Bull. Aust. Math. Soc. **113** (2026), 589–597
doi:10.1017/S0004972725100567

PERFECT CODES IN THE HOMOGENEOUS METRIC

JINJIAO XU, MINJIA SHI✉ and PATRICK SOLÉ

(Received 2 July 2025; accepted 27 July 2025; first published online 30 October 2025)

Abstract

We study perfect codes in the homogeneous metric over the ring $\mathbb{Z}_{2^k}, k \geq 2$. We derive arithmetic nonexistence results from Diophantine equations on the parameters resulting from the sphere packing conditions, and a Lloyd theorem based on the theory of weakly metric association schemes.

2020 *Mathematics subject classification*: primary 94B25; secondary 05E30.

Keywords and phrases: homogeneous metric, perfect code, Lloyd theorem, weakly metric association schemes.

1. Introduction

The homogeneous metric is one of the most important metrics for codes over rings [10]. It was first introduced over residue class rings in [4], but can be defined more generally over quasi-Frobenius rings [9]. Given an alphabet ring, this metric is unique up to scale, depending only on the lattice of ideals of that ring. When the ring is $\mathbb{Z}_4$, the homogeneous weight coincides with the Lee weight and is the Hamming weight of the Gray map [7]. We focus on the ring $\mathbb{Z}_{2^k}$ for an integer $k \geq 2$, in which case, the homogeneous weight of a vector over that ring is the Hamming weight of its image by Carlet's Gray map [3]. The study of perfect codes over $\mathbb{Z}_{2^k}$ was initiated in [12] for $k = 2$ and in [11] for general k.

In this paper, we study unrestricted perfect codes for the homogeneous distance. We derive the sphere packing bound, correcting the expression of the sphere volume in [11] (Remark 2.1). This gives a Diophantine equation on the parameters of a perfect code, a fact which yields nonexistence results on perfect codes (Theorems 3.1, 3.2, 3.4). Following [13], we develop a necessary existence condition, a 'Lloyd theorem', in analogy to the theorem of that name in metric schemes [6, page 64], especially the Hamming scheme [5]. To that end, we define an association scheme on $\mathbb{Z}_{2^k}$, which is weakly metric for the homogeneous distance. This scheme could be generalised to other rings with the homogeneous metric. We expect that our Lloyd theorem can

This research is supported by National Natural Science Foundation of China (Grant no. 12471490).
© The Author(s), 2025. Published by Cambridge University Press on behalf of Australian Mathematical Publishing Association Inc.

https://doi.org/10.1017/S0004972725100567 Published online by Cambridge University Press

give more extensive nonexistence results, at the price of more advanced arithmetic arguments.

The material is arranged as follows. Section 2 collects the notions on codes, homogeneous weight, Gray map and association schemes needed for the other sections. Section 3 studies the Diophantine equation for the parameters of perfect codes. Section 4 derives the Lloyd theorem.

2. Preliminaries

2.1. Codes over $\mathbb{Z}_{2^k}$. For any finite set E, let $|E|$ denote its cardinality. For any positive integer m, let the symbol $[m]$ denote the set $\{1, \ldots, m\}$. For any statement A, let $\mathbb{1}_A$ denote its indicator function. Let $\mathbb{Z}_{2^k}$ be the ring of integers modulo 2^k, where k is a positive integer. Let n be a positive integer and let $\mathbb{Z}_{2^k}^n$ be the set of n-tuples over $\mathbb{Z}_{2^k}$. Then, $\mathbb{Z}_{2^k}^n$ is a module over $\mathbb{Z}_{2^k}$. A set C is said to be a *linear code* with parameters (n, M) if it is a submodule of $\mathbb{Z}_{2^k}^n$ of size M.

For two vectors $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{Z}_{2^k}^n$, the *standard inner product* of $\mathbf{u}$ and $\mathbf{v}$ is defined by $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$. In a natural way, the dual of a linear code $C \subseteq \mathbb{Z}_{2^k}^n$ is defined by $C^\perp = \{\mathbf{v} \in \mathbb{Z}_{2^k}^n \mid \mathbf{u} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{u} \in C\}$.

2.2. Homogeneous weight and Gray map. The homogeneous weight of $x \in \mathbb{Z}_{2^k}$ is given by

$$W_{\mathrm{hom}}(x) = \begin{cases} 0 & \text{if } x = \mathbf{0}, \\ 2^{k-2} & \text{if } x \notin 2^{k-1}\mathbb{Z}_{2^k}, \\ 2^{k-1} & \text{if } x \in 2^{k-1}\mathbb{Z}_{2^k} \backslash \{\mathbf{0}\}. \end{cases}$$

Given two vectors $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{Z}_{2^k}^n$, the homogeneous distance d_{hom} on $\mathbb{Z}_{2^k}^n$ is defined from the homogeneous weight by

$$d_{\mathrm{hom}}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} W_{\mathrm{hom}}(y_i - x_i).$$

The distance d_{hom} is a metric in $\mathbb{Z}_{2^k}^n$. For $\mathbf{u} \in \mathbb{Z}_{2^k}^n$ and an integer $t \geq 0$, define the *sphere* $B_{\mathrm{hom}}(\mathbf{u}, t)$ with *centre* $\mathbf{u}$ and *radius* t by $B_{\mathrm{hom}}(\mathbf{u}, t) := \{\mathbf{v} \in \mathbb{Z}_{2^k}^n \mid d_{\mathrm{hom}}(\mathbf{u}, \mathbf{v}) \leq t\}$.

Define $b_r = |\{\mathbf{u} \in \mathbb{Z}_{2^k}^n \mid d_{\mathrm{hom}}(\mathbf{u}, \mathbf{0}) = r\}|$. Then, $|B_{\mathrm{hom}}(\mathbf{u}, t)| = \sum_r b_r$, where the sum is taken over the terms $b_r = b_{2^{k-2}i}$ for $0 \leq r \leq t = 2^{k-2}t'$. Following [3], the Gray map φ between $\mathbb{Z}_{2^k}$ and $\mathbb{Z}_2^{2^{k-1}}$ is a map from residues to elements of the Reed–Muller code $RM(1, k-1)$ and, for each x in $\mathbb{Z}_{2^k}$, the homogeneous weight of x is equal to the Hamming weight of $\varphi(x)$. Since the homogeneous weight of $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_{2^k}^n$ is the sum of the homogeneous weights of the coordinates of $\mathbf{x}$ and the weight enumerator of $RM(1, k-1)$ is $1 + (2^k - 2)x^{2^{k-2}} + x^{2^{k-1}}$, it follows that the generating function for the numbers b_r for $k \geq 2$ is

$$\sum_r b_r x^r = (1 + (2^k - 2)x^{2^{k-2}} + x^{2^{k-1}})^n. \tag{2.1}$$

https://doi.org/10.1017/S0004972725100567 Published online by Cambridge University Press

From (2.1),

$$b_r = \sum_{s=0}^{\lfloor i/2 \rfloor} \binom{n}{s}\binom{n-s}{i-2s}(2^k-2)^{i-2s}$$

and the volume of the sphere $B_{\hom}(\mathbf{u}, t)$ is

$$|B_{\hom}(\mathbf{u}, t)| = \sum_{r} b_r = \sum_{i=0}^{t'} \sum_{s=0}^{\lfloor i/2 \rfloor} \binom{n}{s}\binom{n-s}{i-2s}(2^k-2)^{i-2s}.$$

REMARK 2.1. We note an error in [11], where the volume of the sphere $B_{\hom}(\mathbf{u}, t)$ with centre $\mathbf{u}$ and radius t is given as $|B_{\hom}(\mathbf{u}, t)| = \sum_{i=0}^{t} \binom{n}{i}(2^k-1)^i$.

A code C over $\mathbb{Z}_{2^k}$ is *t-error correcting* if the following condition holds:

$$\text{for all } \mathbf{u},\ \mathbf{v} \in C, \quad \mathbf{u} \neq \mathbf{v} \Rightarrow B_{\hom}(\mathbf{u},\ t) \cap B_{\hom}(\mathbf{v},\ t) = \emptyset.$$

If for such a t-error correcting code C, the union of these spheres is $\mathbb{Z}_{2^k}^n$, that is, $\mathbb{Z}_{2^k}^n = \bigcup_{\mathbf{u}\in C} B_{\hom}(\mathbf{u}, t)$, then the code is said to be *perfect*. In this situation, by counting the points in $\mathbb{Z}_{2^k}^n$, we have the Diophantine equation $2^{nk} = M(\sum_{r=0}^{t} b_r)$, where M is the size of the perfect code. Thus, M must be a power of 2. Note also that $b_0 = 1$. For later reference, for any integer c with $0 \le c \le nk$, we define the Diophantine equation

$$E(c, k, t): \quad 2^c = \sum_{r=2^{k-2}}^{t} b_r + 1.$$

2.3. Association schemes. Let X be a finite set with at least two elements and let $R = \{R_0, R_1, \ldots, R_s\}$ be a family of $s+1$ relations R_i on X for $s \ge 1$.

DEFINITION 2.2 [6]. The pair (X, R) will be called an association scheme with s classes if the following three conditions are satisfied:

(A_1) the set R is a partition of X^2 and $R_0 = \{(x, x) \mid x \in X\}$ is the diagonal relation;
(A_2) for $i = 0, 1, \ldots, s$, the inverse $R_i^{-1} = \{(y, x) \mid (x, y) \in R_i\}$ of the relation R_i also belongs to R;
(A_3) for any triple of integers $i, j, k = 0, 1, \ldots, s$, there exists a number $p_{ij}^k = p_{ji}^k$ such that $|\{z \in X \mid (x, z) \in R_i,\ (z, y) \in R_j\}| = p_{ij}^k$ for all $(x, y) \in R_k$.

For $i = 0, 1, \ldots, s$, the relation R_i can be described by its *adjacency matrix* D_i, which is defined by

$$D_i(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{if } (x, y) \notin R_i. \end{cases}$$

It is well known that $T = \{\sum_{i=0}^{s} \alpha_i D_i \mid \alpha_i \in \mathbb{C}\}$ constitutes a complex algebra called the Bose–Mesner algebra. There are two 'natural' bases $\{D_i\}$ and $\{J_i\}$ of the Bose–Mesner algebra of a scheme, where $\{J_i\}_{i\in\{0,1,\ldots,s\}}$ is a unique basis of minimal idempotents that satisfy $J_i^2 = J_i,\ J_iJ_j = \mathbf{0}\ (i \neq j),\ \sum_{i=0}^{s} J_i = I_{|X|}$, where $I_{|X|}$ is the identity matrix of

https://doi.org/10.1017/S0004972725100567 Published online by Cambridge University Press

order $|X|$. We consider the linear transformations from J_k to D_k. First, $D_k = \sum_{i=0}^{s} P_k(i)J_i$ for $k = 0, 1, \ldots, s$. The complex numbers $P_k(0), P_k(1), \ldots, P_k(s)$ are the eigenvalues of D_k, also called the *first eigenvalues* of the scheme.

From [13], the *quasi-distance* d on X is a mapping from X^2 to the nonnegative reals satisfying the triangle inequality. An association scheme (X, R) with s classes is a *weakly metric* scheme for the quasi-distance d if the quasi-distance d is constant on the classes of the scheme, that is, for all $(a, b) \in X^2$, $aR_ib \Rightarrow d(a, b) = d(i)$, where $i \in \{0, 1, \ldots, s\}$. The Lloyd polynomial in a weakly metric association scheme is defined by $\Psi_r(l) = \sum_{d(k)\le r} P_k(l)$. In a weakly metric scheme (X, R) with quasi-distance d, the *dispersion function* Π is defined by $\Pi(j) = |\{i \in \{0, 1, \ldots, s\} \mid d(i) \le j\}|$.

3. Diophantine condition

Replacing $x^{2^{k-2}}$ by x in (2.1), we obtain a generating function with a simpler form,

$$\sum_{i=0}^{t'} d_i x^i = (1 + (2^k - 2)x + x^2)^n \quad \text{where } d_i = b_{2^{k-2}i} = b_r, \tag{3.1}$$

and the equation $E(c, k, t)$ becomes $2^c = \sum_{i=1}^{t'} d_i + 1$, where $t = 2^{k-2}t'$.

THEOREM 3.1. *There are no 4-error correcting perfect codes of length n in the homogeneous metric over $\mathbb{Z}_8$ for $n > 1$.*

PROOF. Suppose that C is a 4-error correcting perfect code with parameters (n, M) over $\mathbb{Z}_8$. Then, the parameters of C must satisfy the equation $E(c, k, t)$ when $t' = 2$. According to the generating function (3.1),

$$d_i = \sum_{s=0}^{\lfloor i/2 \rfloor} \binom{n}{s}\binom{n-s}{i-2s}(2^k - 2)^{i-2s}, \quad i = 0, 1, \ldots, 2n.$$

Then, the volume of the sphere of radius 4 in $\mathbb{Z}_8^n$ is

$$\sum_{i=0}^{2} d_i = 1 + (2^k - 2)n + \frac{1}{2}(2^k - 2)^2 n(n-1) + n = (2n-1)(9n-1).$$

Note that for $n > 1$, $2n - 1$ is an odd number, $2n - 1 \ne 2^l$ for any integer l with $0 \le l \le c$ and $t' = 2 < 2n$. However, the left-hand side of $E(c, k, t)$ only has divisors of the form 2^l. Consequently, the parameters of C fail to satisfy the Diophantine condition when $t' = 2$ and there are no such 4-error correcting perfect codes C with respect to the homogeneous weight over $\mathbb{Z}_8$ for $n > 1$. □

THEOREM 3.2. *Let n and k be two positive integers with $n > 1$. There are no $(2^{k-2} \times 3)$-error correcting perfect codes of length n in the homogeneous metric over $\mathbb{Z}_{2^k}$ for $k \ge 2$ if n is even or n is odd and $n \equiv 3 \pmod 4$.*

PROOF. According to the generating function (3.1), the volume of the sphere of radius $2^{k-2} \times 3$ in $\mathbb{Z}_{2^k}^n$ is

https://doi.org/10.1017/S0004972725100567 Published online by Cambridge University Press

$$\sum_{i=0}^{3} d_i = 1 + \binom{n}{1}(2^k-2) + \binom{n}{2}(2^k-2)^2 + \binom{n}{1} + \binom{n}{3}(2^k-2)^3 + \binom{n}{1}\binom{n-1}{1}(2^k-2)$$
$$= 1 + n + (2^k-2)n^2 + \frac{(2^k-2)^2 n(n-1)}{2} + \frac{(2^k-2)^3 n(n-1)(n-2)}{6}$$
$$= 1 + n + 2\left[(2^{k-1}-1)n^2 + (2^{k-1}-1)^2 n(n-1) + \frac{2(2^{k-1}-1)^3 n(n-1)(n-2)}{3}\right].$$

If n is even, then $\sum_{i=0}^{3} d_i > 1$ is odd and $t' = 3 \le 2n$. The equation $E(c,k,t)$ does not hold in this case, which is a contradiction.

If n is odd, let $n+1 = 2^a n_1$, where n_1 is odd and $a \ge 1$. Then,

$$\sum_{i=0}^{3} d_i = 2^a n_1 + 2(2^{k-1}-1)n[n + (2^{k-1}-1)(n-1)] + \frac{4(2^{k-1}-1)^3 n(n-1)(n-2)}{3}$$
$$= 2\left[2^{a-1}n_1 + \frac{2(2^{k-1}-1)^3 n(n-1)(n-2)}{3} + (2^{k-1}-1)n(n+(2^{k-1}-1)(n-1))\right] = 2b,$$

where $b > 1$ is an odd number and $3 < 2n$ under the conditions that n and n_1 are odd and $n \equiv 3 \pmod 4$, noting that $n(n-1)(n-2)/3$ is an integer. Again, this gives a contradiction. The proof of the second part has been finished. □

The following arithmetic lemma plays a crucial role in the proof of Theorem 3.4.

LEMMA 3.3 (Lucas' theorem, [8]). *Let p be a prime number, and let a and b be two nonnegative integers. Then,*

$$\binom{pa + r_a}{pb + r_b} \equiv \binom{a}{b}\binom{r_a}{r_b} \pmod p,$$

where r_a, $r_b \in \{0, 1, \ldots, p-1\}$.

THEOREM 3.4. *Let n, k, t' and l be positive integers with $k \ge 2$. If $n = 2^l$, then there are no $2^{k-2}t'$-error correcting perfect codes of length n in the homogeneous metric over $\mathbb{Z}_{2^k}$ for $1 \le t' < 2n$.*

PROOF. According to the equation $E(c,k,t)$,

$$2^c = \sum_{i=0}^{t'} d_i = 1 + \sum_{i=1}^{t'} \sum_{s=0}^{\lfloor i/2 \rfloor} \binom{n}{s}\binom{n-2}{i-2s}(2^k-2)^{i-2s}$$
$$= 1 + \sum_{i\in[t'],\, i \text{ odd}} \sum_{s=0}^{\lfloor i/2 \rfloor} \binom{n}{s}\binom{n-s}{i-2s}(2^k-2)^{i-2s}$$
$$+ \sum_{i\in[t'],\, i \text{ even}} \sum_{s=0}^{i/2-1} \binom{n}{s}\binom{n-s}{i-2s}(2^k-2)^{i-2s} + \sum_{j=1}^{\lfloor t'/2 \rfloor} \binom{n}{j}.$$

https://doi.org/10.1017/S0004972725100567 Published online by Cambridge University Press

By Lemma 3.3, for $n = 2^l$, the $\binom{n}{j}$ are even for $1 \le j \le \lfloor t'/2 \rfloor$ since $\lfloor t'/2 \rfloor < n$ and so the right-hand side of the equation is odd, which is a contradiction. □

4. A Lloyd theorem

In [13], the Lloyd theorem in a weakly metric association scheme is derived by using the Lloyd polynomial and the first eigenvalues of the scheme.

LEMMA 4.1 [13, Theorem 2]. *Let Y be e-error correcting. Then, $\Psi_e(l)$ vanishes for at least $\Pi(e) - 1$ values of l.*

Bariffi *et al.* [1] introduced a finer partition of the alphabet ring which can be applied to the ring $\mathbb{Z}_{2^k}$: namely, $Z := \{0\}, U := \mathbb{Z}_{2^k}^{\times}, S := \langle 2^{k-1} \rangle, V := \langle 2 \rangle \setminus \langle 2^{k-1} \rangle$. By [14, Lemma], this partition induces an association scheme on $\mathbb{Z}_{2^k}$ with the set of relations $R = \{R_Z, R_U, R_S, R_V\}$, where $xR_Iy \Leftrightarrow x - y \in I$ for any $I \in \{Z, U, S, V\}$. Now, we consider the extension of length n of this association scheme.

DEFINITION 4.2. Let $\pi_I(\mathbf{x}) := |\{j = 1, \ldots, n \mid x_j \in I\}|$, where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_{2^k}^n$ and $I \in \{Z, U, S, V\}$, and let $D_\pi := \{\pi^{(i)} = (\pi_Z^{(i)}, \pi_U^{(i)}, \pi_S^{(i)}, \pi_V^{(i)}) \mid i = 0, 1, \ldots, s\}$. Take $s = \binom{n+3}{3}$. Then, we define the weakly metric association scheme on $\mathbb{Z}_{2^k}^n$ with the set of relations $R = \{R_0, R_1, \ldots, R_s\}$ by

$$\mathbf{x}R_i\mathbf{y} \Leftrightarrow \pi(\mathbf{x} - \mathbf{y}) = \pi^{(i)} \in D_\pi \quad \text{for any } \mathbf{x}, \mathbf{y} \in \mathbb{Z}_{2^k}^n \text{ and } i \in \{0, 1, 2, \ldots, s\},$$

and the quasi-distance d by

$$\mathbf{x}R_i\mathbf{y} \Leftrightarrow d(\mathbf{x}, \mathbf{y}) = d(i) = \pi_U^{(i)} + \pi_V^{(i)} + 2\pi_S^{(i)} \quad \text{for all } (\mathbf{x}, \mathbf{y}) \in (\mathbb{Z}_{2^k}^n)^2.$$

Following [1], for any pair $(\pi, \rho) \in D_\pi^2$, we define the set of all compositions of ρ with respect to π by

$$\mathrm{Comp}_\pi(\rho) := \left\{ \tau = (\tau_Z, \tau_U, \tau_S, \tau_V) \;\middle|\; \tau_I = (\tau_{IZ}, \ldots, \tau_{IV}) \in \{0, \ldots, \pi_i\}^4 \text{ with } \sum_{J \in \{Z,U,S,V\}} \tau_{IJ} = \pi_I \text{ for all } I, \sum_{I \in \{Z,U,S,V\}} \tau_{IJ} = \rho_J \text{ for all } J \right\}.$$

This definition is crucial for our main result in this section.

THEOREM 4.3. *Let n, k and t' be three positive integers with $1 \le t' < 2n$, and let $H := \{\pi = (\pi_Z, \pi_U, \pi_S, \pi_V) \mid \pi \in D_\pi \text{ and } 0 \le \pi_U + \pi_V + 2\pi_S \le t'\}$. Let $f = \mathbb{1}_{\{\tau_{UU} = \tau_{UV} = \tau_{VU} = 0\}}$. If there is a $2^{k-2}t'$-error correcting perfect code of length n in the homogeneous metric over $\mathbb{Z}_{2^k}^n$ for $k \ge 3$, then the Lloyd polynomial $\Phi_{t'}(\mathbf{x})$, namely*

$$\sum_{\pi \in H} \sum_{\tau \in \mathrm{Comp}_{\mathbf{x}}(\pi)} \left(\prod_{I \in \{Z,U,S,V\}} \binom{x_I}{\tau_{IZ}, \ldots, \tau_{IV}} \right) (-1)^{\tau_{US} + \tau_{SU} + \tau_{VV}} 2^{\tau_{VV}} (2^{k-1})^{\tau_{SU} + \tau_{ZU}} (2^{k-1} - 2)^{\tau_{ZV}} f,$$

https://doi.org/10.1017/S0004972725100567 Published online by Cambridge University Press

vanishes for at least $\Pi(t')-1$ *values of* $\mathbf{x}=(x_Z,x_U,x_S,x_V)$*, where the value of* $\Pi(t')$ *is given by one of the following two cases.*

(1) *If* $1\le t'\le n$*, then*

$$\Pi(t')=\begin{cases}\dfrac{1}{2}\binom{t'+4}{3}-\dfrac{1}{8}(t+2)(t'+4) & \text{if } t' \text{ is even},\\[2ex] \dfrac{1}{2}\binom{t'+3}{3}+\dfrac{1}{8}(t'+1)(t'+3) & \text{if } t' \text{ is odd}.\end{cases}$$

(2) *If* $n\le t'\le 2n$*, then*

$$\Pi(t')=\begin{cases}\binom{n+3}{3}-\dfrac{1}{2}\binom{2n-t'+2}{3}-\dfrac{1}{8}(2n-t')(2n-t'+2) & \text{if } t' \text{ is even},\\[2ex] \binom{n+3}{3}-\binom{2n-t'+2}{3}-\dfrac{1}{2}\binom{2n-t'+1}{3} & \\ \qquad\qquad +\dfrac{1}{8}(2n-t'-1)(2n-t'+1) & \text{if } t' \text{ is odd}.\end{cases}$$

PROOF. The proof of this result relies on two steps. The first is to count the number of $\pi^{(i)}\in D_\pi$ satisfying $d(i)\le t'$.

If $1\le t'\le n$, then for any $\pi_S\in\{0,1,\dots,\lfloor t'/2\rfloor\}$, we have $0\le\pi_U+\pi_R\le t'-2\pi_S$. We obtain the number of $\pi^{(i)}\in D_\pi$ satisfying $d(i)\le t'$ by computing $\sum_{\pi_S=0}^{\lfloor t'/2\rfloor}\sum_{i=1}^{t'-2\pi_S+1} i$.

If $n<t'<2n$, then $t'-n\le\lfloor t'/2\rfloor$. Although we still have $\pi_S\in\{0,1,\dots,\lfloor t'/2\rfloor\}$, the maximum value of $\pi_U+\pi_R$ depends on the relationship between π_S and $t'-n$. If $0\le\pi_S\le t'-n$, then $0\le\pi_U+\pi_V\le n-\pi_S$ and the corresponding number of $\pi^{(i)}$ is $\sum_{\pi_S=0}^{t'-n}\sum_{j=1}^{n-\pi_S+1} j=\binom{n+3}{3}-\binom{2n-t'+2}{3}$ for t' both even and odd. If $t'-n<\pi_S\le\lfloor t'/2\rfloor$, we obtain the result by computing $\sum_{\pi_S=t'-n+1}^{\lfloor t'/2\rfloor}\sum_{i=1}^{t'-2\pi_S+1} i$.

By [1, Theorem 30], combined with the general MacWilliams formula for translation association schemes ([6, page 88], [2, Theorem 2.10.12, page 71]), the first eigenvalues of the weakly metric association scheme defined in Definition 4.2 are

$$\sum_{\tau\in\mathrm{Comp}_{\mathbf{x}}(\pi)}\left(\prod_{I\in\{Z,U,S,V\}}\binom{x_I}{\tau_{IZ},\dots,\tau_{IV}}\right)(-1)^{\tau_{US}+\tau_{SU}+\tau_{VV}}2^{\tau_{VV}}(2^{k-1})^{\tau_{SU}+\tau_{ZU}}(2^{k-1}-2)^{\tau_{ZV}}f. \tag{4.1}$$

The proof is completed by Lemma 4.1. □

EXAMPLE 4.4. Let $n=2$, $t'=1$ and $k=3$. Then, $H=\{(2,0,0,0),(1,1,0,0),(1,0,0,1)\}$. For $\pi=(2,0,0,0)$, we have $|\mathrm{Comp}_{\mathbf{x}}(\pi)|=1$ and the value given by (4.1) is 1 for any $\mathbf{x}=(x_Z,x_U,x_S,x_V)\in D_\pi$. We focus on $\mathbf{x}$ such that $\tau_{US},\tau_{SU},\tau_{VV}\neq 0$ for $\tau\in\mathrm{Comp}_{\mathbf{x}}(\pi)$ (to analyse negative terms in (4.1)). There exist seven such $\mathbf{x}$, but we only detail two critical cases due to their similarity.

https://doi.org/10.1017/S0004972725100567 Published online by Cambridge University Press

If $\mathbf{x} = (0, 0, 2, 0)$, then $|\mathrm{Comp}_{\mathbf{x}}(\pi)| = 1$ for both $\pi = (1, 1, 0, 0)$ and $\pi = (1, 0, 0, 1)$. The corresponding τ matrices are

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

respectively, and $\Phi_1(\mathbf{x}) = -5$. If $\mathbf{x} = (1, 0, 1, 0)$, then $|\mathrm{Comp}_{\mathbf{x}}(\pi)| = 2$ for both $\pi = (1, 1, 0, 0)$ and $\pi = (1, 0, 0, 1)$. The τ matrices are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

respectively, and $\Phi_1(\mathbf{x}) = 4$. For the other five $\mathbf{x}$, the values of Φ_1 are found to be $\Phi_1((0, 1, 1, 0)) = -2$, $\Phi_1((0, 0, 1, 1)) = -4, \Phi_1((0, 0, 0, 2)) = -3, \Phi_1((1, 0, 0, 1)) = 5$ and $\Phi_1((0, 1, 0, 1)) = -1$. For all other $\mathbf{x}$, the terms in (4.1) are positive. Since $\Pi(1) - 1 = 2$, there exists no 2-error-correcting perfect code of length 2 in the homogeneous metric over $\mathbb{Z}_8$.

References

[1] J. Bariffi, G. Cavicchioni and V. Weger, 'The existence of MacWilliams-type identities for the Lee, homogeneous and subfield metric', Preprint, 2024, arXiv:2409.11926.

[2] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs* (Springer-Verlag, Berlin, 1989).

[3] C. Carlet, '$\mathbb{Z}_{2^k}$-linear codes', *IEEE Trans. Inform. Theory* **44**(4) (1998), 1543–1547.

[4] I. Constantinescu and W. Heise, 'A metric for codes over residue class rings', *Probl. Inf. Transm.* **33**(3) (1997), 22–28.

[5] D. M. Cvetkovic and J. H. van Lint, 'An elementary proof of Lloyd's theorem', *Indag. Math. (N.S.)* **80**(1) (1977), 6–10.

[6] P. Delsarte, 'An algebraic approach to the association schemes of coding theory', *Philips Res. Rep. Suppl.* **10** (1973), 1–97.

[7] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane and P. Solé, 'The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals and related codes', *IEEE Trans. Inform. Theory* **40**(2) (1994), 301–319.

[8] É. Lucas, 'Sur les congruences des nombres eulériens et des coefficients différentiels des functions trigonométriques suivant un module premier', *Bull. Soc. Math. France* **6** (1878), 49–54.

[9] A. A. Nechaev and T. Khonol'd, 'Weighted modules and representations of codes', *Probl. Inf. Transm.* **35**(3) (1999), 18–39.

[10] M. Shi, A. Alahmadi and P. Solé, *Codes and Rings: Theory and Practice* (Academic Press, Amsterdam, 2017).

[11] I. Şiap, M. Özen and V. Şiap, 'On the existence of perfect linear codes over $\mathbb{Z}_{2^l}$ with respect to homogeneous metric', *Arab. J. Sci. Eng.* **38**(8) (2013), 2189–2192.

[12] V. Şiap and M. Özen, 'On the existence of perfect linear codes over $\mathbb{Z}_4$ with respect to homogeneous weight', *Appl. Math. Sci.* **6**(41) (2012), 2005–2011.

https://doi.org/10.1017/S0004972725100567 Published online by Cambridge University Press

[13] P. Solé, 'A Lloyd theorem in weakly metric association schemes', *European J. Combin.* **10**(2) (1989), 189–196.
[14] V. A. Zinoviev and T. Ericson, 'Fourier-invariant pairs of partitions of finite abelian groups and association schemes', *Probl. Inf. Transm.* **45**(3) (2009), 221–231.

JINJIAO XU,
Key Laboratory of Intelligent Computing Signal Processing, Ministry of Education,
School of Mathematical Sciences, Anhui University, Hefei 230601, PR China
e-mail: xu_jjiao@163.com

MINJIA SHI,
Key Laboratory of Intelligent Computing Signal Processing, Ministry of Education,
School of Mathematical Sciences, Anhui University, Hefei 230601, PR China
e-mail: smjwcl.good@163.com

PATRICK SOLÉ, I2M (Institut de Mathématiques de Marseille),
Aix-Marseille Univ, CNRS, Centrale Marseille, Marseilles, France
e-mail: sole@enst.fr

https://doi.org/10.1017/S0004972725100567 Published online by Cambridge University Press