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A Two-Level Method for Pressure Projection Stabilized P_1 Nonconforming Approximation of the Semi-Linear Elliptic Equations

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Abstract. In this paper, we study a new stabilized method based on the local pressure projection to solve the semi-linear elliptic equation. The proposed scheme combines nonconforming finite element pairs NCP_1-P_1 triangle element and two-level method, which has a number of attractive computational properties: parameter-free, avoiding higher-order derivatives or edge-based data structures, but have more favorable stability and less support sets. Stability analysis and error estimates have been done. Finally, numerical experiments to check estimates are presented.

AMS subject classifications: 35Q30, 74S05, 65N30

Key words: Semi-linear elliptic equations, two-level method, nonconforming finite element method, stabilized method.

1 Introduction

Let Ω be a bounded domain in R^2 with a Lipschitz boundary $\partial\Omega$. In this paper, we will consider the following semi-linear elliptic equations:

$$\begin{cases} -\Delta p - f(p) = g(x) & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p(x)$ is an unknown variable and $g(\mathbf{x}) \in L^2(\Omega)$.

A direct way to solve this model is Galerkin finite element method. However, the regularity of solution is higher if this approach is used. This leads to some difficulties in practical application. An alternative formulation has been used to reduce the need of

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regularity such as mixed finite element method [1]. The classic Galerkin finite element method of mixed formulation require the finite element pairs to satisfy the LBB conditions, but the continuous LBB conditions are not naturally inherited by most mixed finite element spaces. This limits the use of low-order finite element spaces, which do not satisfy LBB condition. Especially, the lowest equal-order pairs, such as $P1-P1$ (or Q_1-Q_1) pairs are of practical important in scientific computation, because they are computationally convenient and efficient in a parallel or multi-grid context [2]. Particular concern is the nonconforming finite element methods, which is even more attractive due to the simplicity and convenience. Compared with the conforming finite element methods, the nonconforming finite element not only have more favorable stability and less support sets, but can also relax the requirement of regularity [3, 4]. Hence, the nonconforming finite element seems more suitable to solve this type of problem.

To overcome the lack of LBB stability, a lot of stabilized techniques have been proposed. Many of these stabilized techniques depend on the stabilized parameter either explicitly or implicitly [5–10]. In practice, these parameter are still being determined by trial and error. Moreover, these is no satisfied answer to the stabilized parameter in all situations. Simultaneously, another class of stabilized method is proposed, which is not require specification of a stabilization parameter [11–14]. This stabilized strategy requires edge-based data structures and a subdivision of grid into patches. Recently, based on polynomial pressure, a new family of stabilized method is presented [15–17], which has some prominent features: parameter free, avoiding higher-order derivatives or edge-based data structures, and stabilization being completely local at the element level. Some of the above techniques have also been extended to transient incompressible flow problems [18, 19].

On the other hand, the numerical solution of a nonlinear system can be very time consuming, two-level methods are considered as an effective alternative method to solve this kind of problem. Two-level methods aim to obtain a discrete approximate solution of a nonlinear partial equation with less computional cost and to preserve the optimal order of convergence. The basic idea is to solve a complicated problem on the coarse grid, then solve a simple symmetric positive or linearized problem on fine grid. This method is first introduced by Xu [20–23] for nonsymmetric linear and linear elliptic problem. Later on, two-grid method is further investigated [24–27]. Due to the validity of two-grid methods in practical computation, this kind of method has been widely applied to the various of nonlinear problem, such as the semi-linear reaction-diffusion model [28, 29], and the semi-linear hyperbolic equation [30].

In this paper, we introduce the stabilized finite element scheme of the semi-linear elliptic equation based on the lowest equal-order pairs of nonconforming mixed finite element pairs (i.e., NCP_1-P_1). Moreover, two-level method is used to reduce the time consuming in the proposed scheme. The stabilized method is based on the local pressure projection [15]. This paper is organized as follows: in next section, an abstract functional setting for the semi-linear elliptic equations is presented, together with some basic notation. Then, in Section 3, the stabilized nonconforming mixed finite element scheme is

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proposed. In Section 4, Stability analysis is done. Error estimates are discussed in Section 5. In Section 6, a series of numerical experiment are given to illustrate the theoretical results. We conclude with a few remarks in the final section.

2 Preliminaries

Throughout this paper, we shall use the standard notation for Sobolev space and their associated norms and seminorms. The symbol c denotes a generic positive constant whose value may change from place to place but remain independent of mesh parameter h.

To introduce the weak formulation of (1.1), we set

$$\mathbf{V}=H(\mathrm{div},\Omega)=\{\mathbf{u}\in(L^2(\Omega))^2|\nabla\cdot\mathbf{u}\in L^2(\Omega)\},\quad W=L^2(\Omega),$$

and define the norm as follows

$$\|\mathbf{v}\|_{\mathbf{V}}=(\|\mathbf{v}\|_0^2+\|\mathrm{div}\mathbf{v}\|_0^2)^{\frac{1}{2}},\quad \|q\|_W=\|q\|_0,\quad \forall\mathbf{v}\in\mathbf{V},\quad q\in W.$$

Taking the flux: $\mathbf{u}=-\nabla p$, a mixed variational formulation of (1.1) is to seek $(\mathbf{u},p)\in V\times W$ such that

$$\begin{cases}(\mathbf{u},\mathbf{v})-(\nabla\cdot\mathbf{v},p)=0, & \forall\mathbf{v}\in\mathbf{V},\\ (\nabla\cdot\mathbf{u},q)-(f(p),q)=(g(x),q), & \forall q\in W.\end{cases}\tag{2.1}$$

For simplicity, we define the continuous bilinear forms as follows

$$a(\mathbf{u},\mathbf{v})=(\mathbf{u},\mathbf{v}),\qquad \forall\mathbf{u},\mathbf{v}\in\mathbf{V},\tag{2.2a}$$

$$b(\mathbf{v},p)=(\nabla\cdot\mathbf{v},p),\qquad \forall\mathbf{v}\in\mathbf{V},\quad p\in W.\tag{2.2b}$$

Based on the above notations, Eq. (2.2) can be rewritten as

$$\begin{cases}a(\mathbf{u},\mathbf{v})-b(\mathbf{v},p)=0, & \forall\mathbf{v}\in\mathbf{V},\\ b(\mathbf{u},q)-(f(p),q)=(g(x),q), & \forall q\in W.\end{cases}\tag{2.3}$$

Furthermore, we introduce the following generalized bilinear form on $(\mathbf{V},W)\times(\mathbf{V},W)$:

$$B((\mathbf{u},p);(\mathbf{v},q))=a(\mathbf{u},\mathbf{v})-b(\mathbf{v},p)+b(\mathbf{u},q).$$

The variational formulation of problem (2.3) can be rewritten as: find a pair of (**u**,p)$\in$ $(\mathbf{V},W)$ such that

$$B((\mathbf{u},p);(\mathbf{v},q))-(f(p),q)=(g(x),q),\quad \forall(\mathbf{v},q)\in(\mathbf{V},W).\tag{2.4}$$

It is obvious that the systems (2.3) is a saddle point problem. The continuity of bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, as well as the coerciveness of bilinear form $a(\cdot,\cdot)$ have been proven [31]. Morever, the bilinear form $b(\cdot,\cdot)$ also satisfies the following conclusions.

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Lemma 2.1 (see [32]). *The bilinear form $b(\cdot,\cdot)$ satisfies the inf-sup conditions, i.e., there is a positive constant α such that*

$$\inf_{p\in W}\sup_{\mathbf{v}\in\mathbf{V}}\frac{b(\mathbf{v},p)}{\|\mathbf{v}\|_{\mathbf{V}}\|q\|_W}\geq\alpha. \tag{2.5}$$

3 The stabilized nonconforming mixed finite element method

To introduce the stabilized scheme, the space discrete is considered. Let $h>0$ be a positive parameter, and K_h be a triangulation of Ω into triangles or quadrilaterals $\{K_j\}$: $\bar{\Omega}=\cup\bar{K}_j$, which is assumed to be regular in the usual sense. Let $\Gamma_j=\partial\Omega\cap\partial K_j$ denotes the boundary of K_j on Ω, and $\Gamma_{jk}=\Gamma_{kj}=\partial K_j\cap\partial K_k$ denotes the interface between elements K_j and K_k. The center of Γ_j and Γ_{jk} is denoted by ξ_j and ξ_{jk}. The finite-element spaces of interest in this paper are defined by setting

$$\begin{aligned}&NCP_1=\{\mathbf{v}\in(L^2(\Omega))^2:\mathbf{v}|_K\in(P_1(K))^2,\mathbf{v}(\xi_{kj})=\mathbf{v}(\xi_{jk}),\mathbf{v}(\xi_j)=0,\ \forall j,k,K\in K_h\},\\&P_1=\{q\in H_1(\Omega):q|_K\in P_1(K),\ \forall K\in K_h\},\end{aligned}$$

where the space NCP_1 is not a subspace of $\mathbf{V}_1$, i.e., $NCP_1\nsubseteq\mathbf{V}_1$. The finite element pairs $NCP_1\times P_1$ satisfy the property: for $\forall(\mathbf{u},p)\in(H^2)^2\times H^1$, there exists $(\mathbf{u}_I,p_I)\in NCP_1\times P_1$ such that

$$\|\mathbf{u}-\mathbf{u}_I\|_0+h(\|\mathbf{u}-\mathbf{u}_I\|_1+\|p-p_I\|_0)\leq ch^2(\|\mathbf{u}\|_2+\|p\|_1), \tag{3.1}$$

where the $\|\|_{1h}$ denote the energy norm:

$$\|\mathbf{v}\|_{\mathbf{1h}}=(\sum_{\mathbf{j}}|\mathbf{v}|^{\mathbf{2}}_{\mathbf{1},\mathbf{K_j}})^{\frac{\mathbf{1}}{\mathbf{2}}}.$$

Note, for $\forall j$ and k, the following compatibility conditions hold:

$$\int_{\Gamma_{jk}}[\mathbf{v}]ds=0\quad\text{and}\quad\int_{\Gamma_j}\mathbf{v}ds=0,$$

where $[\mathbf{v}]=\mathbf{v}_{\Gamma_{jk}}-\Gamma_{kj}$ denote the jump of the function $\mathbf{v}$ across the interface Γ_{jk}.

Set $(\cdot,\cdot)_j=(\cdot,\cdot)_{K_j}$ and $\langle\cdot,\cdot\rangle_j=\langle\cdot,\cdot\rangle_{\partial K_j}$ denote L^2-inner product on K_j, ∂K_j, respectively. So the discrete bilinear forms are defined as follows:

$$\begin{aligned}&a_h(\mathbf{u},\mathbf{v})=\sum_j(\mathbf{u},\mathbf{v})_j, &&\forall\mathbf{u},\mathbf{v}\in NCP_1,\\&b_h(\mathbf{v},p)=\sum_j(\nabla\cdot\mathbf{v},p), &&\forall\mathbf{v}\in NCP_1,\quad p\in P_1.\end{aligned}$$

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Then, the nonconforming finite element approximation of the problem (2.4) can be rewritten as

$$B_h((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))-(f(p_h),q_h)=(g(x),q_h),\quad \forall(\mathbf{v}_h,q_h)\in(NCP_1,P_1), \tag{3.2}$$

where

$$B_h((\mathbf{u},p);(\mathbf{v},q))=a_h(\mathbf{u},\mathbf{v})-b_h(\mathbf{v},p)+b_h(\mathbf{u},q).$$

However, to overcome the limitation of the LBB condition, we need add a stabilized term in variational formulation, where we adopt the the pressure projection stabilized method proposed in [15].

We introduce a standard L^2-projection operator

$$\Pi: L^2(\Omega)\to P_0,$$

with the following properties [15]

$$(p,q_h)=(\Pi p,q_h), \qquad \forall p\in W,\quad q_h\in P_0, \tag{3.3a}$$
$$\|\Pi p\|_0\le c\|p\|_0, \qquad \forall p\in W, \tag{3.3b}$$
$$\|p-\Pi p\|_0\le Ch\|p\|_1, \qquad \forall p\in H_1(\Omega)\cap W, \tag{3.3c}$$

then we can define bilinear form

$$G(p,q)=(p-\Pi p,q-\Pi q).$$

Therefore, the variational stabilized formulation of problem (3.2) reads

$$\mathcal{B}_h((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))-(f(p_h),q_h)=(g(x),q_h),\quad \forall(\mathbf{v}_h,q_h)\in(NCP_1,P_1), \tag{3.4}$$

where

$$\mathcal{B}_h((\mathbf{u},p);(\mathbf{v},q))=B_h((\mathbf{u},p);(\mathbf{v},q))-G(p,q) \tag{3.5}$$

is the new stabilized bilinear form. The following theorem establishes the weak coercivity of the bilinear form $\mathcal{B}_h((\mathbf{u},p);(\mathbf{v},q))=B((\mathbf{u},p);(\mathbf{v},q))-G(p,q)$ for the lowest-order nonconforming mixed finite element pairs.

4 Stability

In this section, we investigate the stability of the problem (3.4), through the norm the variable $\mathbf{v}$ depend the mesh size h, but we omit it. For the sake of simplicity, we suppose the right-hand term $g(x)=0$.

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Theorem 4.1. *Let (NCP_1,P_1) be defined as above, then there exist a positive constant β, independent h, such that for all $\forall(\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)\in(NCP_1,P_1)$,*

$$|\mathcal{B}_h((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))|\leq C(\|\mathbf{u}_h\|_{\mathbf{V}}+\|p_h\|_0)(\|\mathbf{v}_h\|_{\mathbf{V}}+\|q_h\|_0), \tag{4.1a}$$

$$\beta(\|\mathbf{u}_h\|_{\mathbf{V}}+\|p_h\|_0)\leq \sup_{(\mathbf{v}_h,q_h)\in(NCP_1,P_1)}\frac{|\mathcal{B}_h((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))|}{\|\mathbf{v}_h\|_{\mathbf{V}}+\|q_h\|_0}. \tag{4.1b}$$

Proof. The continuous property is obvious:

$$\begin{aligned}|\mathcal{B}_h((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))|=&|a_h(\mathbf{u}_h,\mathbf{v}_h)-b_h(\mathbf{v}_h,p_h)+b_h(\mathbf{u}_h,q_h)-G(p_h,q_h)|\\ \leq& C(\|\mathbf{u}_h\|_0\|\mathbf{v}_h\|_0+\|\nabla\cdot\mathbf{v}_h\|_0\|p_h\|_0+\|\nabla\cdot\mathbf{u}_h\|_0\|q_h\|_0+\|p_h\|_0\|q_h\|_0)\\ \leq& C(\|\mathbf{u}_h\|_{\mathbf{V}}+\|p_h\|_0)(\|\mathbf{v}_h\|_{\mathbf{V}}+\|q_h\|_0).\end{aligned}$$

Next, we will show the coercive property of bilinear form $\mathcal{B}((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))$.

Now, we choose $(\mathbf{v}_h,q_h)=(\mathbf{u}_h,\nabla\cdot\mathbf{u}_h-\alpha p_h)$ for some constant α yet to be determined. According to the formulation (3.5), yields

$$\begin{aligned}&\mathcal{B}_h((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))\\ =&a_h(\mathbf{u}_h,\mathbf{u}_h)-(1+\alpha)b_h(\mathbf{u}_h,p_h)+b_h(\mathbf{u}_h,\nabla\cdot\mathbf{u}_h)-G(p_h,\nabla\cdot\mathbf{u}_h)+\alpha G(p_h,p_h)\\ =&\|\mathbf{u}_h\|_0^2+\|\nabla\cdot\mathbf{u}\|_0^2-(1+\alpha)b_h(\mathbf{u}_h,p_h)-G(p_h,\nabla\cdot\mathbf{u}_h)+\alpha G(p_h,p_h)\\ =&\|\mathbf{u}_h\|_0^2+\|\nabla\cdot\mathbf{u}\|_0^2-(1+\alpha)b_h(\mathbf{u}_h,p_h)-G(p_h,f(p))+\alpha G(p_h,p_h)\\ =&\|\mathbf{u}_h\|_0^2+\|\nabla\cdot\mathbf{u}\|_0^2-(1+\alpha)b_h(\mathbf{u}_h,p_h)+\alpha G(p_h,\alpha p_h-f(p_h)).\end{aligned} \tag{4.2}$$

Under the conditon $\alpha>0$, we bound terms in the right hand side of (4.2),

$$(1+\alpha)b_h(\mathbf{u}_h,p_h)\leq\frac{1}{2}\|\nabla\cdot\mathbf{u}\|_0^2+c_1\|p_h\|_0^2,$$

and satisfies

$$f'(p)<\alpha-1,$$

we have

$$\mathcal{B}_h((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))\geq c_3(\|\mathbf{u}\|_{\mathbf{V}}+\|p_h\|_0)^2. \tag{4.3}$$

Clearly, using the triangle inequality, we have

$$\|\mathbf{u}_h\|_0+\|\nabla\cdot\mathbf{u}_h-\alpha p_h\|_0\leq c_4(\|\mathbf{u}\|_{\mathbf{V}}+\|p_h\|_0). \tag{4.4}$$

Using (4.3) and (4.4), and set $\beta=c_3/c_4$, (4.1) is given. □

Theorem 4.2. *Assume that the function $f,g\in L^2(\Omega)$, we have*

$$\beta(\|\mathbf{u}_h\|_{\mathbf{V}}+\|p_h\|_0)\leq\|f(p)\|_0+\|g(x)\|_0.$$

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Proof. According the Theorem 4.1, we have

$$\begin{aligned}
&\beta(\|\mathbf{u}_h\|_{\mathbf{V}}+\|p_h\|_0)\\
\leq &\sup_{(\mathbf{v}_h,q_h)\in(NCP_1,P_1)}\frac{|\mathcal{B}_h((\mathbf{u}_h,p_h);(\mathbf{v}_h,q_h))|}{\|\mathbf{v}_h\|_{\mathbf{V}}+\|q_h\|_0}\\
\leq &\sup_{(\mathbf{v}_h,q_h)\in(NCP_1,P_1)}\frac{|(f(p_h),q_h)+(g(x),q_h)|}{\|\mathbf{v}_h\|_{\mathbf{V}}+\|q_h\|_0}\\
\leq &\sup_{(\mathbf{v}_h,q_h)\in(NCP_1,P_1)}\frac{|(f(p_h),q_h)+(g(x),q_h)|}{\|q_h\|_0}\\
\leq &\|f(p)\|_0+\|g(x)\|_0.
\end{aligned}$$

Thus, we complete the proof. □

Theorem 4.3. *Suppose the condition $f'(p)\leq 0$ holds, there exists a unique solution of problem* (3.4).

Proof. The proof is similar to the Theorem 2.1 in [33], so we omit it. □

5 Error analysis

In this section, $H>0$ and $h>0$ are the coarse and fine grid parameters, and satisfies $H\gg h>0$. The corresponding nonconforming finite element spaces for the variable $\mathbf{u}$ and conforming finite element spaces for the variable p are denoted by NCP_{1H}, P_{1H} and NCP_{1h}, P_{1h}. Then the following two-level stabilized nonconforming finite element is presented.

The proposed algorithm consists of the two steps:

Step 1 Solve the semi-linear elliptic problem on a coarse mesh: find $(\mathbf{u}_H,P_H)\in(NCP_{1H},P_{1H})$, such that

$$\mathcal{B}_H((\mathbf{u}_H,p_H);(\mathbf{v},q))-(f(p_H),q)=(g,q),\quad\forall(\mathbf{v},q)\in(NCP_{1H},P_{1H}). \tag{5.1}$$

Step 2 Solving the linear elliptic problem on a fine mesh: find $(\mathbf{u}_h,P_h)\in(NCP_{1h},P_{1h})$, such that

$$\begin{aligned}
&\mathcal{B}_h((\mathbf{u}_h,p_h);(\mathbf{v},q))-(f'(p_H)p_h,q)\\
=&(f(p_H)-f'(p_H)p_H,q)+(g,q),\quad\forall(\mathbf{v},q)\in(NCP_{1h},P_{1h}).
\end{aligned} \tag{5.2}$$

To derive error estimates for the finite element solution (u_τ,p_τ), the folowing projection operator $R_\tau,Q_\tau:(\mathbf{V},W)\to(NCP_1,P_1)$ is defined by

$$\mathcal{B}_\tau((R_\tau(\mathbf{u},p),Q_\tau(\mathbf{u},p);(\mathbf{v}_\tau,q_\tau))=B_\tau((\mathbf{u},p);(\mathbf{v}_\tau,q_\tau)) \tag{5.3}$$

for $\forall(\mathbf{u},p)\in(\mathbf{V},W)$, $(\mathbf{v}_\tau,q_\tau)\in(NCP_1,P_1)$ and have the below error estimates, where τ denote h or H.

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Lemma 5.1 (see [34]). *For* $\forall(\mathbf{u},p)\in(H^2(\Omega)\cap\mathbf{V},H^1(\Omega)\cap W)$

$$\|\mathbf{u}-R_\tau(\mathbf{u},p)\|_{\mathbf{V}}+\|p-Q_\tau(\mathbf{u},p)\|_0\leq C\tau(\|\mathbf{u}\|_1+\|\mathrm{div}\mathbf{u}\|_1+\|p\|_1) \tag{5.4}$$

holds.

Next, error estimates are presented for the finite element method (3.4) using the NCP_1-P_1 pair.

Theorem 5.1. *Under the condition Lemma 5.1, and assume that* $f'(p)$ *is continuous and bounded, we have*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}}+\|p-p_h\|_0\leq cH.$$

Proof. Subtracting (5.1) from (3.2), yields

$$\mathcal{B}_H((u-u_H,p-p_H);(v,q))+G(p_H,q)=(f(p)-f(p_H,q).$$

Using the Taylor expansion $f(p)-f(p_H)=f(p')(p-p_H)$, the value p' denote the value p and p_H, we have

$$\mathcal{B}_H((u-u_H,p-p_H);(v,q))+G(p_H,q)=f(p')((p-p_H,q). \tag{5.5}$$

Subtracting (5.5) from (5.3) gives

$$\begin{aligned}
&\mathcal{B}_H((R_H(\mathbf{u},p)-u_H,Q_H(\mathbf{u},p)-p_H);(\mathbf{v}_H,q_H))\\
=&G(p-p_H,q)+f(p')((p-p_H),q_H)\\
=&G(p-Q_H(\mathbf{u},p),q)+G(Q_H(\mathbf{u},p)-p_H,q)+f(p')(p-Q_H(\mathbf{u},p),q_H)\\
&+f(p')(Q_H(\mathbf{u},p)-p_H,q_H).
\end{aligned} \tag{5.6}$$

According to the Theorem 4.1 yields

$$\begin{aligned}
&\beta(\|(R_H(\mathbf{u},p)-u_H\|_{\mathbf{V}}+\|Q_H(\mathbf{u},p)-p_H\|_0)\\
\leq&\sup_{(\mathbf{v}_H,q_H)\in(NCP_1,P_1)}\frac{|\mathcal{B}_H((R_H(\mathbf{u},p)-u_H,Q_H(\mathbf{u},p)-p_H);(\mathbf{v}_H,q_H))|}{\|\mathbf{v}_h\|_{\mathbf{V}}+\|q_h\|_0}\\
\leq&\sup_{(\mathbf{v}_H,q_H)\in(NCP_1,P_1)}\frac{1}{\|\mathbf{v}_H\|_{\mathbf{V}}+\|q_H\|_0}\big[|G(p-Q_H(\mathbf{u},p),q)+G(Q_H(\mathbf{u},p)-p_H,q)\\
&+f(p')(p-Q_H(\mathbf{u},p),q_H)+f(p')(Q_H(\mathbf{u},p)-p_H,q_H)|\big]\\
\leq&c_1\|(I-\Pi)(p-Q_H(\mathbf{u},p)\|_0+c_2\|(I-\Pi)Q_H(\mathbf{u},p)-p_H\|_0\\
&+c_3\|(p-Q_H(\mathbf{u},p)\|_0+c_4\|Q_H(\mathbf{u},p)-p_H\|_0\\
\leq&c_1'\|p-Q_H(\mathbf{u},p)\|_0+c_2'\|Q_H(\mathbf{u},p)-p_H\|,
\end{aligned}$$

where we have used (3.3b), together with Lemma 5.1, we complete the proof. □

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Theorem 5.2. *Under the condition Lemma 5.1, and assume that $f'(p)$, and $f''(p)$ is continuous and bounded for $s\in(-\infty,+\infty)$, it hold that*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}}+\|p-p_h\|_0\leq c(h+H^2).$$

Proof. Subtracting (5.2) from (5.3) gives

$$\begin{aligned}&\mathcal{B}_h((R_h(\mathbf{u},p)-u_h,Q_h(\mathbf{u},p)-p_h);(\mathbf{v}_h,q_h))+(f'(p_H)p_h,q_h)\\&=B_h((\mathbf{u},p);(\mathbf{v}_h,q_h))-(f(p_H)-f'(p_H)p_H,q_h)-(g,q).\end{aligned}\tag{5.7}$$

According to Eq. (2.1), by using the Green formula on each element in K_i, we see that

$$(g,q_h)=B_h((\mathbf{u},p);(\mathbf{v}_h,q_h))+\sum_j\langle v_h\cdot n_j,p\rangle-(f(p),q_h).$$

So, (5.7) can be rewritten as

$$\begin{aligned}&\mathcal{B}_h((R_h(\mathbf{u},p)-u_h,Q_h(\mathbf{u},p)-p_h);(\mathbf{v}_h,q_h))+(f'(p_H)p_h,q_h)\\&=-(f(p_H)-f'(p_H)p_H,q_h)-\sum_j\langle v_h\cdot n_j,p\rangle+(f(p),q_h)\end{aligned}\tag{5.8}$$

for the term $f(p)$, by use of the Taylor expansion, we can get

$$f(p)-(p_H)=f'(p_H)(P-P_H)+f''(\nu)(p-p_H)^2,\tag{5.9}$$

where the value ν is between the value p and p_H.

Combing (5.8) and (5.9), we have

$$\begin{aligned}&\mathcal{B}_h((R_h(\mathbf{u},p)-u_h,Q_h(\mathbf{u},p)-p_h);(\mathbf{v}_h,q_h))\\&=f'(p_H)((Q_h(\mathbf{u},p)-p_h),q_h)+f'(p_H)((p-Q_h(\mathbf{u},p),q_h)\\&\quad+f''(\nu)((p-p_H)^2,q_h)-\sum_j\langle v_h\cdot n_j,p\rangle.\end{aligned}\tag{5.10}$$

From the Theorem 4.1, we have

$$\begin{aligned}&\beta(\|R_h(\mathbf{u},p)-u_h\|_{\mathbf{V}}+\|Q_h(\mathbf{u},p)-p_h\|_0)\\&\leq\sup_{(\mathbf{v}_h,q_h)\in(NCP_1,P_1)}\frac{|\mathcal{B}_h((R_h(\mathbf{u},p)-u_h,p_h);(Q_h(\mathbf{u},p)-p_h,q_h))|}{\|\mathbf{v}_h\|_{\mathbf{V}}+\|q_h\|_0}\\&\leq\sup_{(\mathbf{v}_h,q_h)\in(NCP_1,P_1)}\frac{1}{\|\mathbf{v}_h\|_{\mathbf{V}}+\|q_h\|_0}\Big[\Big|f'(p_H)((Q_h(\mathbf{u},p)-p_h),q_h)+f'(p_H)((p-Q_h(\mathbf{u},p),q_h)\\&\quad+f''(\nu)((p-p_H)^2,q_h)-\sum_j\langle v_h\cdot n_j,p\rangle\Big|\Big]\\&\leq c\|Q_h(\mathbf{u},p)-p_h\|_0+c\|p-Q_h(\mathbf{u},p)\|_0+c\|(p-p_H)^2\|_0+ch\|p\|_1\\&\leq ch(\|\mathbf{u}\|_1+\|\mathrm{div}\mathbf{u}\|_1+\|p\|_1)+c\|(p-p_H)^2\|_0+c\|Q_h(\mathbf{u},p)-p_h\|_0.\end{aligned}\tag{5.11}$$

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Furthermore, when $\beta - c \geq 0$, Eq. (5.11) can be recasted as

$$\begin{aligned} &\|R_h(\mathbf{u},p)-u_h\|_{\mathbf{V}}+\|Q_h(\mathbf{u},p)-p_h\|_0 \\ \leq& ch(\|\mathbf{u}\|_1+\|\mathrm{div}\mathbf{u}\|_1+\|p\|_1)+c\|(p-p_H)^2\|_0. \end{aligned} \tag{5.12}$$

So, we complete the proof. □

It follows from Theorem 5.2 that H is chosen to satisfy $h=\mathcal{O}(H^2)$, then by virtue of the two-level stabilized nonconforming finite element method, the accuracy of numerical solution for semi-linear elliptic problem can be achieve the same as that of the usual stabilized conforming finite element method. But, our method will take less time.

6 Numerical results

In this section, we will give the numerical results to verify the theoretical analysis for the steady reaction diffusion equations. The following semi-linear elliptic problem in the fixed domain $\Omega=[0,1]\times[0,1]$ is considered:

$$-\Delta p+p^3=g \qquad \text{in } \Omega, \tag{6.1a}$$

$$p=0 \qquad \text{on } \partial\Omega. \tag{6.1b}$$

The exact solution is given as follows

$$p(x,y)=(1-x)x^2(1-y)y^2.$$

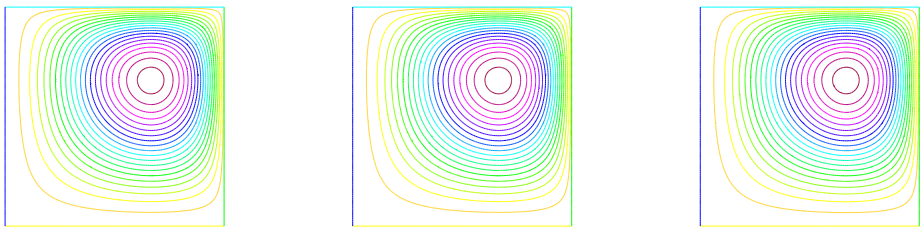
The domain Ω is divided into two families τ_h and τ_H of triangulations with mesh size h and H, respectively. The right-side term g is computed by evaluating problem (6.1) for the given exact solution. The finite element discretization uses the NCP_1-P_1 pairs defined with respect to the same uniform triangulation for variable $\mathbf{u}$ and variable p. Result is show by the below Table and figure. The experimental rates of convergence with respect to the mesh size h are calculated by the formula $\log(E_i/E_{i+1})/\log(h_i/h_{i+1})$, E_i and E_{i+1} are the relative error corresponding to the meshes of size h_i and h_{i+1}, respectively. In two-level methods, h is given, then H is chosen such that $h=H^2$.

We compare the results obtained by our method (NCP_1-P_1) with conforming two-level stabilized finite element method (P_1-P_1) (see Table 1 and Fig. 1). From Fig. 1, we

Table 1: Two-level stabilized finite element method: NCP_1-P_1.

		conforming element		nonconforming element	
$\frac{1}{H}$	$\frac{1}{h}$	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate	$\frac{\|p-p_h\|_0}{\|p\|_0}$	Rate
5	25	6.90053e-3		4.03950e-3	
7	49	1.80234e-3	1.9950	1.05489e-3	1.9952
9	81	6.60063e-4	1.9985	3.86316e-4	1.9986

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(a) (b) (c)

Figure 1: The variable p:(a) exact solution; (b) p_1-p_1 element; (c) NCP_1-p_1 element.

can observe a good agreement of the result obtained with those method reported in the literature. From the Table 1, it is slightly better for pressure approximation if nonconforming finite element discretization is used. Simultaneously, Concerning the difference in computation costs between these schemes, conforming finite element discretization is more expensive than nonconforming finite element discretization.

7 Conclusions

In this paper, we have provided a theoretical analysis for the proposed stabilized scheme. The finite element discretization uses a pair of spaces of finite element NCP_1-P_1 over triangles element. This new method has a number of attractive computational properties. It is free from the stabilization parameter. Compared with some established methods, numerical result shows that new method exhibited good stability behavior, and the consuming time is less.

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