising formal subobjects as m-subobjects.

The 2-adjunction of Theorem 6.1 implies that formal subobjects can also be universally realised as i-subojects. The right 2-adjoint $\mathsf{iCat} \to \mathsf{sLatFib}$ maps every i-category $(\mathbf{C},\mathscr{I})$ to the fibred meet-semilattice $(\mathbf{C},Sub_{\mathscr{I}})$ where

$$Sub_{\mathscr{I}}(A) = \{X | X \hookrightarrow A\}.$$

Acknowledgements

The author is grateful to the anonymous referees for their thoughtful comments and corrections. The paper has benefited in particular from the tip-off that $K_r(\mathbf{X})$ splits inclusively, and that this fact would facilitate a much more conceptual proof of Theorem 7.1 than that of the original version of this paper.

https://doi.org/10.1017/S0960129514000036 Published online by Cambridge University Press

This research received no specific grant from any funding agency, in either the commercial or not-for-profit sectors, and no conflicts of interest arose.

References

Căzănescu, V. E. and Roşu, G. (1997) Weak inclusions systems. *Mathematical Structures in Computer Science* **7** (2) 195–206.

Căzănescu, V. E. and Roşu, G. (2000) Weak inclusions systems; part 2. *Journal of Universal Computer Science* **6** (1) 5–21.

Cockett, J. R. B. and Lack, S. (2002) Restriction categories I: categories of partial maps. *Theoretical Computer Science* **270** (1-2) 223–259.

Cockett, J. R. B. and Lack, S. (2003) Restriction categories II: partial map classification. *Theoretical Computer Science* **294** (1-2) 61–102.

Cockett, J. R. B. and Lack, S. (2007) Restriction categories III: colimits, partial limits and extensitivity. *Mathematical Structures in Computer Science* **17** (4) 775–817.

Cockett, J. R. B. and Manes, E. (2009) Boolean and classical restriction categories. *Mathematical Structures in Computer Science* **19** (2) 357–416.

Diaconescu, R. (2008) *Institution-independent Model Theory 1st*, Birkhäuser.

Diaconescu, R. and Futatsugi, K. (1998) CafeOBJ *Report: The Language, Proof Techniques, and Methodologies for Object-Oriented Algebriac Specification*, AMAST Series in Computing **6**, World Scientific.

Diaconescu, R., Goguen, J. and Stefaneas, P. (1993) Logical Support for Modularisation. In: Huet, G. and Plotkin, G. (eds.) *Logical Environments*, Cambridge University Press 83-130.

Goguen, J. and Burstall, R. (1984) Introducing Institutions. In: Clarke, E. and Kozen, D. (eds.) Proceedings, Logics of Programming Workshop. *Springer-Verlag Lecture Notes in Computer Science* **164** 221–256.

Di Paola, R. A. and Heller, A. (1987) Dominical Categories: Recursion Theory without Elements. *Journal of Symbolic Logic* **52** (3) 594–635.

Robinson, E. and Rosolini, G. (1988) Categories of partial maps. *Information and Computation* **79** (2) 95–130.

Rosolini, G. (1986) *Continuity and effectiveness in topoi*, D. Phil. Thesis, University of Oxford.

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