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A convex structure on sofic embeddings

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Abstract. Nathanial Brown [Topological dynamical systems associated to II_1-factors. *Adv. Math.* **227**(4), 1665–1699] introduced a convex-like structure on the set of unitary equivalence classes of unital *-homomorphisms of a separable type II_1 factor into R^ω (ultrapower of the hyperfinite factor). The goal of this paper is to introduce such a structure on the set of sofic representations of groups. We prove that if the commutant of a representation acts ergodically on the Loeb measure space then that representation is an extreme point.

1. *Introduction*

Abstract convex-like structures are defined in [**Br**]. It is shown that $\mathbb{H}om(N, R^\omega)$ possesses such a structure, where N is a type II_1 factor and R^ω is the ultrapower of the hyperfinite factor. An important result is that $[\pi] \in \mathbb{H}om(N, R^\omega)$ is an extreme point if and only if the relative commutant $\pi(N)' \cap R^\omega$ is a factor.

We are interested in a similar convex structure on the set of sofic representations of a group, which we denote by $\mathrm{Sof}(G, P^\omega)$, where P stands for permutations. The role of von Neumann algebras will be replaced in this paper by ergodic theory.

We start by briefly introducing the objects that we are working with: sofic groups, Loeb measure space, the space of sofic representations. In the second section we recall what an abstract convex-like structure is and introduce such a structure on $\mathrm{Sof}(G, P^\omega)$. Then we prove our main result that ergodicity of the commutant of the sofic representation acting on the Loeb measure space implies that the representation is an extreme point.

1.1. *Ultraproducts of matrix algebras.*

Let ω be a free ultrafilter on $\mathbb{N}$. We shall work with ultraproducts of matrix algebras, which are a particular case of ultraproducts of von Neumann algebras.

Denote by $M_n(\mathbb{C})$ or simply M_n the matrix algebra in dimension n. We shall denote by Tr the normalized trace of a matrix, $\mathrm{Tr}(x) = (1/n)\sum_{i=1}^n x_{ii}$. Recall that for $x \in M_n$

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we have the trace norm, $\|x\|_2 = (\mathrm{Tr}(x^*x))^{1/2}$. Notice that $\|\mathrm{Id}\|_2 = 1$ independent of the matrix dimension.

Let $(n_k)_k \subset \mathbb{N}$ be a sequence such that $\lim_{k\to\infty} n_k = \infty$. Define

$$l^\infty(\mathbb{N}, M_{n_k}) = \left\{ x = (x_k)_k \in \Pi_k M_{n_k} : \sup_k \|x_k\| < \infty \right\},$$

$$\mathcal{N}_\omega = \left\{ x \in l^\infty(\mathbb{N}, M_{n_k}) : \lim_{k\to\omega} \|x_k\|_2 = 0 \right\} \quad \text{and}$$

$$\Pi_{k\to\omega} M_{n_k} = l^\infty(\mathbb{N}, M_{n_k})/\mathcal{N}_\omega.$$

The ultraproduct $\Pi_{k\to\omega} M_{n_k}$ is a von Neumann algebra, though the proof is a little involved. If $x_k \in M_{n_k}$ we shall denote by $\Pi_{k\to\omega} x_k$ the corresponding element in the ultraproduct. Note that this algebra has a faithful trace, namely $\mathrm{Tr}(x) = \lim_{k\to\omega} \mathrm{Tr}(x_k)$, where $x = \Pi_{k\to\omega} x_k$.

1.2. *The Loeb measure space.* The Loeb space was introduced in [**Lo**] (our exposition is from [**El-Sze**]). Inside the algebra M_n, we consider P_n, the subgroup of permutation matrices, and D_n, the subalgebra of diagonal matrices. The group P_n is isomorphic to $\mathrm{Sym}(n)$, the group of all bijections on a set of cardinality n. We interpret $\Pi_{k\to\omega} P_{n_k}$ and $\Pi_{k\to\omega} D_{n_k}$ as subsets in $\Pi_{k\to\omega} M_{n_k}$. By a theorem of Popa (see [**Po**, Proposition 4.3]) $\Pi_{k\to\omega} D_{n_k}$ is a maximal abelian non-separable subalgebra of $\Pi_{k\to\omega} M_{n_k}(\mathbb{C})$. It is isomorphic to $L^\infty(X)$ where X is a Loeb measure space. The construction of the Loeb space is valid for any sequence of probability spaces, but we shall need it just for finite spaces.

Let (X_{n_k}, μ_{n_k}) be a space with n_k points equipped with the normalized counting measure such that $(D_{n_k}, \mathrm{Tr}) \simeq L^\infty(X_{n_k}, \mu_{n_k})$. Let $p \in \Pi_{k\to\omega} D_{n_k}$ be a projection. It is not difficult to see that $p = \Pi_{k\to\omega} p_k$, where p_k is a projection in D_{n_k} and $\mathrm{Tr}(p) = \lim_{k\to\omega} \mathrm{Tr}(p_k)$ by definition. A projection in the algebra is the same information as a measurable subset in the underlying space. This discussion offers a picture of how the Loeb space should be constructed.

Let $(X_{n_k})_\omega$ be the algebraic ultraproduct, that is, $(X_{n_k})_\omega = \Pi_k X_{n_k}/\sim_\omega$, where $\Pi_k X_{n_k}$ is the Cartesian product and $(x_k) \sim_\omega (y_k)$ if and only if $\{k : x_k = y_k\} \in \omega$. If $x_k \in X_{n_k}$ we shall denote by $(x_k)_\omega$ the corresponding element in $(X_{n_k})_\omega$. If $A_k \subset X_{n_k}$ then define $(A_k)_\omega = \{(x_k)_\omega : \{k : x_k \in A_k\} \in \omega\} \subset (X_{n_k})_\omega$. Let $\mathcal{B}^0_\omega = \{(A_k)_\omega : A_k \subset X_{n_k}\}$. Then $\mathcal{B}^0_\omega$ is a Boolean algebra of subsets of $(X_{n_k})_\omega$.

For $(A_k)_\omega \in \mathcal{B}^0_\omega$ define $\mu_\omega((A_k)_\omega) = \lim_{k\to\omega} \mu_{n_k}(A_k)$. Let $\mathcal{B}_\omega$ be the completion of $\mathcal{B}^0_\omega$ with respect to the measure μ_ω. Then $((X_{n_k})_\omega, \mathcal{B}_\omega, \mu_\omega)$ is a non-separable probability space and $L^\infty((X_{n_k})_\omega, \mu_\omega) \simeq (\Pi_{k\to\omega} D_{n_k}, \mathrm{Tr})$.

1.3. *Sofic groups.* Introduced by Gromov in [**Gr**], sofic groups have received considerable attention in recent years. For an introduction to the subject, see the nice survey articles by Pestov [**Pe, Pe-Kw**].

Definition 1.1. A countable group G is called *sofic* if there exist a sequence $\{n_k\}_k \subset \mathbb{N}$, $\lim_k n_k = \infty$, and an injective group morphism $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$.

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The sequence $(n_k)_k$ does not have a special role. If such a morphism exists for some $(n_k)_k$ it will exist for any other $(m_k)_k$ as long as long as $\lim_k m_k = \infty$. The following theorem is due to Elek and Szabo [**El-Sz1**].

THEOREM 1.2. *A countable group G is sofic if and only if there exists a group morphism $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ such that $\mathrm{Tr}(\Theta(g)) = 0$ for any $g \neq e$.*

Proof. A morphism Θ such that $\mathrm{Tr}(\Theta(g)) = 0$ for any $g \neq e$ is clearly injective. For the reverse implication let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be an injective morphism. Because $\mathrm{Tr}(\Theta(g))$ is always a non-negative real, if $|\mathrm{Tr}(\Theta(g))| = 1$ then $\mathrm{Tr}(\Theta(g)) = 1$ and $g = e$. In the end we have $|\mathrm{Tr}(\Theta(g))| < 1$ for any $g \neq e$.

Construct $\Theta^{(m)} = \Theta \otimes \Theta \otimes \cdots \otimes \Theta$ (the m-fold tensor product), that is, $\Theta^{(m)}(g) = \Pi_{k\to\omega} u_g^k \otimes u_g^k \otimes \cdots \otimes u_g^k$, where $\Theta(g) = \Pi_{k\to\omega} u_g^k$. This is a representation of G on $\Pi_{k\to\omega} P_{n_k^m}$. Then $\mathrm{Tr}(\Theta^{(m)}(g)) = \mathrm{Tr}(\Theta(g))^m$. This means that $\mathrm{Tr}(\Theta^{(m)}(g)) \to_{m\to\infty} 0$ for $g \neq e$. A diagonal argument completes the proof. □

Definition 1.3. A *sofic representation of G* is a group morphism $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ such that $\mathrm{Tr}(\Theta(g)) = 0$ for any $g \neq e$, where $\{n_k\}_k$ is any sequence of natural numbers such that $n_k \to_{k\to\infty} \infty$.

Notation 1.4. Let $\{n_k\}_k$ and $\{r_k\}_k$ be two sequences of natural numbers. We denote by $1_{n_k} \otimes \Pi_{k\to\omega} P_{r_k}$ the set of all elements in $\Pi_{k\to\omega} P_{n_k r_k}$ of the type $\Pi_{k\to\omega} 1_{n_k} \otimes u_k$, where $\Pi_{k\to\omega} u_k \in \Pi_{k\to\omega} P_{r_k}$. In the same way $1_{n_k} \otimes \Pi_{k\to\omega} D_{r_k} \subset \Pi_{k\to\omega} D_{n_k r_k}$.

Notation 1.5. Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a group morphism, $\Theta = \Pi_{k\to\omega}\theta_k$. Let $\{r_k\}_k$ be a sequence of natural numbers. Define $\Theta \otimes 1_{r_k} : G \to \Pi_{k\to\omega} P_{n_k r_k}$, $\Theta \otimes 1_{r_k} = \Pi_{k\to\omega}\theta_k \otimes 1_{r_k}$. We shall call $\Theta \otimes 1_{r_k}$ an *amplification* of Θ.

Notation 1.6. There is also a *direct sum* of two sofic representations. If $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$, $\Theta = \Pi_{k\to\omega}\theta_k$, and $\Phi : G \to \Pi_{k\to\omega} P_{m_k}$, $\Phi = \Pi_{k\to\omega}\phi_k$, then define $\Theta \oplus \Phi : G \to \Pi_{k\to\omega} P_{n_k+m_k}$ by $\Theta \oplus \Phi = \Pi_{k\to\omega}\theta_k \oplus \phi_k$.

We shall need the following lemma from [**Pă**]. We also include a short proof for the reader's convenience.

LEMMA 1.7. *Let $\{e_i | i \in \mathbb{N}\}$ be projections in $\Pi_{k\to\omega} D_{n_k}$ such that $\sum_i e_i = 1$. Let $\{u_i | i \in \mathbb{N}\}$ be unitary elements in $\Pi_{k\to\omega} P_{n_k}$ such that $v = \sum_i e_i u_i$ is unitary. Then $v \in \Pi_{k\to\omega} P_{n_k}$.*

Proof. Using the equation $\sum_i e_i = 1$ we can construct projections $e_i^k \in D_{n_k}$ such that:
(1) $e_i = \Pi_{k\to\omega} e_i^k$;
(2) $\sum_i e_i^k = 1_{n_k}$.
By hypothesis we have $u_i = \Pi_{k\to\omega} u_i^k$ where $u_i^k \in P_{n_k}$. If $v^k = \sum_i e_i^k u_i^k$ then $v = \Pi_{k\to\omega} v^k$, but v^k are not necessarily unitary matrices. However, v^k is still a matrix only with 0 and 1 entries and exactly one 1 entry in each row.

We need to estimate the number of columns in v^k having only 0 entries. Denote this number by r_k. Then $v^{k*} v^k$ is a diagonal matrix having r_k entries 0 on the diagonal. This

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implies that

$$\|v^{k*}v^k - \mathrm{Id}\|_2^2 \geq \frac{r_k}{n_k}.$$

Because $\Pi_{k\to\omega} v^{k*}v^k = 1$ we have $r_k/n_k \to_{k\to\omega} 0$. We now construct w^k as follows. The matrix v^k has $n_k - r_k$ columns with at least one non-zero entry. For each such column j chose a row i such that $v^k(i, j) = 1$. Let $w^k(i, j) = 1$. In this way we have $n_k - r_k$ non-zero entries in w^k, all of them distributed on different rows and different columns. Choose a bijection between the remaining r_k rows and r_k columns and complete w^k to form a permutation matrix. Then

$$\|v^k - w^k\|_2^2 = \frac{2r_k}{n_k}.$$

Combined with $r_k/n_k \to_{k\to\omega} 0$, we get $v = \Pi_{k\to\omega} w^k$. This proves the lemma. □

1.4. *The metric space* $\mathrm{Sof}(G, P^\omega)$.

Definition 1.8. For a countable group G define the set $\mathrm{Sof}(G, P^\omega)$ of sofic representations of G, factored by the equivalence relation $(\Theta_1 : G \to \Pi_{k\to\omega} P_{n_k}) \sim (\Theta_2 : G \to \Pi_{k\to\omega} P_{m_k})$ if and only if there are two sequences of natural numbers $\{r_k\}_k$ and $\{t_k\}_k$ such that $n_k r_k = m_k t_k$ for any k and there exists $u \in \Pi_{k\to\omega} P_{n_k r_k}$ such that $\Theta_2 \otimes 1_{t_k} = Adu \circ (\Theta_1 \otimes 1_{r_k})$.

Notation 1.9. For a sofic representation Θ we shall denote by $[\Theta]$ its class in $\mathrm{Sof}(G, P^\omega)$.

Observation 1.10. By Elek and Szabo [**El-Sz2**, Theorem 2], the space $\mathrm{Sof}(G, P^\omega)$ has exactly one point if and only if the group G is amenable.

In order to define a metric on $\mathrm{Sof}(G, P^\omega)$ we need to fix a counting of the group G. So let $G = \{g_0, g_1, \ldots\}$ where $g_0 = e$. For $[\Theta], [\Phi] \in \mathrm{Sof}(G, P^\omega)$, define

$$d([\Theta], [\Phi]) = \inf\left\{\left(\sum_{i=1}^{\infty} \frac{1}{4^i}\|(\Theta \otimes 1)(g_i) - p(\Phi \otimes 1)(g_i)p^*\|_2^2\right)^{1/2} : \{n_k\}_k;\ p \in \Pi_{k\to\omega} P_{n_k}\right\}.$$

The infimum is taken over all the sequences $\{n_k\}_k$ such that the two sofic representations Θ and Φ have amplifications in that dimension. It is clear that this definition does not depend on Θ and Φ, but only on their classes in $\mathrm{Sof}(G, P^\omega)$.

Due to a diagonal argument we can see that the infimum in the definition is attained. This implies that d is indeed a distance. Also $\mathrm{Sof}(G, P^\omega)$ is complete with this metric.

2. *The convex structure on the set of sofic embeddings*

Let us first recall the notion of a metric space with a convex-like structure. The next section is from [**Br**].

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2.1. *Metric spaces with a convex-like structure.* Let (X, d) be a complete metric space which is bounded (there is a constant C such that $d(x, y) \leq C$ for all $x, y \in X$). In order to provide an abstract convex-like structure on X we need to define the element $t_1x_1 + t_2x_2 + \cdots t_nx_n$, where $x_1, \ldots, x_n \in X$ and $0 \leq t_i \leq 1$ such that $\sum_{i=1}^n t_i = 1$. We shall require the following axioms (as Brown put it: 'properties one would expect if X were an honest convex subset of a bounded ball in some normed linear space'):

(1) (Commutativity) $t_1x_1 + \cdots + t_nx_n = t_{\sigma(1)}x_{\sigma(1)} + \cdots + t_{\sigma(n)}x_{\sigma(n)}$ for every permutation $\sigma \in \mathrm{Sym}(n)$.

(2) (Linearity) If $x_1 = x_2$, then $t_1x_1 + t_2x_2 + \cdots + t_nx_n = (t_1 + t_2)x_1 + t_3x_3 + \cdots + t_nx_n$.

(3) (Scalar identity) If $t_i = 1$, then $t_1x_1 + \cdots + t_nx_n = x_i$.

(4) (Metric compatibility) $d(t_1x_1 + \cdots + t_nx_n, s_1x_1 + \cdots s_nx_n) \leq C \sum_i |t_i - s_i|$ and $d(t_1x_1 + \cdots + t_nx_n, t_1y_1 + \cdots + t_ny_n) \leq \sum_i t_i d(x_i, y_i)$.

(5) (Algebraic compatibility)

$$t\left(\sum_{i=1}^n t_i x_i\right) + (1-t)\left(\sum_{j=1}^m s_j y_j\right) = \sum_{i=1}^n t t_i x_i + \sum_{j=1}^m (1-t) s_j y_j.$$

In [**Ca-Fr**], Capraro and Fritz proved that these axioms are enough to deduce that X is a closed convex subset in an abstractly constructed Banach space.

2.2. *Convex combinations of sofic representations.* We now define the convex-like structure on $\mathrm{Sof}(G, P^\omega)$.

Definition 2.1. Let $n \in \mathbb{N}$ and for $i = 1, 2, \ldots, n$ let $\Theta_i : G \to \Pi_{k\to\omega} P_{m_k^i}$ be a sofic representation and $\lambda_i \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$. Define $\sum_{i=1}^n \lambda_i \Theta_i$ as follows: choose natural numbers r_k^i such that $\lim_{k\to\omega} m_k^j r_k^j / \sum_{i=1}^n m_k^i r_k^i = \lambda_j$ for any $j = 1, 2, \ldots, n$ and set $\sum_{i=1}^n \lambda_i \Theta_i : G \to P_{\sum_{i=1}^n m_k^i r_k^i}$, $\sum_{i=1}^n \lambda_i \Theta_i = \bigoplus_{i=1}^n (\Theta_i \otimes 1_{r_k^i})$.

PROPOSITION 2.2. *$[\sum_{i=1}^n \lambda_i \Theta_i]$ is well defined, that is, depends only on $[\Theta_i]$ and λ_i, $i = 1, \ldots, n$. We shall denote this object by $\sum_{i=1}^n \lambda_i [\Theta_i]$.*

PROPOSITION 2.3. *The convex structure defined on $\mathrm{Sof}(G, P^\omega)$ obeys the axioms (1)–(5) of abstract convex-like structures.*

Proof. Verifications are trivial. Perhaps the first part of axiom (4) is a little more technical. □

2.3. *Cutting sofic representations.* One way of constructing new sofic representations from old ones is by cutting a representation with a projection from the commutant. This is actually the reverse operation of the direct sum as introduced in Notation 1.6. Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation, $\Theta = \Pi_{k\to\omega}\theta_k$, where $\theta_k : G \to P_{n_k}$. Let also $p \in \Theta(G)' \cap \Pi_{k\to\omega} D_{n_k}$, such that $\mathrm{Tr}(p) > 0$. Choose projections $p_k \in D_{n_k}$ such that $p = \Pi_{k\to\omega} p_k$. Then $\mathrm{Tr}(p_k) = m_k/n_k$, with $m_k \in \mathbb{N}$. Define $\Theta_p : G \to \Pi_{k\to\omega} P_{m_k}$ by $\Theta_p(g) = \Pi_{k\to\omega} p_k \theta_k(g) p_k$. In fact $\Theta_p(g) = p\Theta(g)p = p\Theta(g)$. By definition Θ_p depends on the choice of projections p_k, but it is easy to see that $[\Theta_p]$ does not depend on this choice.

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Definition 2.4. If $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ and $p \in \Theta(G)' \cap \Pi_{k\to\omega} D_{n_k}$, $p \neq 0$, then define $\Theta_p : G \to p(\Pi_{k\to\omega} P_{n_k})p$ by $\Theta_p(g) = p\Theta(g)$.

Observation 2.5. If $\Theta = \sum_{i=1}^{n} \lambda_i \Phi_i$ then there exists $p \in \Theta(G)' \cap \Pi_{k\to\omega} D_{\sum_{i=1}^{n} m_k^i r_k^i}$ with $\mathrm{Tr}(p) = \lambda_i$ such that $[\Phi_i] = [\Theta_p]$.

2.4. *Actions on the Loeb space.* If $u \in \Pi_{k\to\omega} P_{n_k}$ then $u(\Pi_{k\to\omega} D_{n_k})u^* = \Pi_{k\to\omega} D_{n_k}$. As $\Pi_{k\to\omega} D_{n_k} \simeq L^\infty((X_{n_k})_\omega, \mu_\omega)$, u defines an automorphism of $((X_{n_k})_\omega, \mu_\omega)$. For a sofic representation we are interested in the action of the commutant on this space. Such actions were considered by Kerr and Li in [**Ke-Li**].

Notation 2.6. If $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ is a sofic representation then we shall denote by $\alpha(\Theta)$ the action of $\Theta(G)' \cap \Pi_{k\to\omega} P_{n_k}$ on the Loeb space $((X_{n_k})_\omega, \mu_\omega)$.

The goal of this paper is to link the ergodicity of this action with extreme points in the convex-like structure. We shall now prove that ergodicity is preserved under amplifications. For this purpose we need the following lemmas.

LEMMA 2.7. *Let $n, r \in \mathbb{N}$ and $A_1, \ldots, A_r \subset \{1, \ldots, n\}$. Then there exist $A \subset \{1, \ldots, n\}$ and $p \in \mathrm{Sym}(r)$ such that*

$$\sum_{j=1}^{r} |A \triangle A_j| \leq \sum_{j=1}^{r} |A_{p(j)} \triangle A_j|.$$

Proof. For $i = 1, \ldots, n$ let $a_i = |\{j : i \in A_j\}|$. Define $A = \{i : r < 2a_i\}$ (i is an element of A if and only if more than half of the sets A_j contain i). Then $\sum_{j=1}^{r} |A \triangle A_j| = \sum_{i=1}^{n} \min\{a_i, r - a_i\}$.

For $p \in \mathrm{Sym}(r)$ define $R(p) = \sum_{j=1}^{r} |A_{p(j)} \triangle A_j|$. We shall try to evaluate $\sum_{p\in\mathrm{Sym}(r)} R(p)$. We want to count how many times $i \in A_{p(j)}$ and $i \notin A_j$.

We fix the value of i. For $r - a_i$ different values of j we have $i \notin A_j$. Choose $k \in \{1, \ldots, r\}$ such that $i \in A_k$. There are a_i possible values for k. For $(r-1)!$ permutations p in $\mathrm{Sym}(r)$ we have $p(j) = k$. The number we are looking for is $(r - a_i)a_i(r-1)!$. It may also happen that $i \notin A_{p(j)}$ and $i \in A_j$. In the end we have

$$\sum_{p\in\mathrm{Sym}(r)} R(p) = \sum_{i=1}^{n} 2a_i(r - a_i)(r-1)!.$$

It follows that there exists $p \in \mathrm{Sym}(r)$ such that $R(p) \geq \sum_{i=1}^{n} 2a_i(r - a_i)/r$. It is easy to see that $\min\{a_i, r - a_i\} \leq 2a_i(r - a_i)/r$ so $R(p) \geq \sum_{j=1}^{r} |A \triangle A_j|$. □

LEMMA 2.8. *Let $\{n_k\}_k$ and $\{r_k\}_k$ be two sequences of natural numbers. Let $p \in \Pi_{k\to\omega} D_{n_k r_k}$ be a projection. Assume that p commutes with any element in $1_{n_k} \otimes \Pi_{k\to\omega} P_{r_k}$. Then $p \in \Pi_{k\to\omega} D_{n_k} \otimes 1_{r_k}$.*

Proof. Let $(S_k)_\omega$ be the subset of the Loeb space $(X_{n_r r_k})_\omega$ corresponding to the projection p. Then $p \in (1_{n_k} \otimes \Pi_{k\to\omega} P_{r_k})'$ is equivalent to $u((S_k)_\omega) = (S_k)_\omega$ for any $u \in 1_{n_k} \otimes \Pi_{k\to\omega} P_{r_k}$.

We shall regard $X_{n_k r_k}$ as r_k copies of X_{n_k}. If u_k is of the form $1_{n_k} \otimes p$, with $p \in \mathrm{Sym}(\{1, \ldots, r_k\})$ then u_k will permute these r_k copies of X_{n_k}.

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With respect to this partition of $X_{n_k r_k}$ we have $S_k = \bigsqcup_{j=1}^{r_k} A_k^j$ where $A_k^j \subset X_{n_k}$. Apply the previous lemma to $A_k^1, \ldots, A_k^{r_k}$ to get a set A_k and a permutation q_k. Define $T_k = A_k \times \{1, \ldots, r_k\} \subset X_{n_k r_k}$. Then, by the conclusion of the previous lemma,

$$|T_k \triangle S_k| \leq |(1_{n_k} \otimes q_k)(S_k) \triangle S_k|.$$

Define $u = \Pi_{k\to\omega} 1_{n_k} \otimes p_k$. As $u((S_k)_\omega) = (S_k)_\omega$ by the previous inequality we have $(T_k)_\omega = (S_k)_\omega$. It follows that $p \in \Pi_{k\to\omega} D_{n_k} \otimes 1_{r_k}$. □

PROPOSITION 2.9. *Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation and $\{r_k\}_k$ a sequence of natural numbers. If $\alpha(\Theta)$ is ergodic then also $\alpha(\Theta \otimes 1_{r_k})$ is ergodic.*

Proof. Let $(S_k)_\omega$ be a subset of the Loeb space $(X_{n_r r_k})_\omega$ such that $u((S_k)_\omega) = (S_k)_\omega$ for any $u \in (\Theta \otimes 1_{r_k})' \cap \Pi_{k\to\omega} P_{n_k r_k}$. Assume that $\mu_\omega((S_k)_\omega) \notin \{0, 1\}$. As $1_{n_k} \otimes \Pi_{k\to\omega} P_{r_k} \subset (\Theta \otimes 1_{r_k})' \cap \Pi_{k\to\omega} P_{n_k r_k}$, by the previous lemma there exists a sequence $\{T_k\}_k$, $T_k = A_k \times \{1, \ldots, r_k\} \subset X_{n_k r_k}$, such that $(T_k)_\omega = (S_k)_\omega$.

It is easy to see that $\mu_\omega((T_k)_\omega)$ in $(X_{n_r r_k})_\omega$ is equal to $\mu_\omega((A_k)_\omega)$ in $(X_{n_r})_\omega$. Now the action $\alpha(\Theta)$ is ergodic. So there exists $v \in \Theta' \cap \Pi_{k\to\omega} P_{n_k}$ such that $\mu_\omega(v((A_k)_\omega) \triangle (A_k)_\omega) > 0$. Define $u = v \otimes 1_{r_k}$, $u \in (\Theta \otimes 1_{r_k})' \cap \Pi_{k\to\omega} P_{n_k r_k}$. Again, because we just have an amplification, $\mu_\omega(u((T_k)_\omega) \triangle (T_k)_\omega) = \mu_\omega(v((A_k)_\omega) \triangle (A_k)_\omega)$. This contradicts $u((S_k)_\omega) = (S_k)_\omega$, and we are done. □

2.5. *Extreme points in the convex structure.* We now turn our attention to extreme points in the convex structure.

Definition 2.10. Let A be a convex set. A point $x \in A$ is called *extreme* if whenever $x = ty + (1-t)z$ with $t \in (0, 1)$ and $y, z \in A$ then $x = y = z$.

Observation 2.11. A point $x \in A$ is not extreme if and only if there exists $y, z \in A$ such that $x = (y + z)/2$ and $y \neq z$. This description is classic for convex sets in linear normed spaces. We can either use the result of Capraro and Fritz to transport it to our convex-like structure or we can deduce it directly from the axioms of the convex-like structure.

Our first lemma is similar to [**Br**, Proposition 3.3.4].

LEMMA 2.12. *Let $[\Theta] \in \mathrm{Sof}(G, P^\omega)$. Then $[\Theta]$ is an extreme point if and only if, for any sequence of natural numbers $\{r_k\}_k$ and for any projection $p \in (\Theta \otimes 1_{r_k})' \cap \Pi_{k\to\omega} D_{n_k r_k}$ with $\mathrm{Tr}(p) > 0$, we have $[\Theta] = [(\Theta \otimes 1_{r_k})_p]$.*

Proof. If $p \in (\Theta \otimes 1_{r_k})' \cap \Pi_{k\to\omega} D_{n_k r_k}$ then $\Theta \otimes 1_{r_k} = (\Theta \otimes 1_{r_k})_p \oplus (\Theta \otimes 1_{r_k})_{1-p}$. Then, by the definition of the convex structure, $[\Theta] = \mathrm{Tr}(p)[(\Theta \otimes 1_{r_k})_p] + (1 - \mathrm{Tr}(p))[(\Theta \otimes 1_{r_k})_{1-p}]$. If $[\Theta]$ is an extreme point then $[\Theta] = [(\Theta \otimes 1_{r_k})_p]$ for any p.

For the reverse implication, if $[\Theta] = \lambda_1[\Phi_1] + \lambda_2[\Phi_2]$ then there exists a sequence $\{r_k\}_k$ of natural numbers such that $\Theta \otimes 1_{r_k} = \lambda_1 \Phi_1 + \lambda_2 \Phi_2$ (we replace Φ_i by an amplification if needed). By Observation 2.5 there exists $p \in (\Theta \otimes 1_{r_k})' \cap \Pi_{k\to\omega} D_{n_k r_k}$ with $\mathrm{Tr}(p) = \lambda_1$ such that $[\Phi_1] = [(\Theta \otimes 1_{r_k})_p]$. By hypothesis $[(\Theta \otimes 1_{r_k})_p] = [\Theta]$, so $[\Phi_1] = [\Theta]$. □

THEOREM 2.13. *Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation. Assume that the action $\alpha(\Theta)$ on the Loeb space $((X_{n_k})_\omega, \mu_\omega)$ is ergodic. Then $[\Theta]$ is an extreme point.*

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Proof. If $[\Theta]$ is not an extreme point then there exists $\Theta_1, \Theta_2 : G \to \Pi_{k\to\omega} P_{m_k}$ such that $[\Theta_1] \neq [\Theta_2]$ and $\Theta \otimes 1 = \Theta_1 \oplus \Theta_2$. By hypothesis and Proposition 2.9, $\alpha(\Theta \otimes 1)$ is ergodic.

Let $\{p_i\}_i \subset (\Theta_1(G)' \cap \Pi_{k\to\omega} D_{m_k})$ and $\{q_i\}_i \subset (\Theta_2(G)' \cap \Pi_{k\to\omega} D_{m_k})$ be a maximal family of disjoint projections such that $(\Theta_2)_{q_i} = Adu_i \circ (\Theta_1)_{p_i}$ (no tensor; this implies that $\mathrm{Tr}(p_i) = \mathrm{Tr}(q_i)$ as matrix dimensions have to be the same), where u_i is an ultraproduct of permutations. Assume that $\sum_i p_i < 1$ in $\Pi_{k\to\omega} D_{m_k}$.

Let $p = 1 - \sum_i p_i$ and $q = 1 - \sum_i q_i$. Let $\tilde{p}, \tilde{q} \in (\Theta(G) \otimes 1)' \cap \Pi_{k\to\omega} D_{2m_k}$ defined by $\tilde{p} = p \oplus 0$, $\tilde{q} = 0 \oplus q$. Let A_p, A_q be the subsets of $(X_{2m_k})_\omega$ corresponding to the projections $\tilde{p}$ and $\tilde{q}$. Because the action of $(\Theta(G) \otimes 1)' \cap \Pi_{k\to\omega} P_{2m_k}$ is ergodic there exists $u \in (\Theta(G) \otimes 1)' \cap \Pi_{k\to\omega} P_{2m_k}$ such that $\mu_\omega(u(A_p) \cap A_q) > 0$. This is equivalent to $\tilde{q}u\tilde{p} \neq 0$.

Let $v = \tilde{q}u\tilde{p}$, $p_0 = v^*v = \tilde{p}(u^*\tilde{q}u)$ and $q_0 = vv^* = \tilde{q}(u\tilde{p}u^*)$. Then

$$v(p_0(\Theta \otimes 1)p_0)v^* = vp_0v^*(\Theta \otimes 1) = q_0(\Theta \otimes 1) = q_0(\Theta \otimes 1)q_0.$$

This implies that $v((\Theta_1)_{p_0} \oplus 0)v^* = 0 \oplus (\Theta_2)_{q_0}$, so families $\{p_i\}$ and $\{q_i\}$ are not maximal. Then we must have $\sum_i p_i = 1 = \sum_i q_i$. Recall that $(\Theta_2)_{q_i} = Adu_i \circ (\Theta_1)_{p_i}$.

Define $u = \sum_i u_i p_i$. Then $u \in \Pi_{k\to\omega} P_{m_k}$ by Lemma 1.7 and $\Theta_2 = Adu \circ \Theta_1$, contradicting $[\Theta_1] \neq [\Theta_2]$. □

Using this theorem (and Theorem 2 from [**El-Sz2**]; see also Observation 1.10) we can construct sofic representations such that the commutant acts non-ergodically for any sofic non-amenable group.

Question 2.14. Is a converse of Theorem 2.13 also true?

2.6. *Examples of extreme points.* Theorem 2.13 allows us to provide some examples of extreme points. Kerr and Li proved that $\alpha(\Theta)$ is ergodic for any Θ when G is amenable [**Ke-Li**, Theorem 5.8]. It follows that any element of $\mathrm{Sof}(G, P^\omega)$ is an extreme point. This is possible only if $\mathrm{Sof}(G, P^\omega)$ consists of one point, which is consistent with results from [**El-Sz2**]. The proof of Theorem 5.8 from [**Ke-Li**] contains something more.

PROPOSITION 2.15. [**Ke-Li**, Proof of Theorem 5.8] *There exists a continuous function $f : (0, 1) \to (0, 1)$ such that, for any amenable group H, for any sofic representation $\Theta : H \to \Pi_{k\to\omega} P_{n_k}$ and any $Y = (Y_k)_\omega$, $Z = (Z_k)_\omega$ subsets of $(X_{n_k})_\omega$ of strictly positive measure, there exists $u \in \Theta'(H) \cap \Pi_{k\to\omega} P_{n_k}$ such that*

$$\mu_\omega(u(Y) \cap Z) \geq f(\min(\mu_\omega(Y), \mu_\omega(Z))).$$

Using this proposition we can prove the existence of extreme points for initially sub-amenable groups.

Definition 2.16. A countable group G is called *initially sub-amenable* if, for any finite subset $F \subset G$, there exist an amenable group H and an injective function $\phi : F \to H$ such that $\phi(g)\phi(h) = \phi(gh)$ whenever $g, h, gh \in F$.

THEOREM 2.17. *Let G be an initially sub-amenable group. Then there exists a sofic representation Θ of G such that $\alpha(\Theta)$ is ergodic.*

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Proof. For this proof we need product ultrafilter techniques (see [**Ca-Pă**]). The drawback of the proof is that we start with an ultrafilter ω and get in the end an extreme point for $\omega \otimes \omega$—in other words, an extreme point in the set $\mathrm{Sof}(G, P^{\omega\otimes\omega})$.

Let $G = \bigcup_k F_k$, where $\{F_k\}_k$ is an increasing sequence of finite subsets of G. By hypothesis there exist, for each k, an amenable group H_k and $\phi_k : F_k \to H_k$ such that $\phi(g)\phi(h) = \phi(gh)$ when $g, h, gh \in F_k$. Choose a sofic representation $\Theta_k : H_k \to \Pi_{r\to\omega} P_{n_k,r}$ of H_k, $n_{k,r} \in \mathbb{N}$.

The ultraproduct $\Pi_{r\to\omega} P_{n_k,r}$ is a metric group for each k. We can construct the ultraproduct of these metric groups to get the equality

$$\Pi_{k\to\omega}(\Pi_{r\to\omega} P_{n_k,r}) = \Pi_{(k,r)\to\omega\otimes\omega} P_{n_k,r}.$$

Define $\Theta : G \to \Pi_{(k,r)\to\omega\otimes\omega} P_{n_k,r}$ by

$$\Theta(g) = \Pi_{k\to\omega}\Theta_k(\phi_k(g)).$$

We now prove that $\alpha(\Theta)$ is ergodic. Let $Y = (Y_{k,r})_{\omega\otimes\omega}$, $Z = (Z_{k,r})_{\omega\otimes\omega}$ be subsets of $(X_{n_{k,r}})_{\omega\otimes\omega}$ of strictly positive measure. Define $Y_k = ((Y_{k,r})_r)_\omega$ and $Z_k = ((Z_{k,r})_r)_\omega$. Then $\mu_{\omega\otimes\omega}(Y) = \lim_{k\to\omega} \mu_\omega(Y_k)$, and similarly for Z. Because H_k is amenable, by Proposition 2.15, there exists $u_k \in \Theta'_k(H_k) \cap \Pi_{r\to\omega} P_{n_{k,r}}$ such that

$$\mu_\omega(u_k(Y_k) \cap Z_k) \geq f(\min(\mu_\omega(Y_k), \mu_\omega(Z_k))).$$

Let $u = \Pi_{k\to\omega} u_k$. Then u commutes with Θ and by continuity

$$\mu_{\omega\otimes\omega}(u(Y) \cap Z) \geq f(\min(\mu_{\omega\otimes\omega}(Y), \mu_{\omega\otimes\omega}(Z))) > 0. \qquad \square$$

In [**Cor**], Cornulier constructed a sofic group that is not initially sub-amenable, so the last theorem does not solve the problem of existence of extreme points for sofic groups in general.

Residually finite groups are initially sub-amenable, but in this case there is an easier way of constructing extreme points.

THEOREM 2.18. [**Ke-Li**, Theorem 5.7] *Let G be a residually finite group and let $\{G_i\}_{i\in\mathbb{N}}$ be a sequence of finite index normal subgroups such that $\bigcap_{n\in\mathbb{N}} \bigcup_{i\geq n} G_i = \{e\}$. Let Θ be the sofic representation of G constructed by taking the left action of G on G/G_i. Then $\alpha(\Theta)$ is ergodic.*

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