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On quantum modular forms of non-zero weights

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Abstract

We study functions f on $\mathbb{Q}$ which satisfy a 'quantum modularity' relation of the shape $f(x+1)=f(x)$ and $f(x)-|x|^{-k}f(-1/x)=h(x)$, where $h:\mathbb{R}_{\neq 0}\to\mathbb{C}$ is a function satisfying various regularity conditions. We study the case $\mathrm{Re}(k)\neq 0$. We prove the existence of a limiting function, denoted by $f^{\triangleleft}$ or $f^{\triangleright}$, depending on the sign of $\mathrm{Re}(k)$, which extends continuously f to $\mathbb{R}$ in some sense. This means, in particular, that in the $\mathrm{Re}(k)\neq 0$ case, the quantum modular form itself has to have at least a certain level of regularity. We deduce that the values $\{f(a/q), 1\leqslant a<q, (a,q)=1\}$, appropriately normalized, tend to equidistribute along the graph of $f^{\triangleleft}$ or $f^{\triangleright}$, and we prove that, under natural hypotheses, the limiting measure is diffuse. We apply these results to obtain limiting distributions of values and continuity results for several arithmetic functions known to satisfy the above quantum modularity: higher weight modular symbols associated to holomorphic cusp forms; Eichler integral associated to Maaß forms; a function of Kontsevich and Zagier related to the Dedekind η-function; and generalized cotangent sums.

1. Introduction

In [Zag10], Zagier introduced the concept of quantum modular forms (QMF). Given a Fuchsian cofinite subgroup Γ of $\mathrm{SL}(2,\mathbb{Z})$ whose set of cusps $C(\Gamma)\subset\mathbb{P}^1(\mathbb{Q})$ is non-empty, and $k\in\mathbb{C}$, QMF are defined as functions

$$f: C(\Gamma)\smallsetminus S\to\mathbb{C},$$

for some finite set S, satisfying a form of modularity, in the purposely vague sense that for any $\gamma=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$, the *period function*

$$h_\gamma(x):=f(x)-|cx+d|^{-k}f\left(\frac{ax+b}{cx+d}\right),\quad x\in C(\Gamma)\smallsetminus(S\cup\gamma^{-1}S) \tag{1.1}$$

is somewhat regular with respect to the real topology. The exact assumptions vary widely in the literature; in our case, we will mainly require continuity or bounds on the growth around possible singularities.

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Numerous examples of quantum modular forms are known, in various contexts, and we refer in particular to [Zag10, BFR15, NR17, KLL16, BLZ15, MMY97, JM18]. More references are listed in the introduction of [BD22].

In this paper, we focus on QMF for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, so that $C(\Gamma) = \mathbb{P}^1(\mathbb{Q})$, and which are periodic (i.e. $h_U = 0$ for $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$).[1] By composition, it is sufficient to consider (1.1) for the second generator $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ of $\mathrm{SL}(2,\mathbb{Z})$, so that, in order to prove that f is a QMF, one only needs to verify that

$$h(x) := f(x) - |x|^{-k} f(-1/x) \quad (x \in \mathbb{Q} \setminus \{0\}) \tag{1.2}$$

has some regularity property.

The cases of $\mathrm{Re}(k) = 0$ and $\mathrm{Re}(k) \neq 0$ are different in nature. In the case $k = 0$, by iterating the relation (1.2) and using periodicity, we can express f as a twisted Birkhoff sum of h evaluated along orbits under the Gauss map, see [BD22, Equation (3.1)], the twist being given by the multiplicative automorphic factor. Using this observation, in [BD22] it was then shown that, for a large class of functions h, the multisets $\{f(x) \mid x \in \mathbb{Q} \cap [0,1),\ \mathrm{Den}(x) \leqslant Q\}$ become asymptotically distributed, as $Q \to \infty$, according to a stable law, which is in fact a normal law if h is of moderate growth at 0.[2] As a consequence, one has that, in general, f is nowhere continuous according to the real topology, nor can f be extended by continuity to any point outside of $\mathbb{Q}$. For the same reasons, we expect a similar phenomenon to occur whenever $\mathrm{Re}(k) = 0$.

The purpose of the present paper is to study the case when $\mathrm{Re}(k) \neq 0$.

1.1 Weights with negative real parts

If $\mathrm{Re}(k) < 0$ and h is continuous on $[-1,1] \setminus \{0\}$ with finite right and left limits $h(0^{\pm})$ at 0, then we show that, in fact, f can always be extended by continuity to a bounded function $f^{\triangleleft} : \mathbb{R} \to \mathbb{C}$. Moreover, we prove that $f^{\triangleleft}$ is continuous on $\mathbb{R} \setminus \mathbb{Q}$ and is continuous on the whole real line if $h(0\pm) = f(0)$, a condition that could be morally interpreted as saying that (1.2) holds also at '$0^{\pm}$'. More precisely, the following holds.

Theorem 1.1. *Let $\mathrm{Re}(k) < 0$ and let $f : \mathbb{Q} \to \mathbb{C}$ be a 1-periodic function satisfying (1.2) for a function $h : \mathbb{R} \setminus \{0\} \to \mathbb{C}$ which is continuous on $[-1,1] \setminus \{0\}$ with finite left and right limits $h(0^{\pm})$ at 0. Then the function*

$$f^{\triangleleft}(x) := \begin{cases} f(x) & \text{if } x \in \mathbb{Q}, \\ \lim\limits_{\mathbb{Q} \ni y \to x} f(y) & \text{if } x \notin \mathbb{Q} \end{cases} \tag{1.3}$$

is defined for all $x \in \mathbb{R}$ and is continuous on $\mathbb{R} \setminus \mathbb{Q}$. Moreover, for any rational $x = a/q$ in reduced form, one has $f^{\triangleleft}(y) \to f(x) + q^k(h(0^{\pm}) - f(0))$ as $y \to x^{\pm}$. In particular, $f^{\triangleleft}$ is continuous on $\mathbb{R}$ if and only if $h(0^{\pm}) = f(0)$. Furthermore, in this case, if $h \in \mathcal{C}^m([-1,1],\mathbb{C})$ with $0 \leqslant m < |\mathrm{Re}(k)|/2$, then $f^{\triangleleft} \in \mathcal{C}^m(\mathbb{R},\mathbb{C})$.

If h is continuous at 0 but $h(0) \neq f(0)$, we can modify f by letting $\widetilde{f}(x) := f(x) + \mathrm{Den}(x)^k(h(0) - f(0))$. The map $\widetilde{f}$ is a QMF with h as its period function, and $\widetilde{f}(0) = h(0)$.

As we will see below, in some of the examples of QMF that we will consider, the map f actually admits an expression as an absolutely converging Fourier series, and the differentiability of such series has been studied in detail in many instances, see in particular [Cha04, Pet14, CPRC17].

[1] See [BD22, footnote 2] for a way to handle non-periodic QMF, and a work in progress of the authors with J. Lee for a generalization to other groups in the weight zero case.

[2] The same method yields an analogous result for $\{f(x) \mid x \in \mathbb{Q} \cap [a,b),\ \mathrm{Den}(x) \leqslant Q\}$ for any $0 \leqslant a < b \leqslant 1$.

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Theorem 1.1 instead gives a proof of the continuity and differentiability of f extended to $\mathbb{R}$ which does not rely on an *a priori* knowledge of a Fourier expansion for f.

Theorem 1.1 shows that, in fact, one cannot expect the period function h to be continuous if f does not have some continuity property to start with, so that non-trivial instances of QMF (meaning those cases when h is more regular than f) arise only when h is differentiable at least $\lceil -\mathrm{Re}(k)/2 \rceil$ times.

Our method uses properties of the Gauss map, and is closely related to the works [MMY97, LMPS24, BM19]. These works are concerned with the regularity properties of the Brjuno function. This function is related to linearization problems in holomorphic dynamics [Brj72, Yoc88, Yoc95], and is close to being a QMF of weight -1, see [MMY06]. The methods of [MMY97] likely extend to the setting of Theorem 1.1, and would be relevant to study e.g. the Hölder regularity of the highest-order derivative of f (see Definitions 1.1 to 1.3 in [CPRC17]).

We point out that precise estimates on Hölder regularity are achievable by methods from wavelet theory. In [Pet14] this was carried out using an explicit Fourier expansion, while in [JM18] instead this was performed using properties of the Gauss map, a point of view similar to the one taken in [MMY97] or in the present paper.

The existence of the function $f^{\triangleleft}$ in (1.3) can actually be proved under much weaker hypotheses, which we explain in what follows. This will be required later on, when we will study cotangent sums.

Theorem 1.2. *Suppose that* $\mathrm{Re}(k) < 0$ *and let* $\delta > 0$. *Suppose that* f *satisfies*

$$f(x) = f(x+1) \quad (x \in \mathbb{Q} \smallsetminus [-1, 0]), \tag{1.4}$$

in other words that f *is* 1*-periodic separately on* $(-\infty, 0)$ *and* $(0, \infty)$, *and that Equation (1.2) holds for* $x \in [-1, 1] \smallsetminus \{0\}$, *for a function* h *satisfying*

$$\sup_{\substack{|x-y|<e^{-\varepsilon^{\delta-1}},\\ |x|>2\varepsilon}} |h(x) - h(y)| = o(1) \quad \text{as } \varepsilon \to 0^+, \quad h(x) = O(e^{|x|^{\delta-1}}) \quad \forall x \in [-1, 1] \smallsetminus \{0\}. \tag{1.5}$$

Then there is a full measure set $X \subset \mathbb{R} \smallsetminus \mathbb{Q}$ *such that the limit*

$$f^{\triangleleft}(x) := \lim_{j \to \infty} f(x_j) \tag{1.6}$$

exists for $x \in X$, *where* $(x_j) = ([a_0(x); a_1(x), \dots, a_j(x)])_j$ *denotes the sequence of convergents of* x. *This value coincides with the limit (1.3) when* h *is bounded. In general, for any* $\varepsilon > 0$, *there is a subset* $X_\varepsilon \subset X$ *invariant by* $x \mapsto x + 1$ *and with* $\nu(X_\varepsilon \cap [0, 1]) \geqslant 1 - \varepsilon$, *such that* $f^{\triangleleft}|_{X_\varepsilon}$ *is continuous on* X_ε *equipped with the restricted topology. In particular,* $f^{\triangleleft}$ *is Lebesgue-measurable.*

The set X in this statement does not depend on f, h or δ. It consists of numbers having mildly growing continued fraction coefficients, see Lemma 2.2 below.

The extension of the definition of $f(x)$ also allows us to show that f has a limiting distribution when evaluated at reduced rationals $0 \leqslant a/q < 1$ with $q \to \infty$.

Theorem 1.3. *Suppose that* $\mathrm{Re}(k) < 0$ *and let* $f : \mathbb{Q} \to \mathbb{C}$ *satisfy (1.2) and (1.4), for a function* $h : \mathbb{R} \smallsetminus \{0\} \to \mathbb{C}$ *satisfying (1.5). Then the multiset*

$$\{f(a/q) \colon 1 \leqslant a \leqslant q, (a, q) = 1\}$$

becomes distributed, as $q \to \infty$, *according to the push-forward* $f^{\triangleleft}_*(\nu)$ *of the Lebesgue measure* ν *on* $[0, 1]$ *(see [LG22, p. 13]).*

If, moreover, h *is real-analytic on* $(-1, 1) \smallsetminus \{0\}$ *and* f *is non-constant on* $\mathbb{Q}_{>0}$, *then the measure* $f^{\triangleleft}_*(\nu)$ *is diffuse.*

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In contrast with the case $k=0$ treated in [BV05, BD22], we remark here that we did not need to perform an additional average over q in order to obtain a limiting statement.

We recall that a measure is diffuse if it has no atoms. When $f^{\lhd}$ is real-valued, then $f^{\lhd}_*(\nu)$ is supported on $\mathbb{R}$, and diffuseness is equivalent to the continuity of the associated cumulative distribution function. Under appropriate conditions, we are able to reach the stronger conclusion that for any non-zero linear form $\phi:\mathbb{C}\to\mathbb{R}$, the measure $(\phi\circ f^{\lhd})_*(\nu)$ on $\mathbb{R}$ is diffuse.[3] This is equivalent to the statement that the graph of $f^{\lhd}$ never remains on a given straight line for a positive proportion of time: for any line $D\subset\mathbb{C}$, $\nu((f^{\rhd})^{-1}(D))=0$. Also, the statement that $(\phi\circ f^{\lhd})_*(\nu)$ is diffuse means that its cumulative distribution function is continuous.

1.2 Weights with positive real parts

In the case $\mathrm{Re}(k)>0$, we find that iterating the reciprocity formula (1.2) does not imply the continuity of f, even if h is continuous. It is, however, still possible to extend naturally f if one considers $f(x)$, not as a function of $x=a/q$, but rather as a function of $\overline{x}:=\overline{a}_q/q$, where $\overline{a}_q\in(0,q]$ is the multiplicative inverse of $a \pmod q$.

Theorem 1.4. *Let $\mathrm{Re}(k)>0$ and let f be 1-periodic and satisfy (1.2) with $h(x):\mathbb{R}\smallsetminus\{0\}\to\mathbb{C}$ satisfying $h(x)=O(|x|^{-\mathrm{Re}(k)})$ for $|x|\in(0,1]$. Then the function*

$$f^{\rhd}(x):=\begin{cases} q^{-k}f(\overline{x}) & \text{if } x=a/q\in\mathbb{Q},\\ \lim\limits_{\mathbb{Q}\ni y=a/q\to x} q^{-k}f(\overline{y}) & \text{if } x\notin\mathbb{Q}\end{cases} \tag{1.7}$$

defines a continuous function of $x\in\mathbb{R}\smallsetminus\mathbb{Q}$. Furthermore, if $h(x)=o(|x|^{-\mathrm{Re}(k)})$ as $x\to0$, then $f^{\lhd}$ is continuous on $\mathbb{R}$.

Finally, there exists a full measure set $X\subset\mathbb{R}$ such that $f^{\rhd}$ is α-Hölder continuous at any point of X, for any $\alpha<\frac{1}{2}\mathrm{Re}(k)$. In particular, if $\mathrm{Re}(k)>2$, then $f^{\rhd}$ has derivative zero almost everywhere.

It would be interesting to know what could be said about the L^p or the bounded mean oscillation regularity of $f^{\rhd}$, similarly to [MMY97]; or what could be said about the fractal or multi-fractal property of $f^{\rhd}$ in specific cases, similarly to [JM18].

Also in the case $\mathrm{Re}(k)>0$, the limit (1.7) makes sense under more general hypotheses, which we will require in some of our applications: it suffices that f satisfies (1.4) instead of being periodic on $\mathbb{R}$, and that h merely satisfies $h(x)=O(e^{-|x|^{\delta-1}})$ for $x\in[-1,1]\smallsetminus\{0\}$ and some $\delta>0$. More precisely, the following holds.

Theorem 1.5. *Suppose that $\mathrm{Re}(k)>0$ and let $f:\mathbb{Q}\to\mathbb{C}$ satisfy (1.2) and (1.4) for a function $h:\mathbb{R}\smallsetminus\{0\}\to\mathbb{C}$ with $h(x)\ll e^{|x|^{-1+\delta}}$ for $x\in(-1,1)\smallsetminus\{0\}$ and some $\delta>0$. Then the limit*

$$f^{\rhd}(x):=\lim_{j\to\infty} q(x_{2j+1})^{-k}f(\overline{x_{2j+1}}) \tag{1.8}$$

exists on a full measure subset X of $\mathbb{R}$, and coincides with the value (1.7) under the hypotheses of Theorem 1.4.

Remark 1.6. As in Theorem 1.2 we will show, under the hypotheses stated above, the existence of sets $X_\varepsilon\subset X$ with $\nu(X_\varepsilon)\geqslant 1-\varepsilon$ such that $f^{\rhd}|_{X_\varepsilon}$ is continuous with the restricted topology, and in particular $f^{\rhd}$ is also Lebesgue-measurable.

[3] Such a form ϕ is, of course, proportional to $z\mapsto\mathrm{Re}(e^{i\theta}z)$ for some $\theta\in\mathbb{R}$. This applies in particular to $\phi=\mathrm{Re}$ or $\phi=\mathrm{Im}$.

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Similarly to Theorem 1.2, we can then show that the values of f at rationals converge to a limiting distribution.

Theorem 1.7. *Under the notation and conditions of Theorem 1.5, the multisets*

$$\{q^{-k}f(a/q)\colon 1\leqslant a\leqslant q,(a,q)=1\}$$

become distributed, as $q\to\infty$, according to $f^{\triangleright}_*(\nu)$.

Moreover, if h is not identically zero, and is either continuous on $[-1,1]$ or satisfies $h(x)\ll e^{|x|^{-1+\delta}}$ and $h(x)\sim cx^{-\lambda}$ as $x\to 0^+$ for some $\delta>0$, $c\in\mathbb{C}\smallsetminus\{0\}$ and $\lambda\in\mathbb{R}_{>0}\smallsetminus\{k\}$, then the measure $f^{\triangleright}_*(\nu)$ is diffuse.

Remark 1.8. One can relax considerably the conditions required to ensure the diffuseness of the limiting measures. For example, it is sufficient that h is right continuous at 0 with $h(0^+)\neq 0$ or that $h(x)$ goes to ∞ as $x\to 0$ without staying too close to a multiple of $1-|x|^{-k}$. See § 3.3 for more details.

Similarly as for $\mathrm{Re}(k)<0$, under natural conditions which are, however, more involved, we show that the measure $(\phi\circ f^{\triangleright})_*(\nu)$ on $\mathbb{R}$ is diffuse for any non-zero linear form $\phi\colon\mathbb{C}\to\mathbb{R}$.

1.3 Applications

The above theorems apply to many objects, some of which are described in the following corollaries.

1.3.1 *Eichler integrals of classical holomorphic forms.* The elementary-looking function

$$A_{k,D}(x):=\sum_{\substack{Q(x)=ax^2+bx+c>0\\ a\in-\mathbb{N},\,b,c\in\mathbb{Z},\,b^2-4ac=D}}Q(x)^k,\quad x\in\mathbb{R},\,k\in 2\mathbb{N}+3,\ \square\neq D\equiv 1\ (\mathrm{mod}\ 4)\tag{1.9}$$

was introduced and studied by Zagier in [Zag99];[4] see [Ben15] for more details on the convergence of the sum. With a simple algebraic computation one can verify that $A_{k,D}(x)$ is a 1-periodic QMF of weight $-2k$, satisfying (1.2) with a period function $h=h_{k,D}$ which is a polynomial of degree $2k$ satisfying, by [Zag99, Equation (25)]

$$h_{k,D}(0)=A_{k,D}(0)=\sum_{\substack{0\leqslant b<\sqrt{D}\\ b^2\equiv D\ (\mathrm{mod}\ 4)}}\sigma_k\left(\frac{D-b^2}{4}\right)$$

where $\sigma_k(n)=\sum_{d|n}d^k$. Theorem 1.1 applies to $A_{k,D}$ and gives another proof that $A_{k,D}\in\mathcal{C}^{k-1}(\mathbb{R},\mathbb{R})$. The fact that $A_{k,D}(x)$ is in $\mathcal{C}^{k-1}(\mathbb{R},\mathbb{R})$ was instead obtained by Zagier in [Zag99] by observing that $h_{k,D}$ belongs to the space of period polynomials corresponding to modular forms of weight $2k+2$. This implies, in particular, that $A_{k,D}(x)$ has to coincide with the Eichler integral of a weight $2k+2$ modular form and thus can be written as a Fourier series whose nth Fourier coefficient decays roughly as $n^{-k-1/2}$, see [Zag99, Equation (53)].

From Theorem 1.3 we deduce the following distributional result for $A_{k,D}$.

Corollary 1.9. *Let $k\geqslant 5$ be odd and assume $D\equiv 0,1\ (\mathrm{mod}\ 4)$ is not a square. Then*

$$\{A_{k,D}(a/q)\colon 1\leqslant a\leqslant q,(a,q)=1\}\subset\mathbb{R}$$

becomes distributed, as $q\to\infty$, according to the measure $(A^{\triangleleft}_{k,D})_*(\nu)$. This measure has a continuous cumulative distribution function.

[4] If $k=1,3$ then $A_{k,D}(x)$ can still be defined but it turns out to be constant.

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More generally, let g be in the space $\mathrm{S}_k(1)$ of cusp forms of weight $k \geqslant 12$ and level 1. If $g(z) := \sum_{n\geqslant 1} a_n e(nz)$ for $\mathrm{Im}(z) > 0$, where $e(z) := e^{2\pi i z}$, then the Eichler integral of g

$$\widetilde{g}(z) := \sum_{n\geqslant 1} \frac{a_n}{n^{k-1}} e(nz) \quad (\mathrm{Im}(z) \geqslant 0), \tag{1.10}$$

restricted to $\mathbb{Q}$, is a quantum modular form of weight $2-k$ with period function given by the period polynomials of g.

Corollary 1.10. *For $k \geqslant 12$, let $0 \neq g \in \mathrm{S}_k(1)$. Then*

$$\left\{ \widetilde{g}\left(\frac{a}{q}\right) : 1 \leqslant a \leqslant q, (a,q) = 1 \right\} \subset \mathbb{C}$$

becomes distributed, as $q \to \infty$, according to $\widetilde{g}^{\triangleleft}_*(\nu)$. For any non-zero linear form $\phi : \mathbb{C} \to \mathbb{R}$, the measure $(\phi \circ \widetilde{g}^{\triangleleft})_*(\nu)$ is diffuse.

It follows from Theorem 1.1 that the map $\widetilde{g}$ is $k/2-1$ times differentiable, but in this special case it could be seen immediately from the definition (1.10) and square-root cancellation in averages of a_n, see [Iwa97, Theorem 5.3]. We comment on this after the proofs, see Remark 4.3.

1.3.2 *Period functions of Maaß forms.* In [LZ01], Lewis and Zagier extend in some sense the theory of period polynomial to period functions associated to Maaß forms. Their work then allows us to construct quantum modular forms also from Maaß forms, as was described by Bruggeman [Bru07]. We briefly review their construction. Let u be a Maaß cusp form for $\mathrm{SL}(2,\mathbb{Z})$ of Laplace eigenvalue $s(1-s)$, with $s \in 1/2 + i\mathbb{R}_{>0}$, which we expand as

$$u(x+iy) := \sum_{n\neq 0} a_n \sqrt{y} K_{s-1/2}(2\pi|n|y) e(nx).$$

We define

$$\widetilde{u}(z) := \frac{\pi^s \Gamma(1-s)}{2} \sum_{n\geqslant 1} n^{s-1/2} a_n e(nz) \quad (\mathrm{Im}(z) > 0).$$

We then include $\mathbb{Q}$ in the domain of $\widetilde{u}$ by letting

$$\widetilde{u}(x) := \lim_{y\to 0^+} \widetilde{u}(x+iy) \quad (x \in \mathbb{Q}). \tag{1.11}$$

It is shown in [Bru07, LZ01] that $\widetilde{u}(x)$ thus defined is a quantum modular form of weight $2s$ whose associated period function h is $\mathcal{C}^\infty$ on $\mathbb{R}$ and real-analytic on $\mathbb{R} \smallsetminus \{0\}$. Using a suitable variant of Theorem 1.7, we will deduce the following result.

Corollary 1.11. *Let u be a non-trivial Maaß form of spectral eigenvalue $s(1-s)$. Then*

$$\{q^{-2s}\widetilde{u}(a/q), 1 \leqslant a \leqslant q, (a,q) = 1\}$$

becomes distributed, as $q \to \infty$, according to $\widetilde{u}^{\triangleright}_*(\nu)$. Moreover, for any non-zero linear form $\phi : \mathbb{C} \to \mathbb{R}$, the measure $(\phi \circ \widetilde{u}^{\triangleright})_*(\nu)$ is diffuse.

We also show that $\widetilde{u}^{\triangleright}$ is almost everywhere locally $(1/2-\varepsilon)$-Hölder continuous, but again in this case it can be seen directly from the Fourier expansion of the underlying form, see Remark 4.3.

1.3.3 *A function of Kontsevich and Zagier.* Another example of quantum modular form, described in [Zag01, Zag10], is given by the series

$$\varphi(x) := e(x/24) \sum_{m=0}^{\infty} (1-e(x)) \cdots (1-e(mx)) \quad (x \in \mathbb{Q}). \tag{1.12}$$

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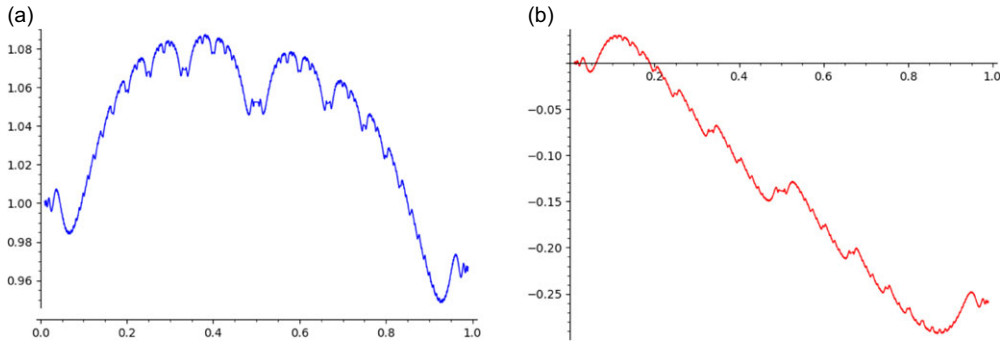


FIGURE 1 (colour online). Approximate plots of $\operatorname{Re}\varphi^{\triangleright}$ (a) and $\operatorname{Im}\varphi^{\triangleright}$ (b) defined in (1.12).

This function was studied by Kontsevich and is related to the Stoimenow numbers, which are used to bound the number of linearly independent Vassiliev invariants of a given degree. In [Zag01], a proof is sketched that φ is a quantum modular form of weight $\frac{3}{2}$ in the generalized meaning that

$$\varphi(x)-e(\pm 1/8)|x|^{-3/2}\varphi(-1/x)=h(x)\quad(\pm x\in\mathbb{Q}_{>0}),\qquad \varphi(x+1)=e(1/24)\varphi(x) \tag{1.13}$$

for a period function $h(x)$ which is smooth on $\mathbb{R}$ (and, in fact, also continues analytically to $\mathbb{C}\smallsetminus i\mathbb{R}_+$).[5] See [BR16] for a proof in the context of period functions for half-integral weight forms, and [GO21] for a generalization to periodic theta functions. In this case, due to the automorphy factors in the reciprocity relation (1.13), the definition of the limiting function $\varphi^{\triangleright}$ is as follows: for $x\in\mathbb{Q}\cap(0,1]$, $x=[0;b_1,\dots,b_r]$ with r odd, denote $\sigma(x)=3+\sum_{1\leqslant j\leqslant r}(-1)^j b_j$. Then

$$\varphi^{\triangleright}(x):=\begin{cases} e(\frac{-1}{24}\sigma(x))\operatorname{Den}(x)^{-3/2}\varphi(\overline{x}) & (x\in\mathbb{Q}\cap(0,1]),\\ \lim_{\mathbb{Q}\ni y\to x}\varphi^{\triangleright}(y) & (x\notin\mathbb{Q}).\end{cases} \tag{1.14}$$

By suitable variants of Theorems 1.4 and 1.7, we obtain the following result.

COROLLARY 1.12. *The map $\varphi^{\triangleright}$ is continuous and, in fact, almost everywhere locally $(3/4-\varepsilon)$-Hölder continuous for any $\varepsilon>0$.*

Moreover, the multiset

$$\{e(\tfrac{-1}{24}\sigma(x))q^{-3/2}\varphi(a/q)\colon 1\leqslant a\leqslant q,(a,q)=1\}$$

becomes distributed, as $q\to\infty$, according to $\varphi^{\triangleright}_*(\nu)$. For any $\theta\in\mathbb{R}$, the cumulative distribution function of $(\operatorname{Re}e^{i\theta}\varphi^{\triangleright})_*(\nu)$ is continuous.

Remark 1.13. The function $\sigma(x)$ can be expressed in terms of the Dedekind sum $s(x)$ [RG72]: using [Hic77, Theorem 1], we find

$$\sigma(x)=x+\overline{x}+12s(x)\quad(x\in(0,1)\cap\mathbb{Q}).$$

The function $\varphi^{\triangleright}:[0,1]\to\mathbb{C}$ is depicted in Figure 1. Note that $\varphi^{\triangleright}(0)=1$.

If we normalize φ by letting $\varphi^{\dagger}(x):=\operatorname{Den}(x)^{-3/2}\varphi(x)$, then as an immediate consequence of Corollary 1.12, we obtain that $\varphi^{\dagger}(\mathbb{Q})$ is equal to the curve $\mathcal{G}$ given by

$$\mathcal{G}:=\bigcup_{n=1}^{24}e^{\pi i n/12}\varphi^{\triangleright}([0,1]),$$

[5] Here h is, up to a constant, naturally interpreted as the period function for the $\frac{1}{2}$-weight modular form given by the Dedekind η function, namely $h(x)=c\int_0^{\infty}\eta(iy)(y+ix)^{-3/2}\,dy$ for some constant $c\in\mathbb{R}$.

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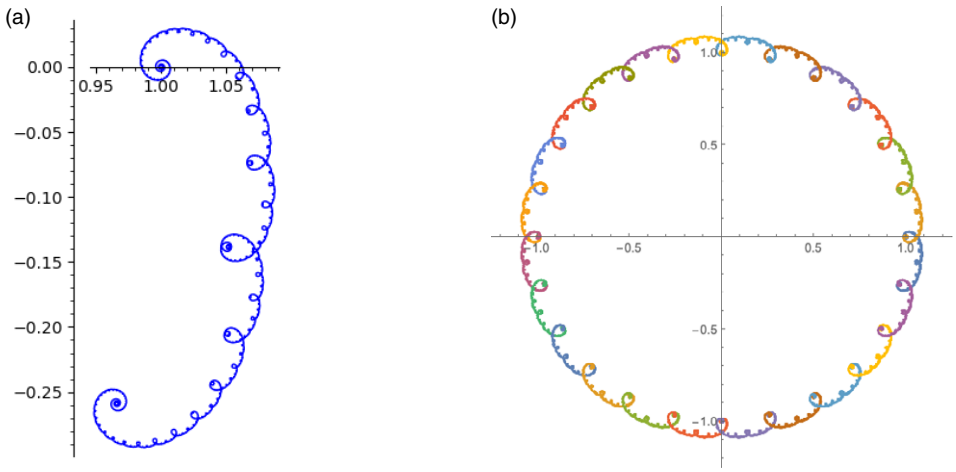


Figure 2 (colour online). Approximate plots of $\varphi^{\triangleright}([0,1])$ (a) and $\varphi^{\dagger}([0,1])$ (b).

which passes through the 24th roots of unity. In Figure 2a, we have plotted the points $\varphi^{\triangleright}(x)$ for $\mathrm{Den}(x) \leqslant 101$. The limiting curve is the graph of $\varphi^{\triangleright}$. In Figure 2b, we have plotted the points $\varphi^{\dagger}(x)$ for $\mathrm{Den}(x) \leqslant 307$, colored depending on $\sigma(x)$. The limiting curve is the curve $\mathcal{G}$.

1.3.4 *Generalized cotangent sums.* Finally we mention the generalized cotangent functions, studied in [BC13a] and defined for $b \in \mathbb{Z}$, $q \in \mathbb{N}_{>0}$, coprime by

$$c_a\left(\frac{b}{q}\right) := q^a \sum_{m=1}^{q-1} \cot\left(\frac{\pi m b}{q}\right) \zeta\left(-a, \frac{m}{q}\right),$$

where $a \in \mathbb{C}$. Here, $\zeta(s,x)$ denotes the Hurwitz zeta function, which is the analytic continuation of $s \mapsto \sum_{n \geqslant 1} (n+x)^{-s}$. When $a = -1$, the poles of ζ cancel out and c_a reduces in that case to the classical Dedekind sum [RG72]. In [BC13a], it was shown that the functions c_a are almost quantum modular forms of weight $1+a$ with period function h_a satisfying, in a neighborhood of 0, the estimate

$$h_a(x) = \frac{\zeta(1-a)}{\zeta(-a)} \frac{i}{\pi x} - \mathrm{sgn}(x) \cot\left(\frac{\pi a}{2}\right) \frac{i}{|x|^{1+a}} + O(1).$$

The meaning of 'almost' here will be made precise later on. The function c_a is very closely related to Eichler integrals of certain real-analytic Eisenstein series, see [BC13a, § 2] and [LZ19]. We will deduce from our main results the following statement on the distribution of values of c_a.

Corollary 1.14. *If* $\mathrm{Re}(a) < -1$, *then the multiset*

$$\{c_a(b/q) \colon 1 \leqslant b \leqslant q, (b,q) = 1\}$$

becomes distributed, as $q \to \infty$, *according to the measure* $\lambda_a := (c_a^{\triangleleft})_*(\nu)$. *If* $\mathrm{Re}(a) > -1$, *then the multiset*

$$\{q^{-1-a} c_a(b/q) \colon 1 \leqslant b \leqslant q, (b,q) = 1\}$$

becomes distributed, as $q \to \infty$, *according to a measure* $\lambda_a := (c_a^{\triangleright})_*(\nu)$. *Moreover:*

(1) *when* $a \in (2\mathbb{Z}_{\geqslant 0} + 1)$, *the measure* λ_a *is supported on* $\{0\}$;
(2) *when* $a \in \mathbb{R} \smallsetminus (2\mathbb{Z}_{\geqslant -1} + 1)$, *the measure* λ_a *is supported inside* $\mathbb{R}$ *and is diffuse;*

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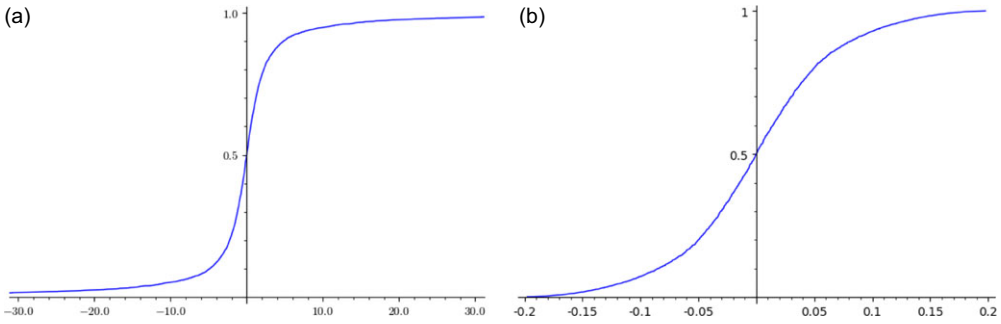


Figure 3 (colour online). CDF of the empirical measures at $q = 5000$ in Corollary 1.14, for $a = -2$ (a) and $a = 1/2$ (b).

(3) *when* $\mathrm{Re}(a) \neq -1$ *and* $a \notin \mathbb{R}$, *then for any non-zero linear form* $\phi : \mathbb{C} \to \mathbb{R}$, *the push-forward measure* $\phi_*(\lambda_a)$ *on* $\mathbb{R}$ *is diffuse.*

When $\mathrm{Re}(a) > 0$, *the map* $c_a^{\triangleright}$ *is continuous at irrationals. Moreover, for* $0 < \mathrm{Re}(a) \leqslant 1$, *the map* $x \mapsto c_a^{\triangleright}(x)$ *is* α*-Hölder-continuous locally almost everywhere, for any* $\alpha < (\mathrm{Re}(a)+1)/2$. *When* $\mathrm{Re}(a) > 1$, *the same is true for the map* $x \mapsto c_a^{\triangleright}(x) + xa\zeta(1-a)/\pi$, *and in particular the map* $c_a^{\triangleright}$ *is then differentiable almost everywhere with derivative* $-a\zeta(1-a)/\pi$.

In this statement we have kept the notation $c_a^{\triangleleft}, c_a^{\triangleright}$; however, we warn that the actual definitions differ from those in Theorems 1.1 and 1.4, mainly due to the fact c_a is only 'almost' a quantum modular form. The precise definitions are given in § 4.3 below in several cases depending on the value of a.

In the case (3) above, we obviously also have that λ_a itself has no atoms.

The empirical cumulative distribution functions (CDFs) in Corollary 1.14 for $q = 5000$ are plotted in Figure 3 for some real values of a. The fact that the corresponding measures are diffuse, stated in item (2) of the previous corollary, translates into the continuity of the limiting functions.

The case $a = 0$ of Corollary 1.14 was studied earlier in [MR16, Bet15]. The fact that the values of c_0 have a limiting distribution was proved in [MR16] with an elaborate argument relying on an explicit expression for $c_0(\bar{b}_q/q)$ in terms of b/q (see e.g. [Bet15, pp. 1–2]). A simpler proof was then given in [Bet15] using an explicit computation of the moments of both the original QMF and the limiting measure. In the generality of Theorem 1.7, and already even in the case $a < -1$ and $a \in (-1, 0)$ of Corollary 1.14, the moments don't exist (see Remark 4.2). The proof of Theorems 1.3 and 1.7 instead identifies directly the limiting function and avoids the computation of moments, thus allowing for more general period functions h.

It might be interesting to know if the CDF of the Cauchy distribution can be obtained as a limit of the distribution of values of $c_a^{\triangleright}, c_a^{\triangleleft}$, appropriately normalized, as $a \to -1$; in other words, if for some scaling factor $\gamma(a) > 0$, we have

$$\lim_{a \to -1} \lambda_a((-\infty, \gamma(a)t]) \stackrel{?}{=} \frac{1}{\pi} \int_{-\infty}^{t} \frac{dv}{1+v^2} \quad (t \in \mathbb{R}).$$

Indeed, this would match the case $a = -1$ (weight zero), for which $c_{-1}(x)$ are the Dedekind sums: it is proved in [Var93] (see [BD22, § 9.4] for another argument) that the values $c_{-1}(x)$ tend to distribute according to a Cauchy distribution when x is picked at random among rationals of denominators at most Q, as $Q \to \infty$.

In Figure 4 we present the plots of the real part of $c_a^{\triangleleft}$ and $c_a^{\triangleright}$ for various values of a. The relevant period function $h(x) = h_a(x)$ roughly satisfies $h(x) \sim \kappa(a)x^{-1} + O(1)$ for some constant $\kappa(a)$,

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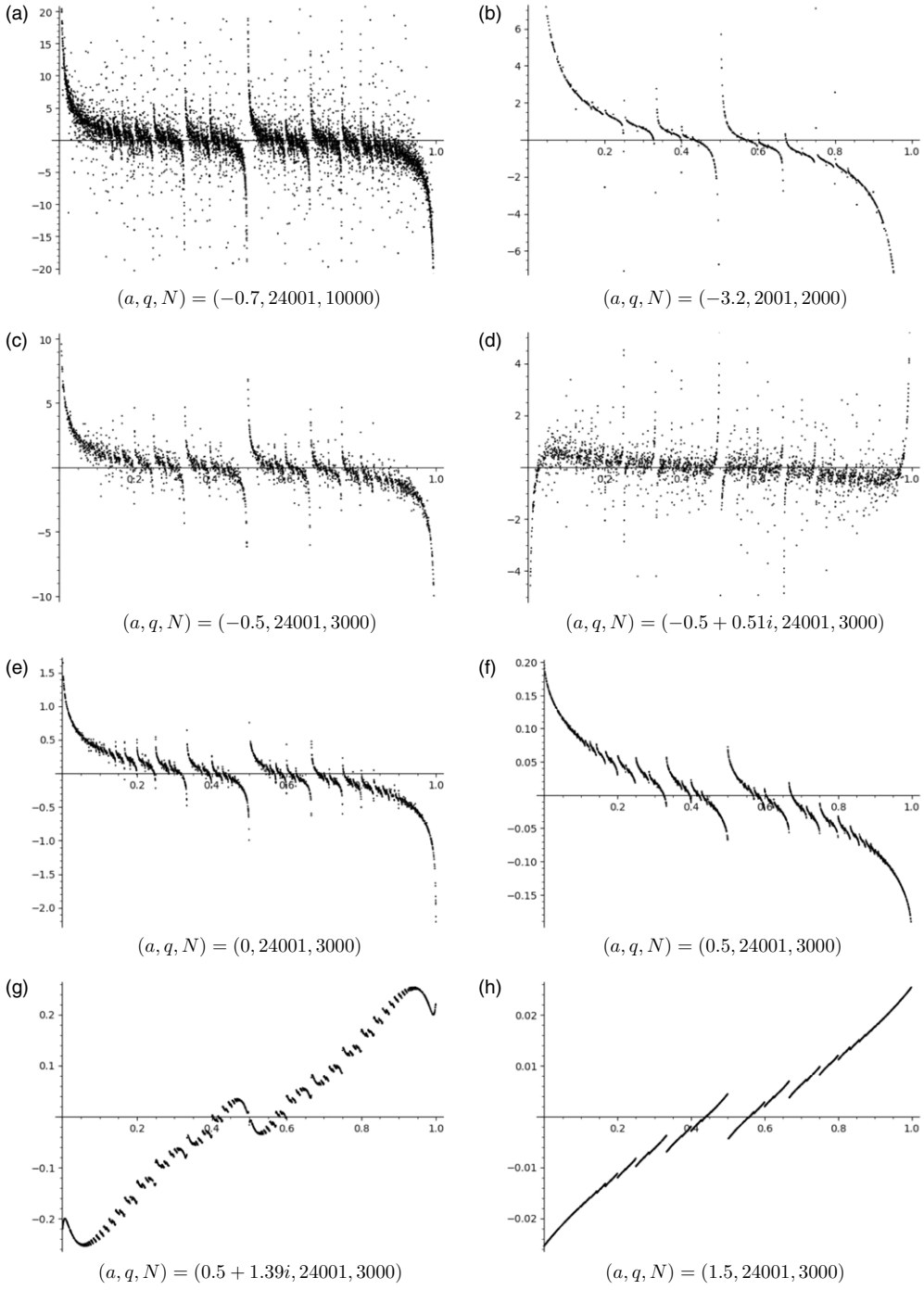


FIGURE 4. Sample points of c_a at N points of denominator q.

see (4.11) later. When $\mathrm{Re}(a) < -1$, we witness a rise in regularity in $c_a^{\triangleleft}$ as $\mathrm{Re}(a)$ decreases, but without reaching full continuity, due to the fact that h_a has a pole at 0. When $\mathrm{Re}(a) > 0$, Corollary 1.14 shows that the map $c_a^{\triangleright}$ is continuous at irrationals, and when $\mathrm{Re}(a) > 1$, that it has derivative $-a\zeta(1-a)/\pi$ almost everywhere. In Figure 4h, we have $a = 1.5$ and $-a\zeta(1-a)/\pi \approx 0.09926$.

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1.4 Notation

1.4.1 *General notation.* We set the following general notation:

- ν is the Lebesgue measure on $[0,1]$;
- $b_j(x)$ is the jth continued fraction coefficient of x;
- $\mathrm{Num}(x)$ and $\mathrm{Den}(x)$ are the numerator and denominator of a rational number x;
- $\mu(c_1,\dots,c_\ell)$ is the Lebesgue measure of the 'cylinder set' $J(c_1,\dots,c_\ell)$ defined at (3.10);
- notation using the letters f, g will typically denote quantum modular forms;
- notation using the letter h will denote period functions;
- the superscripts $f^{\triangleright}$ and $f^{\triangleleft}$ refer to continuous extensions defined in (1.3), (1.6), (1.7) or (1.8);
- T denotes the Gauss map (cf. the next section);
- w and w_j refer to products of iterates of the Gauss map defined in (2.12) and (2.19);
- ϑ_j is a root of unity defined in (2.27);
- the sets $\mathfrak{T}(B)$, $\mathfrak{S}$, $\mathfrak{A}_q$ and $\mathfrak{T}_q$ are defined in Lemmas 2.2, 3.1 and 3.2;
- $V(B,m,x)$ and $\Delta_\lambda(m)$ are defined in (2.6) and (2.7), respectively;
- $\mathscr{S}(\lambda)$ is the property defined in Definition 2.4.

Throughout the paper, we use the Vinogradov notation $A \ll B$ to mean $A = O(B)$. We denote $\varpi = (1+\sqrt{5})/2$ to be the golden ratio.

1.4.2 *The Gauss map, partial quotients, partial denominators.* Let $T : \mathbb{R}\setminus\{0\} \to [0,1)$, $T(x) := \{1/x\}$ be the Gauss map, T^i its ith iterate. For $x \in \mathbb{Q}$, we let $r := r(x)$ be the minimum non-negative integer such that $T^r(x) = 0$ and, for $0 \leqslant j < r$, we write $T^j(x)$ in simplest terms as

$$T^j(x) = \frac{u_{j+1}(x)}{u_j(x)}. \tag{1.15}$$

In particular, we have

$$u_0(x) = \mathrm{Den}(x), \quad u_r(x) = 1,$$

where $\mathrm{Den}(x)$ is the denominator of x. Whenever $u_{r-1}(x) > 1$, we also define

$$u_{r+1}(x) := 1,$$

which will be convenient to change the length of the continued fraction (CF) expansion. In the terminology of [MMY97], with the value $\alpha = 1$, we have $u_j(x) = \mathrm{Den}(x)\beta_{j+1}(x)$. Thus, from [MMY97, Proposition 1.4.(iv)] we deduce

$$\frac{u_j(x)}{u_0(x)} \ll \varpi^{-j}, \quad u_{r-j}(x) \gg \varpi^j \qquad \left(\varpi = \frac{1+\sqrt{5}}{2}\right). \tag{1.16}$$

We will also use the following other decomposition. Given $y \in \mathbb{Q}\cap(0,1)$ and writing uniquely y as its CF expansion

$$y = [0; c_1, \dots, c_s]$$

with s odd, we define

$$v_0(y) := 1, \quad v_j(y) := \mathrm{Den}([0; c_1, \dots, c_j]) \quad (1 \leqslant j \leqslant s). \tag{1.17}$$

Equivalently, $v_j(y) = \mathrm{Den}([0; c_j, \dots, c_1])$. Note that, with $\overline{y} = [0; c_s, \dots, c_1]$, we have

$$v_j(y) = u_{s-j}(\overline{y}),$$

and as such

$$v_j(y) \gg \varpi^j. \tag{1.18}$$

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We refer to Khinchin's book [Khi63] and to the survey [FVV] for additional information about continued fractions.

2. Extending the domain of quantum modular forms

In this section we prove Theorems 1.1, 1.2, 1.4 and 1.5.

2.1 Iteration of the reciprocity formula

First, we notice that by Euclid's algorithm, any QMF f of level 1 is determined by h and its value at 0. Indeed, by repeatedly applying (1.2) and periodicity, we obtain, for $x = \frac{a}{q} \in [0,1)$ in reduced form,

$$f(x) = \sum_{j=0}^{r-1} \left(\prod_{i=0}^{j-1} T^i(x) \right)^{-k} h((-1)^j T^j(x)) + q^k f(0). \tag{2.1}$$

Notice that we used that $\prod_{i=0}^{r-1} T^i(x) = 1/q$.

The above quantities are naturally expressed in terms of continued fractions. Indeed, if $x \in (0,1)$, then the length of its CF expansion $x = [0; b_1, \dots, b_r]$ with minimal length (i.e. $b_r \neq 1$ if $r > 1$) is r. Also, for $0 \leqslant j \leqslant r$ we have $q \prod_{i=0}^{j-1} T^i(x) = (\prod_{i=j}^{r-1} T^i(x))^{-1}$ is the denominator $u_j(x)$ defined in (1.15). Notice also that $u_0(x) = q$. Thus, abbreviating $u_j = u_j(x)$, we can then rewrite (2.1) as

$$f(x) = \sum_{j=0}^{r-1} \left(\frac{u_j}{u_0} \right)^{-k} h\left((-1)^j \frac{u_{j+1}}{u_j} \right) + u_0^k f(0). \tag{2.2}$$

This formula holds also if r is formally replaced by $r+1$, which corresponds to expressing $x = [0; b_1, \dots, b_r]$ rather as $[0; b_1, \dots, b_r - 1, 1]$, since the additional term in (2.2) is

$$u_0^k h((-1)^r) = u_0^k (f(0) - f(0)) = 0, \tag{2.3}$$

by (1.2) and periodicity.

Finally, we deduce another variant of (2.2). First, we notice that if $[0; b_1, \dots, b_r]$ is the CF expansion of $x \neq 0$ with r odd, then $\overline{x} = [0; b_r, \dots, b_1]$. In particular, after the change of variables $r \to r - j$, (2.2) can be rewritten as

$$q^{-k} f(x) = \Psi(\overline{x}) \tag{2.4}$$

where, for $y = [0; c_1, \dots, c_r]$ with r odd, and the notation (1.17),

$$\Psi(y) := \sum_{j=1}^{r} v_j^{-k} h\left((-1)^{j-1} \frac{v_{j-1}}{v_j} \right) + f(0). \tag{2.5}$$

From the expansions (2.2) and (2.5), and the fact that both quantities u_j/u_0 and v_j^{-1} decrease exponentially fast with j, we see the difference of behavior according to the sign of $\mathrm{Re}(k)$, and the relevance of switching x to $\overline{x}$ when $\mathrm{Re}(k) > 0$.

Finally, we need the following simple fact.

Lemma 2.1. *Let $x \in \mathbb{Q}$. Then, there exists a neighborhood I_x of x such that $r(y) \geqslant r(x) + 1$ for all $y \in I_x \smallsetminus \{x\}$.*

Proof. The statement is immediate if $x \in \mathbb{Z}$. Now let $x = [b_0; b_1, \dots, b_r] \in \mathbb{Q} \smallsetminus \mathbb{Z}$ with $r := r(x)$. In particular, $b_r > 1$. Assume r is odd. Then the statement holds for $I_x = (L_x, R_x)$, where $L_x = [0; b_1, \dots, b_r + 1]$, $R_x = [0; b_1, \dots, b_r - 1]$. Indeed, if $y \in (L_x, x)$ then $y = [0; b_1, \dots, b_r, \alpha]$ for some $\alpha \in \mathbb{R}_{>1}$, whereas if $y \in (x, R_x)$ then $y = [0; b_1, \dots, b_r - 1, \alpha]$ for some $\alpha \in \mathbb{R}_{>1}$. In either case

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$T^r(y) = 1/\alpha \neq 0$ and thus $r(y) > r$. The case of r even is identical with the role of L_x and R_x reversed. □

2.2 Continuity almost everywhere and extension

In this section, we state and prove a technical proposition which will be helpful when extending f almost everywhere in Theorems 1.2 and 1.5.

We first need the following lemma, which follows readily from a result of Khinchin [Khi63, Theorem 30]. In the following we will use the notation $r(x) = \infty$ if $x \notin \mathbb{Q}$.

Lemma 2.2. *For each $B > 0$, let*

$$\mathfrak{T}(B) := \{x \in \mathbb{R} \colon b_j(x) \leqslant \max(B, j(\log j)^2) \text{ for all } 1 \leqslant j \leqslant r(x)\}.$$

The set $\mathfrak{T}(B)$ is invariant under $x \mapsto x+1$. Moreover, $\nu(\mathfrak{T}(B) \cap [0,1]) = 1 + o(1)$ as $B \to \infty$ and the set

$$\mathfrak{S} = \bigcup_{B>0} \mathfrak{T}(B)$$

is of full Lebesgue measure.

Proof. By [Khi63, Theorem 30] one deduces that $\mathfrak{S}$ has full measure. Since $\mathfrak{T}(B') \subseteq \mathfrak{T}(B)$ if $0 < B' \leqslant B$, one then deduces that $\nu(\mathfrak{T}(B) \cap [0,1]) = 1 + o(1)$ as $B \to \infty$. Finally, the invariance under $x \mapsto x+1$ is clear since the conditions in $\mathfrak{T}(B)$ don't involve b_0. □

Remark 2.3. Lemma 2.2 holds with the condition $b_j(x) \leqslant \max(B, j(\log j)^2)$ replaced by $b_j(x) \leqslant \max(B, a_j)$ for any non-negative increasing sequence $(a_j)_j$ such that $\sum_{j \geqslant 1} a_j^{-1}$ converges. For our later purposes we will also need that $a_j \ll_\varepsilon j^{1+\varepsilon}$ for all $\varepsilon > 0$. Making the explicit choice $a_j = j(\log j)^2$ simplifies the notation in later computations.

Given $B > 0$, an integer $m \geqslant 1$ and a number $x \in \mathfrak{T}(B)$, define

$$V(B, m, x) := \{x' \in \mathbb{Q} \cap \mathfrak{T}(B) \colon r(x') \geqslant m,\ \forall j \leqslant m, b_j(x') = b_j(x)\}, \tag{2.6}$$

the set of all rationals in $\mathfrak{T}(B)$ whose first m coefficients coincide with those of x.

Definition 2.4 (Property $\mathscr{S}(\lambda)$). We say that $f : \mathbb{Q} \to \mathbb{C}$ has the property $\mathscr{S}(\lambda)$ if the quantity

$$\Delta_\lambda(m) := \sup_{x \in \mathfrak{T}(m^\lambda)} \sup_{x', x'' \in V(m^\lambda, m, x)} |f(x') - f(x'')| \tag{2.7}$$

satisfies

$$\Delta_\lambda(m) \to 0 \tag{2.8}$$

as $m \to \infty$.

Proposition 2.5. *Let $\lambda > 1$, and assume that f has the property $\mathscr{S}(\lambda)$. Then for any $x \in \mathfrak{S} \smallsetminus \mathbb{Q}$, the limit*

$$f^*(x) := \lim_{y \in \mathbb{Q} \cap \mathfrak{T}(B), y \to x} f(y) \tag{2.9}$$

exists for any $B > 0$ such that $x \in \mathfrak{T}(B) \smallsetminus \mathbb{Q}$. Moreover, let $m \in \mathbb{N}$ and $B > 0$ be given with $m \geqslant B^{1/\lambda}$, and define $f^*(x) := f(x)$ for $x \in \mathbb{Q}$. Then uniformly in $x, x' \in \mathfrak{T}(B)$ satisfying $r(x)$, $r(x') \geqslant m$ and $b_j(x) = b_j(x')$ for $j \leqslant m$, we have

$$f^*(x) - f^*(x') = o(1) \quad (m \to \infty), \tag{2.10}$$

where the rate of decay may depend on λ, but not on B.

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Remark 2.6. Let $x \in \mathfrak{T}(B) \setminus \mathbb{Q}$. Since $\mathfrak{T}(B') \subset \mathfrak{T}(B)$ for $B \geqslant B' > 0$, from (2.9) we also get $f^*(x) = \lim_{y \in \mathbb{Q} \cap \mathfrak{T}(B'), y \to x} f(y)$, for any $0 < B' < B$ with $x \in \mathfrak{T}(B') \setminus \mathbb{Q}$. In particular, the existence of the limit implies that its value $f^*(x)$ is independent of B. Notice that the rate of convergence, however, might depend on it.

Proof of Proposition 2.5. Let $x \in \mathfrak{T}(B) \setminus \mathbb{Q}$. We wish to show that the limit (2.9) exists. By Cauchy's criterion, it suffices to show that

$$\lim_{\varepsilon \to 0^+} \sup_{\substack{x', x'' \in \mathbb{Q} \cap \mathfrak{T}(B) \\ |x'-x|, |x''-x| \leqslant \varepsilon}} |f(x') - f(x'')| = 0. \tag{2.11}$$

Let $\varepsilon > 0$. Since $x \notin \mathbb{Q}$, we may find $m \in \mathbb{N}$ such that for any $y \in \mathbb{Q} \cap [x-\varepsilon, x+\varepsilon]$, then $r(y) > m$ and $b_j(y) = b_j(x)$ for all $j \leqslant m$. Clearly, $m \to \infty$ as $\varepsilon \to 0^+$. In particular, we can assume $m \geqslant B^{1/\lambda}$ so that $\mathfrak{T}(B) \subset \mathfrak{T}(m^\lambda)$. Thus, $x \in \mathfrak{T}(m^\lambda)$ and $\mathbb{Q} \cap [x-\varepsilon, x+\varepsilon] \cap \mathfrak{T}(B) \subseteq V(m^\lambda, m, x)$ and (2.11) follows from (2.8).

It remains to show (2.10). Let $x, x' \in \mathfrak{T}(B)$, $m \geqslant B^{1/\lambda}$, and for $0 \leqslant j \leqslant m$, define $b_j := b_j(x) = b_j(x')$. Consider the convergents (x_ℓ), (x'_ℓ) of x and x', respectively, so that, for instance,

$$x_\ell = [b_0; b_1, \dots, b_m, b_{m+1}(x), \dots, b_\ell(x)] \quad (m \leqslant \ell \leqslant r(x)).$$

Also, consider the rational $y := [b_0; b_1, \dots, b_m]$. We have $y \in \mathfrak{T}(m^\lambda)$ by hypothesis on m. We obviously also have $x_\ell \in V(m, m^\lambda, y)$ whenever $m \leqslant \ell \leqslant r(x)$, and similarly $x'_\ell \in V(m, m^\lambda, y)$ for $m \leqslant \ell \leqslant r(x')$. We deduce that $|f(x_\ell) - f(x'_{\ell'})| \leqslant \Delta_\lambda(m)$ with the notation (2.8), and therefore

$$|f^*(x) - f^*(x')| \leqslant \Delta_\lambda(m)$$

by taking $\ell = r(x)$ if $x \in \mathbb{Q}$ or $\ell \to \infty$ if $x \notin \mathbb{Q}$, and similarly for x'. The right-hand side does not depend on B and tends to 0 by (2.8), which is what we claimed. □

2.3 The case of $\mathrm{Re}(k) < 0$

In this section we prove Theorems 1.1 and 1.2. We start by a lemma regarding the size and regularity of products of consecutive iterates of the Gauss map.

Lemma 2.7. *Let $j \in \mathbb{N}$, $\tau \in \mathbb{C}$ and $g : [0,1] \to \mathbb{C}$. For $x \in [0,1)$, define*

$$w(x) = w(x; j, \tau, g) := \mathbf{1}(j \leqslant r(x)) \Big(\prod_{i=0}^{j-1} T^i(x) \Big)^\tau g(T^j(x)), \tag{2.12}$$

where we recall that $r(x) = +\infty$ by convention if $x \notin \mathbb{Q}$ and $\mathbf{1}(j \leqslant r(x))$ indicates the indicator function of the condition $j \leqslant r(x)$.

– *If $\mathrm{Re}(\tau) > 0$ and g is continuous, then w is continuous on $[0,1) \setminus \{x, r(x) \in \{j-1, j\}\}$. Moreover, letting $\pm 1 = (-1)^j$, we have*

$$\begin{aligned} w(x^\pm) &= \mathrm{Den}(x)^{-\tau} g(0) && (r(x) \in \{j-1, j\}), \\ w(x^\mp) &= \mathrm{Den}(x)^{-\tau} g(1) && (r(x) = j), \\ w(x^\mp) &= 0 && (r(x) = j-1). \end{aligned} \tag{2.13}$$

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– *If* $\mathrm{Re}(\tau)>2$ *and* g *is of* $\mathcal{C}^1$ *class, then* w *is of* $\mathcal{C}^1$ *class on* $[0,1)\smallsetminus\{x, r(x)\in\{j-1,j\}\}$. *When* $r(x)\notin\{j-1,j\}$, *we have*

$$w'(x)=\mathbf{1}(j\leqslant r(x))\Big\{\tau g(T^j(x))\sum_{i=0}^{j-1}(-1)^i\Big(\prod_{0\leqslant\ell<i}T^\ell(x)\Big)^{\tau-2}T^i(x)^{\tau-1}\Big(\prod_{i<\ell<j}T^\ell(x)\Big)^{\tau}$$
$$+(-1)^jg'(T^j(x))\Big(\prod_{0\leqslant\ell<j}T^\ell(x)\Big)^{\tau-2}\Big\}. \tag{2.14}$$

Moreover, we have

$$\begin{aligned}
w'(x^\pm)&=\mathrm{Den}(x)^{-\tau+2}\Big(\tau g(0)\sum_{i=0}^{j-1}\frac{(-1)^i}{u_i(x)u_{i+1}(x)}\pm g'(0)\Big) && (r(x)\in\{j-1,j\}),\\
w'(x^\mp)&=\mathrm{Den}(x)^{-\tau+2}\Big(\tau g(1)\sum_{i=0}^{j-1}\frac{(-1)^i}{u_i(x)u_{i+1}(x)}\pm g'(1)\Big) && (r(x)=j), \qquad (2.15)\\
w'(x^\mp)&=0 && (r(x)=j-1).
\end{aligned}$$

– *If* $\mathrm{Re}(\tau)>0$ *and* $g\in\mathcal{C}^0$, *we have*

$$\|w\|_\infty\leqslant\varpi^{\mathrm{Re}(\tau)(1-j)}\|g\|_\infty, \tag{2.16}$$

while if $\mathrm{Re}(\tau)>2$ *and* $g\in\mathcal{C}^1$, *we have*

$$\|w'\|_\infty\leqslant j|\tau|\varpi^{(\mathrm{Re}(\tau)-2)(1-j)}(\|g\|_\infty+\|g'\|_\infty). \tag{2.17}$$

Remark 2.8. The bound (2.16) is essentially contained in the paper [MMY97, Theorem 2.6]. This paper contains also results pertaining to L^p, Hölder and BMO regularity of products of the shape $w(x)$. The proof of Lemma 2.7 and the arguments in [MMY97] both make use of the iterative structure of $w(x)$ (compare (2.18) below and [MMY97, Equation (2.13)]), with the computation of the one-sided limits at rationals (2.13) and (2.15) requiring additional care.

Proof. Value of w'. For the convenience of our argument, we start by noting that if $g\in\mathcal{C}^1$ and $\mathrm{Re}(\tau)>2$, then whenever $r(x)>j$, the right-hand side of (2.12) defines a $\mathcal{C}^1$ function in a neighborhood of x. When $m\geqslant 1$, its derivative can be computed using the expression

$$(T^i)'(x)=(-1)^i\Big(\prod_{0\leqslant\ell<i}T^\ell(x)\Big)^{-2},$$

and this leads to the given formula for $w'(x)$.

Regularity. We treat the continuity and differentiability claims simultaneously, proving the following more general assertion: given $m\in\{0,1\}$, $j\geqslant 1$ and $\mathrm{Re}(\tau)>2m$, if there is a countable set S for which g is $\mathcal{C}^m$ on $[0,1]\smallsetminus S$, then $x\mapsto w(j,\tau,g)$ is $\mathcal{C}^m$ on the set

$$[0,1]\smallsetminus\{x : r(x)\in\{j-1,j\}\text{ or }(r(x)>j\text{ and }T^j(x)\in S)\}.$$

We proceed by induction. Assume first $j=1$. We have $r(x)\geqslant 1\iff x\neq 0$, so that

$$w(x;1,\tau,g)=\mathbf{1}(x\neq 0)x^\tau g(T(x)).$$

This function is easily seen to be $\mathcal{C}^m$ at $x\in[0,1]$ whenever $x>0$, $T(x)\in(0,1)$ and $T(x)\notin S$. These conditions amount to $r(x)>1$ and $T(x)\notin S$, which gives our claim in the case $j=1$.

Let $j>1$, and suppose our claim is proven for $j-1$. Then we note that

$$w(x;j,\tau,g)=w(x;1,\tau,[x\mapsto w(x;j-1,\tau,g)]). \tag{2.18}$$

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By induction, the function $x \mapsto w(x; j-1, \tau, g)$ is $\mathcal{C}^m$ on $[0,1] \smallsetminus S'$, where

$$S' = \{x, r(x) \in \{j-2, j-1\} \text{ or } (r(x) > j-1 \text{ and } T^{j-1}(x) \in S)\}.$$

The set S' is again countable, since T^{j-1} has countably many inverse branches. From the case $j=1$ of our claim, we deduce that $x \mapsto w(x; j, \tau, g)$ is $\mathcal{C}^m$ on $[0,1] \smallsetminus S''$, where

$$\begin{aligned} S'' &= \{x, r(x) \in \{0,1\} \text{ or } (r(x) > 1 \text{ and } T(x) \in S'\} \\ &= \{x, r(x) \in \{0, 1, j-1, j\} \text{ or } (r(x) > j \text{ and } T^j(x) \in S)\} \end{aligned}$$

by definition of S'. It remains to check that w is $\mathcal{C}^m$ at x if $r(x) = 0$ and $j \geqslant 2$, or if $r(x) = 1$ and $j \geqslant 3$. But supposing $r(x) = 1$, we have for $\varepsilon \to 0^+$,

$$T(x-\varepsilon) = O(\varepsilon), \quad T(x+\varepsilon) = 1 - O(\varepsilon) \;\Rightarrow\; T^2(x+\varepsilon) = O(\varepsilon),$$

and therefore, we have in any case $\lim_{y \to x} T(y)T^2(y) = O(|x-y|)$. Thus, if $j \geqslant 3$ and $\mathrm{Re}(\tau) > 2m$, we then have as $y \to x$,

$$w(y; j, \tau, g) = O((T(y)T^2(y))^{\mathrm{Re}(\tau)}) = o(|x-y|^m),$$

and so w is continuous at x, and differentiable there when $m = 1$, with derivative 0. Moreover, if $m = 1$, by the explicit expression (2.14) and the bounds above, we find

$$w'(y) = O(|x-y|^{\mathrm{Re}(\tau)-2}) = o(1),$$

and therefore w is $\mathcal{C}^m$ at x. A similar argument holds when $r(x) = 0$, and concludes the proof of our claim.

Limits of w. Assume $r(x) = j$. Note that for $y \neq x$ in the neighborhood of x, we have $r(y) > r(x)$, and moreover

$$\lim_{y \to x^{\pm}} T^j(y) = 0, \quad \lim_{y \to x^{\mp}} T^j(y) = 1.$$

The expression given for $w(x^{\pm})$ and $w(x^{\mp})$ then follows upon taking the limit in the expression (2.12). The case when $r(x) = j-1$ is similar, using instead the limits

$$\lim_{y \to x^{\pm}} T^j(y) = \lim_{y \to x^{\mp}} T^{j-1}(y) = 0.$$

Limits of w'. The expressions given for $w'(x^{\pm})$ when $r(x) \in \{j-1, j\}$ follow from an argument identical to the computation of the limits $w(x^{\pm})$. In the claimed formulae, we have used the notation (1.15) (for x rational).

Bounds. The bounds (2.16), (2.17) follow immediately from the explicit expressions above, along with the argument of [MMY97, Proposition 1.4.(iv)]. □

2.3.1 *Proof of Theorem 1.1. The domain of $f^{\triangleleft}$.* In this section we show that $f^{\triangleleft}(x)$ is defined for all $x \in \mathbb{R}$. By periodicity, we can assume $x \in (0,1) \smallsetminus \mathbb{Q}$. We also recall that h is bounded on $[-1,1]$ by hypothesis. We let

$$h_j(y) := h((-1)^j y), \quad w_j(y) := w(y, j, -k, h_j), \tag{2.19}$$

for $y > 0$, and extend h_j at $y = 0$ by continuity, so that in particular

$$w_{r(x')}(x') = \mathrm{Den}(x')^k h(0^{\pm}) \quad (x' \in \mathbb{Q},\ \pm = (-1)^{r(x')}). \tag{2.20}$$

Thus, with this notation, (2.1) reads

$$f(x') = \sum_{j \geqslant 0} w_j(x') + \mathrm{Den}(x')^k (f(0) - h(0^{\pm})). \tag{2.21}$$

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It suffices to show that $\sup|f(x')-f(x'')|\to 0$ as $\varepsilon\to 0$, where the sup is over all $x',x''\in\mathbb{Q}$ with $|x-x'|,|x-x''|<\varepsilon$. Since $x\notin\mathbb{Q}$, there exists $m(x,\varepsilon)\to\infty$ as $\varepsilon\to 0$ such that, for all $x^*\in\mathbb{Q}$, $|x-x^*|\leqslant\varepsilon$, we have $r(x^*)>m(x,\varepsilon)$ and $b_j(x^*)=b_j(x)$ for $j\leqslant m(x,\varepsilon)$. It follows that both $\mathrm{Den}(x^*)^k$ and $\sum_{j\geqslant m(x,\varepsilon)}w_j(x^*)$ (by (2.16)) tend to 0 as $\varepsilon\to 0^+$. Thus, for any $\eta>0$, we have

$$|f(x')-f(x'')|\leqslant\eta+\sum_{j=1}^{m(x,\varepsilon)-1}|w_j(x')-w_j(x'')|,\quad\forall x',x''\in(x-\varepsilon,x+\varepsilon)\cap\mathbb{Q}$$

for $\varepsilon>0$ sufficiently small.

By Lemma 2.7, the function $y\to w_j(y)$ is continuous at x and thus for $x',x''\in(x-\varepsilon',x+\varepsilon')\cap\mathbb{Q}$ with $\varepsilon'\in(0,\varepsilon)$ sufficiently small we have $|f(x')-f(x'')|\leqslant 2\eta$. Since η was arbitrary, this yields our claim.

In particular, approximating $x=[0;b_1,b_2,\dots]\in(0,1)\smallsetminus\mathbb{Q}$ by the sequence $x_n=[0;b_1,\dots,b_n]$, we get the normally converging series expression

$$f^{\triangleleft}(x)=\sum_{j\geqslant 0}w_j(x)\tag{2.22}$$

for $x\notin\mathbb{Q}$. Notice that the same expression holds also for $x\in\mathbb{Q}$ if $f(0)=h(0^{\pm})$, by (2.21).

2.3.2 *Proof of Theorem 1.1. The continuity of $f^{\triangleleft}$.* First, assume $x\in(0,1)\smallsetminus\mathbb{Q}$. Then, using the expression (2.22) and an identical argument as above, the continuity of $f^{\triangleleft}$ at x follows immediately.

Assume, then, that $x\in[0,1]\cap\mathbb{Q}$. Let $r=r(x)$ and $q=\mathrm{Den}(x)$. By Lemma 2.1 for all $y\neq x$ in a neighborhood of x, we have $r(y)\geqslant r+1$, and therefore, with $\pm=(-1)^r$,

$$f^{\triangleleft}(y)-f(x)=q^k(h(0^{\pm})-f(0))+O(\mathrm{Den}(y)^{\mathrm{Re}(k)})+\sum_{j\geqslant 0}(w_j(y)-w_j(x))$$

where the term $O(\mathrm{Den}(y)^k)$ is to be ignored if $y\notin\mathbb{Q}$. We let $y\to x$. First note that $\mathrm{Den}(y)^k\to 0$. Also, for $j\notin\{r,r+1\}$, the function w_j is continuous by Lemma 2.7, and so the jth summand in the sum tends to 0. By dominated convergence, we deduce

$$\begin{aligned}f^{\triangleleft}(y)-f(x)&=o(1)+q^k(h(0^{\pm})-f(0))+w_r(y)-w_r(x)+w_{r+1}(y)\\&=o(1)+w_r(y)+w_{r+1}(y)-q^kf(0),\end{aligned}$$

by (2.20). Finally, by (2.13) and since $h_r(1)=h((-1)^r)=0$ by (2.3), we have

$$\begin{aligned}w_r(y)&=o(1)+q^k\begin{cases}h_r(0)&(\mathrm{sgn}(y-x)=(-1)^r),\\0&(\mathrm{sgn}(y-x)=(-1)^{r+1}),\end{cases}\\w_{r+1}(y)&=o(1)+q^k\begin{cases}0&(\mathrm{sgn}(y-x)=(-1)^r),\\h_{r+1}(0)&(\mathrm{sgn}(y-x)=(-1)^{r+1}).\end{cases}\end{aligned}\tag{2.23}$$

Since $h_r(0)=h(0^{\pm})$, $h_{r+1}(0)=h(0^{\mp})$, then splitting in cases depending on the sign $\pm$ and on whether $y\to 0$ from the right or the left, we find

$$f^{\triangleleft}(y)-f(x)\to\begin{cases}q^k(h(0^+)-f(0))&(y\to x^+),\\q^k(h(0^-)-f(0))&(y\to x^-),\end{cases}$$

as claimed. We also notice for later use that if h is continuous at zero, then (2.23) gives

$$\lim_{y\to x^+}w_r(y)=\lim_{y\to x^-}w_{r+1}(y),\quad\lim_{y\to x^-}w_r(y)=\lim_{y\to x^+}w_{r+1}(y).$$

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2.3.3 *Proof of Theorem 1.1. The differentiability of $f^{\triangleleft}$.* We argue by induction, starting from the case $m=1$. Assume that $h\in\mathcal{C}^1([-1,1])$ and $\mathrm{Re}(k)<-2$.

By Lemma 2.7 we have that w_j is of $\mathcal{C}^1$ class on $[0,1)\smallsetminus\{x, r(x)\in\{j-1,j\}\}$ (and thus on any neighborhood of irrational numbers). In particular, by (2.22), (2.16) and (2.17), it follows that $f^{\triangleleft}$ is differentiable on $\mathbb{R}\smallsetminus\mathbb{Q}$. The same argument also gives that $\sum_{j\neq r,r+1} w_j(x)$ is differentiable at rationals. In particular, it suffices to show that the right and left derivatives of $w_r(y)+w_{r+1}(y)$ coincide at rationals. To show this, we start by observing that from the period relation (1.2), proceeding as in [Zag99, Equation (23)], we have

$$h(x+1)-h(x)=-|1+x|^{-k}h(1-1/(x+1)),\quad x\in\mathbb{Q}\smallsetminus\{0,-1\}.$$

In particular, taking the limit as $x\to 0$, we obtain $h(1)=0$ and $h'(1)=kh(0)$. Thus, since by (1.2) we have $h(-1/x)=-|x|^{-k}h(x)$, we find $h'(-1)=-kh(0)$ and thus $h_j'(1)=kh(0)$ for all j. Thus, recalling that $h_j(1)=0$ (and $h_j'(0)=(-1)^j h(0)$) and observing that

$$-k\sum_{i=0}^{r(x)}\frac{(-1)^i}{u_i(x)u_{i+1}(x)}=-k\sum_{i=0}^{r(x)-1}\frac{(-1)^i}{u_i(x)u_{i+1}(x)}-(-1)^r k,\quad x\in\mathbb{Q}\cap(0,1),$$

we obtain from (2.15) that

$$w_r'(x^+)+w_{r+1}'(x^+)=w_r'(x^-)+w_{r+1}'(x^-),\quad x\in\mathbb{Q}\cap(0,1),$$

thus giving that $f^{\triangleleft}$ is differentiable at any $x\in\mathbb{Q}\cap(0,1)$. Finally, by (1.2), we see that $f^{\triangleleft}$ is differentiable at 0 also, with $(f^{\triangleleft})'(0)=h'(0)$. We conclude that $f^{\triangleleft}$ is differentiable on $[0,1)$, hence on $\mathbb{R}$ by periodicity. By (1.2), its derivative satisfies, for $x\neq 0$,

$$h_1(x)=(f^{\triangleleft})'(x)-|x|^{-k-2}(f^{\triangleleft})'(-1/x),$$

where $h_1(x):=h'(x)-k\mathrm{sgn}(x)|x|^{-k-1}f(-1/x)$. The function h_1, extended at 0 by $h_1(0)=h'(0)=(f^{\triangleleft})'(0)$, is continuous on $[-1,1]$. We can then apply the continuity property proven in § 2.3.2 to $(f^{\triangleleft})'$ and deduce that $(f^{\triangleleft})'$ is continuous on $[0,1]$, and hence on $\mathbb{R}$.

If $k<-2m$ and $h\in\mathcal{C}^m$, then this argument can be iterated m times, with functions $(h_n)_{n\leqslant m}$ defined inductively by

$$h_0(x)=h(x),\quad h_{n+1}(x)=h_n'(x)-(k+2n)\mathrm{sgn}(x)|x|^{-k-2n-1}(f^{\triangleleft})^{(n)}(-1/x).$$

2.3.4 *Proof of Theorem 1.2. Extending f when h is unbounded.* We now justify Theorem 1.2. We assume that h satisfies (1.5). First note that, due to the fact that periodicity is assumed separately on $\mathbb{R}_+$ and $\mathbb{R}_-$, we have the following variant of (2.1),

$$f(x)=\sum_{j=0}^{r-1}\left(\prod_{i=0}^{j-1}T^i(x)\right)^{-k}h((-1)^jT^j(x))+q^kf((-1)^r).$$

Lemma 2.9. *Let $\delta>0$ be such that the hypothesis (1.5) holds. Then f has the property $\mathscr{S}(1+\delta)$.*

Proof. Let $x=[b_0;b_1,b_2,\dots]\in\mathfrak{T}(m^{1+\delta})$. By periodicity we can assume $x\in[0,1)$, that is $b_0=0$. If $U:[0,1)\to\mathbb{R}$ is the inverse branch of T^m, given by $U(y)=[0;b_1,\dots,b_m+y]$, then any $x'\in V(m^{1+\delta},m,x)$ is in the range of U. By [MMY97, Proposition 1.14], we have

$$\|(T^j\circ U)'\|_\infty=\sup_{y\in[0,1]}\prod_{i=j}^{m-1}|T^i(U(y))|^2\ll\varpi^{-2(m-j)}$$

uniformly for $0\leqslant j<m$, and therefore, for any $x',x''\in V(m^{1+\delta},m,x)$,

$$|T^j(x')-T^j(x'')|\ll\varpi^{-2(m-j)}\tag{2.24}$$

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for $j<m$. We note also that $T^j(x')\asymp T^j(x)$ for $x'\in V(m^{1+\delta},m,x)$ and $j<m$, with a uniform constant, and that the condition $T^j(x')\gg\min(m^{-1-\delta},j^{-1}(\log j)^{-2})$ holds uniformly over j. Finally, for any $x'\in V(m^{1+\delta},m,x)$, the condition $r(x')\geqslant m$ implies $\mathrm{Den}(x')\geqslant\varpi^m$ by virtue of (1.16) (with $j=r(x')$).

Given $x\in\mathfrak{T}(m^{1+\delta})$ and $x'\in V(m^{1+\delta},m,x)$, with the same notation as in § 2.3.1 we have

$$f(x')=\sum_{0\leqslant j<m/2}w_j(x')+O\Big(\mathrm{Den}(x)^{\mathrm{Re}(k)}+\sum_{j\geqslant m/2}\varpi^{\mathrm{Re}(k)j}\sup_{|y|\gg\min(m^{-1-\delta},j^{-1}(\log j)^{-2})}|h(y)|\Big).$$

By our hypothesis (1.5), the second sum is $\ll\sum_{j\geqslant m/2}\varpi^{\mathrm{Re}(k)j}(e^{j^{1-\delta}(\log j)^2}+e^{O(m^{1-\delta^2})})\ll\varpi^{\mathrm{Re}(k)m/3}$, and therefore

$$f(x')=\sum_{0\leqslant j<m/2}w_j(x')+O(\varpi^{\mathrm{Re}(k)m/3}). \tag{2.25}$$

We now let $x',x''\in V(m^{1+\delta},m,x)$ be given. Taking differences in the expansion (2.25), we obtain

$$|f(x')-f(x'')|\ll\sum_{0\leqslant j<m/2}|w_j(x')-w_j(x'')|+\varpi^{\mathrm{Re}(k)m/3}.$$

Denote temporarily $\Pi_j(y):=\prod_{0\leqslant i<j}T^i(y)$, which is also $u_j(y)/u_0(y)$ in the notation of § 1.4. By the definition (2.12), (2.19) of w_j, splitting the difference in $w_j(x')-w_j(x'')$, we get

$$\begin{aligned}\sum_{0\leqslant j<m/2}|w_j(x')-w_j(x'')|\leqslant\sum_{0\leqslant j<m/2}&(|(h_j(T^j(x'))-h_j(T^j(x'')))\Pi_j(x'')^{-k}|\\&+|h_j(T^j(x'))(\Pi_j(x')^{-k}-\Pi_j(x'')^{-k})|).\end{aligned}$$

Regarding the first term inside the sum, the bound (2.24) and our hypothesis (1.5), applied with some value of $\varepsilon\asymp m^{-1-\delta}$, give $|h_j(T^j(x'))-h_j(T^j(x''))|=o(1)$ as $m\to\infty$ uniformly for all $j<m/2$. By the bound (1.16), we further have $\Pi_j(y)\ll\varpi^{-j}$ uniformly for $y\in[0,1]$ and $j\geqslant0$, and therefore

$$\sum_{0\leqslant j<m/2}|(h_j(T^j(x'))-h_j(T^j(x'')))\Pi_j(x'')^{-k}|\ll o(1)\times\sum_{j\geqslant0}\varpi^{j\,\mathrm{Re}(k)}=o(1)$$

as $m\to\infty$. Regarding the second term, we have, as above, $h_j(T^j(x'))\ll e^{m^{1-\delta^2}}+e^{j^{1-\delta}(\log j)^2}$ for all $j\geqslant0$. We use the inequality $u^{-k}-v^{-k}\ll_k|u-v|^{\min(1,-k)}$, valid for $0\leqslant u,v\leqslant1$, which gives in our case

$$\Pi_j(x')^{-k}-\Pi_j(x'')^{-k}\ll|\Pi_j(x')-\Pi_j(x'')|^{\min(-k,1)}.$$

Splitting the difference, we have

$$\begin{aligned}|\Pi_j(x')-\Pi_j(x'')|&\leqslant\sum_{0\leqslant i\leqslant j}\Pi_{i-1}(x')\Pi_{j-i-1}(T^{i+1}(x''))|T^i(x')-T^i(x'')|\\&\ll\sum_{0\leqslant i\leqslant j}\varpi^{-i}\times\varpi^{-(j-i)}\times\varpi^{-2(m-i)}\\&\ll\varpi^{-2m+j}\end{aligned}$$

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using our bound (2.24) and thus we conclude that

$$\begin{aligned}&\sum_{0\leqslant j<m/2}|h_j(T^j(x'))(\Pi_j(x')^{-k}-\Pi_j(x'')^{-k})|\\&\ll\varpi^{-2m\min(-k,1)}\sum_{j\geqslant 0}\varpi^{j\min(-k,1)}(e^{O(m^{1-\delta^2})}+e^{O(j^{1-\delta}(\log j)^2)})\\&\ll\varpi^{-m\min(-k,1)/2},\end{aligned}$$

which tends to 0 as $m\to\infty$. Grouping our bounds, we deduce $\Delta_{1+\delta}(m)\to 0$ when $m\to\infty$, as claimed. □

We now turn to the proof of Theorem 1.2. Consider $X=(\mathbb{R}\smallsetminus\mathbb{Q})\cap\mathfrak{S}$, where $\mathfrak{S}$ is defined in Lemma 2.2, and let $x\in X$. In particular, $x\in\mathfrak{T}(B)$ for some $B>0$, and therefore so do the convergents x_j of x. Summarizing, we have $x_j\in\mathbb{Q}\cap\mathfrak{T}(B)$ for all j, and $x_j\to x$ as $j\to\infty$. Since f has the property $\mathscr{S}(1+\delta)$ by Lemma 2.9, the limit $f^{\triangleleft}(x):=f^*(x)=\lim_{j\to\infty}f(x_j)$ in (2.9) exists.

Moreover, for any ε, by Lemma 2.2 we can find $B>0$ such that $X_\varepsilon:=(\mathbb{R}\smallsetminus\mathbb{Q})\cap\mathfrak{T}(B)\subset X$ satisfies $\nu(X_\varepsilon\cap[0,1])\geqslant 1-\varepsilon$. This set is also trivially invariant by $x\mapsto x+1$. Let $y\in X_\varepsilon$ and $\eta>0$ be arbitrary. We pick an integer $m\geqslant B^{1/(1+\delta)}$ such that the right-hand side of (2.10) has modulus at most δ. Having chosen m, and since $y\notin\mathbb{Q}$, we may find $\xi>0$ such that any number $x\in(\mathbb{R}\smallsetminus\mathbb{Q})\cap[y-\xi,y+\xi]$ satisfies $b_j(x)=b_j(y)$ for $j\leqslant m$. Now let $x\in X_\varepsilon\cap[y-\xi,y+\xi]$ be arbitrary. Then $x\in\mathfrak{T}(B)$ by definition, and therefore the formula (2.10) gives $|f^{\triangleleft}(x)-f^{\triangleleft}(y)|\leqslant\eta$. We have thus proven that $f^{\triangleleft}|_{X_\varepsilon}$ is continuous at y with the restricted topology, as claimed.

2.4 The case of $\mathrm{Re}(k)>0$.

2.4.1 *Proofs of Theorems 1.4 and 1.5. The domain and continuity of $f^{\triangleright}$.* For the purpose of studying the example of the Kontsevich function (1.12), it will be convenient to generalize the period relations to

$$\begin{cases}f(x+1)=\vartheta f(x) & (x\in\mathbb{Q}),\\ h(x)=f(x)-\vartheta^{\pm 3}|x|^{-k}f(-1/x) & (\pm x\in\mathbb{Q}_{>0}),\end{cases}\tag{2.26}$$

where ϑ is some fixed root of unity. Define

$$\vartheta_j=\vartheta_j(b_1,b_2,\dots)=\begin{cases}\vartheta^{\sum_{i=1}^j(-1)^ib_i} & (j\text{ even}),\\ \vartheta^{3+\sum_{i=1}^j(-1)^ib_i} & (j\text{ odd}),\end{cases}\tag{2.27}$$

where $b_1,b_2,\dots$ are integers. By arguing in the same way as (2.2), we find for $x=[0;b_1,\dots,b_r]$ that

$$f(x)=\sum_{j=0}^{r-1}\vartheta_j\left(\frac{u_j}{u_0}\right)^{-k}h\left((-1)^j\frac{u_{j+1}}{u_j}\right)+\vartheta_r u_0^k f(0).\tag{2.28}$$

Note that by the period relation, we have

$$h((-1)^r)=(\vartheta^{(-1)^r}-\vartheta^{3(-1)^r+(-1)^{r+1}})f(0),$$

which confirms that the expression (2.28) holds for $x=[0;b_1,\dots,b_r]$ regardless of whether this expansion is canonical ($b_r>1$ if $r>1$) or not. Similarly to (2.4), we may then work with r odd, for which we observe that

$$\vartheta_{r-j}(b_1,\dots,b_r)\vartheta_j(b_r,\dots,b_1)=\vartheta_r(b_1,\dots,b_r).$$

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Changing j to $r-j$ in the sum (2.28), and using the notation (1.17), we deduce that for $x=[0;b_1,\dots,b_r]$, r odd, we have

$$\vartheta_r^{-1}(x)q^{-k}f(x)=\Psi(\overline{x}), \tag{2.29}$$

where now $\Psi(y)$ is defined for $y=[0;c_1,\dots,c_r]$, r odd, as

$$\Psi(y)=\sum_{j=1}^{r}\vartheta_j^{-1}(y)v_j(y)^{-k}h\Big((-1)^j\frac{v_{j-1}(y)}{v_j(y)}\Big)+f(0), \tag{2.30}$$

where, whenever the expansion $x=[0;b_1,\dots,b_r]$ is clear from the context, we denote

$$\vartheta_j(x)=\vartheta_j(b_1,\dots), \tag{2.31}$$

but this quantity in fact depends on the tuple (b_j) representing x.

In the case of $\mathrm{Re}(k)>0$ and $h(x)=O(|x|^{-\mathrm{Re}(k)})$ for $x\in[-1,1]\smallsetminus\{0\}$, the existence of $f^{\triangleright}$ for all $x\in\mathbb{R}$ is straightforward. Indeed, by (2.28) we have

$$f^{\triangleright}(x)=\begin{cases}\Psi(x) & \text{if } x\in\mathbb{Q},\\ \lim\limits_{\mathbb{Q}\ni y\to x}\Psi(y) & \text{if } x\notin\mathbb{Q},\end{cases} \tag{2.32}$$

where the limit exists by virtue of the fact that the sum in (2.28) is uniformly convergent and for each j, $v_j(y)$ depends only on finitely many of the functions $y\mapsto b_j(y)=\lfloor 1/T^{j-1}(y)\rfloor$ which are locally constant at irrationals. Note that for irrational $x=[0;c_1,c_2,\dots]$ we simply have

$$f^{\triangleright}(x)=\sum_{j=1}^{\infty}\vartheta_j^{-1}v_j^{-k}h\Big((-1)^{j-1}\frac{v_{j-1}}{v_j}\Big)+f(0). \tag{2.33}$$

The continuity of $f^{\triangleright}$ on $\mathbb{R}\smallsetminus\mathbb{Q}$ when $h(x)=O(|x|^{-\mathrm{Re}(k)})$ is immediate from (2.33). Suppose $x=[0;b_1,\dots,b_r]\in\mathbb{Q}$ with r odd. If $x'\to x^-$ with x' sufficiently close to x, then we have $x'=[0;b_1,\dots,b_r,b',b'_{r+2},\dots]$ with $b'\to\infty$. It follows immediately from the definition that $f^{\triangleright}(x')\to f^{\triangleright}(x)$ if $h(x)=o(|x|^{-\mathrm{Re}(k)})$ as $x\to 0$, and so under this assumption $f^{\triangleright}$ is continuous at x from the left. Next, let $x'\to x^+$, then if $b_r>1$ we have $x'=[0;b_1,\dots,b_r-1,1,b',b'_{r+2},\dots]$ with $b'\to\infty$. Note that

$$\frac{v_{r-1}(x)}{v_r(x)}=\overline{x},\quad \frac{v_{r-1}(x')}{v_r(x')}=\frac{v_{r-1}(x)}{v_r(x)-v_{r-1}(x)}=\frac{\overline{x}}{1-\overline{x}},\quad \frac{v_r(x')}{v_{r+1}(x')}=1-\overline{x}.$$

Then, under the hypothesis $h(x)=o(|x|^{-\mathrm{Re}(k)})$, by (2.30) and the relations (2.26), we obtain

$$\begin{aligned}f^{\triangleright}(x')-f^{\triangleright}(x)&=\vartheta_r^{-1}q^{-k}\Big(\vartheta^{-1}(1-\overline{x})^{-k}h\Big(\frac{\overline{x}}{1-\overline{x}}\Big)+\vartheta h(\overline{x}-1)-h(\overline{x})\Big)+o(1)\\&=\vartheta_r^{-1}q^{-k}\Big(\vartheta^{-1}(1-\overline{x})^{-k}f\Big(\frac{\overline{x}}{1-\overline{x}}\Big)+\vartheta h(\overline{x}-1)-f(\overline{x})\Big)+o(1)\\&=o(1)\end{aligned}$$

as $b'\to\infty$. If $b_r=1$, then $x'=[0;b_1,\dots,b_{r-1}+1,b',b'_{r+1},\dots]$ with $b'\to\infty$. We then obtain in a similar way

$$\begin{aligned}f^{\triangleright}(x')-f^{\triangleright}(x)&=\vartheta_r^{-1}q^{-k}\Big(\vartheta h(\overline{x}-1)-h(\overline{x})-\vartheta^2(\overline{x})^{-k}h\Big(\frac{\overline{x}-1}{\overline{x}}\Big)\Big)+o(1)\\&=o(1),\end{aligned}$$

which goes to 0 as $x'\to x^+$, and concludes the proof that $f^{\triangleright}$ is continuous on $\mathbb{R}$. This proves the first part of Theorem 1.4.

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Assume now that f is not (twisted) periodic, but instead satisfies $f(x)=\vartheta f(x+1)$ only for $x\in\mathbb{Q}\smallsetminus[-1,0]$, and that h satisfies $h(x)=O(e^{|x|^{-1+\delta}})$ for some $\delta>0$. In this situation, we use Proposition 2.5 to extend Ψ.

Lemma 2.10. *Let $\delta>0$, and assume that f satisfies*

$$\begin{cases} f(x+1)=\vartheta f(x) & (x\in\mathbb{Q}\smallsetminus[-1,0]),\\ h(x)=f(x)-\vartheta^{\pm 3}|x|^{-k}f(-1/x) & (\pm x\in\mathbb{Q}_{>0}),\end{cases}$$

and that the map h defined through (1.2) satisfies $h(x)=O(e^{|x|^{-1+\delta}})$. Then the formula (2.29) holds with

$$\Psi(y):=\sum_{j=1}^{r}\vartheta_j^{-1}(y)v_j(y)^{-k}h\left((-1)^j\frac{v_{j-1}(y)}{v_j(y)}\right)+\vartheta^{-1}f(-1)\quad(y=[0;c_1,\dots,c_r];\,r\ \textit{odd}),$$

and the map Ψ has the property $\mathscr{S}(1+\delta)$.

Proof. The verification that Equations (2.29) and (2.30) hold, with $f(0)$ replaced with $\vartheta^{-1}f(-1)$, is straightforward. Let $B=m^{1+\delta}$, $x\in\mathfrak{T}(B)$ and $x',x''\in V(B,m,x)$. Note, in particular, that $b_j(x')=b_j(x'')$ for all $j\leqslant m$, and therefore also $v_j(x')=v_j(x'')$ and $\vartheta_j(x')=\vartheta_j(x'')$. We deduce

$$|\Psi(x')-\Psi(x'')|\leqslant\sum_{j=m+1}^{r(x')}v_j(x')^{-\operatorname{Re}(k)}\left|h\left((-1)^j\frac{v_{j-1}(x')}{v_j(x')}\right)\right|+\sum_{j=m+1}^{r(x'')}v_j(x'')^{-\operatorname{Re}(k)}\left|h\left((-1)^j\frac{v_{j-1}(x'')}{v_j(x'')}\right)\right|.$$

Since $x'\in\mathfrak{T}(B)$, we also have $v_{j-1}(x')/v_j(x')\gg 1/b_j(x')\gg j^{-1}(\log j)^{-2}+B^{-1}$. The same holds for x''. By our hypothesis on h, and combining this with (1.18), we deduce

$$|\Psi(x')-\Psi(x'')|\ll\sum_{j\geqslant m+1}\varpi^{-j\operatorname{Re}(k)}(e^{O(j^{1-\delta}(\log j)^2)}+e^{O(m^{1-\delta^2})})\ll\varpi^{-m\operatorname{Re}(k)/2},$$

which tends to 0 as $m\to\infty$, uniformly in x,x',x''. This shows the property $\mathscr{S}(1+\delta)$ for Ψ. □

Letting $f^{\triangleright}(x):=\Psi^*(x)$ with the notation of Proposition 2.5, the deduction Theorem 1.5 follows by arguments identical to those of § 2.3.4, which were used to prove Theorem 1.2.

2.4.2 *Proof of Theorem 1.4. The differentiability of $f^{\triangleright}$.* We assume here the hypotheses of Theorem 1.4, in particular, that $h(x)=O(|x|^{-\operatorname{Re}(k)})$ for $0<|x|<1$. Let $X=\mathfrak{S}$ be as in Lemma 2.2, let $\varepsilon>0$ and suppose $x=[0;b_1,b_2,\dots]\in\mathfrak{S}$. Let $m:=m(x,\varepsilon)$ be as in § 2.3.1. Let $x'\in(0,1)$ be such that $|x-x'|<\varepsilon$, so that $x'=[b_1,\dots,b_m,b'_{m+1},\dots]$. Let $n\geqslant m$ be the least integer such that $b_{n+1}\neq b'_{n+1}$. We write $x=[0;b_1,\dots,b_n,z]$ and $x'=[0;b_1,\dots,b_n,z']$ with $z=[b_{n+1};b_{n+2},\dots]$, $z'=[b'_{n+1};b'_{n+2},\dots]$ (and $b_{n+1}\neq b'_{n+1}$ by hypothesis). By standard properties of continued fractions, we have

$$|x-x'|=\frac{|z-z'|}{(zv_n+v_{n-1})(z'v_n+v_{n-1})}.$$

This is $\gg b_{n+1}^{-2}v_n^{-2}$, unless $b_{n+1}-b'_{n+1}$ is equal to 1 or -1. In the first case we have $z-z'=1+1/(b_{n+2}+y)-1/(b'_{n+2}+y')\geqslant b_{n+2}^{-1}$ for some $0\leqslant y,y'\leqslant 1$. In the second case we have $z'-z=1-1/(b_{n+2}+y)+1/(b'_{n+2}+y')\geqslant y/2\geqslant-(b_{n+3}+1)^{-1}$. Since $x\in\mathfrak{S}$, in all cases we have $\varepsilon>|x-x'|\gg_x v_n^{-2}n^{-3}(\log n)^{-6}\gg v_n^{-2}(\log v_n)^{-4}$.

Finally, we have

$$f^{\triangleright}(x)-f^{\triangleright}(x')\ll\sum_{j\geqslant n+1}\frac{1}{v_{j-1}^{\operatorname{Re}(k)}}+\sum_{j\geqslant n+1}\frac{1}{v_{j-1}'^{\operatorname{Re}(k)}}\ll v_n^{-\operatorname{Re}(k)}\ll_x\varepsilon^{\operatorname{Re}(k)/2}|\log\varepsilon|^{\operatorname{Re}(k)}$$

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and so it follows that $f^{\triangleright}$ is locally α-Hölder continuous at x, for any $\alpha < \frac{1}{2}\operatorname{Re}(k)$. This completes the proof of Theorem 1.4.

3. Limiting distributions of quantum modular forms

3.1 Convergence in distribution

3.1.1 *Almost sure stability of the CF expansion.* First we need the following lemmas.

Lemma 3.1. *Let $q \geqslant 2$ and let*

$$\mathfrak{A}_q := \{0 \leqslant a < q \mid (a,q) = 1 : a/q \in \mathfrak{T}((\log q)(\log\log q)^2)\}.$$

Then $|\mathfrak{A}_q| = \varphi(q)(1+o(1))$ as $q \to \infty$, where φ denotes Euler's totient function.

Proof. This is a special case of [Ruk06]. □

Lemma 3.2. *Let $q \geqslant 2$ and let*

$$\mathfrak{T}_q := ([0,1) \smallsetminus \mathbb{Q}) \cap \mathfrak{T}((\log q)(\log\log q)^2).$$

Then $\nu(\mathfrak{T}_q) = 1 + o(1)$ as $q \to \infty$.

Proof. This is a consequence of Lemma 2.2 with $B = (\log q)(\log\log q)^2$. □

We can now deduce the following two lemmas, which show that $f^{\triangleright}$ and $f^{\triangleleft}$ can be well approximated by their values at suitable close-by rationals. Here again, we will obtain this as a consequence of the property $\mathscr{S}(\lambda)$ for some $\lambda > 1$.

Lemma 3.3. *Suppose that there exists $\lambda > 1$ such that $f : \mathbb{Q} \to \mathbb{C}$ has the property $\mathscr{S}(\lambda)$. Then the map f^* defined in Proposition 2.5 satisfies*

$$f^*(a/q) = f^*(x) + o(1) \quad \forall a \in \mathfrak{A}_q, \quad \forall x \in \left(\frac{a - q^{1/4}}{q}, \frac{a + q^{1/4}}{q}\right) \cap \left(\mathfrak{T}_q \cup \frac{1}{q}\mathfrak{A}_q\right) \tag{3.1}$$

as $q \to \infty$, uniformly for a and x within the stated sets.

Proof. Let $B := (\log q)(\log\log q)^2$. We assume first $x \in (a/q, (a+q^{1/4})/q)$, and we let $m := 2\lceil B^{1/\lambda}\rceil + 1$, so that, in particular, $m = (\log q)^{1/\lambda + o(1)}$ as $q \to \infty$. Let $a \in \mathfrak{A}_q$, denote $x' := a/q$ and let $x \in (a/q, (a+q^{1/4})/q)$ with either $x \cap \mathfrak{T}_q$ or $x = b/q$, where $b \in \mathfrak{A}_q$. We wish to prove that

$$f^*(x) - f^*(x') = o(1)$$

as $q \to \infty$, with a rate of decay depending at most on h. This follows from Proposition 2.5 if we can prove that $x, x' \in \mathfrak{T}(B)$, $r(x) \geqslant m$ and $b_j(x) = b_j(x')$ for $j \leqslant m$. The fact that $x, x' \in \mathfrak{T}(B)$ follows by definition. Write $x' = a/q = [0; b_1, \dots, b_r]$. Note that $r \ll \log q$ by Euclid's algorithm, so that by definition of $\mathfrak{T}(B)$ and our choice of B, we have $b_j \ll (\log q)(\log\log q)^2$. By induction (see the proof of Theorem 31 in [Khi63]), we have $q^{1/r} \leqslant 2(b_1 \dots b_r)^{1/r} \ll (\log q)(\log\log q)^2$, and therefore $r \gg (\log q)/\log\log q$. We deduce that $r(x') \geqslant m$ for q large enough. Finally, let $a^*/q^* := [0, b_1, \dots, b_m]$ be the mth convergent of a/q. Then we have $(q^*)^{1/m} \ll (\log q)(\log\log q)^2$ as above, and therefore $\log(q^*) \ll m \log\log q = o(\log q)$ as $q \to \infty$. In particular, $q^* < q^{1/3}$ for q large enough. Since m is odd, we deduce

$$\frac{a^*}{q^*} - \frac{a}{q} > \frac{1}{(q^*)^2(b_{m+1}+2)} \gg \frac{1}{q^{2/3}(\log q)^2} \geqslant \frac{q^{1/4}}{q}$$

for q large enough. In particular, any $x \in (a/q, (a+q^{1/4})/q)$ satisfies $a/q < x < a^*/q^*$, and it follows that $b_j(x) = b_j(a/q)$ for $j \leqslant m$. This concludes the proof.

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The remaining case $x \in ((a-q^{1/4})/q, a/q)$ follows by an identical argument taking m even instead of odd. □

3.1.2 *Proof of Theorems 1.3 and 1.7: The convergence in distribution.* Let $\delta > 0$. We now assume that either $\mathrm{Re}(k) < 0$ and h satisfies (1.5), or $\mathrm{Re}(k) > 0$ and h satisfies $h(x) = O(e^{|x|^{-1+\delta}})$. Let f^* be either $f^{\triangleleft}$ or $f^{\triangleright}$, depending on the sign of $\mathrm{Re}(k)$. By Lemmas 2.9 and 2.10, the map f^* satisfies the conclusion of Lemma 3.3.

We will show that for any function $G \in \mathcal{C}_c^\infty(\mathbb{C})$, we have

$$\mathcal{R} := \frac{1}{\varphi(q)} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} G(f^*(a/q)) = \int_0^1 G(f^*(x))\, d\nu + o(1)$$

as $q \to \infty$.

For $0 \leqslant j < q^{3/4}$, let $I_j := [jq^{1/4}, (j+1)q^{1/4}] \cap [0, q]$. If $j < q^{3/4} - 1$, by a standard estimate [FT91, Equations (1.13), (4.11)], we have

$$\sum_{a \in \mathfrak{A}_q \cap I_j} 1 \leqslant \sum_{\substack{(a,q)=1 \\ a \in I_j}} 1 = \frac{\varphi(q)}{q} q^{1/4} + O(q^{1/10}). \tag{3.2}$$

Given $\varepsilon \in (0,1)$, we let $E_\pm = \{j \mid \pm(\#I_j \cap \mathfrak{A}_q - \varphi(q)q^{-3/4}) > \varepsilon\varphi(q)q^{-3/4}\}$ and $E = E_+ \cup E_-$. By (3.2) it is clear that E_+ is empty if we assume q is large enough. Also, summing (3.2) over $j \notin E_-$, we deduce from Lemma 3.1 that $\#E = \#E_- = O(\varepsilon q^{3/4})$. Now, for each $j \notin E$ we fix any $y_j \in I_j \cap \mathfrak{A}_q$. Since $\|G\|_\infty, \|G'\|_\infty < \infty$, by Lemma 3.1 and Equations (3.2) and (3.1), we have

$$\begin{aligned}
\mathcal{R} &= \frac{1}{\varphi(q)} \sum_{a \in \mathfrak{A}_q} G(f^*(a/q)) + o(1) \\
&= \frac{1}{\varphi(q)} \sum_{\substack{0 \leqslant j \leqslant q^{3/4} \\ j \notin E}} \sum_{a \in I_j \cap \mathfrak{A}_q} G(f^*(a/q)) + o(1) + O(\varepsilon) \\
&= \frac{1}{\varphi(q)} \sum_{\substack{0 \leqslant j \leqslant q^{3/4} \\ j \notin E}} G(f^*(y_j)) \sum_{a \in I_j \cap \mathfrak{A}_q} 1 + o(1) + O(\varepsilon).
\end{aligned}$$

For $j \notin E$ the inner sum is $(\varphi(q)/q)q^{1/4}(1 + O(\varepsilon))$ and thus by (3.1) and Lemma 3.2 we deduce

$$\begin{aligned}
\mathcal{R} &= \frac{1}{q^{3/4}} \sum_{\substack{0 \leqslant j \leqslant q^{3/4} \\ j \notin E}} G(f^*(y_j)) + o(1) + O(\varepsilon) \\
&= \sum_{\substack{0 \leqslant j \leqslant q^{3/4} \\ j \notin E}} \int_{(\frac{1}{q} I_j) \cap \mathfrak{T}_q} G(f^*(x))\, dx + o(1) + O(\varepsilon) \\
&= \int_{\mathfrak{T}_q} G(f^*(x))\, dx + o(1) + O(\varepsilon) \\
&= \int_{[0,1]} G(f^*(x))\, dx + o(1) + O(\varepsilon),
\end{aligned}$$

since $\nu(\cup_{j \in E}((1/q)I_j)) = O(\varepsilon)$. Letting $\varepsilon \to 0$ sufficiently slowly, we obtain the claimed result.

This proves the first parts of Theorems 1.3 and 1.7.

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3.2 Proof of Theorem 1.3: Diffuseness when $\mathrm{Re}(k) < 0$

We now focus on the case $\mathrm{Re}(k) < 0$ and aim to show the second part of Theorem 1.3.

Proposition 3.4. *Let k, f, $f^{\triangleleft}$ and h as in Theorem 1.2 and let $\phi : \mathbb{C} \to \mathbb{R}$ be a linear form. Assume further that h is real-analytic on $(-1,1) \smallsetminus \{0\}$. Assume that for some choice of sign $\pm$, there exists a set of positive measure $C \subseteq \pm\mathbb{R}_{>0}$ and a constant $\alpha \in \mathbb{R}$ such that $\phi(f^{\triangleleft}(x)) = \alpha$ for $x \in C$. Then $\phi(f^{\triangleleft}(x)) = \alpha$ for almost all $x \in \pm\mathbb{R}_{>0}$.*

This clearly implies the second part of Theorem 1.3 and concludes its proof.

Proof of Proposition 3.4. We start with the case $k \in \mathbb{R}$. Since the real part of a real-analytic function is real-analytic, we have that $\phi \circ f$ is a quantum modular form of weight k and real-analytic period function $\phi \circ h$. In particular, we can assume that f is real valued and that ϕ restricted to $\mathbb{R}$ is the identity function.

By Theorems 1.1 and 1.2, there exists a full measure set X with $f^{\triangleleft}$ well defined on X. Note that Equation (1.2) holds on $X \smallsetminus \{0\}$ with f replaced by $f^{\triangleleft}$, by taking the limit along convergents and using the definition (1.6). We also recall that in our proof of Theorem 1.2, the set X was taken to be $X = \mathfrak{S}$ from Lemma 2.2. It is clear that $\mathfrak{S}$ is closed under any fixed inverse branch of the Gauss map. We may thus assume that X has this property.

We assume $C \subseteq \mathbb{R}_{>0}$, the other case being analogous. Also, by periodicity we can assume $C \subseteq [0,1] \cap X$. For any $\varepsilon > 0$, by Lebesgue's density theorem we can find an interval $I \subseteq (0,1)$ such that $\nu(C \cap I) \geqslant (1-\varepsilon)\nu(I)$. It follows that there exists an even $g \in \mathbb{N}$ and an inverse branch $U_{\boldsymbol{b}}$ of T^g given by $U_{\boldsymbol{b}}(y) = [0; b_1, \dots, b_g + y]$ for some coefficients $\boldsymbol{b} = (b_1, \dots, b_g) \in \mathbb{N}^g$, such that $D := \{x \in [0,1] \cap X \mid U_{\boldsymbol{b}}(x) \in C\}$ has measure $1 - O(\varepsilon)$. Repeatedly applying the reciprocity formula to $U_{\boldsymbol{b}}(x)$ for any $x \in D$, we have

$$\alpha = f^{\triangleleft}(U_{\boldsymbol{b}}(x)) = \sum_{j=0}^{g-1} \Big(\prod_{i=0}^{j-1} T^i(U_{\boldsymbol{b}}(x)) \Big)^{-k} h((-1)^j T^j(U_{\boldsymbol{b}}(x))) + \Big(\prod_{i=0}^{g-1} T^i(U_{\boldsymbol{b}}(x)) \Big)^{-k} f^{\triangleleft}(x). \tag{3.3}$$

Note that $T^i \circ U_{\boldsymbol{b}}$ is a smooth map, and in fact a homography, for any $i \leqslant g$. In particular, solving for $f^{\triangleleft}(x)$, we obtain that $f^{\triangleleft}$ is given on D by a function which is real-analytic on $(0,1)$. For ε sufficiently small we must have $\nu(C \cap D) \geqslant \nu(C) - \nu([0,1] \smallsetminus D) > 0$, and thus by analytic continuation this implies that $f^{\triangleleft}(x) = \alpha$ for all $x \in D$. Since $\varepsilon > 0$ is arbitrary, we conclude that $f^{\triangleleft}(x) = \alpha$ for a subset of $(0,1)$ of measure 1.

Now, let us assume $k \notin \mathbb{R}$ and write $k = k_1 + ik_2$ with $k_i \in \mathbb{R}$, $k_2 \neq 0$. We assume $\phi(z) = \mathrm{Re}(z)$, the general case being analogous. We let $\alpha \in \mathbb{R}$ such that $\mathrm{Re}(f^{\triangleleft}(x)) = \alpha$ for a set $C \subseteq [0,1]$ of positive measure. As in the case $k \in \mathbb{R}$, given $\varepsilon > 0$ we can find $\boldsymbol{b} \in \mathbb{N}^g$ such that $D := \{x \in [0,1] \cap X \mid U_{\boldsymbol{b}}(x) \in C\}$ has measure $1 - O(\varepsilon)$. Since $(0,1]$ can be written as the disjoint union $(0,1] = \cup_{c \geqslant 1} U_{(c)}((0,1])$, we have

$$\sum_{c \geq 1} \nu(\{x \in U_c((0,1]) \mid U_{\boldsymbol{b}}(x) \notin C\}) = 1 - \nu(D) = O(\varepsilon).$$

In particular, $\nu(\{x \in U_c((0,1]) \mid U_{\boldsymbol{b}}(x) \notin C\}) = O(\varepsilon)$ for all $c \in \mathbb{N}$ and thus also $\nu(\{x \in (0,1] \mid U_{(\boldsymbol{b},c)}(x) \notin C\}) = O(K^2\varepsilon)$ for all $c \leqslant K$. Letting $D_c := \{x \in [0,1] \cap X \mid U_{(\boldsymbol{b},c)}(x) \in C\}$, we then have $\nu(D_c) = 1 - O(\sqrt[3]{\varepsilon})$ for all $c \leqslant K := \varepsilon^{-1/3}$.

Next, we write the formula (3.3) with $\boldsymbol{b}_1 = (\boldsymbol{b}, c_1)$ and $\boldsymbol{b}_2 = (\boldsymbol{b}, c_2)$ in place of $\boldsymbol{b}$ for some $c_1, c_2 \leqslant K$. Taking the real part of the resulting equations, for $x \in D' := D_{c_1} \cap D_{c_2}$ we find

$$\alpha = A_j(x) + \mathrm{Re}(B_j(x)^k f^{\triangleleft}(x)), \quad j = 1, 2, \tag{3.4}$$

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where for $j=1,2$ we have that $B_j(x)=(\prod_{i=0}^{g}T^i(U_{(\boldsymbol{b},c_j)}(x)))^{-1}$ and A_j is real-analytic on $(0,1)$. Notice that $B_j(x)=v_{g-1}+(c_j+x)v_g$, where v_{g-1} and v_g are the partial denominators of $[0;b_1,\dots,b_{g-1}]$ and $[0;b_1,\dots,b_g]$, respectively. In particular, if $c_1,c_2\in[\sqrt{K},K]$ then $B_2(x)/B_1(x)=c_2/c_1+O(\sqrt[6]{\varepsilon})$ uniformly in $x\in[0,1]$.

Writing $L_j(x):=k_2\log(B_j(x))$, Equation (3.4) becomes

$$\alpha=A_1(x)+B_j(x)^{k_1}\operatorname{Re}(f^{\triangleleft}(x))\cos(L_j(x))-B_j(x)^{k_1}\operatorname{Im}(f^{\triangleleft}(x))\sin(L_j(x))\quad(j\in\{1,2\},\ x\in D').$$

This is a system in $\operatorname{Re}(f^{\triangleleft}(x)),\operatorname{Im}(f^{\triangleleft}(x))$ of determinant $B_1(x)^{k_1}B_2(x)^{k_1}\sin(L_1(x)-L_2(x))$ and solving for $\operatorname{Re}(f(x))$ we find

$$\operatorname{Re}(f^{\triangleleft}(x))=\frac{(\alpha-A_2(x))B_1(x)^{k_1}\sin(L_1(x))-(\alpha-A_1(x))B_2(x)^{k_1}\sin(L_2(x))}{B_1(x)^{k_1}B_2(x)^{k_1}\sin(L_1(x)-L_2(x))}\tag{3.5}$$

on $x\in D'$. Now, for ε sufficiently small, we have

$$\sin(L_1(x)-L_2(x))=\sin(k_2\log(c_2/c_1)+O(\sqrt[6]{\varepsilon}))$$

uniformly for $x\in[0,1]$. Thus, for ε sufficiently small we can find $c_1,c_2\in\mathbb{N}\cap[\sqrt{K},K]$ such that $\sin(L_1(x)-L_2(x))$ does not vanish on $(0,1)$. For this choice one has that the right-hand side of (3.5) is analytic on $(0,1)$. Since $\nu(D')=1-O(\sqrt[3]{\varepsilon})$, we then reach the conclusion as in the case $k\in\mathbb{R}$. □

3.3 The continuity of the cumulative distribution function when $\operatorname{Re}(k)>0$

We prove the following generalization of Theorem 1.7. Also, as in § 2.4.1, we allow for a twist by a root of unity ϑ and assume f satisfies (2.26).

Theorem 3.5. *Let $\phi:\mathbb{C}\to\mathbb{R}$ be a linear form, $\operatorname{Re}(k)>0$, and let $h:\mathbb{R}\smallsetminus\{0\}\to\mathbb{R}$ be a function satisfying either of the following two conditions.*

- (1^*) *The function h extends to a continuous function on $[-1,1]$, and either $\operatorname{Im}k=0$ and there exists $p\in\mathbb{Z}$, $\alpha\in(-1,1)$ such that $\phi(\vartheta^p h(\alpha))\neq 0$; or $\operatorname{Im}k\neq 0$ and there exists $\alpha\in(-1,1)$ such that $h(\alpha)\neq 0$.*
- (2^*) *One has $\lim_{x\to0^\pm}|h(x)|=\infty$ for a choice of $\pm$, there exists $\delta\in(0,1)$ such that $h(x)\ll e^{|x|^{-1+\delta}}$ for $|x|\leqslant 1$ and*
  - *either $\operatorname{Im}(k)=0$ and there exists $p\in\mathbb{Z}$ such that for all small $\varepsilon>0$,*

$$\liminf_{x\to0^\pm}\inf\left\{\left|\phi\left(\vartheta^p\frac{h(x)-u^kh(ux)}{|h(x)|}\right)\right|:|\log u|\in(\varepsilon,\varepsilon^{2/3})\right\}>0;\tag{3.6}$$

  - *or $\operatorname{Im}(k)\neq0$ and there exists $\beta\in\mathbb{R}$ such that for all small $\varepsilon>0$,*

$$\liminf_{x\to0^\pm}\inf\left\{\left|\phi\left(e(\alpha)\frac{h(x)-u^kh(ux)}{|h(x)|}\right)\right|:|\log u|\in(\varepsilon,\varepsilon^{2/3}),|\alpha-\beta|<\varepsilon^{2/3}\right\}>0.\tag{3.7}$$

Then the CDF of $(\phi\circ f^{\triangleright})_*(\nu)$ is continuous.

Remark 3.6.

- The hypotheses could be relaxed somehow. For instance, instead of (1^*), it suffices to ask that h has finite left and right limits at certain rationals along with a non-vanishing condition. This will be clear from our arguments. One could also deal with some cases when the limits (3.6) and (3.7) are zero, as long as one can control the asymptotic of these quantities as $x\to0$. We refer to Remark 3.15 below, as well as to the proof of Corollary 1.14 when $a=0$ for an example where this consideration is relevant.

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– When h is continuous on $[-1,1]$, the condition (1^*) is clearly necessary, since otherwise $\phi\circ f^{\triangleright}$ vanishes identically on $[0,1]$.
– Some condition of the type (2^*) is necessary in order to prevent examples such as $h(x)=1-|x|^{-k}$, for which the function f is constant.

Proof of the second part of Theorem 1.7. Assume first that h is continuous on $[-1,1]$ and non-zero. Then the hypothesis (1^*) is clearly satisfied with either $\phi=\mathrm{Re}$ or $\phi=\mathrm{Im}$. We deduce that $(\phi\circ f^{\triangleright})_*(\nu)$ is diffuse, and therefore so is $f^{\triangleright}_*(\nu)$.

Assume next that $h(x)\ll e^{|x|^{-1+\delta}}$ on $[-1,1]\smallsetminus\{0\}$ and $h(x)\sim cx^{-\lambda}$ as $x\to 0^+$. Then for $|\log u|\leqslant 1$, we deduce that as $x\to 0^+$,

$$\frac{h(x)-u^k h(ux)}{|h(x)|}=\frac{c}{|c|}(1+o(1))(1-u^{k-\lambda})\gg_{\varepsilon,k,\lambda}1$$

if additionally $|\log u|\geqslant\varepsilon$. Similarly as above, we deduce that hypothesis (2^*) holds for $\phi=\mathrm{Re}$ or $\phi=\mathrm{Im}$ and we deduce that $f^{\triangleright}_*(\nu)$ is diffuse.

This shows that the second part of Theorem 1.7 holds, and concludes its proof. □

Remark 3.7. When $h(x)=cx^{-\lambda}$ with λ satisfying $\mathrm{Re}(k)>0$ but $\lambda\notin\mathbb{R}$, hypothesis (2^*) no longer holds, but it is plausible that our arguments could be adapted by further localizing the values of x involved in (3.6) and (3.7). We do not pursue this here.

We first give the following lemma.

Lemma 3.8. *Let $F:[0,1]\to\mathbb{R}$ be measurable. Assume that for each $\varepsilon>0$, we can find a countable collection $(S_j)_j$ of disjoint measurable subsets of $[0,1]$, such that*

$$\sum_j \nu(S_j)\geqslant 1-\varepsilon,$$

and for each j and $y\in S_j$,

$$\nu(\{x\in S_j, F(x)=F(y)\})\leqslant\varepsilon\nu(S_j). \tag{3.8}$$

Then $F_*(\nu)$ is continuous.

Proof. Let $a\in\mathbb{R}$, and $X=F^{-1}(\{a\})$. Let $\varepsilon>0$ be arbitrary, and $(S_j)_j$ given by the hypothesis. For each j, either $S_j\cap X$ is empty, or there exists $y\in S_j$ such that $F(y)=a$ and then

$$\nu(X\cap S_j)=\nu(\{x\in S_j, F(x)=F(y)\})\leqslant\varepsilon\nu(S_j)$$

by hypothesis. In both cases we have $\nu(X\cap S_j)\leqslant\varepsilon\nu(S_j)$. Summing over j, we find

$$\nu(X)\leqslant\nu([0,1]\smallsetminus\cup_j S_j)+\sum_j\nu(X\cap S_j)\leqslant 2\varepsilon,$$

and letting $\varepsilon\to 0$ gives the conclusion. □

We divide the proof depending on whether (1^*) or (2^*) holds.

3.3.1 *The case of bounded h.* Let $N\in\mathbb{Z}_{>0}$ be the order of ϑ as a root of unity, so that

$$\vartheta^N=1,$$

where ϑ is the automorphy factor in the generalized period relation (2.26). We recall that ϑ_j was defined in (2.27). Recall also from (2.31) that the notation $\vartheta_j(x)$ for a number $x\in\mathbb{Q}$ depends on how we expand x in CF when x is rational. In what follows, we will work with the expansion

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of odd length. For each $g \geqslant 1$, we define $e_g(x) \in \mathbb{Z}/N\mathbb{Z}$ through $\vartheta_g(x) = \vartheta^{e_g(x)}$ and notice that by definition we have

$$e_g(x) \equiv e_{g-1}(x) + (-1)^g b_g + 3(-1)^{g+1} \pmod{N}. \tag{3.9}$$

Given $c_1, \dots, c_\ell \geqslant 1$, we denote

$$\begin{aligned} J(c_1, \dots, c_\ell) &:= \{x \in [0,1], b_i(x) = c_i \text{ for } 1 \leqslant i \leqslant \ell\}, \\ \mu(c_1, \dots, c_\ell) &:= \nu(J(c_1, \dots, c_\ell)). \end{aligned} \tag{3.10}$$

By [Khi63, p. 57], we have

$$\mu(c_1, \dots, c_\ell) = \frac{1}{v_\ell(x)(v_\ell(x) + v_{\ell-1}(x))} \asymp \frac{1}{v_\ell(x)^2} \quad (x = [0; c_1, \dots, c_\ell]). \tag{3.11}$$

Lemma 3.9. *Let $m \in \mathbb{N}$, $K > 1$ and $(L_1, \dots, L_m) \in \mathbb{N}^m$ be fixed. Let $V \geqslant 1$, and for each $e \in \mathbb{Z}/N\mathbb{Z}$ and $\omega > 0$, let $\mathscr{B}(e, \omega) \subset \mathbb{Z} \cap [\sqrt{V}, K\sqrt{V}]$ be given, and*

$$T := \inf_{e \in \mathbb{Z}/N\mathbb{Z},\, \omega > 0} \#\mathscr{B}(e, \omega).$$

Also, for $x \in (0,1) \smallsetminus \mathbb{Q}$, let

$$G_V(x) := \{g \in [V, 2V] \mid b_g(x) \in \mathscr{B}(e_{g-1}(x), v_{g-1}(x)),\ b_{g+i}(x) = L_i\ \forall i = 1, \dots, m\}$$

and let $X_{V,\mathscr{B}} = X_{V,\mathscr{B}}(K, L_1, \dots, L_m)$ be the subset of $[0,1] \smallsetminus \mathbb{Q}$ such that $G_V(x)$ contains at least an even and an odd integer. Then $\nu(X_{V,\mathscr{B}}) = 1 + o(1)$ as $V, T \to \infty$, where the rate of decay of $o(1)$ depends at most on $K, L_1, \dots, L_m$.

Proof. We show that $G_V(x)$ contains an even integer for x in a set of measure $1 + o(1)$, the odd integer case being analogous.

By [Khi63, Equation (57)] we have uniformly for $\ell \geqslant 1$ and $c_1, \dots, c_{\ell+1} \geqslant 1$,

$$\tfrac{1}{3} c_{\ell+1}^{-2} \mu(c_1, \dots, c_\ell) \leqslant \mu(c_1, \dots, c_{\ell+1}) \leqslant 2c_{\ell+1}^{-2} \mu(c_1, \dots, c_\ell). \tag{3.12}$$

In particular, writing $L := \max(L_1, \dots, L_m)$, we have for any $c \in \mathbb{Z}_{>0}$,

$$\mu(c_1, \dots, c_\ell, c, L_1, \dots, L_m) \geqslant 3^{-m-1} c^{-2} L^{-2m} \mu(c_1, \dots, c_\ell).$$

Since $\sum_{c \in \mathscr{B}(e_\ell, v_\ell)} c^{-2} \geqslant (1/K^2 V) \#\mathscr{B}(e_\ell, v_\ell)$, where $e_\ell = e_\ell([0; c_1, \dots, c_\ell])$, it then follows that the measure of $x \in [0,1] \smallsetminus \mathbb{Q}$ such that none of the integers $g \in \{2\lfloor V/2 \rfloor + \ell(2m+2) \mid 1 \leqslant \ell \leqslant V/(2m+2)\}$ satisfy $b_g(x) \in \mathscr{B}$ is

$$\ll (1 - K^{-2}(3L)^{-3m} T V^{-1})^{\lfloor V/(2m+2) \rfloor} = o(1)$$

as $V, T \to \infty$ with m, L fixed. The lemma then follows. □

We now prove Theorem 3.5. To illustrate Remark 3.6, we assume only that h is bounded on $[-1,1]$, not necessarily continuous, and that it has finite left or right limit at 0. We assume it is the former, the latter case being analogous, as we will comment after the proof. If $\mathrm{Im}(k) = 0$, we suppose that $\phi(\vartheta^p h(0^-)) \neq 0$ for some $p \in \mathbb{Z}$, whereas if $\mathrm{Im}(k) \neq 0$, we assume that $h(0^-) \neq 0$ and set $p := 0$. Moreover, we let

$$\kappa \in \mathbb{R}, \quad \phi(e(-\kappa)\vartheta^p h(0^-)) \text{ is maximal.} \tag{3.13}$$

This means, in other terms, that for any $z \in \mathbb{C}$,

$$\phi(\vartheta^p h(0^-) z) = |h(0^-)| \operatorname{Re}(e(\kappa) z). \tag{3.14}$$

If $\mathrm{Im}(k) = 0$, then setting $z = 1$ we see that $\kappa \not\equiv \pm\frac{1}{2} \pmod 1$ by hypothesis.

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We define our choice of sets $\mathscr{B}=\mathscr{B}(e,\omega)$. We let $\xi\in(0,1)$ be a parameter, on which these sets will depend. For $e\in\mathbb{Z}/N\mathbb{Z}$, $\omega>0$, let $K=2$ if $\mathrm{Im}(k)=0$ and $K=\max(2,e^{3\pi/|\mathrm{Im}(k)|})$ otherwise. Also, let $\mathscr{B}(e,\omega)=\mathscr{B}_{V,\xi,N,k}(e,\omega)$ be defined by

$$\mathscr{B}(e,\omega):=\Bigg\{b\in\mathbb{Z}\cap[\sqrt{V},K\sqrt{V}]\colon b\equiv 3-(p+e)\pmod N \\ \text{and}\ \begin{cases}-\xi\leqslant\left\{\dfrac{\mathrm{Im}(k)}{2\pi}\log(\omega\sqrt{V})-\kappa\right\}+\dfrac{\mathrm{Im}(k)}{2\pi}\log(b/\sqrt{V})\leqslant\xi & \text{if } k\notin\mathbb{R}\\ 0\leqslant\log(b/\sqrt{V})\leqslant\xi & \text{if } k\in\mathbb{R}\end{cases}\Bigg\}, \tag{3.15}$$

where $\{\cdot\}$ denotes the fractional part, and κ was defined at (3.13). Here our choice of K ensures that the set $\mathscr{B}$ is not void for V large enough, and in fact

$$\#\mathscr{B}\gg_{k,N}\xi\sqrt{V} \tag{3.16}$$

for V large enough in terms of k, N and ξ. Moreover, on the one hand, by (3.9) the congruence condition ensures that whenever a number $x\in[0,1]\smallsetminus\mathbb{Q}$ satisfies $b_g(x)\in\mathscr{B}(e_{g-1}(x),v_{g-1}(x))$ with g even, we have $e_g(x)\equiv -p\pmod N$, and therefore we deduce

$$\vartheta_g(x)=\vartheta^{-p}\quad(g\in G_V(x)\cap 2\mathbb{Z}). \tag{3.17}$$

On the other hand, the condition involving $\|\cdot\|_{\mathbb{R}/\mathbb{Z}}$ ensures that

$$\mathrm{Re}(e(\kappa)(v_{g-1}(x)b_g(x))^{-i\,\mathrm{Im}\,k})=1+O(\xi^2)\quad(\mathrm{Im}\,k\neq 0, g\in G_V(x)) \tag{3.18}$$

by the Taylor expansion of the cosine, with an absolute implicit constant.

We let $L_1,\dots,L_m\in\mathbb{N}$ be chosen later and take $X_{V,\mathscr{B}}$, $G_V(x)$ as in Lemma 3.9. For all even g and all $c_1,\dots,c_{g-1}\in\mathbb{N}$, we let

$$S_{g,(c_i),\mathscr{B}}:=\{x\in(0,1)\smallsetminus\mathbb{Q}:\forall i<g, b_i(x)=c_i; g\in G_V(x)\cap 2\mathbb{Z};\forall j\geqslant 1, g-2j\notin G_V(x)\}, \tag{3.19}$$

that is, $S_{g,(c_i),\mathscr{B}}$ is the subset of $J(c_1,\dots,c_{g-1})$ such that g is the smallest even element of $G_V(x)$. We will drop the subscript $\mathscr{B}$ from the notation.

Lemma 3.10. *Assume that $L_i<\sqrt{V}$ for all $1\leqslant i\leqslant m$ and that $g\in 2\mathbb{Z}\cap[V,2V]$. For all $c_1,\dots,c_{g-1},c\geqslant 1$ and $\varepsilon>0$, we have*

$$\nu(\{x\in S_{g,(c_i)}\colon |b_g(x)-c|\leqslant\varepsilon\sqrt{V}\})\ll\varepsilon\xi^{-1}\nu(S_{g,(c_i)})$$

where the constant depends on m, k and N at most.

Proof. We start by observing that the condition $g-2j\notin G_V(x)$ in the definition of $S_{g,(c_i),\mathscr{B}}$ reduces to a condition on the first $g-1$ partial quotients of x, $c_1,\dots,c_{g-1}$, if $j>m/2$. In particular, we can assume this condition is satisfied, since otherwise $S_{g,(c_i)}=\emptyset$ and the result is trivial. Similarly, we can assume $c\in[(1-\varepsilon)\sqrt{V},(K+\varepsilon)\sqrt{V}]$.

Next, we observe that if $1\leqslant j<m/2$, then the condition $g-2j\notin G_V(x)$ follows from the truth of the other conditions in the definition of $S_{g,(c_i)}$. Indeed, for $g\in G_V(x)$ we have $b_g(x)\in\mathscr{B}$ and $b_{g+i}(x)=L_i$ for $1\leqslant i\leqslant m$, so that $b_{g-2j+2j}(x)=b_g(x)\geqslant\sqrt{V}>L_{2j}$ and thus $g-2j\notin G_V(x)$. Recalling the notation (3.10), we thus have

$$\{x\in S_{g,(c_i)}\colon b_g(x)=c'\}=J(c_1,\dots,c_{g-1},c',L_1,\dots,L_m)$$

for all $c'\in\mathscr{B}(e_{g-1}(x'),v_{g-1}(x'))$ with $x'=[0;c_1,\dots,c_{g-1}]$. By (3.11), for all $c'\in[(1-\varepsilon)\sqrt{V},(K+\varepsilon)\sqrt{V}]$, we have

$$\mu(0;c_1,\dots,c_{g-1},c',L_1,\dots,L_m)\asymp\mu(0;c_1,\dots,c_{g-1},c,L_1,\dots,L_m)$$

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with a constant depending at most on m and K. Then, using (3.16) we deduce on the one hand

$$\nu(S_{g,(c_i)}) \gg \xi\sqrt{V}\mu([0; c_1, \dots, c_{g-1}, c, L_1, \dots, L_m])$$

and on the other hand

$$\nu(\{x \in S_{g,(c_i)}, |b_g(x) - c| \leqslant \varepsilon\sqrt{V}\}) \ll \varepsilon\sqrt{V}\mu([0; c_1, \dots, c_{g-1}, c, L_1, \dots, L_m]).$$

Grouping these two bounds yields our claim. □

Let $\varepsilon \in (0,1)$ be fixed. We set $m = 1$ and $L_1 := L \in \mathbb{N}_{>0}$ be a parameter. We let $X_{V,\mathscr{B}}$ as in Lemma 3.9 and assume that $L \to \infty$ as $V \to \infty$ sufficiently slowly with respect to V, so that we still have $\nu(X_{V,\mathscr{B}}) \to 1$ as well as $L < \sqrt{V}$. In particular, assuming $V \geqslant 1$ is large enough, we have

$$\nu(X_{V,\mathscr{B}}) \geqslant 1 - \varepsilon.$$

For $g \in [V, 2V] \cap 2\mathbb{Z}$ and $c_1, \dots, c_{g-1} \geqslant 1$, the sets $(S_{g,(c_i)})$ are disjoint and, by construction, their union over all g and (c_i) contains $X_{V,\mathscr{B}}$ and thus has measure $\geqslant 1 - \varepsilon$. The collection of sets $(S_{g,(c_i)})$ will play the rôle of (S_j) in Lemma 3.8.

Recall that, since h is assumed to be bounded, we have the expression (2.33). Consider one of the sets $S_{g,(c_j)}$, and $x, y \in S_{g,(c_j)}$ satisfying $\phi(f^{\triangleright}(x)) = \phi(f^{\triangleright}(y))$. Since the terms $j < g$ in (2.33) depend only on $c_1, \dots, c_{g-1}$, we have $f^{\triangleright}(x) - f^{\triangleright}(y) = F_g(x) - F_g(y)$, where

$$F_g(x) = \sum_{j=g}^{\infty} \vartheta_j^{-1}(x) v_j(x)^{-k} h\left((-1)^{j-1}\frac{v_{j-1}(x)}{v_j(x)}\right) \tag{3.20}$$

and similarly for y. If $x \in S_{g,(c_j)}$ we have $v_{g+j}(x) \gg v_g(x) L\, \varpi^j$ for $j \geqslant 1$, whence since g is even and h is bounded, we have

$$\begin{aligned} F_g(x) &= \frac{\vartheta_g(x)^{-1} + o(1)}{v_g(x)^k} h(0^-) + O\left(\frac{L^{-\operatorname{Re}(k)}}{v_g(x)^{\operatorname{Re}(k)}} \sum_{j \geqslant 1} \frac{\|h\|_\infty}{\varpi^{\operatorname{Re}(k) j}}\right) \\ &= \frac{\vartheta_g(x)^{-1} + o(1)}{v_g(x)^k} h(0^-), \end{aligned} \tag{3.21}$$

as $V \to \infty$.

Consider first the case $\operatorname{Im} k = 0$. By (3.17), we have

$$\vartheta_g(x) = \vartheta_g(y) = \vartheta^{-p}. \tag{3.22}$$

Also, since $b_j(x) = b_j(y)$ for $j < g$ and $b_g(x) \asymp b_g(y) \asymp \sqrt{V}$, we have $v_{g-1}(x) \asymp v_{g-1}(y)$ and $v_g(x) \asymp v_g(y)$ with constants depending only on k. Thus, by (3.14) and the mean value theorem, we deduce that for $\operatorname{Im} k = 0$ we have

$$\begin{aligned} 0 = |\phi(F_g(x) - F_g(y))| &= |h(0^-)\cos(2\pi\kappa)||v_g(x)^{-k} - v_g(y)^{-k}| + o(v_g(x)^{-\operatorname{Re} k}) \\ &\gg |h(0^-)\cos(2\pi\kappa)| \frac{|v_g(x) - v_g(y)|}{v_g(x)^{k+1}} + o(v_g(x)^{-\operatorname{Re} k}) \\ &\gg |h(0^-)\cos(2\pi\kappa)| \frac{|b_g(x) - b_g(y)| V^{-1/2} + o(1)}{v_g(x)^k}. \end{aligned}$$

We deduce that $|b_g(x) - b_g(y)| = o(\sqrt{V})$ as $V \to \infty$.

We reach a similar conclusion when $\operatorname{Im} k \neq 0$. Indeed, in this case we still have (3.22), with $p = 0$. Furthermore, since $v_g(x) = v_{g-1} b_g(x) + v_{g-2}(x)$, we have

$$\phi(h(0^-) v_g(x)^{-i \operatorname{Im} k}) = |h(0^-)| \operatorname{Re}(e(\kappa)(v_{g-1} b_g(x))^{-i \operatorname{Im} k})(1 + O(V^{-1/2})) = 1 + O(\xi^2 + V^{-1/2}),$$

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where in the last equality we used (3.18). Similarly to the above, as $V \to \infty$, we have

$$\begin{aligned}0 &= |\phi(F_g(x) - F_g(y))| \\ &= |\phi(\vartheta_g(x)^{-1} h(0^-) v_g(x)^{-i\operatorname{Im} k}) v_g(x)^{-\operatorname{Re} k} - \phi(\vartheta_g(y)^{-1} h(0^-) v_g(y)^{-i\operatorname{Im} k}) v_g(y)^{-\operatorname{Re} k}| + o(v_g(x)^{-\operatorname{Re} k}) \\ &= |h(0^-)||v_g(x)^{-\operatorname{Re} k} - v_g(y)^{-\operatorname{Re} k}| + (o(1) + O(\xi^2)) v_g(x)^{-\operatorname{Re} k} \\ &\gg |h(0^-)| \frac{|b_g(x) - b_g(y)| V^{-1/2} + o(1) + O(\xi^2)}{v_g(x)^{\operatorname{Re} k}}.\end{aligned}$$

As above, we deduce

$$|b_g(x) - b_g(y)| = (o(1) + O(\xi^2))\sqrt{V}.$$

We now pick $\xi = c\sqrt{\varepsilon}$ with a sufficiently small constant $c > 0$ and assume V is large enough so that the right-hand side above is at most $\varepsilon\sqrt{V}$ in modulus. We get

$$\nu(\{x \in S_{g,(b_i)}, F_g(x) = F_g(y)\}) \leqslant \nu(\{x \in S_{g,(b_i)}, |b_g(x) - b_g(y)| \leqslant \varepsilon\sqrt{V}\}) \ll \sqrt{\varepsilon}\nu(S_{g,(b_i)})$$

by Lemma 3.10. Since $\varepsilon > 0$ was arbitrary, this verifies the hypothesis (3.8), thus Lemma 3.8 applies and yields the desired conclusion.

The case when the non-vanishing hypothesis involves $h(0^+)$ is similar, taking g to be odd instead of even, and picking $b \equiv 3 + p + e \pmod N$ in (3.15).

Now, assume that $\alpha \in (-1, 1)$ satisfies condition (1*) of the theorem. We may assume that $\alpha \in \mathbb{Q}$ by continuity. We pick α such that the length $r(|\alpha|)$ of the continued fraction expansion of $|\alpha|$ is minimal. We may assume that $\alpha \neq 0$, or in other words $r(|\alpha|) \geqslant 1$, since the complementary case was treated above. We write $|\alpha| = [0; c_1, \dots, c_r]$. By minimality of r, we have $h(\pm[0; c_s, \dots, c_r]) = 0$ for $1 < s \leqslant r$ if $\operatorname{Im} k \neq 0$; and likewise $\phi(\pm\vartheta^p h([0; c_s, \dots, c_r])) = 0$ if $\operatorname{Im} k = 0$. We repeat the arguments above with $m = r + 1$,

$$L_i = c_{r+1-i} \text{ for } 1 \leqslant i \leqslant r, \quad L_{r+1} = L,$$

with $L \to \infty$ as $V \to \infty$. Note that if $b_g \asymp \sqrt{V}$ and $b_{g+j} = L_j = c_{r+1-j}$ for $1 \leqslant j \leqslant r$, then

$$\frac{v_{g+j-1}}{v_{g+j}} = [0; b_{g+j}, b_{g+j-1}, \dots, b_g + O(1)] \to [0; c_{r+1-j}, \dots, c_r]$$

as $V \to \infty$. Thus, if the parity of g was chosen such that $(-1)^{g+r-1} = \operatorname{sgn}(\alpha)$, then by hypothesis we have

$$\begin{cases} h((-1)^{g+j-1} v_{g+j-1}/v_{g+j}) = o(1) & (j = 1, \dots, r-1), \\ h((-1)^{g+r-1} v_{g+r-1}/v_{g+r}) = h(\alpha) + o(1). \end{cases}$$

The rest of the argument follows *mutatis mutandis*, with the estimate (3.21) being replaced by

$$F_{g,N}(x) = \frac{\vartheta_{g+r}(x)^{-1} + o(1)}{v_{g+r}(x)^k} h(\alpha).$$

Remark 3.11. By choosing the parity of g appropriately, it is clear that the arguments above hold under milder hypotheses, namely one-sided continuity of h at α along with the vanishing of the values $h(T^j(\alpha)^{\pm})$ for $1 < j \leqslant r$.

3.3.2 *The case of unbounded h.* The following lemma provides a substitute for Lemma 3.9 in the case when h is unbounded.

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Lemma 3.12. *Let $K>1$, $\delta\in(0,1)$, and $\rho>(2+2\delta)^{-1}$ be fixed, $V\geqslant 1$, and $\psi:\mathbb{R}_{\geqslant 1}\to\mathbb{R}_{\geqslant 1}$ be such that $\lim_{x\to\infty}\psi(x)=+\infty$. Also, for each $e\in\mathbb{Z}/N\mathbb{Z}$ and $\omega>0$, let $\mathscr{B}(e,\omega)\subset\mathbb{Z}\cap[\sqrt{V},K\sqrt{V}]$ be given, and assume*

$$T:=\inf_{e\in\mathbb{Z}/N\mathbb{Z},\,\omega>0}\#\mathscr{B}(e,\omega)\gg V^{\rho}.$$

Finally, for $x\in[0,1]\smallsetminus\mathbb{Q}$, let

$$G_V(x):=\{g\in[V,2V]\mid b_g(x)\in\mathscr{B}(e_{g-1}(x),v_{g-1}(x)),\ b_{g+j}(x)<j^{1+\delta}\psi(V)\ \forall j\geqslant 1\}$$

and $X_{V,\mathscr{B}}$ be the subset of $[0,1]\smallsetminus\mathbb{Q}$ such that $G_V(x)$ contains at least an even and an odd integer. Then $\nu(X_{V,\mathscr{B}})=1+o(1)$ as $V\to\infty$, where the rate of decay of $o(1)$ depends at most on K,ψ,δ and ρ.

Proof. We prove that $G_V(x)$ contains an even integer asymptotically almost surely, the odd case being identical. We fix $\varepsilon>0$ and observe we can assume $1\leqslant\psi(V)<V^{\varepsilon}$. We then observe that by Lemma 2.2 one has that $b_{g+j}(x)<j^{1+\delta}\psi(V)$ for all $j>D:=V^{1/(1+\delta)+\varepsilon}$, all $g\in[V,2V]$ and all x in a subset of $[0,1)$ of measure $1+o(1)$ as $V\to\infty$. In particular, it suffices to show that the larger set

$$G'_V(x):=\{g\in[V,2V]\mid b_g(x)\in\mathscr{B}(e_{g-1}(x),v_{g-1}(x)),\ b_{g+j}(x)<j^{1+\delta}\psi(V)\ \forall 1\leqslant j\leqslant D\}$$

contains an even integer asymptotically almost surely.

Now, for $m\geqslant 0$ and $c_1,\dots,c_m\in\mathbb{N}$, $I\subseteq\mathbb{N}$, let

$$M_I:=\{x\in[0,1]\smallsetminus\mathbb{Q}\colon 2\mathbb{Z}\cap I\cap G'_V(x)\neq\emptyset\},\quad M_I(c_1,\dots,c_m):=M_I\cap J(c_1,\dots,c_m),$$

so that we need to prove that $\nu(M_{[V,2V]})\to 1$ as $V\to\infty$. For $m=\ell-1$ and any $\ell\in 2\mathbb{Z}\cap[V,2V]$, by (3.12) we have

$$\nu(M_{\{\ell\}}(c_1,\dots,c_m))\geqslant\mu(c_1,\dots,c_m)\frac{T}{3K^2V}\prod_{j=1}^{\infty}\left(1-\frac{1}{3j^{1+\delta}\psi(V)}\right)\geqslant\mu(c_1,\dots,c_m)\frac{T}{4K^2V}$$

for V large enough. Splitting into subintervals we then have that the same bound holds for any $m\leqslant\ell-1$.

Next, we let $C:=[V^{1/(2+2\delta)-\varepsilon}]\leqslant D$ and notice that if $0<\ell_2-\ell_1\leqslant C$, then $M_{\{\ell_1\}}(c_1,\dots,c_m)\cap M_{\{\ell_2\}}(c_1,\dots,c_m)=\emptyset$. Indeed, if $x\in M_{\{\ell_1\}}(c_1,\dots,c_m)$, then

$$b_{\ell_2}(x)\leqslant(\ell_2-\ell_1)^{1+\delta}\psi(V)\leqslant C^{1+\delta}V^{\varepsilon}<\sqrt{V}$$

and thus $x\notin M_{\{\ell_2\}}(c_1,\dots,c_m)$. It follows that for any $W\in\mathbb{N}$ with $[W-C,W)\subseteq[V,2V]$, we have

$$\nu(M_{[W-C,W)}(c_1,\dots,c_m))=\sum_{\ell\in I\cap 2\mathbb{Z}}\nu(M_{\{\ell\}}(c_1,\dots,c_m))\geqslant\mu(c_1,\dots,c_m)\frac{T\lfloor C/2\rfloor}{4K^2V}\tag{3.23}$$

for any $m<W-C$.

Let $W\in\mathbb{N}$ with $V<W-2D$ and $W\leqslant 2V$, and let $m=W-C-1\geqslant W-D$. We observe that the condition $\ell\in G'_V(x)$ depends only on the first $\ell+D$ partial quotients of x. Thus, since $m\geqslant W-D$, we have that $M_{[V,W-2D)}(c_1,\dots,c_m)$ is either empty or is equal to $J(c_1,\dots,c_m)$.

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By (3.23), we deduce

$$\begin{aligned}\nu(M_{[V,W)})-\nu(M_{[V,W-2D)}) &= \nu(M_{[W-2D,W)}\smallsetminus M_{[V,W)})\\ &\geqslant \nu(M_{[W-C,W)}\smallsetminus M_{[V,W-2D)})\\ &= \sum_{\substack{c_1,\dots,c_m\geqslant 1\\ M_{[V,W-2D)}(c_1,\dots,c_m)=\emptyset}} \nu(M_{[W-C,W)}(c_1,\dots,c_m))\\ &\geqslant \frac{TC}{12K^2V}\sum_{\substack{c_1,\dots,c_m\geqslant 1\\ M_{[V,W-2D)}(c_1,\dots,c_m)=\emptyset}} \mu(c_1,\dots,c_m)\\ &= \frac{TC}{12K^2V}(1-\nu(M_{[V,W-2D)})).\end{aligned}$$

We have used $\lfloor C/2\rfloor\geqslant C/3$ in the fourth line. We thus obtain

$$1-\nu(M_{[V,W)})\leqslant(1-\nu(M_{[V,W-2D)}))(1-TC/12V).$$

Iterating, we then obtain

$$1-\nu(M_{[V,2V)})\leqslant(1-TC/12K^2V)^{[V/2D]}=o(1)$$

since $TC/K^2D\asymp TV^{-1/(2+2\delta)-2\varepsilon}\to\infty$ if ε is small enough. □

For $j\in\mathbb{N}$ and K,ψ as in the lemma, we let

$$Q_j(V)=\begin{cases}j^{1+\delta}\psi(V) & \text{if } j^{1+\delta}\psi(V)\notin[\sqrt{V},K\sqrt{V}],\\ K\sqrt{V}, & \text{otherwise.}\end{cases}$$

We then define

$$G_V^*(x):=\{g\in[V,2V]\mid b_g(x)\in\mathscr{B}(e_{g-1}(x),v_{g-1}(x)),\ b_{g+j}(x)<Q_j(V)\ \forall j\geqslant 1\}$$

and the corresponding set $X_{V,\mathscr{B}}$. Clearly, $G_V^*(x)\supseteq G_V(x)$ and thus, under the hypothesis of Lemma 3.12, $\nu(X_{V,\mathscr{B}}^*)=1+o(1)$ as $V\to\infty$.

We let

$$\kappa:=\begin{cases}-\beta & \text{if } \operatorname{Im}k\neq 0,\\ 0 & \text{if } \operatorname{Im}k=0,\end{cases}$$

and define the sets $\mathscr{B}$ as in (3.15). Also, we define $S_{g,(c_i)}^*=S_{g,(c_i),\mathscr{B}}^*$ as in (3.19), but with $G_V^*(x)$ in place of $G_V(x)$. The following lemma is an analogue of Lemma 3.10.

Lemma 3.13. *Assume that $g\in 2\mathbb{Z}\cap[V,2V]$ with $V\geqslant 1$ sufficiently large. For all $c_1,\dots,c_{g-1},c\geqslant 1$ and $\varepsilon>0$, we have*

$$\nu(\{x\in S_{g,(c_i)}^*\colon |b_g(x)-c|\leqslant\varepsilon\sqrt{V}\})\ll\varepsilon\xi^{-1}\nu(S_{g,(c_i)}^*),$$

where the constant depends on k, δ and N at most.

Proof. We may assume $S_{g,(c_i)}^*\neq\emptyset$. Let $x'\in S_{g,(c_i)}^*$. First we prove that for all $c'\in\mathscr{B}(e_{g-1}(x'),v_{g-1}(x'))$, we have

$$\{x\in S_{g,(c_i)}^*\colon b_g(x)=c'\}=\{x\in J(c_1,\dots,c_{g-1},c')\mid b_{g+j}(x)<Q_j(V),\ j\geqslant 1\}. \tag{3.24}$$

Any x in the left-hand side satisfies $g\in G_V^*(x)$, and therefore $b_{g+j}(x)<Q_j(V)$. The inclusion $\subseteq$ follows trivially. Consider then a number x in the right-hand side. The condition $b_g(x)=c'$ is evident, and in order to prove the inclusion $\supseteq$, there remains to show $x\in S_{g,(c_i)}^*$. Since $x\in J(c_1,\dots,c_{g-1},c')$, it suffices to prove that $g\in G_V^*(x)$ and $\forall j\geqslant 1$, $g-2j\notin G_V^*(x)$. The condition $g\in G_V^*(x)$ holds since $c'\in\mathscr{B}(e_{g-1}(x'),v_{g-1}(x'))$ by hypothesis and $(e_{g-1}(x),v_{g-1}(x))=(v_{g-1}(x'),e_{g-1}(x'))$.

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Next, let $j \geqslant 1$. We wish to show that $g-2j \notin G_V^*(x)$. We assume that $g-2j \in [V, 2V]$; it suffices to prove that for such j, we have $b_{g-2j}(x) \notin \mathscr{B}(e_{g-2j-1}(x), v_{g-2j-1}(x))$ or that $b_{g-2j+\ell}(x) \geqslant Q_\ell(V)$ for some $\ell \geqslant 1$. We first prove that

$$b_{g-2j}(x') \notin \mathscr{B}(e_{g-2j-1}(x'), v_{g-2j-1}(x')) \quad \text{or} \quad \exists \ell \in \{1, \dots, 2j\}, b_{g-2j+\ell}(x') \geqslant Q_\ell(V). \tag{3.25}$$

Indeed, from the condition $g-2j \notin G_V^*(x')$, it suffices to prove that $b_{g-2j+\ell}(x') < Q_\ell(V)$, if $\ell > 2j$, but this is immediate since the condition $g \in G_V^*(x')$ implies $b_{g-2j+\ell}(x') < Q_{\ell-2j}(V) \leqslant Q_\ell(V)$. Now we argue that the conditions (3.25) hold with x' replaced by x. Since x, x' share the same first $g-1$ CF coefficients, this is trivial except for the condition at $\ell = 2j$, which involves b_g. It suffices to check that $b_g(x) \geqslant Q_{2j}(V) \iff b_g(x') \geqslant Q_{2j}(V)$. By construction, we have $Q_{2j}(V) \notin [\sqrt{V}, K\sqrt{V}]$, so both conditions are in fact equivalent to $Q_{2j}(V) \leqslant \sqrt{V}$. This shows that the conditions (3.25) hold with x' replaced by x, and therefore $g-2j \notin G_V^*(x)$. This concludes the proof of the inclusion $\supseteq$ in (3.24).

From there, the rest of the proof follows closely that of Lemma 3.10. By (3.12) and since $Q_j(V) \geqslant j^{1+\delta}\psi(V)$, we deduce that

$$\begin{aligned}\nu(\{x \in S^*_{g,(c_i)} : b_g(x) = c'\}) &= \mu(c_1, \dots, c_{g-1}, c') \prod_{j \geqslant 1} \left(1 + O\left(\frac{1}{j^{1+\delta}\psi(V)}\right)\right) \\ &= \mu(c_1, \dots, c_{g-1}, c')(1 + o(1))\end{aligned}$$

as $V \to \infty$. We conclude as for Lemma 3.10. □

The following technical lemma allows us to extract values of h which will give a dominant contribution to $f^\triangleright$.

Lemma 3.14. *Let $R > 1, C \geqslant 1$ and $\delta \in (0,1)$. Let $h : [-1,1] \smallsetminus \{0\} \to \mathbb{R}$ be such that $h(x) \ll e^{|x|^{-1+\delta}}$ for $x \neq 0$, and assume $|h(x)| \to \infty$ as $x \to 0^\pm$ for a choice of $\pm$. Finally, let $\xi : (0,1] \to (0,1]$ with $\lim_{x \to 0^+} \xi(x) = 0$. Then there exist $\nu > 0$ and $\psi : [1, \infty) \to \mathbb{R}_{>0}$ with $\lim_{+\infty} \psi = +\infty$ such that*

$$\sup_{(Cj^{-1-\delta}\psi(1/z))^{-1} < |x| < 1} \frac{|h(x)|}{R^j} < \inf_{0 < \pm y \leqslant \xi(z)} \frac{|h(y)|}{\psi(1/z)} \tag{3.26}$$

for all $j \geqslant 1$ and all $z \in (0, \nu]$.

Proof. Assume $|h(x)| \to \infty$ as $x \to 0^+$; the complementary case follows by changing $h(x)$ to $h(-x)$. We fix a function $\psi_1 : (0,1] \to \mathbb{R}_{>0}$ going to ∞ at 0^+. By hypothesis, we have $|h(x)| < \exp(C^{1-\delta} j^{1-\delta^2} \psi_1(z)^{1-\delta})$ for $z \in (0, \nu]$, $(Cj^{1+\delta}\psi_1(z))^{-1} < |x| < 1$ and $\nu > 0$ sufficiently small. Now, for any $z \in (0,1]$ the maximum $\psi_2(z) := \max_{j \in \mathbb{N}}(R^{-j} \exp(C^{1-\delta} j^{1-\delta^2} \psi_1(z)^{1-\delta}))$ exists. Moreover, since $\psi_1(z)$ goes to infinity as $z \to 0^+$, then clearly so does $\psi_2(z)$. Also, by hypothesis we have $\inf_{0<y<\xi(z)} |h(y)| \to \infty$ as $z \to 0^+$. It follows that we can ensure that $\psi_2(z)^2 \leqslant \inf_{0<y\leqslant\xi(z)} |h(y)|$ for $z \in (0, \nu]$ by taking ψ_1 that goes to infinity sufficiently slowly and ν small enough. By construction, for $z \in (0, \nu)$ and all $j \geqslant 1$, we then have

$$\sup_{(Cj^{1+\delta}\psi_1(z))^{-1} < |x| < 1} \frac{|h(x)|}{R^j} < R^{-j} \exp(j^{1-\delta^2} \psi_1(z)^{1-\delta}) \leqslant \psi_2(z) \leqslant \frac{\inf_{0<y\leqslant\xi(z)} |h(y)|}{\psi_2(z)}.$$

It is then sufficient to take $\psi(1/z) := \min(\psi_1(z), \psi_2(z))$. □

We shall now show that if (2*) is satisfied then the CDF of $(\phi \circ f^\triangleright)_*(\nu)$ is continuous. We assume $\lim_{x \to 0^-} |h(x)| \to \infty$, the other case being analogous.

We take $R = \varpi^{\mathrm{Re}(k)/2}$, $C = 2K$, where K was defined at (3.15), and $\xi(w) := w^{-1/2}$, and we apply Lemma 3.14, thus finding a constant $\nu > 0$ and a function $\psi(x)$ with the properties claimed in this lemma. We then apply Lemma 3.12 with the same function ψ. As in the bounded case, for

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any $g \in [V, 2V] \cap 2\mathbb{Z}$ and $c_1, \dots, c_{g-1} \geqslant 1$, the sets $(S^*_{g,(c_i)})$ are disjoint and, given any fixed $\varepsilon > 0$, their union has measure $\geqslant 1 - \varepsilon$, if V is large enough in terms of ε.

For any $x \in S^*_{g,(c_i)}$ and any $j \geqslant 1$, we have

$$\frac{v_{g+j-1}}{v_{g+j}} \geqslant \frac{1}{2b_{g+j}} \geqslant \frac{1}{2Q_j(V)} \geqslant \frac{1}{2K\psi(V)j^{1+\delta}},$$

where we dropped the dependency on x in the inequalities for ease of notation. Thus, assuming $V > 1/\nu$, by (3.26) (with $z = 1/V$) we have

$$\frac{|h((-1)^{j-1}v_{g+j-1}/v_{g+j})|}{\varpi^{\mathrm{Re}(k)j}} < \inf_{0<z\leqslant 1/\sqrt{V}} \frac{|h(-z)|}{\varpi^{\mathrm{Re}(k)j/2}\psi(V)} \leqslant \frac{|h(-v_{g-1}/v_g)|}{\varpi^{\mathrm{Re}(k)j/2}\psi(V)},$$

since $v_{g-1}/v_g \leqslant 1/b_g \leqslant 1/\sqrt{V}$. With $F_g(x)$ as in (3.20) we then deduce the following analogue of (3.21),

$$\begin{aligned} F_g(x) &= \frac{\vartheta^p h(-v_{g-1}(x)/v_g(x))}{v_g(x)^k} + O\Big(\sum_{j\geqslant 1} \frac{|h((-1)^{j-1}v_{g+j-1}(x)/v_{g+j}(x))}{v_g(x)^{\mathrm{Re}(k)}\varpi^{\mathrm{Re}(k)j}}\Big) \\ &= \frac{\vartheta^p + o(1)}{v_g(x)^k} h(-v_{g-1}(x)/v_g(x)). \end{aligned}$$

We let $x, y \in S^*_{g,(c_i)}$ and write

$$u_g = \frac{v_g(x)}{v_g(y)} = \frac{b_g(x)v_{g-1} + v_{g-2}}{b_g(y)v_{g-1} + v_{g-2}}$$

and $\omega_g := v_{g-1}/v_g(x)$, so that we have

$$F_g(x) - F_g(y) = \vartheta^p \frac{|h(-\omega_g)|}{v_g(x)^k}\Big(\frac{h(-\omega_g) - u_g^k h(-u_g\omega_g)}{|h(-\omega_g)|} + o(1)\Big).$$

By the definition of $\mathscr{B}$ we have

$$v_g(x)^{-i\,\mathrm{Im}(k)} = e(\alpha_g)(1 + o(1)),$$

with $\alpha_g = \alpha_g(x) = 0$ if $\mathrm{Im}(k) = 0$ and otherwise $\alpha_g(x)$ satisfying $|\alpha_g(x) - \beta| \leqslant \xi$. Moreover, $\log u_g = \log(b_g(x)/b_g(y)) + o(1) \ll_k \xi$ and, if $|b_g(x) - b_g(y)| > \varepsilon\sqrt{V}$, then also $|\log u_g| \geqslant \log(1 + \varepsilon/K) + o(1) \gg_k \varepsilon$ for V large enough. Taking $\xi = \varepsilon^{2/3}$ we then have

$$\begin{aligned} \phi(F_g(x) - F_g(y)) &= \frac{|h(-\omega_g)|}{|v_g(x)|^k}\phi\Big(\vartheta^p e(\alpha_g)\Big(\frac{h(-\omega_g) - u_g^k h(-u_g\omega_g)}{|h(-\omega_g)|}\Big) + o(1)\Big) \\ &\neq 0 \end{aligned} \tag{3.27}$$

by hypothesis (2*), provided that $|b_g(x) - b_g(y)| > \varepsilon\sqrt{V}$. We then conclude in the same way as we did in § 3.3.1.

Remark 3.15. From the above proof, it is clear that the term $o(1)$ in (3.27) can be replaced by $O(1/\sqrt{V} + 1/\psi(V))$. This will be used in one special case of our applications to the cotangent sums.

Remark 3.16. From the above proof, it is also immediate to see that we can modify the condition (2*), $k \notin \mathbb{R}$, of Theorem 3.5 as follows. Given $\beta \in \mathbb{R}$ and any small $\varepsilon > 0$, we denote by $\mathcal{R} = \mathcal{R}_{k,\varepsilon,\beta}$ the set or reduced rationals $x = p/q$ with $|q^{-i\,\mathrm{Im}(k)}e(-\beta) - 1| \leqslant 10\varepsilon^{2/3}$. Then, we can weaken the condition $\lim_{x\to 0^\pm} |h(x)| = \infty$ in (2*) by introducing the restriction $x \in \mathcal{R}$ in the limit[6]

[6]This amounts to observing that Lemma 3.14 also holds under this weaker hypothesis, provided that one adds the condition $y \in \mathcal{R}$ in the inf on the right-hand side of (3.26).

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and replace (3.7) by

$$\liminf_{\substack{x\to 0^{\pm}\\ x\in\mathcal{R}}} \inf\left\{\left|\phi\left(e(\alpha)\frac{h(x)-u^k h(ux)}{|h(x)|}\right)\right| : \begin{array}{l} |\log u|\in(\varepsilon,\varepsilon^{2/3}), |\alpha-\beta|<\varepsilon^{2/3}\\ ux\in\mathcal{R}, \mathrm{Num}(ux)=\mathrm{Num}(x)\end{array}\right\}>0.$$

4. Arithmetic applications

4.1 Eichler integrals of holomorphic cusp forms

Proof of Corollary 1.10. The map $f(x)=\widetilde{g}(x)$ satisfies the relation (1.2) with weight $2-k$ and with h being a non-zero polynomial of degree at most $k-2$, see [Eic57, p. 273]. Since h is Lipschitz and bounded on $[-1,1]$, the estimates (1.5) hold trivially. We may therefore apply Theorem 1.3, which gives the claimed convergence in distribution. Notice that in this case $\widetilde{g}^{\triangleleft}$ coincides on the whole real line with $\widetilde{g}$ as defined in (1.10).

Finally, assume $(\phi\circ\widetilde{g}^{\triangleleft})_*(\nu)$ has an atom for some non-zero linear form ϕ and write $\phi(z)=\mathrm{Re}(e^{i\theta}z)$ for some $\theta\in\mathbb{R}$ and all $z\in\mathbb{C}$. Then, by Proposition 3.4, we obtain that $\phi\circ\widetilde{g}^{\triangleleft}$ is constant on $[0,1]$. By (1.10), $\phi\circ\widetilde{g}^{\triangleleft}$ is given by the Fourier series

$$x\mapsto\sum_{n\geqslant 1}\frac{\mathrm{Re}(e^{i\theta}a_n)}{n^{k-1}}\cos(nx)-\sum_{n\geqslant 1}\frac{\mathrm{Im}(e^{i\theta}a_n)}{n^{k-1}}\sin(nx),$$

which is constant on $[0,1]$ only if $\mathrm{Re}(e^{i\theta}a_n)=\mathrm{Im}(e^{i\theta}a_n)=0$, i.e. $a_n=0$, for all $n\in\mathbb{N}_{>0}$. Thus $g=0$ which was excluded by hypothesis. □

Proof of Corollary 1.9. The map $A_{k,D}$ defined in (1.9) coincides with $F_{k+1,D}$ defined in [Zag99, Equation (15)]. Consider the even cusp form $g=f_{k+1,D}\in S_{2k+2}(1)$ defined in [Zag99, Equation (53)]. By [Zag99, Equation (55)], we have $A_{k,D}(x)=c_{k,D}+\frac{1}{2}\widetilde{g}(x)$ for all $x\in\mathbb{Q}$ and some $c_{k,D}\in\mathbb{R}$. Therefore, Corollary 1.9 follows from the just proved Corollary 1.10. □

4.2 Kontsevich's function

Proof of Corollary 1.12. Define h by formula (1.13). In [Zag01, Theorem, p. 958],[7] it is proved that $h\in C^\infty(\mathbb{R})$ and that h is real-analytic except at 0, and moreover $h(0)=1$. Thus, φ and h satisfy the hypotheses of the first part of Theorem 1.7, with the generalized hypotheses (2.26) (with $\vartheta=e(1/24)$) which were adopted in the proof. This proves the existence of the limiting distribution. The limit (1.14) arises from the relations (2.29) and (2.32).

Let $\theta\in\mathbb{R}$ and $\phi: z\mapsto\mathrm{Re}(e^{i\theta}z)$. To see that $(\phi\circ\varphi^{\triangleright})_*(\nu)$ does not have atoms, we apply Theorem 3.5. The period function h is continuous on $[-1,1]$, and $h(0)=1$. Since $\vartheta\notin\mathbb{R}$, certainly one of $\phi(1)$ or $\phi(\vartheta)$ is non-zero, and therefore hypothesis (1^*) is satisfied for some $p\in\{0,1\}$. Theorem 3.5 applies and yields the desired conclusion.

Finally, the continuity of $\varphi^{\triangleright}$ follows immediately from (the twisted version of) Theorem 1.4. □

4.3 Cotangent sums

For $b\in\mathbb{Z}$, $q\geqslant 1$ and $(b,q)=1$, the cotangent sums we are interested in are defined by

$$c_a\left(\frac{b}{q}\right):=q^a\sum_{m=1}^{q-1}\cot\left(\frac{\pi mb}{q}\right)\zeta\left(-a,\frac{m}{q}\right),$$

see [BC13a, p. 226]. The special case $a=-1$ corresponds to the classical Dedekind sums [Rie92, p. 466], while the case $a=0$ corresponds to the cotangent sum from [BC13b]. These sums were

[7] The factors $\zeta_8^{\pm 3}$ there should be read $\zeta_8^{\pm 1}$.

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described as 'imperfect' quantum modular forms, due to an irregular term arising in the period relation, namely the last term on the right-hand side of [BC13a, Equation (17)]; see the discussion in Example 0 in [Zag10]. The point of the upcoming definition is that we can relax this lack of regularity at the cost of weakening the periodicity hypothesis to (1.4). Let

$$\rho\Big(\frac{b}{q}\Big)=\begin{cases}\Big\{\frac{\bar b}{q}\Big\} & (q>1,(b,q)=1),\\ 1 & (q=1,b>0),\\ 0 & (q=1,b<0).\end{cases}$$

By Bezout's theorem, we have for $b\neq 0$, $q\geqslant 1$, $(b,q)=1$,

$$\rho\Big(\frac{b}{q}\Big)-\rho\Big(\frac{-q}{b}\Big)=\frac{1}{bq}.$$

For $x\in\mathbb{Q}$, let

$$\widetilde{c}_a(x):=c_a(x)+a\kappa_1(a)\,\mathrm{Den}(x)^{1+a}\rho(x)\quad(x\neq 0),$$

extended arbitrarily at 0, and with $\kappa_1(a):=(\zeta(1-a))/\pi$. Note that c_a is 1-periodic, and that ρ satisfies the weak periodicity (1.4), from which we deduce that $\widetilde{c}_a$ also satisfies (1.4).

From [BC13a, Theorem 4], we have

$$c_a(x)-|x|^{-1-a}c_a(-1/x)=h_a(x)-a\kappa_1(a)\frac{\mathrm{Den}(x)^a}{\mathrm{Num}(x)}\quad(x\in\mathbb{Q}\smallsetminus\{0\}),\tag{4.1}$$

and thus

$$\widetilde{c}_a(x)-|x|^{-1-a}\widetilde{c}_a(-1/x)=h_a(x)\quad(x\in\mathbb{Q}\smallsetminus\{0\}),\tag{4.2}$$

for $h_a(x):=-\mathrm{sgn}(x)i\zeta(-a)\psi_a(|x|)$ with $\psi_a(x)$ as in [BC13a, Theorem 4]. In particular, h_a is real analytic on $\mathbb{R}_{\neq 0}$. Also, by the same theorem, for $a\neq 0$, we have[8]

$$h_a(x)=\kappa_2(a)\mathrm{sgn}(x)|x|^{-1-a}+\kappa_1(a)x^{-1}+O_a(|x|)\tag{4.3}$$

as $x\to 0$, for $\kappa_2(a)=-\zeta(-a)\cot(\pi a/2)$, and where the error has to be replaced by $O(|x|\log|1/x|)$ if $a=-2$. For $a=0$, we instead have

$$h_0(x)=-\frac{\log(2\pi|x|)-\gamma}{\pi x}+O(|x|).\tag{4.4}$$

We prove the distributional statements in Corollary 1.14, considering several cases depending on the value of a. The statement in Corollary 1.14 about the continuity of $c_a^{\triangleright}$ when $\mathrm{Re}(a)>0$ will follow immediately from Theorem 1.4 and the behavior around 0 of the period functions.

4.3.1 *The case* $\mathrm{Re}(a)<-1$. By [BC13a, Theorem 1], it follows that both h_a and h_a' are bounded by $|x|^{O(1)}$ for all $x\neq 0$. Thus, the conditions (1.5) are satisfied, and by Theorem 1.2, the function

$$\widetilde{c}_a^{\triangleleft}(x)=\lim_{j\to\infty}\widetilde{c}_a([b_0;b_1,\dots,b_j])\quad(x=[b_0;b_1,b_2,\dots])\tag{4.5}$$

exists for x in a full measure set $X\subset\mathbb{R}$. By Theorem 1.3, we deduce that the multisets

$$\{\widetilde{c}_a(b/q),\ 0<b<q,(b,q)=1\}$$

become equidistributed, as $q\to\infty$, according to $(\widetilde{c}_a^{\triangleleft})_*(\nu)$. Since

$$\widetilde{c}_a(x)-c_a(x)=O(\mathrm{Den}(x)^{1+\mathrm{Re}(a)})=o_a(1)\tag{4.6}$$

[8]For $a\in\mathbb{Z}$ the result is obtained by continuity, cf. the explanation after Theorem 1 in [BC13a].

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as $\mathrm{Den}(x)\to\infty$, uniformly in the numerator $\mathrm{Num}(x)$, it follows that the same conclusion holds for the multisets

$$\left\{c_a\left(\frac{b}{q}\right),\ 0<b<q,(b,q)=1\right\}.$$

Next we prove that if $a\in\mathbb{R}_{<-1}$, then $(\widetilde{c}_a^{\lhd})_*(\nu)$ is diffuse on $\mathbb{R}$. Assume, for the sake of contradiction, the existence of $\lambda\in\mathbb{R}$ and $C\subset X\cap\mathbb{R}_{>0}$ of positive Lebesgue measure such that $\widetilde{c}_a^{\lhd}(x)=\lambda$ for all $x\in C$. By Proposition 3.4, we deduce that $\widetilde{c}_a^{\lhd}(x)=\lambda$ for almost all $x>0$. Using the fact that c_a is odd, which transfers to $\widetilde{c}_a^{\lhd}$ almost everywhere by (4.6), we obtain by the period relation (4.2) that $h_a(x)=\lambda(1+|x|^{-1-a})$ for almost all $x>0$. However, this contradicts (4.3) as $x\to 0^+$, regardless of the value of a.

Now, let $a\notin\mathbb{R}$ and let $\phi:\mathbb{C}\to\mathbb{R}$ a non-zero linear form. We assume by contradiction that $(\phi\circ\widetilde{c}_a^{\lhd})_*(\nu)$ is not diffuse. As above, we deduce that there exists $\lambda\in\mathbb{R}$ such that $\phi(\widetilde{c}_a^{\lhd}(x))=\lambda$ for almost all $x>0$. Composing the period relation (4.2) with ϕ, we obtain $\phi(h_a(x))=\lambda-\phi(|x|^{-1-a}\widetilde{c}_a^{\lhd}(-1/x))$. We pick $y\in(0,1)$ so that $\widetilde{c}_a^{\lhd}(-y)$ and $\widetilde{c}_a^{\lhd}(1/(n+y))$ are defined in (4.5) for all $n\in\mathbb{N}_{>0}$, and $\phi(\widetilde{c}_a^{\lhd}(1/(n+y)))=\lambda$. Taking $x=1/(n+y)$, by periodicity we then obtain

$$\phi(h_a(x))=\lambda-\phi(|n+y|^{1+a}\widetilde{c}_a^{\lhd}(-y)). \tag{4.7}$$

Now, by [BC13a, Theorem 1],[9] the asymptotic in (4.3) can be extended to

$$h_a(x)=\frac{\kappa_2(a)\mathrm{sgn}(x)}{|x|^{1+a}}+\frac{\kappa_1(a)}{x}+\sum_{m=1}^{M}(-1)^m\frac{2B_{2m}}{(2m)!}\zeta(1-2m-a)(2\pi x)^{2m-1}+O_{a,M}(|x|^{2M+1}), \tag{4.8}$$

for any $M\in\mathbb{N}$, where $B_{2n}\in\mathbb{Q}_{\neq 0}$ denotes the $2n$th Bernoulli number. By the functional equation we have

$$\frac{\zeta(1-2m-a)}{\zeta(1-2(m+1)-a)}=-\frac{2\pi}{(2m+a)(2m+a+1)}\frac{\zeta(2m+a)}{\zeta(2m+2+a)}=\frac{2\pi}{4m^2}\left(2-\frac{1+2a}{m}+O(m^{-2})\right)$$

as $m\to\infty$. In particular, $\zeta(1-2m-a)$ cannot be a real multiple of $\zeta(1-2(m+1)-a)$ for m large enough and thus at least one of these terms in the expansion (4.8) survives once composed with ϕ. Letting $n\to\infty$, we see that this is not compatible with (4.7) and thus we reach a contradiction. This finishes the proof of Corollary 1.14 when $\mathrm{Re}(a)<-1$.

4.3.2 *The case* $\mathrm{Re}(a)>-1$ *and* $a\zeta(a)\neq 0$. It is remarked in [BC13a, p. 227] that $c_a(x)\equiv 0$ whenever a is a positive odd integer. In this case, Corollary 1.14 holds for trivial reasons and thus we assume that $a\notin 2\mathbb{Z}+1$ throughout the section. We also assume $\mathrm{Re}(a)>-1$ and $a\zeta(a)\neq 0$ and notice that by the functional equation we have $\kappa_1(a)\neq 0$.

Since $h_a(x)=O(|x|^{-\max(1,1+\mathrm{Re}(a))})$ for $|x|<1$, the hypotheses of Theorem 1.7 are satisfied and it follows that the limit

$$\widetilde{c}_a^{\rhd}(x):=\lim_{j\to\infty}\mathrm{Den}(x_{2j+1})^{-1-a}\widetilde{c}_a(\overline{x_{2j+1}})\quad(x_{2j+1}=[b_0(x);b_1(x),\dots,b_{2j+1}(x)]) \tag{4.9}$$

exists for x in a full measure set $X\subset\mathbb{R}\smallsetminus\mathbb{Q}$. By Theorem 1.7, we have that the multiset

$$\{q^{-1-a}\widetilde{c}_a(b/q):\,0<b<q,(b,q)=1\}=\{q^{-1-a}\widetilde{c}_a(\overline{b}/q):\,0<b<q,(b,q)=1\}$$

becomes distributed, as $q\to\infty$, according to $(\widetilde{c}_a^{\rhd})_*(\nu)$. But by definition of $\widetilde{c}_a$, we have $q^{-1-a}\widetilde{c}_a(\overline{b}/q)=q^{-1-a}c_a(\overline{b}/q)+a\kappa_1(a)\{b/q\}$ for $q>1$. Letting

$$c_a^{\rhd}(x):=\widetilde{c}_a^{\rhd}(x)-a\kappa_1(a)\{x\}, \tag{4.10}$$

[9] We remark that there is a typo in [BC13a, Equation (5)], as the minus sign in front of the sum should be removed.

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it follows that the multiset

$$\{q^{-1-a}c_a(b/q)\colon 0<b<q,(b,q)=1\}$$

becomes distributed according to $(c_a^{\triangleright})_*(\nu)$, as claimed.

We now turn to showing that the relevant measures are diffuse.

It is convenient to make a further simple modification to c_a and define $\breve{c}_a(x):=\widetilde{c}_a(x)-\kappa_2(a)\mathrm{sgn}(x)$. Clearly, one has that $\breve{c}_a(x)$ still satisfies (1.4), whereas (4.2) holds with $h_a(x)$ replaced by $\breve{h}_a(x)=h_a(x)-\kappa_1(a)(1+|x|^{-1-a})$. In particular, (4.3) becomes

$$\breve{h}_a(x)=-\kappa_2(a)\mathrm{sgn}(x)+\kappa_1(a)x^{-1}+O_a(x),\quad x\to 0. \tag{4.11}$$

The hypothesis of Theorem 1.7 is thus still satisfied. Taking the limit as in (4.9), one deduces that $\breve{c}_a^{\triangleright}(x)=\widetilde{c}_a^{\triangleright}(x)$ for $x\in X$. Also, by (4.11) for $\varepsilon<|\log u|<\varepsilon^{2/3}$, we have

$$\frac{\breve{h}_a(x)-u^{1+a}\breve{h}_a(ux)}{|\breve{h}_a(x)|}=\frac{\kappa_2(a)(1-u^{1+a})+\kappa_1(a)(1-u^a)x^{-1}+o(1)}{\kappa_2(a)+\kappa_1(a)x^{-1}+o(1)}=1-u^a+o(1) \tag{4.12}$$

as $x\to 0^+$, uniformly in sufficiently small $\varepsilon>0$. Notice that $1-u^a=-a\log u+O_a(\varepsilon^{4/3})\gg\varepsilon$. When $a\in\mathbb{R}$, then clearly $c_a(x)\in\mathbb{R}$ for all $x\in\mathbb{Q}$, and therefore $h_a(x)\in\mathbb{R}$ for $x\neq 0$ as well. By (3.6) one can then deduce that $(\breve{c}_a^{\triangleright})_*(\nu)$ is diffuse, and thus so is $(\widetilde{c}_a^{\triangleright})_*(\nu)$. If $a\notin\mathbb{R}$ one can show in the same way that $(\phi(\widetilde{c}_a^{\triangleright}))_*(\nu)$ is diffuse for any non-zero linear form $\phi\colon\mathbb{C}\to\mathbb{R}$, upon choosing $\beta:=-\arg(-a)-\kappa$ in (3.7), where κ is such that $\phi(z)=\mathrm{Re}(e^{i\kappa}z)$ for all $z\in\mathbb{C}$.

We now wish to prove that the measure $(\phi_a(c_a^{\triangleright}))_*(\nu)$ is also diffuse (with $\phi=\mathrm{Re}$ if $a\in\mathbb{R}$). Notice that we can assume $\phi(a\kappa_1(a))\neq 0$ since otherwise $\phi\circ c_a^{\triangleright}=\phi\circ\widetilde{c}_a^{\triangleright}$.

We start with the case $\mathrm{Re}(a)>1$. Suppose by contradiction that $\phi\circ\widetilde{c}_a^{\triangleright}(x)=\lambda$ for all $x\in C\subseteq(0,1)$ for a set C of positive measure. Since $\mathrm{Re}(a)>1$, by Theorem 1.4 we can assume that $\widetilde{c}_a$ is α-Hölder continuous at any point of C for any $\alpha\in(1,\frac{1}{2}(1+\mathrm{Re}(a)))$. Also, since C is uncountable, it contains one of its accumulation points, i.e. there exists a sequence $(z_m)_m$ in C converging to $z\in C$. Then, on the one hand we have

$$\phi(\widetilde{c}_a(z))-\phi(\widetilde{c}_a(z_m))=\phi(\widetilde{c}_a(z)-\widetilde{c}_a(z_m))=O(|z-z_m|^{\alpha})=o(|z-z_m|)$$

as $m\to\infty$, and on the other hand by (4.10) we have

$$\phi(\widetilde{c}_a^{\triangleright}(z))-\phi(\widetilde{c}_a^{\triangleright}(z_m))=\phi(c_a^{\triangleright}(z))-\phi(c_a^{\triangleright}(z_m))-\phi(a\kappa_1(a))(z-z_m)=-\phi(a\kappa_1(a))(z-z_m).$$

Since these equations are not compatible we reach the desired contradiction.

Now, assume $\mathrm{Re}(a)\in(-1,1]$. First we notice that in this case we can proceed directly with $\widehat{c}_a(x)=c_a(x)-\kappa_2(a)\mathrm{sgn}(x)$. Indeed, if we let $\widehat{h}_a(x):=\breve{h}_a(x)-a\kappa_1(a)(\mathrm{Den}(x)^a/\mathrm{Num}(x))\mathbf{1}_{\mathbb{Q}}(x)$, where $\mathbf{1}_{\mathbb{Q}}$ is the indicator function of the rationals, we have that $\widehat{h}_a(x)$ still satisfies the hypothesis of Theorem 1.7. Clearly, the function $c_a^{\triangleright}(x)$ obtained this way coincides with the $c_a^{\triangleright}(x)$ defined above for almost all x. Also, if $\mathrm{Re}(a)<1$ we have that $\widehat{h}_a(x)\sim\breve{h}_a(x)$ as $x\to 0$, and thus we can show as in (4.12) that $(\widehat{h}(x)-u^{1+a}\widehat{h}(ux))/|\widehat{h}(x)|=1-u^a+o(1)$ as $x\to 0$. The same argument then gives that $(\phi_a(c_a^{\triangleright}))_*(\nu)$ is diffuse.

Finally, we assume $\mathrm{Re}(a)=1$ (we recall $a\neq 1$). We apply Lemma 3.14 in the form of Remark 3.16. Using the notation of the remark, we have $\widehat{h}_a(x)=a\kappa_1(a((\mathrm{Den}(x)-\mathrm{Den}(x)^a)/\mathrm{Num}(x))+O(1)\asymp 1/x\to\infty$ with $x\in\mathcal{R}$, for $\beta\notin\mathbb{Z}$ and ε sufficiently small. Then, for $x,ux\in\mathcal{R}$, $x=p/q$, $\mathrm{Num}(ux)=p$, $\varepsilon<|\log u|<\varepsilon^{2/3}$, as $x\to 0^+$ we

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have

$$\begin{aligned}\frac{\widehat{h}_a(x)-u^{1+a}\widehat{h}_a(ux)}{|\widehat{h}_a(x)|} &= a\kappa_1(a)\frac{(1-u^a)q/p-(1-u)q^a/p}{|\breve{h}_a(x)|}+o(1)\\ &= a\kappa_1(a)\frac{(q^{a-1}-a)\log u+O(\varepsilon^{4/3})}{|\breve{h}_a(x)|x}+o(1)\\ &= a\kappa_1(a)\frac{(e(\beta)-a)\log u+O(\varepsilon^{4/3})}{|\breve{h}_a(x)|x}+o(1),\end{aligned}$$

from which we can once again conclude that $(\phi_a(c_a^{\triangleright}))_*(\nu)$ is diffuse by choosing β appropriately.

4.3.3 *The case of* $\zeta(a)=0$. We assume $\zeta(a)=0$, so that $\kappa_1(a)=0$, $\mathrm{Re}(a)\in(0,1)$ and $\mathrm{Im}(a)\neq 0$. In particular, with the same notation as in the previous section, we have that $\widehat{h}_a(x)=-\kappa_2(a)\mathrm{sgn}(x)+o(1)$ as $x\to 0$. Thus, $\widehat{h}_a(x)$ is continuous on $[-1,1]\smallsetminus\{0\}$ with non-zero right and left limits $\widehat{h}_a(0^{\pm})=\mp\kappa_2(a)$ at $x=0$.[10] Since $\kappa_2(a)\neq 0$ and the weight $1+a\notin\mathbb{R}$, we immediately deduce from the condition (1^*) in Theorem 3.5 that $(\phi(c_a^{\triangleright}))_*(\nu)$ is diffuse for any non-zero linear form ϕ.

4.3.4 *The case of* $a=0$. We let $\widehat{h}_0(x):=h_0(x)+1/(\pi\mathrm{Num}(x))\mathbf{1}_{\mathbb{Q}}(x)$. Using (4.4) instead of (4.3), we obtain

$$\frac{\widehat{h}_0(x)-u\widehat{h}_0(ux)}{|\widehat{h}_0(x)|}=-\frac{u\log u+O(|x|)}{\log(2\pi|x|)+\gamma},$$

which is not quite sufficient for hypothesis (3.6) to hold. However, we observe that for $h=\widehat{h}_0$ (and $\delta=1/2$, $C=4$, $R=\varpi^{1/2}$, and $\xi(w):=w^{-1/2}$), the conclusion of Lemma 3.14 holds with $\psi(x)=x^{-1/5}$. In view of this, and following Remark 3.15, we obtain

$$\frac{\widehat{h}_0(x)-u\widehat{h}_0(ux)}{|\widehat{h}_0(x)|}+O(|x|^{-1/5})=-\frac{u\log u+O(|x|)}{\log(2\pi|x|)+\gamma}+O(|x|^{-1/5}),$$

which is clearly non-zero for $x>0$ small enough and $\varepsilon<|\log u|<\varepsilon^{2/3}$. Substituting this estimate inside (3.27) and completing the arguments in § 3.3.2, we conclude that $(c_0^{\triangleright})_*(\nu)$ is diffuse.

Remark 4.1. We have $c_0^{\triangleright}(x)=2\pi^{-2}D(x)$, where $D(x)=\sum_{n\geqslant 1}d(n)\sin(2\pi nx)/n$ whenever the series converges (see e.g. [Bet15]). Using [BM19, Equation (40)], we deduce that $W(x)=\pi c_0^{\triangleright}(x)+2G(x)$, where W is the Wilton function defined in [LMPS24], and where G has the properties stated in [BM19, Section 5.2]. In particular, G is bounded and continuous at irrationals. This explains the similarity between the graph of $c_0^{\triangleright}$ in Figure 4e and the graph of the Wilton function as depicted on [LMPS24, p. 19]. We thank the referee for remarking on this similarity.

Remark 4.2.

(i) For $a<-1$, we note that the function $\widetilde{c}_a^{\triangleleft}$ is not in $L^1([0,1])$, and therefore not in L^p for every $p\geqslant 1$. Indeed, otherwise, using (4.6), we would have

$$\int_{[0,1]}|h_a|\ll 1+\int_{[0,1]}(|\widetilde{c}_a(x)|+|\widetilde{c}_a(T(x))|)\,dx<\infty,$$

whereas by (4.3) we have $h_a(x)\sim\kappa_1(a)/x$ as $x\to 0$ and thus $\int_{[0,1]}|h_a|=\infty$.

(ii) For $a\in(-1,0)$, we note that as $q\to\infty$, the random variable $q^{-1-a}c_a(b/q)$, where b is taken at random in $\{1,\dots,q\}$ and coprime with q, does not have moments of order larger than

[10] In fact, one has that $\widehat{h}_a(x)+\kappa_2(a)\mathrm{sgn}(x)$ extends to a function in $\mathcal{C}^\infty(\mathbb{R})$.

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$1/|a|$: indeed, from (4.1) and (4.3), the sole contribution of $b=1$ is $q^{-1-a}c_a(1/q)=\kappa_2(a)+\kappa_1(a)q^{-a}+O(1/q)$, which diverges as $q\to\infty$. Similarly, when $a\leqslant -1$, the random variable $c_a(b/q)$ does not have moments of order larger than 1, as then $c_a(1/q)\sim\kappa_1(a)q$.

4.4 Eichler integrals of Maaß cusp forms

Proof of Corollary 1.11. For simplicity, we assume u is even or odd and let $j\in\{0,1\}$ be such that $u(-\overline{z})=(-1)^j u(z)$. The case where u is neither is believed not to be possible, but it could also be handled easily by splitting u into its even or odd components. In [Bru07], it is shown that the map $\widetilde{u}$ defined in (1.11) is a quantum modular form of weight $2s$, whose associated period function h is defined by[11]

$$h(x)=c(s)\mathrm{sgn}(x)^{1-j}\psi(|x|)\quad (x\in\mathbb{R}_{\neq 0}),$$

and by continuity at $x=0$, where $c(s)=i\pi^{-s}/\Gamma(1-s)$ is a non-zero proportionality constant [LZ01, Equation (1.12)]. The map h is real-analytic separately on $\mathbb{R}_{\leqslant 0}$ and $\mathbb{R}_{\geqslant 0}$ and is $\mathcal{C}^\infty$ on $\mathbb{R}$. Thus, Theorem 1.4 applies and yields the first part of Corollary 1.11. The expansion of ψ at 0 is implicit in [LZ01] and can be immediately deduced by shifting the line of integration to the left in the first display of [LZ01, p. 205], from which one deduces that $h(x)$ is not identically zero on $[-1,1]$. Thus, since the weight is $2s\notin\mathbb{R}$, Theorem 3.5 applies in the case (1^*), and we deduce that $(\phi\circ\widetilde{u}^{\triangleright})_*(\nu)$ is diffuse. This concludes the proof of Corollary 1.11. □

Remark 4.3. By the functional equation for twists of Maaß-form L-functions, [KMV02, (A.12) and (A.13)], we have

$$\widetilde{u}(\overline{a}/q)=q^{2s}\sum_{n\geqslant 1}\frac{a_n}{n^{1/2+s}}(c_1\cos(2\pi na/q)+c_2\sin(2\pi na/q))\quad (a\overline{a}\equiv 1\pmod q),$$

where c_1,c_2 are numbers depending on u. With this formulation, the regularity of $a/q\mapsto q^{-2s}\widetilde{u}(\overline{a}/q)$ relates to the differentiability properties for $\widetilde{g}$ which we mentioned in § 1.3.1.

An important classical problem is the analogous question when $a_n=d(n)$ is the divisor function, see [Rie68, Cho31, Win37]. This can be seen as the case when u is replaced by a certain weight- 0, real-analytic, non-cuspidal Eisenstein series [Iwa02, p. 62]. Regularity in this case can be studied from the expression above as a Fourier series; recent general results can be found in [CPRC17].

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Conflicts of interest

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[11] This corrects a typo in [Bru07, § 1.1.3], as the equation for $\widetilde{\psi}$ should be $-|x|^{-2s}\widetilde{\psi}(-1/x)=\widetilde{\psi}(x)$.

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