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DYADIC SHIFT RANDOMIZATION IN CLASSICAL DISCREPANCY THEORY

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Abstract. Dyadic shifts $D \oplus T$ of point distributions D in the d-dimensional unit cube U^d are considered as a form of randomization. Explicit formulas for the L_q-discrepancies of such randomized distributions are given in the paper in terms of Rademacher functions. Relying on the statistical independence of Rademacher functions, Khinchin's inequalities, and other related results, we obtain very sharp upper and lower bounds for the mean L_q-discrepancies, $0 < q \leqslant \infty$. The upper bounds imply directly a generalization of the well-known Chen theorem on mean discrepancies with respect to dyadic shifts (Theorem 2.1). From the lower bounds, it follows that for an arbitrary N-point distribution D_N and any exponent $0 < q \leqslant 1$, there exist dyadic shifts $D_N \oplus T$ such that the L_q-discrepancy $\mathcal{L}_q[D_N \oplus T] > c_{d,q}(\log N)^{(1/2)(d-1)}$ (Theorem 2.2). The lower bounds for the L_∞-discrepancy are also considered in the paper. It is shown that for an arbitrary N-point distribution D_N, there exist dyadic shifts $D_N \oplus T$ such that $\mathcal{L}_\infty[D_N \oplus T] > c_d(\log N)^{(1/2)d}$ (Theorem 2.3).

§1. *Dyadic shifts and the mean discrepancies.* The classical problem in discrepancy theory deals with the distribution of finite point sets in rectangular sub-boxes in the unit cube with sides parallel to the coordinate axes. A detailed discussion of numerous methods and results known in the field can be found in [**1**, **2**, **8**, **9**, **12**]. We recall only the main definitions and facts necessary for the purposes of our paper.

Let D be an arbitrary finite subset, or distribution, in the unit cube $U^d = [0, 1)^d$. The *local discrepancy* $\mathcal{L}[D, Y]$, $Y = (y_1, \dots, y_d) \in U^d$, is defined by

$$\mathcal{L}[D, Y] = |D \cap B_Y| - |D| \operatorname{vol} B_Y, \tag{1.1}$$

where $B_Y = [0, y_1) \times \cdots \times [0, y_d)$ is a rectangular box of volume $\operatorname{vol} B_Y = y_1, \dots, y_d$, and $|\cdot|$ denotes the cardinality of a set.

The *L_q-discrepancies* are defined by

$$\mathcal{L}_q[D] = \left(\int_{U_d} |\mathcal{L}[D, Y]|^q \, dY \right)^{1/q}, \quad 0 < q < \infty, \tag{1.2}$$

and

$$\mathcal{L}_\infty[D] = \sup_{Y \in U^d} |\mathcal{L}[D, Y]|. \tag{1.3}$$

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We write $\mathbb{N}$ for the set of all positive integers, $\mathbb{N}_0$ for the set of all non-negative integers, $\mathbb{N}^d$ and $\mathbb{N}_0^d$ for the product of d copies of the corresponding sets. For $s \in \mathbb{N}_0$, we put

$$\mathbb{Q}(2^s) = \{x = m2^{-s} \in [0, 1) : m = 0, 1, \dots, 2^s - 1\}$$

and

$$\mathbb{Q}^d(2^s) = \{X = (x_1, \dots, x_d) \in U_d : x_j \in \mathbb{Q}(2^s),\ j = 1, \dots, d\}.$$

Furthermore, we put

$$\mathbb{Q}(2^\infty) = \bigcup_{s \geqslant 0} \mathbb{Q}(2^s) \quad \text{and} \quad \mathbb{Q}^d(2^\infty) = \bigcup_{s \geqslant 0} \mathbb{Q}^d(2^s).$$

The points of $\mathbb{Q}^d(2^\infty)$ are called *dyadic rational points*.

Any $y \in [0, 1)$ can be represented in the form

$$y = \sum_{a \geqslant 1} \eta_a(y) 2^{-a}, \tag{1.4}$$

where $\eta_a(y) \in \{0, 1\} \simeq \mathbb{F}_2$, $a \in \mathbb{N}$. Here $\mathbb{F}_2$ is the field of two elements identified with the set of residues $\{0, 1\}$ mod 2.

The dyadic expansion (1.4) is unique if we agree that for each dyadic rational point, the sum in (1.4) contains finitely many non-zero terms. Under this convention, $\eta_a(y) = 0$ for $a > s$ if $y \in \mathbb{Q}(2^s)$ or, in other words, for each point $y \in [0, 1)$, the sequence $\{\eta_a(y) : a \in \mathbb{N}\}$ contains infinitely many zeros.

In a natural way, the set of dyadic rational points can be endowed with the structure of a vector space over the finite field $\mathbb{F}_2$. For any two points x and y in $\mathbb{Q}(2^\infty)$, we define their sum $x \oplus y$ by

$$\eta_a(x \oplus y) = \eta_a(x) + \eta_a(y) \bmod 2, \quad a \in \mathbb{N}, \tag{1.5}$$

and for any two points $X = (x, \dots, x_d)$ and $Y = (y_1, \dots, y_d)$ in $\mathbb{Q}^d(2^\infty)$, we define

$$X \oplus Y = (x_1 \oplus y_1, \dots, x_d \oplus y_d). \tag{1.6}$$

With respect to the addition $\oplus$ defined in this way, each set $\mathbb{Q}^d(2^s)$ is a vector space over the field $\mathbb{F}_2$, and $\dim \mathbb{Q}^d(2^s) = ds$.

Note that (1.5) and (1.6) consistently define the addition $\oplus$ for all pairs of points X and Y, whenever only one of the points, say Y, belongs to $\mathbb{Q}^d(2^\infty)$, while the other is an arbitrary point $X \in U^d$.

The above shows that, for an arbitrary distribution D and any point $T \in \mathbb{Q}^d(2^\infty)$, we can define the dyadic shift $D \oplus T = \{X \oplus T : X \in D\}$ and view it as a new distribution. For each $s \in \mathbb{N}$, we can consider the family $\{D \oplus T : T \in \mathbb{Q}^d(2^s)\}$ as a randomization of D and the corresponding discrepancies $\mathcal{L}_q[D \oplus T]$ as random variables.

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The aim of the present paper is to study the *mean L_q-discrepancies*

$$\mathcal{M}_{s,q}[D] = \left(2^{-ds} \sum_{T \in \mathbb{Q}^d(2^s)} \mathcal{L}_q[D \oplus T]^q\right)^{1/q} \quad 0 < q < \infty, \tag{1.7}$$

and

$$\mathcal{M}_{s,\infty}[D] = \max_{T \in \mathbb{Q}^d(2^s)} \mathcal{L}_\infty[D \oplus T]. \tag{1.8}$$

Our results are given in the next section in Theorems 2.1–2.3. In Theorem 2.1, we will consider the upper bounds for $\mathcal{M}_{s,q}[D]$, $0 < q < \infty$, and specific distributions D, the so-called (δ, s, d)-nets. The lower bounds for $\mathcal{M}_{s,q}[D]$ and arbitrary distributions D will be given in Theorems 2.2 and 2.3 for exponents $0 < q \leqslant 1$ and $q = \infty$, respectively.

We now recall the definition of dyadic (δ, s, d)-nets. Consider *elementary intervals* $\Delta_a^m \subset [0, 1)$ of the form

$$\Delta_a^m = [m2^{-a}, (m+1)2^{-a}), \quad a \in \mathbb{N}_0 \text{ and } m = 0, 1, \dots, 2^a - 1, \tag{1.9}$$

and *elementary boxes* $\Delta_A^M \subset U^d$ of the form

$$\Delta_A^M = \Delta_{a_1}^{m_1} \times \cdots \times \Delta_{a_d}^{m_d}, \quad m_j = 0, 1, \dots, 2^{a_j} - 1 \text{ and } j = 1, \dots, d, \tag{1.10}$$

where $A = (a_1, \dots, a_d)$, $M = (m_1, \dots, m_d) \in \mathbb{N}_0^d$. Every such box has volume $\operatorname{vol} \Delta_A^M = 2^{-a_1 - \cdots - a_d}$.

Let $0 \leqslant \delta \leqslant s$ be integers. A subset $D_{2^s} \subset U^d$ consisting of $N = 2^s$ points is called a *dyadic* (δ, s, d)*-net of deficiency* δ if each elementary box Δ_A^M of volume $2^{\delta - s}$ contains exactly 2^δ points of D_{2^s}.

It follows from the definition that any (δ, s, d)-net D_{2^s} has zero discrepancy in all elementary boxes of large volume. Specifically,

$$|D_{2^s} \cap \Delta_A^M| \begin{cases} = 2^s \operatorname{vol} \Delta_A^M & \text{if } \operatorname{vol} \Delta_A^M \geqslant 2^{\delta - s}, \\ \leqslant 2^\delta & \text{if } \operatorname{vol} \Delta_A^M < 2^\delta. \end{cases} \tag{1.11}$$

Indeed, in the first case, each box Δ_A^M is a disjoint union of elementary boxes of volume $2^{\delta - s}$, and in the second, each box Δ_A^M is contained in an elementary box of volume $2^{\delta - s}$.

Notice also that for any (δ, s, d)-net D_{2^s}, its shift $D_{2^s} \oplus T$, $T \in \mathbb{Q}^d(2^\infty)$, is a net with the same parameters.

Indeed, $|(D \oplus T) \cap \Delta_A^M| = |D \cap (\Delta_A^M \oplus T)|$, $T \in \mathbb{Q}^d(2^\infty)$, and $\Delta_A^M \oplus T = \Delta_A^{M(T)}$ with an index $M(T)$.

Replacing the base 2 in the definitions (1.9) and (1.10) by an arbitrary prime p, we arrive at (δ, s, d)-nets in base p. In arbitrary dimensions d, the first constructions of dyadic (δ, s, d)-nets with $\delta \leqslant d \log d$ were given by Sobol. Later, other constructions of nets in arbitrary base p were proposed by Faure. For details and further references, see [**1**, **12**].

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It is significant that for each base p, the deficiency δ increases with the growth of the dimension d. Furthermore, $(0, s, d)$-nets in the base p and with arbitrary large s exist if and only if $d \leqslant p + 1$. In particular, infinite sequences of dyadic nets with $\delta = 0$ exist only in dimensions $d = 1, 2$ and 3.

It is known that (δ, s, d)-nets D_{2^s} fill the unit cube very uniformly, and the L_∞-discrepancies admit the bounds

$$\mathcal{L}_\infty[D_{2^s}] < C_d 2^\delta s^{d-1}, \quad s \to \infty, \tag{1.12}$$

with a constant C_d depending only on dimension d. Furthermore, for arbitrary (δ, s, d)-nets, the order of this bound as $s \to \infty$ cannot be improved.

We recall that for an arbitrary N-point distribution $D_N \subset U^d$, the bound

$$\mathcal{L}_q[D_N] > c_{d,q}(\log N)^{(1/2)(d-1)}, \quad 1 < q < \infty, \tag{1.13}$$

holds with positive constants $c_{d,q}$ depending only on d and q.

These classical bounds are due to Roth for $2 \leqslant q \leqslant \infty$ and Schmidt for $1 < q < 2$. In two dimensions, it is known that (1.13) is also true for $q = 1$, a result due to Halász.

The order of the bound (1.13) is best possible as $N \to \infty$. In the most general form, in all dimensions $d \geqslant 2$ and for all exponents $1 < q < \infty$, this fundamental fact was established by Chen. Previously, for $1 < q \leqslant 2$, this fact was established by Davenport, Roth and other authors.

We remark that Chen gave two different proofs of his theorem. In the first proof [**6**], averages of the L_q-discrepancies were considered with respect to the usual Euclidean translations of point distributions. The original idea of the p-adic shifts was introduced and exploited in the second proof in the paper [**7**].

We refer the reader to [**1**, **2**, **8**, **9**, **12**] for detailed discussion of all these questions.

§2. *Main results.* Our first result concerns upper bounds for the mean L_q-discrepancies.

THEOREM 2.1. *Let D_{2^s} be an arbitrary dyadic (δ, s, d)-net. Then, for each $0 < q < \infty$, we have*

$$\mathcal{M}_{s,q}[D_{2^s}] < 2^{-d+\delta+1}(\lceil \tfrac{1}{2}q \rceil (s+1))^{(1/2)(d-1)} + d2^\delta. \tag{2.1}$$

In particular, there exist dyadic shifts $T \in \mathbb{Q}^d(2^s)$ such that

$$\mathcal{L}_q[D_{2^s} \oplus T] \leqslant 2^{-d+\delta+1}(\lceil \tfrac{1}{2}q \rceil (s+1))^{(1/2)(d-1)} + d2^\delta. \tag{2.2}$$

Theorem 2.1 shows that, in all dimensions, there exist dyadic (δ, s, d)-nets which meet the lower bound (1.13).

For the first time, results of such type were established by Chen for nets of deficiency $\delta = 0$ in an arbitrary prime base $p \geqslant 2$.

Chen's original approach relies on an elaborate combinatorial analysis involving simultaneous induction on the parameters d, s, and even integers q.

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In this approach, the assumption $\delta = 0$ turns out to be essential. As a result, for each fixed prime base p, Chen's theorem could only be established in dimensions $d \leqslant p + 1$, and for dyadic nets only in dimensions 1, 2 and 3. In other words, to establish Chen's theorem in dimension d, a prime $p \geqslant d - 1$ needs to be chosen.

In the author's paper [**14**], a new approach to the study of the mean L_q-discrepancies was proposed. In this approach, the value of the deficiency δ turns out to be completely irrelevant. This approach relies on the theory of lacunary function series. In the case of dyadic nets, these are series of Rademacher functions, which form a lacunary subsystem of the Walsh functions. In the case of nets in an arbitrary base p, these series form a lacunary subsystem of the corresponding Chrestenson–Levy functions. The detailed description of such functional systems can be found in [**11**].

A result similar to Theorem 2.1 was established previously in [**14**] (see also [**15**]), but with worse constants in the bounds. As functions of q, the constants given above in (2.1) and (2.2) are optimal in the following sense. It can be shown that

$$\mathcal{L}_q[D_{2^s}] \leqslant \mathcal{L}_\infty[D_{2^s}] \leqslant 2^{d/\varepsilon}(\mathcal{L}_q[D_{2^s}] + d2^{\delta+1}), \tag{2.3}$$

where $q = \varepsilon s \to \infty$ and $\varepsilon > 0$ is an arbitrary constant; see Lemma 6.2.

Therefore, (2.1) and (2.2) imply (1.12). Furthermore, if the order of the constants in (2.1) and (2.2) could be improved as $q \to \infty$, then the order of (1.12) could be also improved as $s \to \infty$ for a subsequence of (δ, s, d)-nets.

We now consider lower bounds for the mean L_q-discrepancies. In what follows, log denotes the logarithm in base 2.

THEOREM 2.2. *Let $D_N \subset U^d$, $d \geqslant 2$, be an arbitrary N-point distribution and an exponent $0 < q \leqslant 1$ be arbitrary and fixed. Suppose that an integer s is chosen to satisfy*

$$s \geqslant \log N + \frac{2d+1}{q} + \frac{1}{2}(d-1)\log(d-1) + d + 1 + \log d. \tag{2.4}$$

Then

$$\mathcal{M}_{s,q}[D_N] > \gamma_q(d)(\log N)^{(1/2)(d-1)}, \tag{2.5}$$

where

$$\gamma_q(d) = 2^{-(2d+1)/q-d-1}(d-1)^{-(1/2)(d-1)}. \tag{2.6}$$

In particular, there exist dyadic shifts $T \in \mathbb{Q}^d(2^s)$ such that

$$\mathcal{L}_q[D_N \oplus T] > \gamma_q(d)(\log N)^{(1/2)(d-1)}. \tag{2.7}$$

Certainly, (2.5) and (2.7) hold also for $1 < q < \infty$ but, in this case, these bounds follow at once from (1.13).

In dimensions $d \geqslant 3$, even the exact order of the L_1-discrepancy is not known. The L_q-discrepancies with $0 < q < 1$ were never considered at all for any dimension d.

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Theorem 2.2 shows that, in contrast to the L_q-discrepancies of individual distributions, the problem of the mean L_q-discrepancies can be resolved completely for all exponents $0 < q \leqslant 1$.

It is worth noting that Theorems 2.1 and 2.2 can be extended to the *conditional mean L_q-discrepancies*

$$M_{s,q}[D, V] = \left(|V|^{-1} \sum_{T \in V} \mathcal{L}_q[D \oplus T]^q\right)^{1/q}, \quad 0 < q < \infty, \tag{2.8}$$

where V is a subset in $\mathbb{Q}^d(2^s)$.

It turns out that the conditional means (2.8) can meet the bounds of order (2.1) and (2.5) with very small averaging subsets V of cardinality $|V| = O(s^{\omega_q(d)})$ as $s \to \infty$; here $\omega_q(d)$ is a constant independent of s.

Certainly, such subsets V should be rather specific. Some results in this direction were obtained in [**15**], and further studies of these intriguing questions will continue in forthcoming papers.

Our result on the mean L_∞-discrepancy can be stated as follows.

THEOREM 2.3. *Let $D_N \subset U^d$, $d \geqslant 3$, be an arbitrary N-point distribution. Suppose that an integer s is chosen to satisfy*

$$s \geqslant \log N + \tfrac{1}{2}(d-2)\log(d-2) + 2d + \log d. \tag{2.9}$$

Then

$$\mathcal{M}_{s,\infty}[D_N] > \gamma_\infty(d)(\log N)^{(1/2)d}, \tag{2.10}$$

where

$$\gamma_\infty(d) = 2^{-2d-1}(d-2)^{-(1/2)(d-2)}. \tag{2.11}$$

In particular, there exist dyadic shifts $T \in \mathbb{Q}^d(2^s)$ such that

$$\mathcal{L}_\infty[D_N \oplus T] > \gamma_\infty(d)(\log N)^{(1/2)d}. \tag{2.12}$$

In dimensions $d \geqslant 3$, the exact order of the L_∞-discrepancy still remains an open question.

In two dimensions, the answer is known: Schmidt's lower bound

$$\mathcal{L}_\infty[D_N] > c \log N, \quad D_N \subset U^2,$$

is best possible. In higher dimensions, Beck's lower bound

$$\mathcal{L}_\infty[D_N] > c_\varepsilon \log N (\log\log N)^{(1/8)-\varepsilon}, \quad D_N \subset U^3, \tag{2.13}$$

where $\varepsilon > 0$ is arbitrary small, for three-dimensional distributions remained the only known result for many years. Rather recently, the stronger lower bound

$$\mathcal{L}_\infty[D_N] > c_d(\log N)^{(1/2)(d-1)+\eta_d}, \tag{2.14}$$

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with small constants $\eta_d \gtrsim d^{-2}$ depending only on d, was established in all dimensions $d \geqslant 3$. These deep results are due to Bilyk and Lacey [3] in dimension $d = 3$ and Bilyk *et al* [5] in dimensions $d \geqslant 4$; see also the surveys [2, 4].

For many years, a few specialists in discrepancy theory have believed that in all dimensions $d \geqslant 3$, the best possible lower bound is of the form

$$\mathcal{L}_\infty[D_N] > c_d(\log N)^{d-1}.$$

However, contrary to such popular belief, it has recently been conjectured that the best possible lower bound should have the form

$$\mathcal{L}_\infty[D_N] > c_d(\log N)^{(1/2)d}. \tag{2.15}$$

This latest conjecture is inspired by some highly non-trivial parallels between discrepancy theory and the theory of stochastic processes. The reader can consult the papers [2–5] for a more detailed discussion of these questions.

Theorem 2.3 shows that the conjectured bound (2.15) is valid for the mean L_∞-discrepancy.

We will see that the mean L_q-discrepancies can be represented in terms of the Rademacher series; see §4. For such series, very sharp upper and lower L_q-bounds for any $0 < q < \infty$ can be given by Khinchin's inequality. In fact, Theorems 2.1 and 2.2 are corollaries of this inequality. At the same time, Theorem 2.3 is a corollary of a suitably modified Khinchin inequality, adapted to the L_∞-norm; see Lemma 3.2.

The lower bounds (1.13), (2.13) and (2.14) are obtained with the help of different variations of Roth's orthogonal function method; cf. [1, 2]. It is interesting to note that, in the proofs of Theorems 2.2 and 2.3, we will not use any auxiliary orthogonal functions. The corresponding lower bounds (2.5) and (2.10) will be derived directly from the explicit formulas for discrepancies given in Lemma 4.3.

§3. *Rademacher functions and related inequalities.* In this section all necessary facts on Rademacher functions and related topics are collected.

In the one-dimensional case, the Rademacher functions $r_a(y)$, $y \in [0, 1)$, $a \in \mathbb{N}$, can be defined by

$$r_a(y) = (-1)^{\eta_a(y)} = 1 - 2\eta_a(y), \tag{3.1}$$

where $\eta_a(y)$ are the coefficients in the dyadic expansion (1.4). It is convenient to put $r_0(y) \equiv 1$. Notice immediately that expansion (1.4) takes the form

$$y = \frac{1}{2} - \frac{1}{2}\sum_{a \geqslant 1} 2^{-a} r_a(y). \tag{3.2}$$

The Rademacher functions $r_a(\cdot)$, $a \in \mathbb{N}$, form a sequence of independent random variables taking the values ± 1 with probability $1/2$. This fact can be expressed by the relations

$$\operatorname{mes}\{y \in [0, 1) : r_{a_1}(y) = \varepsilon_1, \dots, r_{a_l}(y) = \varepsilon_l\} = 2^{-l} \tag{3.3}$$

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which hold for any $1 \leqslant a_1 < \cdots < a_l$, $l \in \mathbb{N}$, and any $\varepsilon_j = \pm 1$, $j = 1, \ldots, l$; see, for example [10, 13].

Each function $r_a(y)$, $a \in \mathbb{N}$, is piecewise constant on elementary intervals $\Delta_a^m = [m2^{-a}, (m+1)2^{-a})$, $m = 0, 1, \ldots, 2^a - 1$. Therefore, the relations (3.3) are equivalent to their discrete analogs

$$|\{y \in \mathbb{Q}(2^s) : r_{a_1}(y) = \varepsilon_1, \ldots, r_{a_l}(y) = \varepsilon_l\}| = 2^{s-l} \tag{3.4}$$

for any $1 \leqslant a_1 < \cdots < a_l \leqslant s$, $s \in \mathbb{N}$, and any $\varepsilon_j = \pm 1$, $j = 1, \ldots, l$.

The k-dimensional Rademacher functions $r_A(Y)$, $Y = (y_1, \ldots, y_k) \in U^k$, $A = (a_1, \ldots, a_k) \in \mathbb{N}_0^k$, are defined by

$$r_A(Y) = \prod_{j=1}^{d} r_{a_j}(y_j). \tag{3.5}$$

In some formulas, we write k for dimension, because the formulas will be used in the subsequent text with $k = d$ and $k = d - 1$.

We introduce the linear space $\mathcal{R}_s^k$, $s \in \mathbb{N}_0$, consisting of all functions of the form

$$f(Y) = \sum_{A \in I_s^k} \lambda_A r_A(Y), \tag{3.6}$$

with real coefficients λ_A. Here $I_s = \{0, 1, \ldots, s\}$, and I_s^k denotes the product of k copies of I_s.

It follows from (3.4) that the set of functions $\{r_a(\cdot) : a \in I_s\}$ is linearly independent on $\mathbb{Q}(2^s)$, and therefore, the set $\{r_A(\cdot) : A \in I_s^d\}$ is linearly independent on $\mathbb{Q}^d(2^s)$. Thus, $\dim \mathcal{R}_s^k = (s+1)^k$, and $\mathcal{R}_s^k$ is a very small subspace in the large space $\mathcal{B}_s^k$ of dimension 2^{ks} consisting of all real-valued functions that are piecewise constant on elementary cubes

$$\Delta_s^M = [m_1 2^{-s}, (m_1+1)2^{-s}) \times \cdots \times [m_k 2^{-s}, (m_k+1)2^{-s}),$$

where $m_j = 0, 1, \ldots, 2^s - 1$ and $j = 1, \ldots, k$.

Each function $f \in \mathcal{B}_s^k$ is determined by its values on dyadic rational points $\mathbb{Q}^k(2^s)$, and we have

$$\|f\|_q = \|f\|_{s,q}, \quad 0 < q \leqslant \infty, \tag{3.7}$$

where

$$\|f\|_q = \left(\int_{U^k} |f(Y)|^q \, dY \right)^{1/q}, \quad 0 < q < \infty,$$

$$\|f\|_\infty = \sup_{Y \in U^k} |f(Y)|,$$

$$\|f\|_{s,q} = \left(2^{-ks} \sum_{Y \in \mathbb{Q}^k(2^s)} |f(Y)|^q \right)^{1/q}, \quad 0 < q < \infty,$$

$$\|f\|_{s,\infty} = \max_{Y \in \mathbb{Q}^k(2^s)} |f(Y)|.$$

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The k-dimensional Khinchin inequality: For each function $f \in \mathcal{R}_s^k$ and all $0 < q < \infty$, we have

$$\alpha_q^k Q_2[f] \leqslant \|f\|_{s,q} \leqslant \beta_q^k Q_2[f], \tag{3.8}$$

where

$$Q_2[f] = \left(\sum_{A \in I_s^k} \lambda_A^2 \right)^{1/2}. \tag{3.9}$$

The constants α_q^k and β_q^k are independent of f and s. They are the kth powers of the constants α_q and β_q respectively, with

$$\alpha_q \geqslant \begin{cases} 2^{-(2-q)/q} & \text{if } 0 < q < 2, \\ 1 & \text{if } 2 \leqslant q < \infty, \end{cases} \tag{3.10}$$

and

$$\beta_q \leqslant \lceil \tfrac{1}{2} q \rceil^{1/2}. \tag{3.11}$$

In the one-dimensional case, (3.8) is a corollary of the independence of Rademacher functions; see (3.3) and (3.4). Its proof can be found in many texts on harmonic analysis and probability theory; see, for example, [**10**, §10.3, Theorem 1], [**13**], [**17**, Ch. 5, Theorem 8.4].

The extension of Khinchin's inequality to higher dimensions can easily be given by induction on k; we refer the reader to [**16**, Appendix D] for details.

In the subsequent text, we shall use corollaries of Khinchin's inequality given below in Lemmas 3.1 and 3.2.

For $Y = (y_1, \dots, y_d) \in U^d$ and $A = (a_1, \dots, a_d) \in I_s^d$, $d \geqslant 2$, we put

$$\begin{cases} Y = (\mathbf{Y}, y), & \mathbf{Y} = (y_1, \dots, y_{d-1}) \in U^{d-1} \text{ and } y = y_d \in [0, 1), \\ A = (\mathbf{A}, a), & \mathbf{A} = (a_1, \dots, a_{d-1}) \in I_s^{d-1} \text{ and } a = a_d \in I_s. \end{cases} \tag{3.12}$$

Then any function $f \in \mathcal{R}_s^d$ can be written in the form

$$f(Y) = f(\mathbf{Y}, y) = \sum_{\mathbf{A} \in I_s^{d-1}} \Phi_{\mathbf{A}}(y) r_{\mathbf{A}}(\mathbf{Y}), \tag{3.13}$$

where

$$\Phi_{\mathbf{A}}(y) = \sum_{a \in I_s} \lambda_A r_a(y), \tag{3.14}$$

as well as in the form

$$f(Y) = f(\mathbf{Y}, y) = \sum_{a \in I_s} \varphi_a(\mathbf{Y}) r_a(y), \tag{3.15}$$

where

$$\varphi_a(\mathbf{Y}) = \sum_{\mathbf{A} \in I_s^{d-1}} \lambda_A r_{\mathbf{A}}(\mathbf{Y}). \tag{3.16}$$

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LEMMA 3.1. *For each function* $f \in \mathcal{R}_s^d$, *we have*

$$\|f\|_{s,q} \leqslant \beta_q^{d-1} Q_{\infty,2}[f], \quad 0 < q < \infty, \tag{3.17}$$

where

$$Q_{\infty,2}[f] = \max_{y \in \mathbb{Q}(2^s)} \left(\sum_{\mathbf{A} \in I_s^{d-1}} \Phi_{\mathbf{A}}(y)^2 \right)^{1/2}, \tag{3.18}$$

and

$$\|f\|_{s,q} \geqslant \alpha_q^d Q_2[f], \tag{3.19}$$

where $Q_2[f]$ *is defined in (3.9).*

Proof. Applying the right inequality in (3.8) with $k = d - 1$ to (3.13), we obtain (3.17). The bound (3.19) is just the left inequality in (3.8) with $k = d$. □

Lemma 3.1 will be used in the proof of Theorems 2.1 and 2.2. For the proof of Theorem 2.3 the following more specific result will be needed. This result can be thought of as a modification of Khinchin's inequality for the L_∞-norm.

LEMMA 3.2. *For each function* $f \in \mathcal{R}_s^d$, *we have*

$$\|f\|_{s,\infty} \geqslant \alpha_1^{d-1} Q_{1,2}[f], \tag{3.20}$$

where

$$Q_{1,2}[f] = \sum_{a \in I_s} Q_2[\varphi_a], \tag{3.21}$$

$$Q_2[\varphi_a] = \left(\sum_{\mathbf{A} \in I_s^{d-1}} \lambda_A^2 \right)^{1/2}. \tag{3.22}$$

Proof. First of all, we observe that relations (3.4) imply the following identity for each one-dimensional function $\varphi \in \mathcal{R}_s$. Let

$$\varphi(y) = \sum_{a \in I_s} \varphi_a r_a(y), \quad y \in [0, 1).$$

Then

$$\|\varphi\|_{s,\infty} = \sum_{a \in I_s} |\varphi_a|. \tag{3.23}$$

Indeed, we can always assume that $\varphi_0 \geqslant 0$, and in view of relations (3.4), there exists a point $y_0 \in \mathbb{Q}(2^s)$ such that $r_a(y_0) = \operatorname{sign} \varphi_a$ if $\varphi_a \neq 0$, $a \in I_s$. Therefore

$$\|\varphi\|_{s,\infty} \geqslant |\varphi(y_0)| = \sum_{a \in I_s} |\varphi_a|.$$

The opposite inequality is obvious, and (3.23) follows.

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Applying (3.23) to (3.15), we obtain

$$\begin{aligned}\|f\|_{s,\infty} &= \max_{\mathbf{Y}\in\mathbb{Q}^{d-1}(2^s)} \max_{y\in\mathbb{Q}(2^s)} |f(\mathbf{Y}, y)| \\ &= \max_{\mathbf{Y}\in\mathbb{Q}^{d-1}(2^s)} \sum_{a\in I_s} |\varphi_a(\mathbf{Y})| \\ &\geqslant 2^{-(d-1)s} \sum_{\mathbf{Y}\in\mathbb{Q}^{d-1}(2^s)} \sum_{a\in I_s} |\varphi_a(\mathbf{Y})| = \sum_{a\in I_s} \|\varphi_a(\cdot)\|_{s,1} \\ &\geqslant \alpha_1^{d-1} \sum_{a\in I_s} Q_2[\varphi_a] = \alpha_1^{d-1} Q_{1,2}[f],\end{aligned}$$

where, in the last step, we use the left inequality in (3.8) with $k = d - 1$ and $q = 1$. The proof of Lemma 3.2 is complete. □

§4. *Rademacher functions and explicit formulas for discrepancies.* For an arbitrary point $y \in [0, 1)$ with dyadic expansion (1.4), we denote by

$$y^{(s)} = \sum_{a=1}^{s} \eta_a(y) 2^{-a}, \quad s \in \mathbb{N}, \tag{4.1}$$

its projection to $\mathbb{Q}(2^s)$. For $s = 0$, we put $y^{(0)} = 0$, so that

$$y = y^{(s)} + \theta_s(y) 2^{-s}, \quad s \in \mathbb{N}_0, \tag{4.2}$$

where $\theta_s(y) \in [0, 1)$ for all $y \in [0, 1)$.

We put

$$\delta^{(s)}(x, y) = \begin{cases} 1 & \text{if } x^{(s)} = y^{(s)}, \\ 0 & \text{if } x^{(s)} \neq y^{(s)}. \end{cases} \tag{4.3}$$

It follows immediately from (1.4) and (4.1) that the elementary intervals Δ_s^m, $m = 0, 1, \ldots, 2^s - 1$ (see (1.9)), can be written in the form

$$\Delta_s^m = [m2^{-s}, (m + 1)2^s) = \{z \in [0, 1) : z^{(s)} = m2^{-s}\}.$$

Therefore

$$\delta^{(s)}(x, y) = \delta^{(s)}(x^{(s)} \oplus y^{(s)}) = \chi(\Delta_s^0, x^{(s)} \oplus y^{(s)}) \tag{4.4}$$

and

$$\delta^{(s)}(x^{(s)} \oplus y^{(s)}) = \sum_{m=0}^{2^s-1} \chi(\Delta_s^m, x) \chi(\Delta_s^m, y). \tag{4.5}$$

In the sequel, we write $\chi(\mathcal{E}, \cdot)$ for the characteristic function of a set $\mathcal{E}$. Notice that

$$\chi(\Delta_s^m, x) = \chi(\Delta_s^m, x^{(s)}) = \chi(\Delta_s^m, x^{(a)}) \tag{4.6}$$

for any $a \geqslant s$.

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It follows from (4.4) and (4.5) that

$$\delta^{(s)}(x^{(s)} \oplus y^{(s)}) = \sum_{z \in \mathbb{Q}(2^s)} \delta^{(s)}(x^{(s)} \oplus z)\delta^{(s)}(z \oplus y^{(s)}).$$

Furthermore, $\delta^{(s)}(x^{(s)} \oplus y^{(s)})$ is the reproducing kernel for the space $\mathcal{B}_s$; in other words,

$$\begin{aligned} f(x) &= \sum_{y \in \mathbb{Q}(2^s)} \delta^{(s)}(x^{(s)} \oplus y^{(s)}) f(y) \\ &= 2^s \int_0^1 \delta^{(s)}(x^{(s)} \oplus y^{(s)}) f(y)\, dy, \quad f \in \mathcal{B}_s. \end{aligned} \tag{4.7}$$

Consider the elementary intervals

$$\Pi_a = \Delta_a^1 = [2^{-a}, 2^{1-a}), \quad a \in \mathbb{N}. \tag{4.8}$$

It is convenient to put $\Pi_0 = [0, 1)$.

In terms of the dyadic expansion (1.4), the intervals (4.8) can be described by

$$\Pi_a = \{z \in [0, 1) : \eta_a(z) = 1, \eta_i(z) = 0 \text{ for } i < a\}. \tag{4.9}$$

Notice that for each $s \in \mathbb{N}$, the set of intervals $\{\Pi_a : a > s\}$ form a partition of the open interval $(0, 2^{-s})$.

The following result is of crucial importance later in the paper.

LEMMA 4.1. *For each $s \in \mathbb{N}$, the characteristic function $\chi([0, y), \cdot)$ of the interval $[0, y)$, $y \in [0, 1)$, has the representation*

$$\chi([0, y), x) = \chi^{(s)}([0, y), x) + \varepsilon^{(s)}(x, y), \tag{4.10}$$

where

$$\chi^{(s)}([0, y), x) = \frac{1}{2} - \frac{1}{2} \sum_{a=1}^{s} \chi(\Pi_a, x^{(s)} \oplus y^{(s)}) r_a(y). \tag{4.11}$$

Furthermore, for all $x, y \in [0, 1)$, we have

$$0 \leqslant \chi^{(s)}([0, y), x) \leqslant 1 \tag{4.12}$$

and

$$|\varepsilon^{(s)}(x, y)| \leqslant \tfrac{1}{2}\delta^{(s)}(x^{(s)} \oplus y^{(s)}). \tag{4.13}$$

Proof. We shall check the statements of the lemma for all possible arrangements of points x and y.

If $x = y$, then $\chi([0, y), y) = 0$, $\chi^{(s)}([0, y), y) = 1/2$, $\varepsilon^{(s)}(y, y) = -1/2$, and the bounds (4.12) and (4.13) hold.

If $x \neq y$, we put

$$\nu = \nu(x, y) = \min\{a \in \mathbb{N} : \eta_a(x) \neq \eta_a(y)\}.$$

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In view of (4.2), we obtain

$$y - x = (\eta_\nu(y) - \eta_\nu(x))2^{-\nu} + (\theta_\nu(y) - \theta_\nu(x))2^{-\nu}, \tag{4.14}$$

where $\eta_\nu(x) \neq \eta_\nu(y)$ and $0 \leqslant |\theta_\nu(y) - \theta_\nu(x)| < 1$. From (4.14), we conclude that:

(i) $x < y$ if and only if $\eta_\nu(y) = 1$ and $\eta_\nu(x) = 0$;
(ii) $x > y$ if and only if $\eta_\nu(y) = 0$ and $\eta_\nu(x) = 1$.

Furthermore, we conclude from (4.9) that

$$\chi(\Pi_a, x^{(a)} \oplus y^{(a)}) = \begin{cases} 1 & \text{if } a = \nu, \\ 0 & \text{if } a \neq \nu. \end{cases} \tag{4.15}$$

The above can be expressed by the explicit formulas

$$\begin{aligned} \chi([0, y), x) &= \chi(\Pi_\nu, x^{(\nu)} \oplus y^{(\nu)})\eta(y) \\ &= \tfrac{1}{2}\chi(\Pi_\nu, x^{(\nu)} \oplus y^{(\nu)})(1 - r_\nu(y)) \\ &= \tfrac{1}{2} - \chi(\Pi_\nu, x^{(\nu)} \oplus y^{(\nu)})r_\nu(y). \end{aligned} \tag{4.16}$$

Now, taking (4.16) and (4.15) into account, we consider the following two possibilities.

(i) If $\nu \leqslant s$, then (4.10) holds with $\varepsilon^{(s)}(x, y) = 0$, and the bounds (4.12) and (4.13) are obvious.

(ii) If $\nu > s$, then (4.10) holds with $\chi^{(s)}([0, y), x) = \frac{1}{2}$ and

$$\varepsilon^{(s)}(x, y) = -\tfrac{1}{2}\chi(\Pi_\nu, x^{(\nu)} \oplus y^{(\nu)})r_\nu(y),$$

and the bound (4.12) is obvious. The bound (4.13) holds because $\Pi_\nu \subset \Delta_s^0$ and, therefore,

$$\chi(\Pi_\nu, x^{(\nu)} \oplus y^{(\nu)}) \leqslant \chi(\Delta_s^0, x^{(s)} \oplus y^{(s)}) = \delta^{(s)}(x^{(s)} \oplus y^{(s)});$$

cf. (4.4), (4.6).

The proof of Lemma 4.1 is complete. □

We emphasize that (4.16) and (4.15) imply the explicit formula

$$\begin{aligned} \chi([0, y), x) &= \sum_{a \in \mathbb{N}} \chi(\Pi_a, x^{(a)} \oplus y^{(a)})\eta_a(y) \\ &= \frac{1}{2} - \sum_{a \in \mathbb{N}} \chi(\Pi_a, x^{(a)} \oplus y^{(a)})r_a(y) - \delta(x, y), \end{aligned} \tag{4.17}$$

where $\delta(x, y)$ is equal to 1 if $x = y$ and to 0 otherwise.

Furthermore, for any x and y the sums in (4.17) contain at most one non-zero term. In this sense, one can say that series in (4.17) converge for all x and y, while

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the convergence is not uniform. Lemma 4.1 shows how we may deal with such series. Although the error terms $\varepsilon^{(s)}$ in (4.10) are not small, they are concentrated on small subsets.

Consider the multi-dimensional extension of the above result. For an arbitrary point $Y = (y_1, \dots, y_d) \in U^d$, we denote by $Y^{(s)} = (y_1^{(s)}, \dots, y_d^{(s)})$ its projection to $\mathbb{Q}^d(2^s)$, so that

$$Y = Y^{(s)} + \Theta_s(Y)2^{-s}, \quad s \in \mathbb{N}_0,$$

where

$$\Theta_s(Y) = (\theta_s(y_1), \dots, \theta_s(y_d)) \in U^d. \tag{4.18}$$

Introduce elementary boxes of the form

$$\Pi_A = \Pi_{a_1} \times \cdots \times \Pi_{a_d}, \quad A = (a_1, \dots, a_d) \in \mathbb{N}_0^d. \tag{4.19}$$

Each such box has volume $\operatorname{vol} \Pi_A = 2^{-a_1 - \cdots - a_d}$.

Write $\varkappa(A)$ for the number of non-zero elements in $A = (a_1, \dots, a_d) \in \mathbb{N}_0^d$.

Multiplying the formulas (4.10) written for $x = x_j$, $y = y_j$, $j = 1, \dots, d$ (recall that $r_0(y) \equiv 1$ and $\Pi_0 = [0, 1)$), we obtain the following result.

LEMMA 4.2. *For each $s \in \mathbb{N}$, the characteristic function $\chi(B_Y, X)$ of the rectangular box $B_Y = [0, y_1) \times \cdots \times [0, y_d)$, $Y \in U^d$, has the representation*

$$\chi(B_Y, X) = \chi^{(s)}(B_Y, X) + \varepsilon^{(s)}(X, Y), \tag{4.20}$$

where

$$\chi^{(s)}(B_y, X) = 2^{-d} \sum_{A \in I_s^d} (-1)^{\varkappa(A)} \chi(\Pi_A, X^{(s)}) r_A(Y). \tag{4.21}$$

Furthermore, for all $X = (x_1, \dots, x_d)$, $Y = (y_1, \dots, y_d) \in U^d$, we have

$$0 \leqslant \chi^{(s)}(B_Y, X) \leqslant 1 \tag{4.22}$$

and

$$|\varepsilon^{(s)}(X, Y)| \leqslant \frac{1}{2} \sum_{j=1}^{d} \delta^{(s)}(x_j^{(s)} \oplus y_j^{(s)}). \tag{4.23}$$

Proof. By definition

$$\chi^{(s)}(B_y, X) = \prod_{j=1}^{d} \chi^{(s)}([0, y_j), x_j),$$

and (4.22) follows from (4.12).

Using (3.12), we obtain

$$\begin{aligned} \chi(B_Y, X) &= \chi(B_{\mathbf{Y}}, \mathbf{X}) \chi([0, y), x) \\ &= \chi^{(s)}(B_{\mathbf{Y}}, \mathbf{X}) + \varepsilon^{(s)}(\mathbf{X}, \mathbf{Y})(\chi^{(s)}([0, y), x) + \varepsilon^{(s)}(x, y)) \\ &= \chi^{(s)}(B_Y, X) + \varepsilon^{(s)}(X, Y), \end{aligned}$$

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where

$$\varepsilon^{(s)}(X,Y)=\varepsilon^{(s)}(\mathbf{X},\mathbf{Y})\chi^{(s)}([0,y),x)+\varepsilon^{(s)}(x,y)\chi(B_Y,X).$$

Therefore

$$|\varepsilon^{(s)}(X,Y)|\leqslant|\varepsilon^{(s)}(\mathbf{X},\mathbf{Y})|+|\varepsilon^{(s)}(x,y)|. \tag{4.24}$$

In the one-dimensional case, the bound (4.23) is given in (4.13). Using (4.24), we obtain (4.23) in all dimensions by induction on d. □

Multiplying the formulas (3.2) written for $y=y_j$, $j=1,\dots,d$, we obtain

$$y_1\dots y_d=2^{-d}\sum_{A\in\mathbb{N}_0^d}(-1)^{\varkappa(A)}2^{-a_1-\dots-a_d}r_A(Y).$$

Since $\operatorname{vol}B_Y=y_1\dots y_d$ and $\operatorname{vol}\Pi_A=2^{-a_1-\dots-a_d}$, this can be rewritten in the form

$$\begin{aligned}\operatorname{vol}B_Y&=2^{-d}\sum_{A\in\mathbb{N}_0^d}(-1)^{\varkappa(A)}\operatorname{vol}\Pi_A r_A(Y)\\&=\operatorname{vol}^{(s)}B_Y+\varepsilon^{(s)}(Y),\quad s\in\mathbb{N}_0,\end{aligned} \tag{4.25}$$

where

$$\operatorname{vol}^{(s)}B_Y=2^{-d}\sum_{A\in I_s^d}(-1)^{\varkappa(A)}\operatorname{vol}\Pi_A r_A(Y), \tag{4.26}$$

and $\varepsilon^{(s)}(Y)$ satisfies the bound

$$|\varepsilon^{(s)}(Y)|\leqslant d2^{-s-1},\quad Y\in U^d, \tag{4.27}$$

easily proved by induction on d.

The local discrepancy (1.1) can be written in the form

$$\mathcal{L}[D,Y]=\sum_{X\in D}\mathcal{L}(X,Y),\qquad\mathcal{L}(X,Y)=\chi(B_Y,X)-\operatorname{vol}B_Y. \tag{4.28}$$

Substituting (4.20) and (4.25) into (4.28), we obtain

$$\mathcal{L}(X,Y)=\mathcal{L}^{(s)}(X,Y)+\mathcal{E}^{(s)}(X,Y), \tag{4.29}$$

where

$$\mathcal{L}^{(s)}(X,Y)=2^{-2}\sum_{A\in I_s^d}(-1)^{\varkappa(A)}\lambda_A(X^{(s)}\oplus Y^{(s)})r_A(Y),$$

$$\lambda_A(X^{(s)}\oplus Y^{(s)})=\chi(\Pi_A,X^{(s)}\oplus Y^{(s)})-\operatorname{vol}\Pi_A,$$

and

$$\mathcal{E}^{(s)}(X,Y)=\varepsilon^{(s)}(X,Y)-\varepsilon^{(s)}(Y).$$

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In view of (4.23) and (4.27), we have

$$|\mathcal{E}^{(s)}(X,Y)| \leqslant \frac{1}{2}\Big(\sum_{j=1}^{d} \delta^{(s)}(x_j^{(s)} \oplus y_j^{(s)}) + d2^{-s}\Big), \quad X, Y \in U^d.$$

For an arbitrary distribution $D \subset U^d$, we denote by

$$D^{(s)} = \{X^{(s)} : X \in D\}, \quad s \in \mathbb{N}_0,$$

its projection onto $\mathbb{Q}^d(2^s)$, so that $|D^{(s)}| = |D|$, where some points of $D^{(s)}$ may coincide.

We define the *micro-local discrepancies* by

$$\begin{aligned}\lambda_A[D^{(s)} \oplus Y^{(s)}] &= \sum_{X \in D} \lambda_A(X^{(s)} \oplus Y^{(s)}) \\ &= \sum_{X \in D} (\chi(\Pi_A, X^{(s)} \oplus Y^{(x)}) - \operatorname{vol} \Pi_A) \\ &= |(D^{(s)} \oplus Y^{(s)}) \cap \Pi_A| - |D| \operatorname{vol} \Pi_A. \end{aligned} \tag{4.30}$$

Substituting (4.29) into (4.28), we arrive at the following result summarizing the above discussion.

LEMMA 4.3. *For each $s \in \mathbb{N}$, the local discrepancy $\mathcal{L}[D,Y]$ has the representation*

$$\mathcal{L}[D,Y] = \mathcal{L}^{(s)}[D,Y] + \mathcal{E}^{(s)}[D,Y], \tag{4.31}$$

where

$$\mathcal{L}^{(s)}[D,Y] = 2^{-d} \sum_{A \in I_s^d} (-1)^{\varkappa(A)} \lambda_A[D^{(s)} \oplus Y^{(s)}] r_A(Y), \tag{4.32}$$

and the term $\mathcal{E}^{(s)}[D,Y]$ satisfies the bound

$$|\mathcal{E}^{(s)}[D,Y]| \leqslant \frac{1}{2}\Big(\sum_{j=1}^{d} \delta_j^{(s)}[D^{(s)} \oplus Y^{(s)}] + d|D|2^{-s}\Big), \tag{4.33}$$

where

$$\delta_j^{(s)}[D^{(s)} \oplus Y^{(s)}] = \sum_{X \in D} \delta_j^{(s)}(x_j^{(s)} \oplus y_j^{(s)}). \tag{4.34}$$

§5. *Explicit formulas and preliminary bounds for the mean discrepancies.* Applying Lemma 4.3 to a shifted distribution $D \oplus T$, $T \in \mathbb{Q}^d(2^s)$, we obtain

$$\mathcal{L}[D \oplus T, Y] = \mathcal{L}^{(s)}[D \oplus T, Y] + \mathcal{E}^{(s)}[D \oplus T, Y], \tag{5.1}$$

where the term $\mathcal{L}^{(s)}[D \oplus T, Y]$ can be written in the form

$$\mathcal{L}^{(s)}[D \oplus T, Y] = \mathcal{F}^{(s)}[D, T \oplus Y^{(s)}, Y^{(s)}], \tag{5.2}$$

$$\mathcal{F}^{(s)}[D,Z,Y] = 2^{-d} \sum_{A \in I_s^d} (-1)^{\varkappa(A)} \lambda_A[D \oplus Z] r_A(Y), \tag{5.3}$$

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and

$$\lambda_A[D \oplus Z] = \sum_{X \in D} (\chi(\Pi_A, X^{(s)} \oplus Z) - \operatorname{vol} \Pi_A)$$
$$= |(D \oplus Z) \cap \Pi_A| - |D| \operatorname{vol} \Pi_A, \quad Z \in \mathbb{Q}^d(2^s). \tag{5.4}$$

Let $L_q(\mathbb{Q}^d(2^s) \times U^d)$, $0 < q \leqslant \infty$, be the space consisting of all functions $f(T, Y)$, $T \in \mathbb{Q}^d(2^s)$, $Y \in U^d$, with $|||f|||_q < \infty$, where

$$|||f|||_q = \left(2^{-ds} \sum_{T \in \mathbb{Q}^d(2^s)} \int_{U^d} |f(T, Y)|^q \, dY\right)^{1/q}, \quad 0 < q < \infty,$$

and

$$|||f|||_\infty = \max_{T \in \mathbb{Q}^d(2^s)} \sup_{Y \in U^d} |f(T, Y)|.$$

For any two functions $f_1, f_2 \in L_q(\mathbb{Q}^d(2^s) \times U^d)$, we have

$$|||f_1 + f_2|||_q \leqslant |||f_1|||_q + |||f_2|||_q, \quad 1 \leqslant q \leqslant \infty, \tag{5.5}$$
$$|||f_1 + f_2|||_q^q \leqslant |||f_1|||_q^q + |||f_2|||_q^q, \quad 0 < q \leqslant 1. \tag{5.6}$$

For $1 \leqslant q < \infty$, (5.5) is the standard Minkowski inequality, while (5.6) is its modification for $0 < q < 1$; see [**17**, Ch. 1, (9.11), (9.13)].

Now write

$$\mathcal{M}_q^{(s)}[D] = |||\mathcal{L}^{(s)}[D \oplus ., .]|||_q, \quad 0 < q \leqslant \infty, \tag{5.7}$$

and

$$\mathcal{E}_q^{(s)}[D] = |||\mathcal{E}^{(s)}[D \oplus ., .]|||_q, \quad 0 < q \leqslant \infty. \tag{5.8}$$

Substituting (5.1) into (1.7) and using (5.5), we obtain the upper bound

$$\mathcal{M}_{s,q}[D] \leqslant \mathcal{M}_q^{(s)}[D] + \mathcal{E}_q^{(s)}[D], \quad 1 \leqslant q < \infty. \tag{5.9}$$

For $0 < q \leqslant 1$, we can simply put

$$\mathcal{M}_{s,q}[D] \leqslant \mathcal{M}_{s,1}[D] \leqslant \mathcal{M}_1^{(s)}[D] + \mathcal{E}_1^{(s)}[D], \quad 0 < q \leqslant 1. \tag{5.10}$$

Similarly, using (5.6), we obtain the lower bound

$$\mathcal{M}_{s,q}[D]^q \geqslant \mathcal{M}_q^{(s)}[D]^q - \mathcal{E}_q^{(s)}[D]^q, \quad 0 < q \leqslant 1. \tag{5.11}$$

The bounds (5.9)–(5.11) will be used in the proofs of Theorems 2.1 and 2.2.

It follows from (5.2) and (5.3) that $\mathcal{L}^{(s)}[D \oplus T, Y]$, as a function of $Y \in U^d$, belongs to the space $\mathcal{B}_s^d$. Hence we can use (3.7) and write (5.7) in the form

$$\mathcal{M}_q^{(s)}[D] = \left(2^{-ds} \sum_{T \in \mathbb{Q}^d(2^s)} \|\mathcal{L}^{(s)}[D \oplus T, .]\|_{s,q}^q\right)^{1/q}$$
$$= \left(2^{-2ds} \sum_{T, Y \in \mathbb{Q}^d(2^s)} |\mathcal{L}^{(s)}[D \oplus T, Y]|^q\right)^{1/q}, \quad 0 < q < \infty. \tag{5.12}$$

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The following simple observation explains why the mean L_q-discrepancies can be expressed in terms of Rademacher series.

In the vector space of pairs $(T, Y) \in \mathbb{Q}^d(2^s) \times \mathbb{Q}^d(2^s) \simeq \mathbb{F}_2^{2ds}$, we consider the linear mapping

$$\tau : (T, Y) \to (T \oplus Y, Y). \tag{5.13}$$

Obviously, $\tau^2 = \mathbb{1}$, $\tau^{-1} = \tau$. Hence, τ is a one-to-one mapping, and in the double sum in (5.12), the variables $Z = T \oplus Y$ and Y can be viewed as independent. As a result, we have

$$\mathcal{M}_q^{(s)}[D] = \left(2^{-ds} \sum_{Z \in \mathbb{Q}^d(2^s)} \mathcal{F}_q^{(s)}[D, Z]^q\right)^{1/q}, \quad 0 < q < \infty, \tag{5.14}$$

where

$$\mathcal{F}_q^{(s)}[D, Z] = \left(2^{-ds} \sum_{Y \in \mathbb{Q}^d(2^s)} |\mathcal{F}[D, Z, Y]|^q\right)^{1/q}. \tag{5.15}$$

Formulas (5.14) and (5.15) will be used in the proofs of Theorems 2.1 and 2.2.

In the case of the mean L_∞-discrepancy, the above argument needs to be slightly modified. First of all, using (1.8) and (1.3), we can write

$$\mathcal{M}_{s,\infty}[D] = \max_{T \in \mathbb{Q}^d(2^s)} \sup_{Y \in U^d} |\mathcal{L}[D \oplus T, Y]| \geqslant \max_{T, Y \in \mathbb{Q}^d(2^s)} |\mathcal{L}[D \oplus T, Y]|. \tag{5.16}$$

For $Z, Y \in \mathbb{Q}^d(2^s)$, we put $T = Z \oplus Y$ and

$$\mathcal{F}[D, Z, Y] = \mathcal{L}[D \oplus Z \oplus Y, Y]. \tag{5.17}$$

With this notation, (5.1) takes the form

$$\mathcal{F}[D, Z, Y] = \mathcal{F}^{(s)}[D, Z, Y] + \mathcal{E}^{(s)}[D, Z, Y], \tag{5.18}$$

where $\mathcal{F}^{(s)}[D, Z, Y]$ is defined in (5.3) and

$$\mathcal{E}^{(s)}[D, Z, Y] = \mathcal{E}^{(s)}[D \oplus Z \oplus Y, Y]. \tag{5.19}$$

Since τ defined in (5.13) is a one-to one mapping, we have

$$\max_{T, Y \in \mathbb{Q}^d(2^s)} |\mathcal{L}[D \oplus T, Y]| = \max_{Z, Y \in \mathbb{Q}^d(2^s)} |\mathcal{F}[D, Z, Y]|. \tag{5.20}$$

This relation can be continued as follows. We have

$$\begin{aligned} \max_{Z, Y \in \mathbb{Q}^d(2^s)} |\mathcal{F}[D, Z, Y]| &= \max_{Z \in \mathbb{Q}^d(2^s)} \max_{Y \in \mathbb{Q}^d(2^s)} |\mathcal{F}[D, Z, Y]| \\ &\geqslant 2^{-ds} \sum_{Z \in \mathbb{Q}^d(2^s)} \max_{Y \in \mathbb{Q}^d(2^s)} |\mathcal{F}[D, Z, Y]| \\ &\geqslant \mathcal{F}_{1,\infty}^{(s)}[D] - \mathcal{E}_{1,\infty}^{(s)}[D], \end{aligned} \tag{5.21}$$

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where

$$\mathcal{F}^{(s)}_{1,\infty}[D] = 2^{-ds} \sum_{Z \in \mathbb{Q}^d(2^s)} \mathcal{F}^{(s)}_{\infty}[D, Z], \tag{5.22}$$

$$\mathcal{F}^{(s)}_{\infty}[D, Z] = \max_{Y \in \mathbb{Q}^d(2^s)} |\mathcal{F}^{(s)}[D, Z, Y]|, \tag{5.23}$$

and

$$\mathcal{E}^{(s)}_{1,\infty}[D] = 2^{-ds} \sum_{Z \in \mathbb{Q}^d(2^s)} \max_{Y \in \mathbb{Q}^d(2^s)} |\mathcal{E}^{(s)}[D, Z, Y]|. \tag{5.24}$$

Comparing (5.16), (5.20) and (5.21), we obtain the lower bound

$$\mathcal{M}_{s,\infty}[D] \geqslant \mathcal{F}^{(s)}_{1,\infty}[D] - \mathcal{E}^{(s)}_{1,\infty}[D]. \tag{5.25}$$

This bound will be used in the proof of Theorem 2.3.

We shall call the quantities $\mathcal{M}^{(s)}_q[D]$ and $\mathcal{F}^{(s)}_{1,\infty}[D]$ the *principal terms*, and the quantities $\mathcal{E}^{(s)}_q[D]$ and $\mathcal{E}^{(s)}_{1,\infty}[D]$ the *error terms*.

§6. *Bounds for the error terms and some auxiliary bounds.*

LEMMA 6.1. (i) *Let D_{2^s} be an arbitrary dyadic (δ, s, d)-net. Then*

$$\mathcal{E}^{(s)}_q[D_{2^s}] \leqslant d2^{\delta}, \quad 0 < q \leqslant \infty. \tag{6.1}$$

(ii) *Let $D_N \subset U^d$ be an arbitrary N-point distribution. Then*

$$\mathcal{E}^{(s)}_q[D_N] \leqslant dN2^{-s}, \quad 0 < q \leqslant 1, \tag{6.2}$$

and

$$\mathcal{E}^{s}_{1,\infty}[D_N] \leqslant dN2^{-s}. \tag{6.3}$$

Proof. The functions $\delta^{(s)}_j[D^{(s)} \oplus Y^{(s)}]$, $j = 1, \ldots, d$, defined in (4.34), belong to the space $\mathcal{B}^d_s$ and satisfy (3.7). We put

$$\delta^{(s)}_{j,q}[D] = \|\delta^{(s)}_j[D^{(s)} \oplus .]\|_q = \|\delta^{(s)}_j[D^{(s)} \oplus .]\|_{s,q}, \quad 0 < q \leqslant \infty. \tag{6.4}$$

Obviously,

$$\delta^{(s)}_{j,q}[D \oplus Z] = \delta^{(s)}_{j,q}[D], \quad Z \in \mathbb{Q}^d(2^s). \tag{6.5}$$

Applying (4.5) to (4.34), we obtain

$$\delta^{(s)}_j[D^{(s)} \oplus Z] = \sum_{m=0}^{2^s-1} N_{j,m} \chi(\Delta^m_{s,j}, z_j), \tag{6.6}$$

where

$$N_{j,m} = \sum_{X \in D} \chi(\Delta^m_s, x^{(s)}_j) = |D \cap \Delta^m_{s,j}|,$$

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and $\Delta_{s,j}^m$ denotes the elementary box

$$\Delta_{s,j}^m = \{X = (x_1, \dots, x_d) \in U^d : x_j \in \Delta_s^m \text{ and } x_i \in [0,1), i \neq j\}.$$

Notice that $\operatorname{vol} \Delta_{s,j}^m = 2^{-s}$. Also, for each $j = 1, \dots, d$, the boxes $\Delta_{s,j}^m$, $m = 0, 1, \dots, 2^s - 1$, form a partition of the unit cube U^d. Therefore,

$$\sum_{m=0}^{2^s-1} N_{j,m} = N = |D|. \tag{6.7}$$

(i) From (6.6), we obtain the bound

$$\delta_{j,q}^{(s)}[D] \leqslant \delta_{j,\infty}^{(s)} \leqslant \max_m N_{j,m}, \quad 0 < q \leqslant \infty. \tag{6.8}$$

Using (5.8), (4.33), and (6.5), we obtain

$$\mathcal{E}_q^{(s)}[D \oplus T] \leqslant \frac{1}{2}\left(\sum_{j=1}^d \delta_{j,\infty}^{(s)}[D] + d|D|2^{-s}\right), \quad 0 < q \leqslant \infty. \tag{6.9}$$

If D_{2^s} is an arbitrary (δ, s, d)-net, then $N = 2^s$ and $N_{j,m} \leqslant 2^\delta$ for all j and m; see (1.11). Comparing the bounds (6.8) and (6.9) for such a net, we obtain (6.1).

(ii) From (6.6) and (6.7), we obtain the bound

$$\delta_{j,q}^{(s)}[D] \leqslant \delta_{j,1}^{(s)}[D] = \sum_{j=1}^{2^s-1} N_{j,m} 2^{-s} = N 2^{-s}, \quad 0 < q \leqslant 1. \tag{6.10}$$

Using (5.8), (4.33) and (6.5), we obtain

$$\begin{aligned} \mathcal{E}_q^{(s)}[D \oplus T] &\leqslant \mathcal{E}_1^{(s)}[D \oplus T] \\ &\leqslant \frac{1}{2}\left(\sum_{j=1}^d \delta_{j,1}^{(s)}[D] + d|D|2^{-s}\right), \quad 0 < q \leqslant 1. \end{aligned} \tag{6.11}$$

If D_N is an arbitrary N-point distribution, then the bounds (6.10) and (6.11) imply (6.2).

For the function (5.19), the bound (4.33) takes the form

$$\begin{aligned} |\mathcal{E}^{(s)}[D, Z, Y]| &= |\mathcal{E}^{(s)}[D \oplus Z \oplus Y, Y]| \\ &\leqslant \frac{1}{2}\left(\sum_{j=1}^d \delta_j^{(s)}[D^{(s)} \oplus Z] + d|D|2^{-s}\right), \end{aligned} \tag{6.12}$$

where the right-hand side is independent of Y. Substituting (6.12) into (5.24), we obtain

$$\mathcal{E}_{1,\infty}^{(s)}[D] \leqslant \frac{1}{2}\left(\sum_{j=1}^d \delta_{j,1}^{(s)}[D] + d|D|2^{-s}\right). \tag{6.13}$$

If D_N is an arbitrary N-point distribution, then the bounds (6.10) and (6.13) imply (6.3). The proof of Lemma 6.1 is complete. □

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Next, we establish the bound (2.3) mentioned in our earlier discussion of Theorem 2.1.

LEMMA 6.2. *For an arbitrary distribution $D \subset U^d$, we have*

$$\mathcal{L}_q[D] \leqslant \mathcal{L}_\infty[D] \leqslant 2^{ds/q}(\mathcal{L}_q[D] + 2\mathcal{E}_\infty^{(s)}[D]), \quad 1 \leqslant q < \infty, \tag{6.14}$$

where the term $\mathcal{E}_\infty^{(s)}[D]$ is defined in (5.8). In particular, for an arbitrary (δ, s, d)-net D_{2^s} and $q = \varepsilon s$, $\varepsilon > 0$, the bound (6.14) takes the form

$$\mathcal{L}_q[D_{2^s}] \leqslant \mathcal{L}_\infty[D_{2^s}] \leqslant 2^{d/\varepsilon}(\mathcal{L}_q[D_{2^s}] + d2^{\delta+1}). \tag{6.15}$$

Proof. It follows from (4.7) that the function

$$\delta^{(s)}(X^{(s)} \oplus Y^{(s)}) = \prod_{j=1}^{d} \delta^{(s)}(x_j^{(x)} \oplus y_j^{(s)})$$

is the reproducing kernel for the space $\mathcal{B}_s^d$; in other words,

$$f(X) = \sum_{Y \in \mathbb{Q}^d(2^s)} \delta^{(s)}(X^{(s)} \oplus Y^{(s)}) f(Y), \quad f \in \mathcal{B}_s^d. \tag{6.16}$$

Applying Hölder's inequality to the sum in (6.16) and taking (3.7) into account, we obtain

$$\begin{aligned} \|f\|_\infty = \|f\|_{s,\infty} &\leqslant \left(\sum_{Y \in \mathbb{Q}^d(2^s)} |f(Y)|^q \right)^{1/q} \\ &= 2^{ds/q} \|f\|_{s,q} = 2^{ds/q} \|f\|_q, \quad 1 \leqslant q < \infty. \end{aligned}$$

In particular,

$$\|\mathcal{L}^{(s)}[D, .]\|_\infty \leqslant 2^{ds/q} \|\mathcal{L}^{(s)}[D, .]\|_q, \tag{6.17}$$

where $\mathcal{L}^{(s)}[D, .]$ is defined in (4.32).

On the other hand, we deduce from (4.31) and (5.8) that

$$\|\mathcal{L}^{(s)}[D, .]\|_\infty \geqslant \|\mathcal{L}[D, .]\|_\infty - \|\mathcal{E}[D, .]\|_\infty \geqslant \mathcal{L}_\infty[D] - \mathcal{E}_\infty^{(s)}[D]$$

and

$$\|\mathcal{L}^{(s)}[D, .]\|_q \leqslant \|\mathcal{L}[D, .]\|_q + \|\mathcal{E}[D, .]\|_\infty \leqslant \mathcal{L}_q[D] + \mathcal{E}_\infty^{(s)}[D].$$

Comparing these inequalities with (6.17), we obtain

$$\begin{aligned} \mathcal{L}_\infty[D] &\leqslant 2^{ds/q}(\mathcal{L}_q[D] + \mathcal{E}_\infty^{(s)}[D]) + \mathcal{E}_\infty^{(s)}[D] \\ &\leqslant 2^{ds/q}(\mathcal{L}_q[D] + 2\mathcal{E}_\infty^{(s)}[D]). \end{aligned}$$

This proves the right bound in (6.14). The left bound is obvious.

If D_{2^s} is a (δ, s, d)-net and $q = \varepsilon s$, $\varepsilon > 0$, then using the bound (6.1), we obtain (6.15).

The proof of Lemma 6.2 is complete. □

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To conclude this section, we give one further auxiliary result that will be used in the proofs of Theorems 2.2 and 2.3.

Consider the subset

$$J_\sigma^k(s) = \{\Pi_A : A \in I_s^k \text{ and } \operatorname{vol}\Pi_A = 2^{-\sigma}\}, \quad \sigma \in \mathbb{N}, \tag{6.18}$$

of the k-dimensional elementary boxes $\Pi_A \subset U^k$, $k \geqslant 2$; see (4.19).

LEMMA 6.3. *If $s \geqslant \sigma$, then the subset $J_\sigma^k(s) = J_\sigma^k$ is independent of s, and*

$$|J_\sigma^k| \geqslant \left(\frac{\sigma}{k-1}\right)^{k-1}. \tag{6.19}$$

Proof. Since $\operatorname{vol}\Pi_A = 2^{-a_1-\cdots-a_k}$, the subset (6.18) consists of boxes Π_A with $A = (a_1, \dots, a_k) \in I_s^k$, where

$$a_1 + \cdots + a_k = \sigma. \tag{6.20}$$

Each solution of (6.20) satisfies $0 \leqslant a_j \leqslant \min\{\sigma, s\}$, $j = 1, \dots, k$. For $s \geqslant \sigma$, the set of all solutions is independent of s.

If $s \geqslant \sigma$, then for any $(a_1, \dots, a_{k-1}) \in \mathbb{N}_0^{k-1}$ with $0 \leqslant a_j \leqslant \lfloor \sigma/(k-1) \rfloor$, $j = 1, \dots, k-1$, the integer $a_k = \sigma - a_1 - \cdots - a_{k-1}$ satisfies $0 \leqslant a_k \leqslant \sigma$. Therefore, $A = (a_1, \dots, a_k)$ is a solution of (6.20), and

$$|J_\sigma^k| \geqslant (1 + \lfloor \sigma/(k-1) \rfloor)^{k-1} \geqslant \left(\frac{\sigma}{k-1}\right)^{k-1}. \qquad \square$$

§7. *Proofs of Theorems 2.1–2.3.* The proof of each of Theorems 2.1–2.3 consists of two steps. First, relying on the bounds for sums of Rademacher functions given in Lemmas 3.1 and 3.2, we establish very good bounds for the principal terms $\mathcal{M}_q^{(s)}[D]$ and $\mathcal{F}_{1,\infty}^{(s)}[D]$. Next, relying on the upper bounds for the error terms $\mathcal{E}_q^{(s)}[D]$ and $\mathcal{E}_{1,\infty}^{(s)}$ given in Lemma 6.1, we compare the principal terms with the corresponding mean discrepancies $\mathcal{M}_{s,q}[D]$.

Proof of Theorem 2.1. Let D_{2^s} be a (δ, s, d)-net. We first study the quantity (5.15). Applying (3.17) to (5.3), we have

$$\mathcal{F}_q^{(s)}[D_{2^s}, Z] \leqslant \beta_q^{d-1} Q_{\infty,2}[\mathcal{F}^{(s)}], \tag{7.1}$$

where

$$Q_{\infty,2}[\mathcal{F}^{(s)}] = 2^{-d} \max_{y \in \mathbb{Q}(2^s)} \left(\sum_{\mathbf{A} \in I_s^{d-1}} \Phi_{\mathbf{A}}(Z, y)^2 \right)^{1/2}, \tag{7.2}$$

$$\Phi_{\mathbf{A}}(Z, y) = \sum_{a \in I_s} \lambda_A[D_{2^s} \oplus Z] r_a(y), \tag{7.3}$$

and the coefficients $\lambda_A[D_{2^s} \oplus Z]$ are defined in (5.4).

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For each $Z \in \mathbb{Q}^d(2^s)$, the shift $D_{2^s} \oplus Z$ is also a (δ, s, d)-net, and it follows from (1.11) that

$$\lambda_A[D_{2^s} \oplus Z] = 0 \quad \text{if vol } \Pi_A \geqslant 2^{\delta - s}.$$

The condition on volumes can be written in the form

$$\text{vol } \Pi_A = \text{vol } \Pi_{\mathbf{A}} \text{ vol } \Pi_a = 2^{-a_1 - \cdots - a_{d-1} - a} \geqslant 2^{\delta - s}$$

or $a \leqslant s - \delta - a_1 - \cdots - a_{d-1}$. Therefore, the summation in (7.3) is extended to

$$s \geqslant a \geqslant l, \quad l = \max\{0, s - \delta - a_1 - \cdots - a_{d-1} + 1\}.$$

The elementary boxes Π_A are mutually disjoint. For a given $\mathbf{A}$, all the boxes $\Pi_A = \Pi_{\mathbf{A}} \times \Pi_a$, $s \geqslant a \geqslant l$, are embedded in the elementary box $\Pi_{\mathbf{A}} \times \Delta$, where $\Delta = \Delta^0_{l-1}$ if $l \geqslant 1$, and $\Delta = [0, 1)$ if $l = 0$. In both cases, vol $\Pi_{\mathbf{A}} \times \Pi_a \leqslant 2^{\delta - s}$. Hence $|(D_{2^s} \oplus Z) \cap (\Pi_{\mathbf{A}} \times \Delta)| \leqslant 2^{\delta}$ by the definition of (δ, s, d)-nets; see (1.11).

With these bounds, we now estimate the function (7.3). We have

$$\begin{aligned} |\Phi_{\mathbf{A}}(Z, y)| &\leqslant \sum_{a=l}^{s} |\lambda_A[D_{2^s} \oplus Z]| \\ &\leqslant \sum_{a=l}^{s} |(D_{2^s} \oplus Z) \cap \Pi_A| + |D_{2^s}| \sum_{a=l}^{s} \text{vol } \Pi_A \\ &\leqslant |(D_{2^s} \oplus Z) \cap (\Pi_{\mathbf{A}} \times \Delta)| + 2^s \text{ vol}(\Pi_{\mathbf{A}} \times \Delta) \leqslant 2^{\delta+1}. \end{aligned}$$

Substituting this into (7.2), we obtain

$$Q_{\infty,2}[\mathcal{F}^{(s)}] \leqslant 2^{-d+\delta+1} |I_s^{d-1}|^{1/2} = 2^{-d+\delta+1} (s+1)^{(1/2)(d-1)},$$

and therefore

$$\mathcal{F}_q^{(s)}[D_{2^s, Z}] \leqslant \beta_q^{d-1} 2^{-d+\delta+1} (s+1)^{(1/2)(d-1)}.$$

Thus, for the principal term (5.14), we have

$$\mathcal{M}_q^{(s)}[D_{2^s}] \leqslant 2^{-d+\delta+1} \lceil \tfrac{1}{2} q \rceil^{(1/2)(d-1)} (s+1)^{(1/2)(d-1)}, \tag{7.4}$$

where the bound (3.11) for the constant β_q has also been used.

Substituting (7.4) and (6.1) into (5.9) and (5.10), we obtain

$$\mathcal{M}_{s,q}[D_{2^s}] < 2^{-d+\delta+1} (\lceil \tfrac{1}{2} q \rceil (s+1))^{(1/2)(d-1)} + d 2^{\delta}, \quad 0 < q < \infty.$$

The proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. Let $D_N \subset U^d$, $d \geqslant 2$, be an N-point distribution. We first study the quantity (5.15). Applying (3.19) to (5.3), we have

$$\mathcal{F}_q^{(s)}[D_N, Z] \geqslant \alpha_q^d Q_2[\mathcal{F}^{(s)}], \tag{7.5}$$

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where

$$Q_2[\mathcal{F}^{(s)}] = 2^{-d}\Big(\sum_{A\in I_s^d}\lambda_A[D_N\oplus Z]^2\Big)^{1/2}. \tag{7.6}$$

The coefficients $\lambda_A[D_N\oplus Z]$ are defined in (5.4), and it is clear that

$$|\lambda_A[D_N\oplus Z]| \geqslant \langle\langle N \operatorname{vol}\Pi_A\rangle\rangle, \tag{7.7}$$

where $\langle\langle t\rangle\rangle = \min\{|t-n| : n\in\mathbb{Z}\}$ is the distance of $t\in\mathbb{R}$ from the nearest integer. Thus

$$Q_2[\mathcal{F}^{(s)}] \geqslant 2^{-d}\Big(\sum_{A\in I_s^d}\langle\langle N \operatorname{vol}\Pi_A\rangle\rangle^2\Big)^{1/2}. \tag{7.8}$$

Let $\sigma\in\mathbb{N}$ be chosen to satisfy

$$2^{-2} < N2^{-\sigma} \leqslant 2^{-1}.$$

Then $\langle\langle N \operatorname{vol}\Pi_A\rangle\rangle > 2^{-2}$ for all boxes Π_A with $\operatorname{vol}\Pi_A = 2^{-\sigma}$.

Let $s\in\mathbb{N}$ be chosen to satisfy

$$s \geqslant \sigma = \lceil\log N\rceil + 1 \geqslant \log N + 1. \tag{7.9}$$

Then, using Lemma 6.3 with $k=d$, we have

$$\sum_{A\in I_s^d}\langle\langle N \operatorname{vol}\Pi_A\rangle\rangle^2 \geqslant \sum_{A\in J_\sigma^d}\langle\langle N \operatorname{vol}\Pi_A\rangle\rangle^2 > 2^{-4}|J_\sigma^d| \geqslant 2^{-4}\Big(\frac{\log N+1}{d-1}\Big)^{d-1}.$$

Substituting this into (7.8), we have

$$Q_2[\mathcal{F}^{(s)}] > 2^{-d-2}(d-1)^{-(1/2)(d-1)}(\log N+1)^{(1/2)(d-1)},$$

and therefore

$$\mathcal{F}^{(s)}[D_N, Z] > \alpha_q^d 2^{-d-2}(d-1)^{-(1/2)(d-1)}(\log N+1)^{(1/2)(d-1)}.$$

Thus, for the principal term (5.14), we have

$$\begin{aligned}\mathcal{M}_q^{(s)}[D_N] &> c_q(d)(\log N+1)^{(1/2)(d-1)}, \quad 0<q\leqslant 1,\\ c_q(d) &= 2^{-2d/q-d-1}(d-1)^{-(1/2)(d-1)},\end{aligned} \tag{7.10}$$

where the bound (3.10) for the constant α_q has also been used.

Substituting (7.10) and (6.2) into (5.11), we have

$$\begin{aligned}\mathcal{M}_{s,q}[D_N]^q &> c_q(d)^q(\log N+1)^{(1/2)(d-1)q} - (dN2^{-s})^q\\ &\geqslant c_q(d)^q(\log N+1)^{(1/2)(d-1)q}(1-\xi_q(s)), \quad 0<q\leqslant 1,\end{aligned}$$

where

$$\xi_q(s) = c_q(d)^{-q}(dN2^{-s})^q.$$

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Let s be chosen sufficiently large to satisfy $\xi_q(s) \leqslant 1/2$. To do this, we put

$$s \geqslant \log N + \frac{2d+1}{q} + \frac{1}{2}(d-1)\log(d-1) + d + 1 + \log d,$$

and in this case condition (7.9) is also satisfied.

As a result, we have

$$\mathcal{M}_{s,q}[D_N] > \gamma_q(d)(\log N + 1)^{(1/2)(d-1)}, \quad 0 < q \leqslant 1,$$

where

$$\gamma_q(d) = 2^{-1/q} c_q(d) = 2^{-(2d+1)/q - d - 1}(d-1)^{-(1/2)(d-1)}$$

The proof of Theorem 2.2 is complete. □

Proof of Theorem 2.3. Let $D_N \subset U^d$, $d \geqslant 3$, be an N-point distribution. We first study the quantity (5.23). Applying (3.20) to (5.3), we have

$$\mathcal{F}_\infty^{(s)}[D_N, Z] \geqslant \alpha_1^{d-1} Q_{1,2}[\mathcal{F}^{(s)}], \tag{7.11}$$

where

$$Q_{1,2}[\mathcal{F}^{(s)}] = 2^{-d} \sum_{a \in I_s} Q_2[\varphi_a], \tag{7.12}$$

and

$$Q_2[\varphi_a] = \left(\sum_{\mathbf{A} \in I_s^{d-1}} \lambda_A[D_N \oplus Z]^2 \right)^{1/2}. \tag{7.13}$$

Using (7.7), we deduce that

$$Q_2[\varphi_a] \geqslant \left(\sum_{\mathbf{A} \in I_s^{d-1}} \langle\langle N \operatorname{vol} \Pi_A \rangle\rangle^2 \right)^{1/2}. \tag{7.14}$$

Notice that $\operatorname{vol} \Pi_A = \operatorname{vol} \Pi_A \operatorname{vol} \Pi_a = \operatorname{vol} \Pi_A 2^{-a}$. Define $\sigma_a \in \mathbb{N}$ by

$$2^{-2} < N 2^{-\sigma_a - a} \leqslant 2^{-1}.$$

Then $\langle\langle N \operatorname{vol} \Pi_A \rangle\rangle > 2^{-2}$ for all boxes $\Pi_{\mathbf{A}}$ with $\operatorname{vol} \Pi_{\mathbf{A}} = 2^{-\sigma_a}$.

It is clear that $\sigma_a = \sigma - a$, $0 \leqslant a \leqslant \sigma$, where

$$\sigma = \lceil \log N \rceil + 1 \geqslant \log N + 1.$$

Assume now that

$$0 \leqslant a \leqslant \tfrac{1}{2}\sigma \quad \text{and} \quad \sigma \geqslant \sigma_a \geqslant \tfrac{1}{2}\sigma.$$

Let $s \in \mathbb{N}$ be chosen to satisfy $s \geqslant \sigma$. Then

$$s \geqslant \sigma = \sigma_0 > \sigma_1 > \sigma_2 > \cdots. \tag{7.15}$$

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Using Lemma 6.3 with $k = d - 1$, we have

$$\sum_{\mathbf{A}\in I_s^{d-1}} \langle\langle N \operatorname{vol} \Pi_A\rangle\rangle^2 \geqslant \sum_{\mathbf{A}\in J_{\sigma_a}^{d-1}} \langle\langle N \operatorname{vol} \Pi_A\rangle\rangle^2 > 2^{-4}|J_{\sigma_a}^{d-1}|$$
$$\geqslant 2^{-4}\left(\frac{\sigma_a}{d-2}\right)^{d-2} \geqslant 2^{-4}\left(\frac{\sigma/2}{d-2}\right)^{d-2}.$$

Hence, for the quantity (7.13), we have

$$Q_2[\varphi_a] > 2^{-2}(d-2)^{-(1/2)(d-2)}(\sigma/2)^{(1/2)d-1}, \quad 0 \leqslant a \leqslant \sigma/2.$$

Substituting this into (7.12), we have

$$Q_{1,2}[\mathcal{F}^{(s)}] \geqslant 2^{-d} \sum_{0\leqslant a\leqslant \sigma/2} Q_2[\varphi_a] > 2^{-d-2}(d-2)^{-(1/2)(d-2)}(\sigma/2)^{(1/2)d}$$
$$\geqslant 2^{-(3/2)d-2}(d-2)^{-(1/2)(d-2)}(\log N + 1)^{(1/2)d},$$

and therefore

$$\mathcal{F}_\infty^{(s)}[D_N, Z] > \alpha_1^{d-1}2^{-(3/2)d-2}(d-2)^{-(1/2)(d-2)}(\log N + 1)^{(1/2)d}.$$

Thus, for the principal term (5.22), we have

$$\mathcal{F}_{1,\infty}^{(s)}[D_N] > c_\infty(d)(\log N + 1)^{(1/2)d}, \quad c_\infty(d) = 2^{-2d}(d-2)^{-(1/2)(d-2)}, \tag{7.16}$$

where the bound (3.10) for the constant α_1 has also been used.

Substituting (7.16) and (6.3) into (5.25), we have

$$\mathcal{M}_{s,\infty}[D_N] > c_\infty(d)(\log N + 1)^{(1/2)d} - (1/2)(d+1)N2^{-s}$$
$$\geqslant c_\infty(d)(\log N + 1)^{(1/2)d}(1 - \xi_\infty(s)),$$

where

$$\xi_\infty(s) = c_\infty(d)^{-1}(dN2^{-s}).$$

Let s be chosen sufficiently large to satisfy $\xi_\infty(s) \leqslant \frac{1}{2}$. To do this, we put

$$s \geqslant \log N + \tfrac{1}{2}(d-2)\log(d-2) + 2d + \log d,$$

and in this case condition (7.15) is also satisfied.

As a result, we have

$$\mathcal{M}_{s,\infty}[D_N] > \gamma_\infty(d)(\log N + 1)^{(1/2)d},$$

where

$$\gamma_\infty(d) = \tfrac{1}{2}c_\infty(d) = 2^{-2d-1}(d-2)^{-(1/2)(d-2)}.$$

The proof of Theorem 2.3 is complete. □

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References

1. J. Beck and W. W. L. Chen, *Irregularities of Distribution* (*Cambridge Tracts in Mathematics* **89**), Cambridge University Press (1987).
2. D. Bilyk, Roth's orthogonal function method in discrepancy theory and some new connections. In *A Panorama of Discrepancy Theory* (*Lecture Notes in Mathematics* **2107**) (eds W. W. L. Chen, A. Srivastav and G. Travaglini), Springer (New York, 2014), 65–149.
3. D. Bilyk and M. T. Lacey, On the small ball inequality in three dimensions. *Duke Math. J.* **143** (2008), 81–115.
4. D. Bilyk and M. T. Lacey, The supremum norm of the discrepancy function: recent results and connections. In *Monte Carlo and Quasi-Monte Carlo Methods 2012* (*Springer Proceedings in Mathematics and Statistics* **65**) (eds J. Dick, F. Y. Kuo, G. W. Peters and I. H. Sloan), Springer (New York, 2013), 23–38.
5. D. Bilyk, M. T. Lacey and A. Vagharshakyan, On the small ball inequality in all dimensions. *J. Funct. Anal.* **254** (2008), 2470–2502.
6. W. W. L. Chen, On irregularities of distribution. *Mathematika* **27** (1980), 153–170.
7. W. W. L. Chen, On irregularities of distribution II. *Quart. J. Math.* **34** (1983), 257–279.
8. W. W. L. Chen, A. Srivastav and G. Travaglini (Eds.), *A Panorama of Discrepancy Theory* (*Lecture Notes in Mathematics* **2107**), Springer (New York, 2014).
9. W. W. L. Chen and M. M. Skriganov, Upper bounds in classical discrepancy theory. In *A Panorama of Discrepancy Theory* (*Lecture Notes in Mathematics* **2107**) (eds W. W. L. Chen, A. Srivastav and G. Travaglini), Springer (New York, 2014), 3–64.
10. Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, 3rd edn., Springer (New York, 1997).
11. B. I. Golubov, A. V. Efimov and V. A. Skvortsov, *Walsh Series and Transformations: Theory and Applications*, Nauka (Moscow, 1987); Engl. transl., *Mathematics and its Applications (Soviet Ser.* **64**), Kluwer (Dordrecht, 1991).
12. J. Matoušek, *Geometric Discrepancy: An Illustrated Guide* (*Algorithms and Combinatorics* **18**), 2nd edn., Springer (New York, 2010).
13. G. Peshkir and A. N. Shiryaev, Khinchin inequality and a martingale extention of the sphere of their action. *Uspekhi Mat. Nauk* **50**(5) (1995), 3–62; Engl. transl., *Russian Math. Surveys* **50**(5) (1995), 849–904.
14. M. M. Skriganov, Khinchin's inequality and Chen's theorem. *Algebr. Anal.* **23**(4) (2011), 179–204; Engl. transl., *St. Petersburg Math. J.* **23**(4) (2012), 761–778.
15. M. M. Skriganov, On mean values of the L_q-discrepancies of point distributions. *Algebr. Anal.* **24**(6) (2012), 196–225; Engl. transl., *St. Petersburg Math. J.* **24**(6) (2013), 991–1012.
16. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions* (*Princeton Mathematical Series* **30**), Princeton University Press (Princeton, NJ, 1970).
17. A. Zygmund, *Trigonometric Series, Vols. 1 and 2*, 3rd edn., Cambridge University Press (Cambridge, 2002).

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