Math. Proc. Camb. Phil. Soc. (1987), **102**, 173
Printed in Great Britain

A note on Hamiltonian structure

By S. BOWMAN
Department of Mathematics, University of Chicago, Chicago, IL 60637, *U.S.A.*

(*Received* 20 *May* 1986; *revised* 31 *October* 1986)

§1. *Introduction*

The aim of this paper is to give some results on the determination of Hamiltonian structure for a set of evolutionary equations, either differential or non-local. There are several reasons why the recognition of Hamiltonian structure is important. Firstly, as in the case of Lagrangian systems, conservation laws are often connected with symmetry groups of the equations. This point is addressed in §4. Secondly, variational principles for steady solutions can often be found by restricting attention to the steady equations. Connected to the second point, methods have been developed by Arnol'd [**2**] and more recently by Marsden, Weinstein etc. (see, e.g. [**24**]) to study the nonlinear stability of steady solutions, on lines similar to the proof of nonlinear stability for the solitary-wave solution of the Korteweg–de Vries equation by Benjamin [**4**] and Bona [**9**].

Three papers giving conditions for Hamiltonian structure of a system of partial differential equations

$$\mathbf{u}_t = J\mathscr{E}H, \tag{1·1}$$

where $\mathbf{u}$ is an n-component vector, J is skew-symmetric operator, and $\mathscr{E}H$ denotes the Euler derivative (gradient) of the Hamiltonian density $H(\mathbf{u})$, are by Olver [**16**] and Gel'fand and Dorfman [**11**, **12**]. The latter concentrated on the case when H depends on $\mathbf{u}$ and finitely many derivatives with respect to one spatial derivative. The results of Olver apparently applied only to the case when the density H and operator J depend on $\mathbf{u}$ and finitely many derivatives of $\mathbf{u}$ and J^{-1} can be defined. In this case he gave two conditions that imply Hamiltonian structure, which he proved to be equivalent, one involving J^{-1} and one not. This second condition implies the one given by Gel'fand and Dorfman on the Schouten–Nijenhuis bracket of J with itself. The mechanisms developed by Olver [**16**], moreover, give a highly practicable procedure to check the conditions. In fact they have been applied, without apparent justification, for systems where the density H does not depend locally on $\mathbf{u}$, see for example, Olver [**17**] and Benjamin and Olver [**7**]. A major purpose of this paper is therefore to show that these conditions indeed imply Hamiltonian structure. The cases above are covered, and further examples are presented in §6.

The papers of Olver [**16**] and Gel'fand and Dorfman [**11**, **12**] are phrased in the language of differential geometry, which does not seem to be immediately applicable to many non-local problems. In §2 therefore, the definitions of vector fields, Lie brackets, Euler derivatives, skew-symmetric operators and Poisson brackets, that are adopted in this paper are given. A condition for the Poisson bracket to satisfy the Jacobi identity is presented, as is a condition for the Lie and Poisson brackets generally to satisfy the Lie–Poisson relation (2·20). These conditions are related to each

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

other, and then compared to Olver's condition in §3 where the differential geometric language is recalled.

For completeness, §5 considers necessary and sufficient conditions for a vector field $\mathbf{P}(\mathbf{u})$ to be the Euler derivative (gradient) of some density $F(\mathbf{u})$. Such results are clearly important, for determining both variational principles and Hamiltonian structure.

The theory presented here is designed to test given sets of equations for Hamiltonian structure, and cannot suggest how to derive equations ensuring Hamiltonian formulations. Such methods have been developed and applied to various problems in classical mechanics and fluid dynamics, for example by Abraham, Marsden etc. (see [**15**]). One may use the theory described here to check that the formulations so obtained are indeed Hamiltonian.

§2. *Definitions and conditions for Hamiltonian structure*

Let D be a simply connected domain in $\mathbb{R}^n$, X the space of smooth vector-valued functions $\mathbf{u}: D \to \mathbb{R}^m$ and $\mathscr{R}$ the ring of functions $\phi: D \to \mathbb{R}$. The function $\mathbf{u}$ will be the independent variable. A *vector field* $\mathbf{P}$ on X is an m-component vector whose components $P_j\,(j = 1, \ldots, m)$ are C^∞-Gateaux differentiable functions on X. The components of $\mathbf{P}$ may depend *non-locally* on $\mathbf{u}$. (A simple example is afforded by supposing $D = \mathbb{R}$, $m = 1$ and $P(u) = Hu$, where Hu is the Hilbert transform of u.)

Definition 1. The *Lie bracket* of two vector fields $\mathbf{P}$ and $\mathbf{Q}$ is the vector field $[\mathbf{P}, \mathbf{Q}]$ given by

$$[\mathbf{P}, \mathbf{Q}]\,(\mathbf{u}) = \frac{d}{d\epsilon}\{\mathbf{P}(\mathbf{u}+\epsilon\mathbf{Q}(\mathbf{u})) - \mathbf{Q}(\mathbf{u}+\epsilon\mathbf{P}(\mathbf{u}))\}_{\epsilon=0}. \tag{2·1}$$

This definition reduces to the standard definition of the Lie bracket, up to a minus sign, when $\mathbf{P}$ and $\mathbf{Q}$ depend only on $\mathbf{u}$ and not on any spatial derivatives of $\mathbf{u}$, namely,

$$[\mathbf{P}, \mathbf{Q}] = (\mathbf{Q}\,.\,\nabla)\,\mathbf{P} - (\mathbf{P}\,.\,\nabla)\,\mathbf{Q}, \tag{2·2}$$

where ∇ denotes the operator $(\partial/\partial u^1, \ldots, \partial/\partial u^m)^T$ and u^j, $j = 1, \ldots, m$, are the components of $\mathbf{u}$. For the particular case when $\mathbf{P}$ and $\mathbf{Q}$ depend on $\mathbf{u}$ and finitely many spatial derivatives of $\mathbf{u}$, let $v_{\mathbf{P}}$ denote the operator

$$v_{\mathbf{P}} = \sum_{j,J} D^J P_j\, \partial/\partial u^j_J, \tag{2·3}$$

where D^J denotes the total derivative with respect to the spatial variable $\mathbf{x}$ with multi-index J and u^j_J is short for $D^J u^j$. Then

$$[\mathbf{P}, \mathbf{Q}] = v_{\mathbf{Q}}(\mathbf{P}) - v_{\mathbf{P}}(\mathbf{Q}), \tag{2·4}$$

which expression appears in theorem 4·2 of [**16**] and in [**11**] (equation 2·3). The equivalence of (2·1) and (2·4) in this case is clear on appeal to Taylor's theorem. No such neat expression exists, however, in the case of non-local vector fields.

It follows from Definition 1 that the set of vector fields is a Lie algebra. This result is expressed in Theorem 1.

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

THEOREM 1. *For all vector fields* $\mathbf{P}$, $\mathbf{Q}$ *and* $\mathbf{R}$, *the Jacobi identity*

$$[\mathbf{P}, [\mathbf{Q}, \mathbf{R}]]+[\mathbf{Q}, [\mathbf{R}, \mathbf{P}]]+[\mathbf{R}, [\mathbf{P}, \mathbf{Q}]] = 0 \tag{2·5}$$

is satisfied.

It is important to note that the result is valid in the case of non-local vector fields.

Remark. The theorem is well known in the case when $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ depend only on $\mathbf{u}$ and no spatial derivatives of $\mathbf{u}$ (see, for example, [3]). It is included here to illustrate the methods used later. In the following proof, where the argument of a vector field is $\mathbf{u}$ it is suppressed.

Proof of Theorem 1. By Definition 1 we have

$$\begin{aligned}
[\mathbf{P}, [\mathbf{Q}, \mathbf{R}]] &= \frac{d}{d\mu}\mathbf{P}\left(\mathbf{u}+\mu\frac{d}{d\epsilon}\{\mathbf{Q}(\mathbf{u}+\epsilon\mathbf{P})-\mathbf{R}(\mathbf{u}+\epsilon\mathbf{Q})\}_{\epsilon=0}\right)_{\mu=0} \\
&\quad -\frac{d}{d\mu}\left[\frac{d}{d\epsilon}\{\mathbf{Q}(\mathbf{u}+\mu\mathbf{P}+\epsilon\mathbf{R}(\mathbf{u}+\mu\mathbf{P}))-\mathbf{R}(\mathbf{u}+\mu\mathbf{P}+\epsilon\mathbf{Q}(\mathbf{u}+\mu\mathbf{P}))\}\right]_{\epsilon,\mu=0} \\
&= \frac{d}{d\mu}\mathbf{P}\left(\mathbf{u}+\mu\frac{d}{d\epsilon}\mathbf{Q}(\mathbf{u}+\epsilon\mathbf{P})_{\epsilon=0}\right)_{\mu=0} - \frac{d}{d\mu}\mathbf{P}\left(\mathbf{u}+\mu\frac{d}{d\epsilon}\mathbf{R}(\mathbf{u}+\epsilon\mathbf{Q})_{\epsilon=0}\right)_{\mu=0} \\
&\quad -\frac{\partial^2}{\partial\epsilon\,\partial\mu}\mathbf{Q}(\mathbf{u}+\mu\mathbf{P}+\epsilon\mathbf{R})_{\epsilon,\mu=0}+\frac{\partial^2}{\partial\epsilon\,\partial\mu}\mathbf{R}(\mathbf{u}+\mu\mathbf{P}+\epsilon\mathbf{Q})_{\epsilon,\mu=0} \\
&\quad +\frac{d}{d\lambda}\mathbf{R}\left(\mathbf{u}+\lambda\frac{d}{d\lambda}\mathbf{Q}(\mathbf{u}+\mu\mathbf{P})_{\mu=0}\right)_{\mu=0} - \frac{d}{d\lambda}\mathbf{Q}\left(\mathbf{u}+\lambda\frac{d}{d\mu}\mathbf{R}(\mathbf{u}+\mu\mathbf{P})_{\mu=0}\right)_{\lambda=0}
\end{aligned} \tag{2·6}$$

Each line follows an appeal to Taylor's theorem. For example, the result has been used that, for a twice Gateaux-differentiable function ϕ of $\mathbf{u}$,

$$\frac{\partial^2\phi}{\partial\epsilon\,\partial\mu}(\mathbf{u}+\epsilon\mathbf{a}+\mu\mathbf{b}+\epsilon\mu\mathbf{c})_{\epsilon,\mu=0} = \frac{\partial^2\phi}{\partial\epsilon\,\partial\mu}(\mathbf{u}+\epsilon\mathbf{a}+\mu\mathbf{b})_{\epsilon,\mu=0}+\frac{d}{d\lambda}\phi(\mathbf{u}+\lambda\mathbf{c})_{\lambda=0}. \tag{2·7}$$

The cyclic nature of the expression (2·6) proves the identity (2·5).

Definition 2. The *Euler derivative* F of a Gateaux differentiable density $F: X\to\mathscr{R}$ is *defined* here by

$$\frac{d}{d\epsilon}F(\mathbf{u}+\epsilon\dot{\mathbf{u}})_{\epsilon=0} \sim \dot{\mathbf{u}}\,.\,\mathscr{E}F, \tag{2·8}$$

where $\dot{\mathbf{u}}$ is an arbitrary smooth variation of $\mathbf{u}$ and $\sim$ denotes equality up to a non-trivial divergence of some vector. (Note that any function v can be written trivially in the form $v = \partial/\partial x\int^x v\,dx$. In the definition of $\sim$, it is implied that such an identity is not being used.) For example when $n = m = 1$ and F depends on u and u_x $(= \partial u/\partial x)$, we have

$$\frac{d}{d\epsilon}F(u+\epsilon\dot{u})_{\epsilon=0} = \frac{\partial F}{\partial u}\dot{u}+\frac{\partial F}{\partial u_x}\dot{u}_x = \left(\frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right)\right)\dot{u}+\left(\dot{u}\frac{\partial F}{\partial u_x}\right)_x,$$

so that

$$\mathscr{E}F = \frac{\partial F}{\partial u}-\frac{d}{dx}\left(\frac{\partial F}{\partial u_x}\right), \tag{2·9}$$

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

the usual expression from the calculus of variations. A less trivial example is given by considering for $n = 2$, $m = 1$, the non-local function F of ω defined by

$$F(\omega) = \omega\psi - \tfrac{1}{2}|\nabla\psi|^2,$$

where ψ is determined from ω by solving the Poisson equation

$$\left.\begin{aligned} -\Delta\psi &= \omega \quad \text{in} \quad D, \\ \psi &\quad \text{given on} \quad \partial D. \end{aligned}\right\} \tag{2·10}$$

Then

$$\frac{d}{d\epsilon}F(\omega + \epsilon\dot{\omega})_{\epsilon=0} = \omega\dot{\psi} + \dot{\omega}\psi - \nabla\psi \,.\, \nabla\dot{\psi}, \tag{2·11}$$

where $\dot{\psi}$ is obtained from $\dot{\omega}$ by solving

$$\left.\begin{aligned} -\Delta\dot{\psi} &= \dot{\omega} \quad \text{in} \quad D, \\ \dot{\psi} &= 0 \quad \text{on} \quad \partial D. \end{aligned}\right\} \tag{2·12}$$

It follows from (2·10) that the expression (2·11) may be written in the form

$$\dot{\omega}\psi - \nabla \,.\, (\dot{\psi}\nabla\psi), \tag{2·13}$$

so that

$$\mathscr{E}F = \psi. \tag{2·14}$$

The vector field ψ is a non-local function of the variable ω.

Definition 3. An operator J taking vector fields to vector fields is *skew-symmetric* if, for any pair of vector fields $\mathbf{P}$ and $\mathbf{Q}$, we have

$$\mathbf{P} \,.\, J\mathbf{Q} \sim -\mathbf{Q} \,.\, J\mathbf{P}. \tag{2·15}$$

Definition 4. The *Poisson bracket* (F, G) of two Gateaux-differentiable densities F and G which may depend non-locally on $\mathbf{u}$ is *defined* by

$$(F, G) = \frac{d}{d\epsilon}F(\mathbf{u} + \epsilon J\mathscr{E}G)_{\epsilon=0}. \tag{2·16}$$

It follows from Definition 2 that

$$(F, G) \sim \mathscr{E}F \,.\, J\mathscr{E}G, \tag{2·17}$$

and hence from (2·15) that

$$(F, G) \sim -\mathscr{E}G \,.\, J\mathscr{E}F \sim -(G, F). \tag{2·18}$$

Definition 5. The skew-symmetric operator J induces *Hamiltonian structure* if the Poisson bracket satisfies the Jacobi identity in the sense that, for any smooth densities F, G and H, we have

$$(F, (G, H)) + (G, (H, F)) + (H, (F, G)) \sim 0. \tag{2·19}$$

Related to the condition (2·19) on the Poisson bracket is the condition that the Lie and Poisson brackets satisfy the relation

$$[J\mathscr{E}F, J\mathscr{E}G] = J\mathscr{E}(F, G), \tag{2·20}$$

for all densities F and G for which each side is defined. The relation (2·20) will be called the Lie–Poisson relationship.

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

This relationship need not be true for arbitrary skew-symmetric J, but it is shown in Theorem 2 that when J does *not* depend on $\mathbf{u}$, (2·20) is automatically satisfied in the sense that for all densities F and G

$$\dot{\mathbf{u}} \,.\, \{[J\mathscr{E}F, J\mathscr{E}G] - J\mathscr{E}(F, G)\} \sim 0, \tag{2·21}$$

where the vector $\dot{\mathbf{u}}$ is arbitrary. One generally deduces relation (2·20) from (2·21).

THEOREM 2. *If J is a skew-symmetric operator* independent *of* $\mathbf{u}$, *then* (2·21) *is satisfied by all smooth densities F and G.*

Proof. Consider any vector $\dot{\mathbf{u}}$. Then Definition 3 implies that

$$\dot{\mathbf{u}} \,.\, J\mathscr{E}(F, G) \sim -\mathscr{E}(F, G) \,.\, J\dot{\mathbf{u}}. \tag{2·22}$$

Hence

$$\dot{\mathbf{u}} \,.\, J\mathscr{E}(F, G) \underset{\text{(Def. 2)}}{\sim} \frac{d}{d\mu}(F, G)(\mathbf{u} - \mu J\dot{\mathbf{u}})_{\mu = 0}$$

$$\underset{\text{(Def. 4)}}{\sim} \frac{\partial^2}{\partial\mu\,\partial\epsilon} F(\mathbf{u} - \mu J\dot{\mathbf{u}} + \epsilon(J\mathscr{E}G)(\mathbf{u} - \mu J\dot{\mathbf{u}}))_{\epsilon, \mu = 0} \tag{2·23}$$

$$\underset{\text{(Def. 2)}}{\sim} \frac{\partial^2}{\partial\mu\,\partial\epsilon} F(\mathbf{u} - \mu J\dot{\mathbf{u}} + \epsilon J\mathscr{E}G)_{\epsilon, \mu = 0} + \mathscr{E}F \,.\, \frac{d}{d\mu}(J\mathscr{E}G)(\mathbf{u} - \mu J\dot{\mathbf{u}})_{\mu = 0}$$

$$\underset{\text{(Def. 3)}}{\sim} \frac{\partial^2}{\partial\mu\,\partial\epsilon} F(\mathbf{u} - \mu J\dot{\mathbf{u}} + \epsilon J\mathscr{E}G)_{\epsilon, \mu = 0} - \frac{d}{d\mu}(J\mathscr{E}F \,.\, \mathscr{E}G(\mathbf{u} - \mu J\dot{\mathbf{u}}))_{\mu = 0} \tag{2·24}$$

$$\underset{\text{(Def. 2)}}{\sim} \frac{\partial^2}{\partial\mu\,\partial\epsilon} F(\mathbf{u} - \mu J\dot{\mathbf{u}} + \epsilon J\mathscr{E}G)_{\epsilon, \mu = 0} - \frac{\partial^2}{\partial\mu\,\partial\epsilon} G(\mathbf{u} - \mu J\dot{\mathbf{u}} + \epsilon J\mathscr{E}F)_{\epsilon, \mu = 0}. \tag{2·25}$$

Moreover, we have

$$\dot{\mathbf{u}} \,.\, [J\mathscr{E}F, J\mathscr{E}G] = \dot{\mathbf{u}} \,.\, \frac{d}{d\epsilon}\{(J\mathscr{E}F)(\mathbf{u} + \epsilon J\mathscr{E}G) - (J\mathscr{E}G)(\mathbf{u} + \epsilon J\mathscr{E}F)\}_{\epsilon = 0} \tag{2·26}$$

$$\underset{\text{(Def. 2)}}{\sim} \frac{d}{d\epsilon}\{-J\dot{\mathbf{u}} \,.\, \mathscr{E}F(\mathbf{u} + \epsilon J\mathscr{E}G) + J\dot{\mathbf{u}} \,.\, \mathscr{E}G(\mathbf{u} + \epsilon J\mathscr{E}F)\}_{\epsilon = 0}$$

$$\underset{\text{(Def. 3)}}{\sim} \frac{\partial^2}{\partial\mu\,\partial\epsilon} F(\mathbf{u} + \epsilon J\mathscr{E}G - \mu J\dot{\mathbf{u}})_{\epsilon, \mu = 0} - \frac{\partial^2}{\partial\mu\,\partial\epsilon} G(\mathbf{u} + \epsilon J\mathscr{E}F - \mu J\dot{\mathbf{u}})_{\epsilon, \mu = 0}. \tag{2·27}$$

Subtracting (2·25) from (2·27) yields the result. ∎

We now give a condition on J to ensure that (2·21) is satisfied for the case when J depends on $\mathbf{u}$. This condition will be related to the conditions of Olver [**16**] and Gel'fand and Dorfman [**11**].

THEOREM 3. *The relation* (2·21) *is satisfied for given densities F and G if and only if*

$$\dot{\mathbf{u}} \,.\, \frac{d}{d\epsilon} J(\mathbf{u} + \epsilon J\mathscr{E}G)_{\epsilon = 0} \mathscr{E}F - \dot{\mathbf{u}} \,.\, \frac{d}{d\epsilon} J(\mathbf{u} + \epsilon J\mathscr{E}F)_{\epsilon = 0} \mathscr{E}G - \mathscr{E}F \,.\, \frac{d}{d\mu} J(\mathbf{u} - \mu J\dot{\mathbf{u}})_{\mu = 0} \mathscr{E}G \sim 0. \tag{2·28}$$

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

Proof. The calculations in the proof of Theorem 2 produce an extra term

$$\mathscr{E}F \cdot \frac{d}{d\mu} J(\mathbf{u} - \mu J\dot{\mathbf{u}})_{\mu=0} \mathscr{E}G \tag{2·29}$$

from (2·23), and two extra terms from (2·26), namely,

$$\dot{\mathbf{u}} \cdot \frac{d}{d\epsilon} J(\mathbf{u} + \epsilon J\mathscr{E}G)_{\epsilon=0} \mathscr{E}F - \dot{\mathbf{u}} \cdot \frac{d}{d\epsilon} J(\mathbf{u} + \epsilon J\mathscr{E}F)_{\epsilon=0} \mathscr{E}G. \tag{2·30}$$

The relation is therefore satisfied if the difference between the expressions (2·29) and (2·30) is equivalent to 0, i.e. if the condition of the theorem is satisfied. ∎

The next theorem shows that if (2·21) holds, then the operator J defines Hamiltonian structure.

THEOREM 4. *If the relationship* (2·21) *holds, then the Poisson bracket satisfies the Jacobi identity.*

Proof. Consider ((F, G), H), an expression for which is by Definition 4 given by

(†) $$((F, G), H) = \frac{d}{d\epsilon}(F, G)(\mathbf{u} + \epsilon J\mathscr{E}H)_{\epsilon=0}$$

$$\underset{\text{(Def. 4)}}{\sim} \frac{\partial^2}{\partial\mu\,\partial\epsilon} F(\mathbf{u} + \epsilon J\mathscr{E}H + \mu(J\mathscr{E}G)(\mathbf{u} + \epsilon J\mathscr{E}H))_{\epsilon,\mu=0}$$

$$= \frac{\partial^2}{\partial\mu\,\partial\epsilon} F(\mathbf{u} + \epsilon J\mathscr{E}H + \mu J\mathscr{E}G)_{\epsilon,\mu=0}$$

$$+ \frac{d}{d\lambda} F\left(\mathbf{u} + \lambda \frac{d}{d\epsilon}(J\mathscr{E}G(\mathbf{u} + \epsilon J\mathscr{E}H))_{\epsilon=0}\right)_{\lambda=0}$$

$$\underset{\text{(Def. 2)}}{\sim} \frac{\partial^2}{\partial\mu\,\partial\epsilon} F(\mathbf{u} + \epsilon J\mathscr{E}H + \mu J\mathscr{E}G)_{\epsilon,\mu=0}$$

$$+ \mathscr{E}F \cdot \frac{d}{d\epsilon}(J\mathscr{E}G)(\mathbf{u} + \epsilon J\mathscr{E}H)_{\epsilon=0}$$

$$= \frac{\partial^2}{\partial\mu\,\partial\epsilon} F(\mathbf{u} + \epsilon J\mathscr{E}H + \mu J\mathscr{E}G)_{\epsilon,\mu=0} + \mathscr{E}F \cdot J\frac{d}{d\epsilon} \cdot G(\mathbf{u} + \epsilon J\mathscr{E}H)_{\epsilon=0}$$

$$+ \mathscr{E}F \cdot \frac{d}{d\epsilon} J(\mathbf{u} + \epsilon J\mathscr{E}H)_{\epsilon=0} \mathscr{E}G$$

$$\underset{\text{(Def. 3)}}{\sim} \frac{\partial^2}{\partial\mu\,\partial\epsilon} F(\mathbf{u} + \epsilon J\mathscr{E}H + \mu J\mathscr{E}G)_{\epsilon,\mu=0}$$

$$- J\mathscr{E}F \cdot \frac{d}{d\epsilon} G(\mathbf{u} + \epsilon J\mathscr{E}H)_{\epsilon=0} + \mathscr{E}F \cdot \frac{dJ}{d\epsilon}(\mathbf{u} + \epsilon J\mathscr{E}H)\mathscr{E}G$$

$$\underset{\text{(Def. 2)}}{\sim} \frac{\partial^2}{\partial\mu\,\partial\epsilon} F(\mathbf{u} + \epsilon J\mathscr{E}H + \mu J\mathscr{E}G)_{\epsilon,\mu=0}$$

$$- \frac{\partial^2}{\partial\mu\,\partial\epsilon} G(\mathbf{u} + \epsilon J\mathscr{E}H + \mu J\mathscr{E}F)_{\epsilon,\mu=0}$$

$$+ \mathscr{E}F \cdot \frac{dJ}{d\epsilon}(\mathbf{u} + \epsilon J\mathscr{E}H)_{\epsilon=0}\mathscr{E}G.$$

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

It follows that

$$((F, G), H)+((G, H), F)+((H, F), G)$$
$$\sim \mathscr{E}F \cdot \frac{d}{d\epsilon} J(\mathbf{u}+\epsilon J\mathscr{E}H)_{\epsilon=0}\mathscr{E}G+\mathscr{E}G \cdot \frac{d}{d\epsilon} J(\mathbf{u}+\epsilon J\mathscr{E}F)_{\epsilon=0}\mathscr{E}H$$
$$+\mathscr{E}H \cdot \frac{d}{d\epsilon} J(\mathbf{u}+\epsilon J\mathscr{E}G)_{\epsilon=0}\mathscr{E}F \sim 0. \quad (2{\cdot}31)$$

Since (2·21) is assumed, Theorem 3 with $\dot{\mathbf{u}} = \mathscr{E}H$ implies that

$$\mathscr{E}H \cdot \frac{d}{d\epsilon} J(\mathbf{u}+\epsilon J\mathscr{E}G)_{\epsilon=0}\mathscr{E}F-\mathscr{E}H \cdot \frac{d}{d\epsilon} J(\mathbf{u}+\epsilon\mathscr{E}F)_{\epsilon=0}\mathscr{E}G$$
$$+\mathscr{E}F \cdot \frac{d}{d\epsilon} J(\mathbf{u}+\epsilon J\mathscr{E}H)_{\epsilon=0}\mathscr{E}G \sim 0. \quad (2{\cdot}32)$$

Using the skew-symmetry of J on the second term in (2·32), we conclude that the expression (2·31) is a non-trivial divergence. It should be stressed that the results and proofs presented here do not require any operator or density to depend locally on $\mathbf{u}$. (Note also that the term (†) may be written in the equivalent form

$$J\mathscr{E}H \cdot \mathscr{E}(F, G) \sim -\mathscr{E}H \cdot J\mathscr{E}(F, G),$$

and corresponds to the first term of equation (2) in [**12**] (with $H = K = J$). Thus the condition there on the Schouten–Nijenhuis bracket may be considered as precisely the Jacobi condition for the Poisson bracket, without any simplification made based upon the truth of (2·20) when J does not depend on $\mathbf{u}$.)

§3. *The conditions of Olver and of Gel'fand and Dorfman*

In this section we recall the conditions for Hamiltonian structure given by Olver[**16**] and by Gel'fand and Dorfman[**11**, **12**], the work of Olver being stressed here, since the methods he developed seem to be practicable in examples. Throughout this section the skew-symmetric operator J will be assumed to depend on $\mathbf{u}$ and finitely many spatial derivatives of $\mathbf{u}$.

In preparation for the discussion of these conditions the differential-geometric notions that will be needed are defined. The paper of Olver[**16**] concentrated on the use of differential forms. The works of Gel'fand and Dorfman and Olver[**18**] used multi-vectors, alternating maps on multiples of one-forms. The latter approach is better when changes of variables are considered. A one-form is taken here to be a linear map taking vector fields to $\mathscr{R}$, defined up to divergences. For example, let F be a function of $\mathbf{u}$, possibly non-local. Then

$$dF(\mathbf{R}) = \frac{d}{d\epsilon} F(\mathbf{u}+\epsilon\mathbf{R})_{\epsilon=0} \sim \mathscr{E}F \cdot \mathbf{R} = \mathscr{E}F \cdot d\mathbf{u}(\mathbf{R}), \quad (3{\cdot}1)$$

(see Definition 2) for each vector field $\mathbf{R}$, so that dF and $\mathscr{E}F \cdot d\mathbf{u}$ are identified. We will consider only one-forms that can be written, up to a divergence, in the form $\bar{\mathbf{P}} = \sum_{j=1}^{m} P_j\, du_j$. A one-vector is a linear function taking one-forms to $\mathscr{R}$, for example the one-vector ∂w^j is defined by

$$\partial w^j(\bar{\mathbf{P}}) = P_j. \quad (3{\cdot}2)$$

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

The two-vector $\omega_1 \wedge \omega_2$ obtained from two one-vectors ω_1 and ω_2 is an anti-symmetric bilinear map defined on one-forms $\bar{\mathbf{P}}$ and $\bar{\mathbf{Q}}$ by

$$\omega_1 \wedge \omega_2(\bar{\mathbf{P}}, \bar{\mathbf{Q}}) = \omega_1(\bar{\mathbf{P}})\,\omega_2(\bar{\mathbf{Q}}) - \omega_2(\bar{\mathbf{P}})\,\omega_1(\bar{\mathbf{Q}}). \tag{3·3}$$

The extension to multi-vectors is clear.

Now consider the two-vector ω given by

$$\omega = \frac{1}{2} \sum_{i,j=1}^{m} \partial u^i \wedge J_{ij}\, \partial u^j, \tag{3·4}$$

where J_{ij} is the ij element of the operator J. It follows from the skew-symmetry of J that

$$\omega(\bar{\mathbf{P}}, \bar{\mathbf{Q}}) \sim \mathbf{P} \,.\, J\mathbf{Q},$$

and so in particular that

$$\omega(\overline{\mathscr{E}F}, \overline{\mathscr{E}G}) \sim (F, G), \tag{3·5}$$

for any pair of smooth densities F and G. Olver[**18**] called ω the cosymplectic two-vector.

Following Olver([**16**], lemma 4·5), let ∂_J be the operator taking k-vectors to $k+1$-vectors, defined by

$$\partial_J D^I = D^I \partial_J, \quad \partial_J \partial = 0,$$

$$\partial_J(\omega_1 \wedge \omega_2) = \partial_J \omega_1 \wedge \omega_2 + (-1)^n \omega_1 \wedge \partial_J \omega_2$$

$$\partial_J u^i = \sum_{j=1}^{m} J_{ij}\, \partial u^j,$$

and

$$\partial_J f(\mathbf{u}) = \sum_{k,K} \frac{\partial f}{\partial u^k_K} D^K(\partial_J u^k), \tag{3·6}$$

where ω_2 is an arbitrary multi-vector, ω_1 an n-vector and f a scalar function depending on $\mathbf{u}$ and finitely many derivatives of $\mathbf{u}$.

Applying the operator ∂_J to the two-vector (3·4) gives

$$\partial_J \omega = -\tfrac{1}{2} \sum_{i,j,K,l,k} \partial u^i \wedge D^K(J_{kl}\, \partial u^l) \wedge \left(\frac{\partial J_{ij}}{\partial u^k_K} \partial u^j \right). \tag{3·7}$$

We now relate this three-vector to the condition in §2 for Hamiltonian structure.

THEOREM 5. *The conditions of Theorem* 3 *are satisfied if and only if, for all* $\dot{\mathbf{u}}$ *and smooth densities* F *and* G, *we have*

$$\partial_J \omega(\bar{\dot{\mathbf{u}}}, \overline{\mathscr{E}F}, \overline{\mathscr{E}G}) \sim 0. \tag{3·8}$$

Proof. It follows from the expression (3·7) that

$$\begin{aligned} -\partial_J \omega(\bar{\dot{\mathbf{u}}}, \overline{\mathscr{E}F}, \overline{\mathscr{E}G}) = -\Big\{ & \dot{\mathbf{u}} \,.\, \frac{d}{d\epsilon} J(\mathbf{u} + \epsilon J \mathscr{E}F)_{\epsilon=0} \mathscr{E}G - \bar{\dot{\mathbf{u}}} \,.\, \frac{d}{d\epsilon} J(\mathbf{u} + \epsilon J \mathscr{E}G)_{\epsilon=0} \mathscr{E}F \\ & + \mathscr{E}F \,.\, \frac{d}{d\mu} J(\mathbf{u} - \mu J \dot{\mathbf{u}})_{\mu=0} \mathscr{E}G - \mathscr{E}G \,.\, \frac{d}{d\mu} J(\mathbf{u} - \mu J \dot{\mathbf{u}})_{\mu=0} \mathscr{E}F \\ & + \mathscr{E}F \,.\, \frac{d}{d\epsilon} J(\mathbf{u} + \epsilon J \mathscr{E}G)_{\epsilon=0} \dot{\mathbf{u}} - \mathscr{E}G \,.\, \frac{d}{d\epsilon} J(\mathbf{u} + \epsilon J \mathscr{E}F)_{\epsilon=0} \dot{\mathbf{u}} \Big\} \end{aligned} \tag{3·9}$$

$$\sim \dot{\mathbf{u}} \,.\, \frac{d}{d\epsilon} J(\mathbf{u} + \epsilon J \mathscr{E}F)_{\epsilon=0} \mathscr{E}G - \dot{\mathbf{u}} \,.\, \frac{d}{d\epsilon} J(\mathbf{u} + \epsilon J \mathscr{E}G)_{\epsilon=0} \mathscr{E}F + \mathscr{E}F \,.\, \frac{d}{d\mu} J(\mathbf{u} - \mu J \dot{\mathbf{u}})_{\mu=0} \mathscr{E}G. \tag{3·10}$$

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

The expression (3·10) follows from (3·9) on appeal to the skew-symmetry of J. Thus the result is proven. ∎

Hence Hamiltonian structure is guaranteed by the condition $\partial_J \omega \sim 0$, even when the densities depend non-locally on $\mathbf{u}$. If the operator J depends non-locally on $\mathbf{u}$ also, then the conditions in Theorem 3 need to be checked directly. One useful consequence of Olver's formulation is that if the two-vector ω is equivalent to another two-vector ω', and if $\partial_J \omega \sim 0$, then automatically $\partial_J \omega' \sim 0$. This fact will be used in the examples in §6.

§4. *Noether's Theorem*

A system of equations

$$\mathbf{u}_t = \mathbf{Q}(\mathbf{u}) \tag{4·1}$$

is said to be *Hamiltonian* if

$$\mathbf{Q} = J\mathscr{E}H, \tag{4·2}$$

for some density H called the Hamiltonian density with J some skew-symmetric operation that induces Hamiltonian structure. The differential equation

$$\mathbf{u}_\epsilon = \mathbf{P}(\mathbf{u}(\epsilon), t), \quad u(\epsilon = 0) = \mathbf{u}_0 \tag{4·3}$$

generates a one-parameter group of symmetries for equation (4·1) if, for each sufficiently small ϵ, the solution $\mathbf{u}(\epsilon)$ of (4·3) satisfies (4·1) whenever $\mathbf{u}_0$ is a solution. Systematic methods to determine symmetry groups of differential equations are described in the books by Bluman and Cole[**8**], Ovsjannikov[**20**], and Olver[**19**], for example. It should be stressed here that the equations considered here may be *local* or *non-local*. We start the analysis with a lemma.

LEMMA 1. *If equation* (4·3) *generates a one-parameter symmetry group of* (4·1), *then*

$$\frac{\partial \mathbf{P}}{\partial t} = [\mathbf{Q}, \mathbf{P}], \tag{4·4}$$

where $\partial/\partial t$ *denotes the derivative with respect to* explicit *dependence on* t.

For a proof, see for example [**3**].

Now suppose that the operator J does not depend explicitly on t and that there exists a density T such that $\mathbf{P} = J\mathscr{E}T$; then the Lemma shows that

$$J\mathscr{E}\left\{\frac{\partial T}{\partial t} + (T, H)\right\} = 0. \tag{4·9}$$

The term in curly brackets is the total derivative T_t of T along trajectories of (4·1). Hence we deduce that T_t is a Casimir, that is, it belongs to the kernel of the operator $J\mathscr{E}$. Such a quantity ϕ that does not depend explicitly on t satisfies

$$\phi_t = (\phi, H) \sim 0, \tag{4·10}$$

and so defines a conservation law. (This definition is equivalent to that given by Marsden *et al.* [**15**], for example.)

Equation (4·9) indicates that the symmetry generates a conservation law

$$T_t = \frac{\partial T}{\partial t} + (T, H) \sim 0. \tag{4·11}$$

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

The result is not guaranteed, however, and the following equation provides a simple example:

$$u_t = -u_x. \tag{4·12}$$

This equation is Hamiltonian with $J = -(\,.\,)_x$ and $H = \frac{1}{2}u^2$. Consider the symmetry generated by the equation

$$u_\epsilon = -1 = -\left(\frac{\partial}{\partial u}(xu)\right)_x, \tag{4·13}$$

so that the corresponding density is $T = xu$ for which $T_t \sim u$ and not zero. Note that u belongs to the kernel of $J\mathscr{E}$ and so is a Casimir.

On the other hand, if we know that a density T is conserved by solutions of (4·1) in the sense that

$$\frac{\partial T}{\partial t} + (T, H) \sim 0, \tag{4·14}$$

then it follows that with $\mathbf{P} = J\mathscr{E}T$ equation (4·4) is satisfied, which is therefore a necessary condition that $\mathbf{P}$ should generate a one-parameter symmetry group. Thus conservation laws are generally indicated by symmetries. This relationship is sometimes called an extension of Noether's Theorem for Lagrangian systems.

§5. *Conditions for an Euler derivative*

It is clearly important to know when a local or non-local vector field $\mathbf{f}(\mathbf{u})$ can be written in the form $\mathscr{E}F(\mathbf{u})$ for some density F. So first suppose that $\mathbf{f} = \mathscr{E}F$ and that F is twice continuously differentiable. Then, for any vectors $\mathbf{v}$ and $\mathbf{w}$,

$$\begin{aligned}\mathbf{v}\,.\,\frac{d}{d\epsilon}\mathbf{f}(\mathbf{u}+\epsilon\mathbf{w})_{\epsilon=0} &= \mathbf{v}\,.\,\frac{d}{d\epsilon}F(\mathbf{u}+\epsilon\mathbf{w})_{\epsilon=0} \\ &\sim \frac{\partial^2}{\partial\epsilon\,\partial\mu}F(\mathbf{u}+\epsilon\mathbf{w}+\mu\mathbf{v})_{\epsilon,\mu=0} \sim \mathbf{w}\,.\,\frac{d}{d\mu}\mathbf{f}(\mathbf{u}+\mu\mathbf{v})_{\mu=0}.\end{aligned} \tag{5·1}$$

The following theorem, a slight variation of a result given by Lax [**14**], shows that the necessary condition (5·1) is sufficient, even in the case when $\mathbf{f}$ depends non-locally on $\mathbf{u}$. The condition is also similar to that given by Vainberg [**22**].

THEOREM 6. *Suppose that, for all vectors* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$,

$$\mathbf{v}\,.\,\frac{d}{d\epsilon}\mathbf{f}(\mathbf{u}+\epsilon\mathbf{w})_{\epsilon=0} \sim \mathbf{w}\,.\,\frac{d}{d\epsilon}\mathbf{f}(\mathbf{u}+\epsilon\mathbf{v})_{\epsilon=0}. \tag{5·2}$$

Then there exists a density F *for which* $\mathbf{f} = \mathscr{E}F$.

Proof. The proof is basically as given by Lax ([**14**]; see also Vainberg [**22**]). One first defines the integral $\int_\Gamma \mathbf{f}\,.\,d\mathbf{u}$ along a smooth path Γ in function space by

$$\int_\Gamma \mathbf{f}\,.\,d\mathbf{u} = \int_{t_0}^{t_1} \mathbf{f}(\mathbf{u}(t))\,.\,\frac{d\mathbf{u}}{dt}\,dt. \tag{5·3}$$

This integral has the same properties as the usual Riemann integral in Euclidean space, although here the integral is an element of $\mathscr{R}$.

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

For the particular vector field $\mathbf{f}$ of the theorem, define $F(\mathbf{u})$ by

$$F(\mathbf{u}) = \int_{\mathbf{u}_0}^{\mathbf{u}} \mathbf{f} \,.\, d\mathbf{u}, \tag{5·4}$$

for some fixed vector $\mathbf{u}_0$. It can be shown that the function F is well-defined in the sense that the integral is independent of path up to a non-trivial divergence, whence it is easy to see that $\mathscr{E}F = \mathbf{f}$.

We conclude this section with a couple of examples, the second of which will be encountered in §6.

Example 1. Consider the non-local function $f(\omega) = \psi$, where ψ is obtained from ω by solving the two-dimensional Poisson equation

$$\left.\begin{aligned} \omega &= -\Delta\psi \quad \text{in} \quad D, \\ \psi &\quad \text{given on} \quad \partial D, \end{aligned}\right\} \tag{5·5}$$

on a domain $D \subseteq \mathbb{R}^2$. We may ask whether f is the Euler derivative of some function $H(\omega)$. By Theorem 6, it is sufficient to show that, for any functions v and w,

$$v\frac{d}{d\epsilon}\psi(\omega + \epsilon w)_{\epsilon=0} \sim w\frac{d}{d\epsilon}\psi(\omega + \epsilon v)_{\epsilon=0}.$$

Let ψ^w and ψ^v be the solutions of

$$v = -\Delta\psi^v, \quad w = -\Delta\psi^w \quad \text{in} \quad D,$$

$$\psi^v = 0, \quad \psi^w = 0 \quad \text{on} \quad \partial D.$$

Then

$$\begin{aligned} v\frac{d}{d\epsilon}\psi(\omega + \epsilon w)_{\epsilon=0} - w\frac{d}{d\epsilon}\psi(\omega + \epsilon v)_{\epsilon=0} &= -\Delta\psi^v \,.\, \psi^w + \Delta\psi^2 \,.\, \psi^v \\ &\sim \nabla\psi^v \,.\, \nabla\psi^w - \nabla\psi^w \,.\, \nabla\psi^v = 0, \end{aligned} \tag{5·6}$$

as required. To determine the density H, use the formula (5·4), so that

$$\begin{aligned} H(\omega) = \int^{\omega} \psi d\omega = \psi\omega - \int^{\psi} \omega \, d\psi = \psi\omega + \int^{\psi} \Delta\psi \, d\psi \\ \sim \psi\omega - \int^{\nabla\psi} \nabla\psi \,.\, d(\nabla\psi) = \psi\omega - \tfrac{1}{2}|\nabla\psi|^2. \end{aligned} \tag{5·7}$$

It has already been checked that $\mathscr{E}H = \psi$ (see the second example of the Euler derivative). Equally $H' = \frac{1}{2}|\nabla\psi|^2$, which is equivalent to H, will suffice.

A slightly more complicated example is the following.

Example 2. Here functions are dependent on two variables ρ and σ. Consider the vector field

$$\mathbf{f}(\rho, \sigma) = \begin{pmatrix} g\rho - \frac{1}{2}|\nabla\psi|^2 \\ \psi \end{pmatrix}, \tag{5·8}$$

where ψ is determined in terms of ρ and σ by solving the elliptic problem

$$\left.\begin{aligned} -\nabla \,.\, (\rho\nabla\psi) &= \sigma \quad \text{in} \quad D, \\ \psi &\quad \text{given on} \quad \partial D. \end{aligned}\right\} \tag{5·9}$$

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

It is shown next that $\mathbf{f} = \mathscr{E}H$ for a density H. So first check that the conditions of Theorem 6 are satisfied. Let

$$\psi^{\mathbf{v}} = \frac{d}{d\epsilon}\psi(\rho+\epsilon v_1, \sigma+\epsilon v_2)_{\epsilon=0} \quad \text{and} \quad \psi^{\mathbf{w}} = \frac{d}{d\epsilon}\psi(\rho+\epsilon w_1, \sigma+\epsilon w_2)_{\epsilon=0}.$$

Then $\psi^{\mathbf{v}}$ and $\psi^{\mathbf{w}}$ are the solutions of the equations

$$\left.\begin{aligned} -\nabla.(\rho\nabla\psi^{\mathbf{v}}) &= v_2+\nabla.(v_1\nabla\psi), \\ -\nabla.(\rho\nabla\psi^{\mathbf{w}}) &= w_2+\nabla.(w_1\nabla\psi), \end{aligned}\right\} \quad \text{in} \quad D \tag{5·10}$$

$$\psi^{\mathbf{v}} = \psi^{\mathbf{w}} = 0 \quad \text{on} \quad \partial D.$$

Hence

$$\begin{aligned}(v_1, v_2)^T.\frac{d}{d\epsilon}\mathbf{f}(\rho+\epsilon w_1, \sigma+\epsilon w_2)_{\epsilon=0} &- (w_1, w_2)^T.\frac{d}{d\epsilon}\mathbf{f}(\rho+\epsilon v_1, \sigma+\epsilon v_2)_{\epsilon=0} \\ &= -v_1\nabla\psi.\nabla\psi^{\mathbf{w}}+v_2\psi^{\mathbf{w}}+w_1\nabla\psi.\nabla\psi^{\mathbf{v}}-w_2\psi^{\mathbf{v}} \\ &= -\mathbf{v}_1\nabla\psi.\nabla\psi^{\mathbf{w}}-(\nabla.(\rho\nabla\psi^{\mathbf{v}})+\nabla.(v_1\nabla\psi))\psi^{\mathbf{w}}+w_1\nabla\psi^{\mathbf{v}}+(\nabla.(\rho\nabla\psi^{\mathbf{w}}) \\ &\qquad +\nabla.(w_1\nabla\psi))\psi^{\mathbf{v}} \\ &\sim -v_1\nabla\psi^{\mathbf{w}}+\rho\nabla\psi^{\mathbf{v}}.\nabla\psi^{\mathbf{w}}+v_1\nabla\psi.\nabla\psi^{\mathbf{w}}+w_1\nabla\psi.\nabla\psi^{\mathbf{v}}-\rho\nabla\psi^{\mathbf{w}}.\nabla\psi^{\mathbf{v}} \\ &\qquad -w_1\nabla\psi.\nabla\psi^{\mathbf{v}} = 0,\end{aligned}$$

as required. To determine H, again consider the formula (5·4), which gives

$$\begin{aligned} H(\rho,\sigma) &= \int\{(gy-\tfrac{1}{2}|\nabla\psi|^2)\,d\rho+\psi\,d\sigma\} = \int(gy-\tfrac{1}{2}|\nabla\psi|^2)\,d\rho+\sigma\psi-\int\sigma\,d\psi \\ &\sim \int(gy-\tfrac{1}{2}|\nabla\psi|^2)\,d\rho+\sigma\psi-\int\rho\nabla\psi.\,d\nabla\psi \\ &= \int gy\,d\rho+\sigma\psi-\int d(\tfrac{1}{2}\rho|\nabla\psi|^2) = gy\rho+\sigma\psi-\tfrac{1}{2}\rho|\nabla\psi|^2. \end{aligned} \tag{5·11}$$

Equally $H' = \frac{1}{2}\rho|\nabla\psi|^2+gy\rho$ has the same property.

§6. *Examples of Hamiltonian structure*

The following examples of Hamiltonian structure are taken from fluid mechanics. The first two are concerned with equations for the motion of a continuously stratified inviscid fluid in two dimensions. In the first case the skew-symmetric operator J is fairly complicated, whereas the structure of the second example is simpler, though there is a restriction on its applicability, being valid only for flows without reversal.

The third example is presented by the shallow-water approximation for three-dimensional waves in a homogeneous fluid. In the final example the structure of the first example is extended to cover three-dimensional flows.

Example 1. Consider the two-dimensional flow of an incompressible, stratified fluid in a domain $D \subseteq \mathbb{R}^2$. The incompressibility of the fluid implies the existence of a stream function ψ such that the velocity $(u, v) = (\psi_y, -\psi_x)$. The equations of motion, as presented by Benjamin [**5**, **6**], are given in terms of the density ρ and the quantity $\sigma = -\nabla.(\rho\nabla\psi)$. They are

$$\left.\begin{aligned} \rho_t &= -\partial(\rho,\psi), \\ \sigma_t &= -\partial(\sigma,\psi)-\partial(\rho, gy-\tfrac{1}{2}|\nabla\psi|^2), \end{aligned}\right\} \tag{6·1}$$

and

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

where $\partial(u, v) = u_x v_y - u_y v_x$, for any pair of differentiable functions u and v. Equations (6·1) may be written in the form

$$\begin{pmatrix}\rho\\ \sigma\end{pmatrix}_t = \begin{pmatrix}0 & -\partial(\rho, .)\\ -\partial(\rho, .) & -\partial(\sigma, .)\end{pmatrix}\begin{pmatrix}\mathscr{E}_\rho H\\ \mathscr{E}_\sigma H\end{pmatrix}, \tag{6·2}$$

where $H(\rho, \sigma)$ is given by (5·19). It may be readily checked that the matrix operator J in (6·2) is skew-symmetric. The question to be settled is whether J satisfies the condition for Hamiltonian structure. So consider the two-vector

$$\tilde{\omega} = \tfrac{1}{2}(\partial\rho\,\partial\sigma)\wedge\begin{pmatrix}0 & -\partial(\rho, .)\\ -\partial(\rho, .) & -\partial(\sigma, .)\end{pmatrix}\begin{pmatrix}\partial\rho\\ \partial\sigma\end{pmatrix}$$

$$\sim \rho(\partial\rho)_x\wedge(\partial\sigma)_y - \rho(\partial\rho)_y\wedge(\partial\sigma)_x + \sigma(\partial\sigma)_x\wedge(\partial\sigma)_y. \tag{6·3}$$

It follows from the facts that $\partial_J\rho = -\rho_x(\partial\sigma)_y + \rho_y(\partial\sigma)_x$ and

$$\partial_J\sigma = -\rho_x(\partial\rho)_y + \rho_y(\partial\rho)_x - \sigma_x(\partial\sigma)_y + \sigma_y(\partial\sigma)_x,$$

that

$$\partial_J\tilde{\omega} \sim \rho_y(\partial\sigma)_x\wedge(\partial\rho)_x\wedge(\partial\sigma)_y + \rho_x(\partial\sigma)_y\wedge(\partial\rho)_y\wedge(\partial\sigma)_x$$

$$-\rho_x(\partial\rho)_y\wedge(\partial\sigma)_x\wedge(\partial\sigma)_y + \rho_y(\partial\rho)_x\wedge(\partial\sigma)_x\wedge(\partial\sigma)_y = 0, \tag{6·4}$$

as required for Hamiltonian structure of these equations.

Example 2. This example is also concerned with two-dimensional flows of a continuously stratified inviscid fluid. The difference between this and the previous example is the choice of dependent and independent variables. We assume that the flow is along a channel and that there is no reversal of flow. It is then possible to take as independent variables the horizontal coordinate x and a vertical Lagrangian coordinate λ, defined so that there exists a single-valued function $\eta(x, t, \lambda)$ for which the density $\rho(x, y, t) = \bar{\rho}(\lambda)$, a known function, on $y = \eta(x, t, \lambda)$. In the Hamiltonian formulation the independent variables are η and τ defined by

$$\tau(x, t, \lambda) = \sigma(x, \eta(x, t, \lambda), t)\,\eta_\lambda. \tag{6·5}$$

One may readily verify that equations (6·1) imply

$$\left.\begin{aligned}\eta_t &= -\psi_x\\ \text{and}\qquad \tau_t &= -(\tfrac{1}{2}\bar{\rho}_\lambda(u^2+v^2) + u\tau - g\bar{\rho}_\lambda\eta)_x.\end{aligned}\right\} \tag{6·6}$$

The system (6·6) is equivalent to the Hamiltonian system

$$\begin{pmatrix}\eta\\ \tau\end{pmatrix}_t = \begin{pmatrix}0 & -\partial/\partial x\\ -\partial/\partial x & 0\end{pmatrix}\begin{pmatrix}\mathscr{E}_\eta H\\ \mathscr{E}_\tau H\end{pmatrix}, \tag{6·7}$$

where the Hamiltonian density H is the energy density

$$H = \tfrac{1}{2}\rho(u^2+v^2)\,\eta_\lambda - \tfrac{1}{2}g\bar{\rho}_\lambda\,\eta^2, \tag{6·8}$$

and u and v are the horizontal and vertical velocity components evaluated on $y = \eta(x, t, \lambda)$. In terms of η and the stream-function these velocities may be written in the form

$$u = \psi_\lambda/\eta_\lambda \quad\text{and}\quad v = (\eta_x\psi_\lambda - \eta_\lambda\psi_x)/\eta_\lambda. \tag{6·9}$$

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

Moreover, we have

$$\tau = -(\bar{\rho}(u+v\eta_x))_\lambda + (\bar{\rho}v\eta_\lambda)_x. \tag{6·10}$$

It may be shown that the first variation $\dot{H}$ of H is given by

$$\dot{H} \sim \psi\dot{\tau} + (u\tau + \tfrac{1}{2}\bar{\rho}_\lambda(u^2+v^2) - g\bar{\rho}_\lambda\eta)\dot{\eta}. \tag{6·11}$$

Thus we deduce that

$$\mathscr{E}_\eta H = u\tau + \tfrac{1}{2}\bar{\rho}_\lambda(u^2+v^2) - g\bar{\rho}_\lambda\eta, \tag{6·12}$$

$$\mathscr{E}_\tau H = \psi, \tag{6·13}$$

and equations (6·6) and (6·7) are equivalent systems.

The matrix operator in (6·7) is clearly skew-symmetric and determines Hamiltonian structure, having constant coefficients. This is a non-trivial example of a simple Hamiltonian structure. (For more details and implications concerning steady waves, see [**10**].)

Example 3. Consider a shallow-water model for three-dimensional motion of a homogeneous inviscid fluid, with unit density say. The dependent variables will be the depth of fluid $D(x, y, t)$ and the horizontal velocities $u(x, y, t)$ and $v(x, y, t)$. The equations of motion have a canonical Hamiltonian representation if Lagrangian coordinates are used (see Salmon[**21**]), but the usual Eulerian formulation is considered here. This formulation gives equations that may be written in the form

$$\left.\begin{aligned} u_t &= -(\tfrac{1}{2}(u^2+v^2)+gD)_x + v(v_x-u_y), \\ v_t &= -(\tfrac{1}{2}(u^2+v^2)+gD)_y - u(v_x-u_y), \\ \text{and} \quad D_t &= -(uD)_x - (vD)_y, \end{aligned}\right\} \tag{6·14}$$

and are obtained from the Euler equations and the kinematic condition on the free surface $z = D(x, y, t)$. The system (6·14) may be written

$$\begin{pmatrix} D \\ u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & -\partial/\partial x & -\partial/\partial y \\ -\partial/\partial x & 0 & \xi \\ -\partial/\partial y & -\xi & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial D \\ \partial H/\partial u \\ \partial H/\partial v \end{pmatrix}, \tag{6·15}$$

where the Hamiltonian density H is given by

$$H = \tfrac{1}{2}(u^2+v^2)\,D + \tfrac{1}{2}gD^2, \tag{6·16}$$

and $\xi = (v_x - u_y)/D$ is the potential vorticity (see, for example, Gill[**13**] for discussions about this quantity ξ). The operator J in (6·15) is clearly skew-symmetric. It remains to prove that this operator determines Hamiltonian structure. So consider the two-form

$$\tilde{\omega} = \tfrac{1}{2}(\partial D\,\partial u\,\partial v) \wedge J(\partial D\,\partial u\,\partial v)^T \sim \partial u_x \wedge \partial D + \partial v_y \wedge \partial D + \xi\,\partial u \wedge \partial v. \tag{6·17}$$

Now as

$$\partial_J D = -(\partial u)_x - (\partial v)_y, \quad \partial_J u = -(\partial D)_x + \xi\partial v, \quad \text{and} \quad \partial_J v = -(\partial D)_y - \xi\partial u, \tag{6·18}$$

it follows that

$$\partial_J \xi = \partial_J((v_x-u_y)/D) = (\partial_J v)_x/D - (\partial_J u)_y/D - \frac{(v_x-u_y)}{D^2}\partial_J D = -(\xi_x\,\partial u + \xi_y\,\partial v)/D, \tag{6·19}$$

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

after some cancellation. Finally, equations (6·17) and (6·19) imply that

$$\partial_J \tilde{\omega} \sim 0, \tag{6·20}$$

as required for Hamiltonian structure, and hence for the Poisson bracket to satisfy the Jacobi identity. For more details see Bowman[**10**]. The same Hamiltonian structure for compressible flows has been studied, for example, by Verosky[**23**].

Example 4. This example is an extension of example 1 to three-dimensional flows of an incompressible, inviscid, stratified fluid. The dependent variables are the density ρ and a vector quantity σ related to the vorticity and given by

$$\boldsymbol{\sigma} = \nabla \times (\rho \mathbf{u}), \tag{6·21}$$

where $\mathbf{u}$ denotes the velocity. The axes are chosen so that the z-direction points vertically upwards. The unit vector in this direction will be called $\mathbf{k}$.

The Euler equations and the equation for conservation of mass may be written in the form

$$\rho(\mathbf{u}_t + \tfrac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u} \times \boldsymbol{\omega}) = -\nabla p - \rho g \mathbf{k} \tag{6·22}$$

and

$$\rho_t + \nabla . (\rho \mathbf{u}) = 0, \tag{6·23}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity and p denotes pressure. After taking the curl of equation (6.22) and appealing to equation (6·23) and the incompressibility condition $\nabla . \mathbf{u} = 0$, we find that

$$\boldsymbol{\sigma}_t + \nabla \times (\boldsymbol{\sigma} \times \mathbf{u}) + \nabla \times (\rho \nabla (gz - \tfrac{1}{2}|\mathbf{u}|^2) = 0. \tag{6·24}$$

With the Hamiltonian density H given by

$$H \sim \boldsymbol{\sigma} . \boldsymbol{\psi} - \tfrac{1}{2}\rho|\mathbf{u}|^2 + g\rho z \sim \tfrac{1}{2}\rho|\mathbf{u}|^2 + g\rho z, \tag{6·25}$$

where $\boldsymbol{\psi}$ is any vector stream-function for $\mathbf{u}$ (so that $\mathbf{u} = \nabla \times \boldsymbol{\psi}$), equations (6·23) and (6·24) may be written in the form

$$\begin{pmatrix} \rho \\ \boldsymbol{\sigma} \end{pmatrix}_t = \begin{pmatrix} 0 & -\nabla . (\rho \nabla \times \, . \,) \\ -\nabla \times (\rho \nabla \, . \,) & -\nabla \times (\boldsymbol{\sigma} \times \nabla \times \, . \,) \end{pmatrix} \begin{pmatrix} \mathscr{E}_\rho H \\ \mathscr{E}_\sigma H \end{pmatrix}. \tag{6·26}$$

If the two systems of equations are to be equivalent, then it should be true that

$$\begin{pmatrix} \mathscr{E}_\rho H \\ \mathscr{E}_\sigma H \end{pmatrix} = \begin{pmatrix} gz - \tfrac{1}{2}|\mathbf{u}|^2 \\ \boldsymbol{\psi} \end{pmatrix}. \tag{6·27}$$

Here $\boldsymbol{\psi} = \mathscr{E}_\sigma H$ belongs to an equivalence class, since we may add the gradient $\nabla\phi$ of any scalar function ϕ to $\boldsymbol{\psi}$ so that $\mathbf{u} = \nabla \times (\boldsymbol{\psi} + \nabla\phi)$. Equations (6·26) are well defined for this equivalence class, as is the definition (6·25) of H since, for any scalar field ϕ,

$$\boldsymbol{\sigma} . \nabla\phi \sim -(\nabla . \boldsymbol{\sigma})\phi = 0.$$

It can be checked by use of the methods of the previous section that the vector on the right side of (6·27) is the Euler derivative of some scalar density, which we now show to be equivalent to H. The scalar in question is given by

$$H' \sim \int (gz - \tfrac{1}{2}|\mathbf{u}|^2)\, d\rho + \int \boldsymbol{\psi} . d\boldsymbol{\sigma} = \int (gz - \tfrac{1}{2}|\mathbf{u}|^2)\, d\rho + \boldsymbol{\sigma} . \boldsymbol{\psi} - \int \boldsymbol{\sigma} . d\boldsymbol{\psi}. \tag{6·28}$$

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

Now as $\boldsymbol{\sigma} = \nabla \times (\rho \mathbf{u})$ and $\mathbf{u} = \nabla \times \boldsymbol{\psi}$, it follows that

$$H' \sim \int d(g\rho z - \tfrac{1}{2}\rho|\mathbf{u}|^2) + \boldsymbol{\sigma} \cdot \boldsymbol{\psi} = \boldsymbol{\sigma} \cdot \boldsymbol{\psi} - \tfrac{1}{2}\rho|\mathbf{u}|^2 + g\rho z \sim H. \tag{6·29}$$

The demonstration that the matrix operator J in (4·38) is skew-symmetric and defines Hamiltonian structure is straightforward, but tedious. The reader is referred to Bowman [**10**] for details. Of course, these results do not indicate why such a structure is possible in this problem, but the formulations may be related to an alternative given by Abarbanel *et al.*[**1**] who apply geometrical methods to yield Hamiltonian formulations (based on the idea of semi-direct products).

I thank Professors Benjamin and Bona for many helpful discussions, and the support of the S.E.R.C. while part of this work was carried out is gratefully acknowledged.

REFERENCES

[1] H. ABARBANEL, D. HOLM, J. MARSDEN and T. RATIU. Nonlinear stability of stratified flow. To appear in *Phil. Trans. R. Soc. Lond. A*.

[2] V. I. ARNOL'D. Sur la géometrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier* **16** (1966), 319–361.

[3] V. I. ARNOL'D. *Mathematical Methods of Classical Mechanics*. Graduate Texts in Maths, vol. 60 (Springer-Verlag, 1978).

[4] T. B. BENJAMIN. The stability of solitary waves. *Proc. Roy. Soc. London Ser. A* **328** (1972) 153–183.

[5] T. B. BENJAMIN. Impulse, flow force and variational principles. *IMA J. Appl. Math.* **32** (1984), 3–68.

[6] T. B. BENJAMIN. On the Boussinesq model for two-dimensional wave motions in heterogeneous fluids. *J. Fluid Mech.* **165** (1986), 445–474.

[7] T. B. BENJAMIN and P. J. OLVER. Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.* **125**, (1982), 137–185.

[8] G. W. BLUMAN and J. D. COLE. Similarity methods for differential equations. *Appl. Math. Sci.* (Springer-Verlag, 1974).

[9] J. L. BONA. On the stability of surface waves. *Proc. Roy. Soc. London Ser. A* **344** (1975), 363–374.

[10] S. BOWMAN. Mathematical aspects of wave motions in stratified fluids. D.Phil. thesis, Univ. of Oxford (1985).

[11] I. M. GEL'FAND and I. YA. DORFMAN. Hamiltonian operators and related algebraic structures. *Funktsional. Anal. i Prilozhen* **13** (1979), 13–30.

[12] I. M. GEL'FAND and I. YA. DORFMAN. The Schouten bracket and Hamiltonian operators. *Funktsional. Anal. i Prilozhen* **14** (1980), 223–226.

[13] A. E. GILL. The hydraulics of rotating channel flow. *J. Fluid Mech.* **80** (1977), 641–671.

[14] P. D. LAX. Almost periodic solutions of the K–dV equation. *SIAM Rev.* **18** (1976), 351–375.

[15] J. E. MARSDEN, T. RATIU and A. WEINSTEIN. Reduction and Hamiltonian structures on duals of semi-direct product Lie algebras. *Contemp. Math.* **28** (Amer. Math. Soc., 1984), 55–100.

[16] P. J. OLVER. On the Hamiltonian structure of evolution equations. *Math. Proc. Cambridge Philos. Soc.* **88** (1980), 71–88.

[17] P. J. OLVER. A nonlinear Hamiltonian structure for the Euler equation. *J. Math. Anal. Appl.* **89** (1982), 233–250.

[18] P. J. OLVER. Hamiltonian perturbation theory and water waves. *Contemp. Math.* **28** (Amer. Math. Soc., 1984), 231–249.

[19] P. J. OLVER. *Applications of Lie groups to differential equations*. Graduate Texts in Maths, vol. 107 (Springer-Verlag, 1986).

[20] L. V. OVSJANNIKOV. *Group Analysis of Differential Equations* (Academic Press, 1982).

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press

[21] R. Salmon. New equations for nearly geostrophic flow. *J. Fluid Mech.* **153** (1985), 461–477.
[22] M. Vainberg. *Variational Methods in the Study of Nonlinear Operators* (Holden-Day, 1964).
[23] J. Verosky. The Hamiltonian structure of generalized fluid equations. *Lett. Math. Phys.* **9** (1985), 51–58.
[24] A. Weinstein. Stability of Poisson–Hamilton equilibria. *Contemp. Math.* **28** (Amer. Math. Soc., 1984), 3–13.

https://doi.org/10.1017/S0305004100067165 Published online by Cambridge University Press