—Technical Note—

PARAMETRIC STUDIES ON BENDING OF TWISTED TIMOSHENKO BEAMS UNDER COMPLEX LOADINGS

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ABSTRACT

Static bending of a twisted Timoshenko beam subjected to combined transverse and axial loadings is studied. The equilibrium equations are established in the twist coordinates by applying the principle of minimum potential energy. The governing equations are then reduced into solvable algebraic equations using a finite element approach. The effects of the twist angle, thickness-to-width ratio, length-to-thickness ratio, loading and boundary conditions on the static bending characteristics of the twisted beams are investigated. The present parametric analyses will provide engineers a good insight into the influence of various structural aspects of the twisted beam on its response to different static loads.

Keywords: Static bending, Timoshenko beam, Finite element, Twist angle.

1. INTRODUCTION

Structural analyses of the twisted structures, such as turbine blades, propeller blades and drill bits, have attracted many researchers' attention for the past years. The reasons result from their important application in many engineering fields and the complex behavior and modeling required. A general review of the structural aspects of twisted beams can be found in review paper by Rosen [1]. Basically, when the twisted structures are modeled structurally as twisted beams, they have been analyzed based on Euler-Bernoulli beam theory [2], Timoshenko beam theory [3,4] or helical beam approach [5]. Recently, the geometrically exact finite strain beam theory, one of the well-known advanced beam theories, has been widely used to derive the governing equations for naturally curved and twisted beams [6-12].

The previous studies on static bending of twisted beams were mainly concerned with the cantilevered beams under transverse loads. Little study was devoted to establish the static equilibrium equations of a twisted beam including the shear effect and subjected to the simultaneous axial force, distributed transverse force and couple in the twist coordinates. Hence, in the present analysis the equilibrium equations of static bending for a twisted Timoshenko beam under distributed transverse forces, distributed bending couples and the axial force are derived in the twist frame. The finite element method is then used to put the discretized equilibrium equations into the solvable matrix form of algebraic equations. Parametric studies are conducted to demonstrate the effects of the twist angle, thickness-to-width ratio, length-to-thickness ratio, loading type and boundary conditions on the static bending behaviors of the twisted beams. Such an analysis will provide engineers a good insight into the influence of various parameters of the twisted beam on its static bending.

2. PROBLEM FORMULATION

Figure 1 shows the twisted beam configuration and coordinate systems. The twist angle per unit length β_o (ϕ/L, ϕ = total twist angle) of the beam is constant. Coordinate system XYZ represents the inertial system and the twist coordinates $\xi\eta Z$ are fixed to the beam and moves along the beam pretwist angle such that axes ξ and η are in the principal directions of the beam cross section. By applying the principle of minimum potential energy [13] to the total potential energy of the beam system and using the transformation relationship [14] between the inertial and twist coordinates, the static equilibrium equations and associated boundary conditions at $Z = 0$ and L for a twisted Timoshenko beam subjected to distributed transverse forces (f_X, f_Y), distributed bending couples (q_X, q_Y) and the axial force (F_Z) in the twist frame can be obtained as

$$\bar{K}_1 d'' + (\bar{K}_2 + \bar{K}_3)\, d' + \bar{K}_4 d + \bar{F} = 0 \tag{1}$$

$$(\bar{K}_1 d' + \bar{K}_2 d) = 0 \;\; or \;\; d = 0 \tag{2}$$

where

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DOI : 10.1017/jmech.2012.23
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https://doi.org/10.1017/jmech.2012.23 Published online by Cambridge University Press

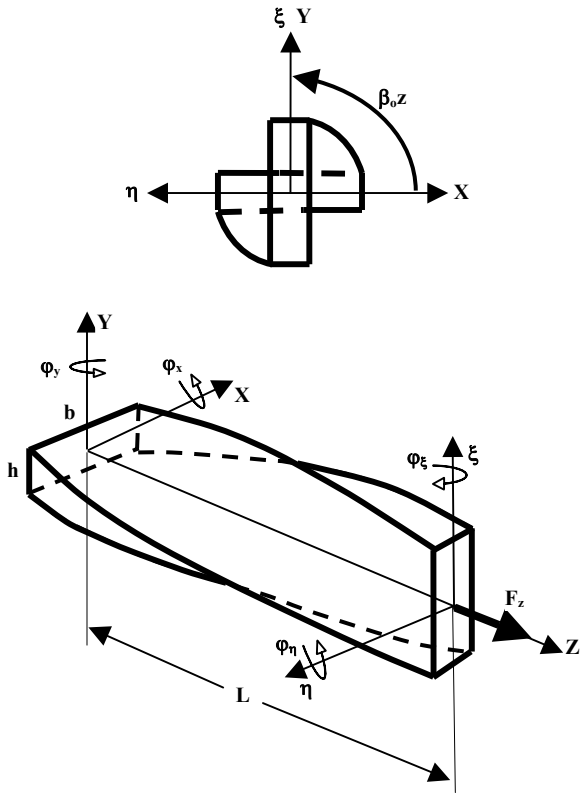


Fig. 1 Beam configuration and coordinate systems

$$\bar{K}_1 = \bar{K}_{1B} + \bar{K}_{1F} = \begin{bmatrix} -\kappa GA & 0 & 0 & 0 \\ 0 & -\kappa GA & 0 & 0 \\ 0 & 0 & -EI_\xi & 0 \\ 0 & 0 & 0 & -EI_\eta \end{bmatrix} + \begin{bmatrix} -F_Z & 0 & 0 & 0 \\ 0 & -F_Z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{K}_2 = \bar{K}_{2B} + \bar{K}_{2F} = \begin{bmatrix} 0 & \beta_o \kappa GA & 0 & \kappa GA \\ -\beta_o \kappa GA & 0 & \kappa GA & 0 \\ 0 & 0 & 0 & -EI_\xi \beta_o \\ 0 & 0 & EI_\eta \beta_o & 0 \end{bmatrix} + \begin{bmatrix} 0 & \beta_o F_Z & 0 & 0 \\ -\beta_o F_Z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{K}_3 = \bar{K}_{3B} + \bar{K}_{3F} = \begin{bmatrix} 0 & \beta_o \kappa GA & 0 & 0 \\ -\beta_o \kappa GA & 0 & 0 & 0 \\ 0 & -\kappa GA & 0 & -EI_\eta \beta_o \\ -\kappa GA & 0 & EI_\xi \beta_o & 0 \end{bmatrix} + \begin{bmatrix} 0 & \beta_o F_Z & 0 & 0 \\ -\beta_o F_Z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{K}_4 = \bar{K}_{4B} + \bar{K}_{4F} = \begin{bmatrix} \beta_o^2 \kappa GA & 0 & -\beta_o \kappa GA & 0 \\ 0 & \beta_o^2 \kappa GA & 0 & \beta_o \kappa GA \\ -\beta_o \kappa GA & 0 & \beta_o^2 EI_\eta + \kappa GA & 0 \\ 0 & \beta_o \kappa GA & 0 & \beta_o^2 EI_\xi + \kappa GA \end{bmatrix} + \begin{bmatrix} \beta_o^2 F_Z & 0 & 0 & 0 \\ 0 & \beta_o^2 F_Z & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{d} = \{u_\xi \quad u_\eta \quad \phi_\xi \quad \phi_\eta\}^T, \quad \bar{\mathrm{F}} = \{-f_\xi \quad -f_\eta \quad -q_\xi \quad q_\eta\}^T$$

Then, following a similar finite element approach [15], we can discretize the above governing differential equations into the algebraic equations using the Mindlin-type linear beam element [16]. By putting the displacement function of the beam element, $d^{(e)}(z) = N(z)p^{(e)}$, into the weak form of the equilibrium Eqs. (1) and (2), and applying Galerkin's criterion, the resulting beam element equations are obtained as follows.

$$\left(K_B^{(e)} + K_F^{(e)}\right) p^{(e)} = f_D^{(e)} + f_P^{(e)} \tag{3}$$

where

$$\mathrm{K}_B^{(e)} = -\int_0^{L_e} \mathrm{B}^{\mathrm{T}} \bar{\mathrm{K}}_{1B} \, \mathrm{B} dz - \int_0^{L_e} \mathrm{B}^{\mathrm{T}} \bar{\mathrm{K}}_{2B} \, \mathrm{N} dz + \int_0^{L_e} \mathrm{N}^{\mathrm{T}} \bar{\mathrm{K}}_{3B} \, \mathrm{B} dz + \int_0^{L_e} \mathrm{N}^{\mathrm{T}} \bar{\mathrm{K}}_{4B} \, \mathrm{N} dz$$

$$K_F^{(e)} = -\int_0^{L_e} B^T \bar{K}_{1F} \, B dz - \int_0^{L_e} B^T \bar{K}_{2F} \, N dz + \int_0^{L_e} N^T \bar{K}_{3F} \, B dz + \int_0^{L_e} N^T \bar{K}_{4F} \, N dz$$

$$f_D^{(e)} = -\int_0^{L_e} N^T \bar{F}^{(e)} dZ, \quad f_P^{(e)} = -(N^T \bar{K}_1 B + N^T \bar{K}_2 N) \, p^{(e)} \Big|_0^{L_e}$$

$$N = \begin{bmatrix} N_1 & 0 & 0 & 0 & N_2 & 0 & 0 & 0 \\ 0 & N_1 & 0 & 0 & 0 & N_2 & 0 & 0 \\ 0 & 0 & N_1 & 0 & 0 & 0 & N_2 & 0 \\ 0 & 0 & 0 & N_1 & 0 & 0 & 0 & N_2 \end{bmatrix}, \quad B = N'$$

Here shape functions $N_1 = 1 - z/L_e$ and $N_2 = z/L_e$, where L_e is the length of the twisted beam element. During the derivation of the element stiffness of a Timoshenko beam by using elements of C^0-order, the selective reduced integration method [17] is used here. The stiffness matrix due to bending effect is computed using the normal quadrature rule, while the stiffness matrix due to shear effect is integrated with one-point quadrature scheme to alleviate shear locking. Directly assembling these element matrices and imposing the essential boundary conditions, the general form of the global system of equations can be obtained

$$(K_B + K_F) p = f_D + f_P \tag{4}$$

https://doi.org/10.1017/jmech.2012.23 Published online by Cambridge University Press

3. NUMERICAL RESULTS AND DISCUSSIONS

The material and geometric properties used in the following examples are E = 199.95Gpa, G = 81.95Gpa, b = 25.4mm, and κ = 5/6. Three values of thickness-to-width ratio, h/b = 0.125, 0.25, 0.5, and of length-to-thickness ratio, L/h = 6, 12, 100, are considered. For each case, the beam is subjected to unit loading applied in the flexible- or stiff-plane direction. To ensure the convergence and accuracy of the displacements of twisted beams, different mesh sizes ranging from 40 to thousand elements are used depending on how large the twist angle is. For purposes of comparison, the dimensionless displacement U is used, which is defined as the ratio of the maximum deflection of the twisted beam evaluated by the Timoshenko beam theory to the theoretical maximum deflection of the corresponding untwisted Euler-Bernoulli beam bent in its most flexible plane. This quantity along the direction of applied load is called the relative maximum deflection, but is called the relative maximum lateral deflection when its direction is perpendicular to the direction of applied load.

Figures 2(a) ~ 2(d) present the dimensionless deformation plots of centroidal axis of various cantilevered twisted beams subjected to unit tip load at the free end along the flexible-plane direction. As the twist angle is increased, the coupling effect of the pretwist becomes significant. The pretwist of the cantilevered beam increases the resistance to bending by comparing with the untwisted one. The cantilevered beam becomes much stiffer in the thickness (or Y) direction when it has a greater twist angle.

Figure 3 shows the relative displacements at the tip versus the twist angle for the cantilevered beams with L/h = 100 and h/b = 0.25 subjected to various types of loadings. For the beam under tip force, distributed force or distributed couple, the beam loaded in flexible-plane direction becomes stiffer with the increasing twist angle but it becomes more flexible while loaded in stiff-plane direction. Meanwhile, the two relative maximum deflections in the direction of applied loads become closer to each other as the twist angle increases. The increasing pretwist tends to strengthen the weaker plane and weaken the stronger plane. The pretwist has a net beneficial effect on the bending stiffness of a twisted cantilever beam subjected to tip force, distributed force or distributed couple. Hence, the deflections are independent of the orientation of the applied load as the twist angle becomes very large. Unlike the previous cases, the relative maximum deflection decreases for the beam under the applied tip moment in flexible-plane direction and increases for the beam loaded in stiff-plane direction with the increasing twist angle up to 180°. Thereafter, an interesting phenomenon occurs for pretwists of 180° and greater that the two relative maximum deflections oscillate and form local maxima and minima, and have the same value at certain twist angles. However, the difference of the two relative maximum deflections is diminished as the twist angle continues to increase. Thus, the effect of the direction of the applied tip moment on the

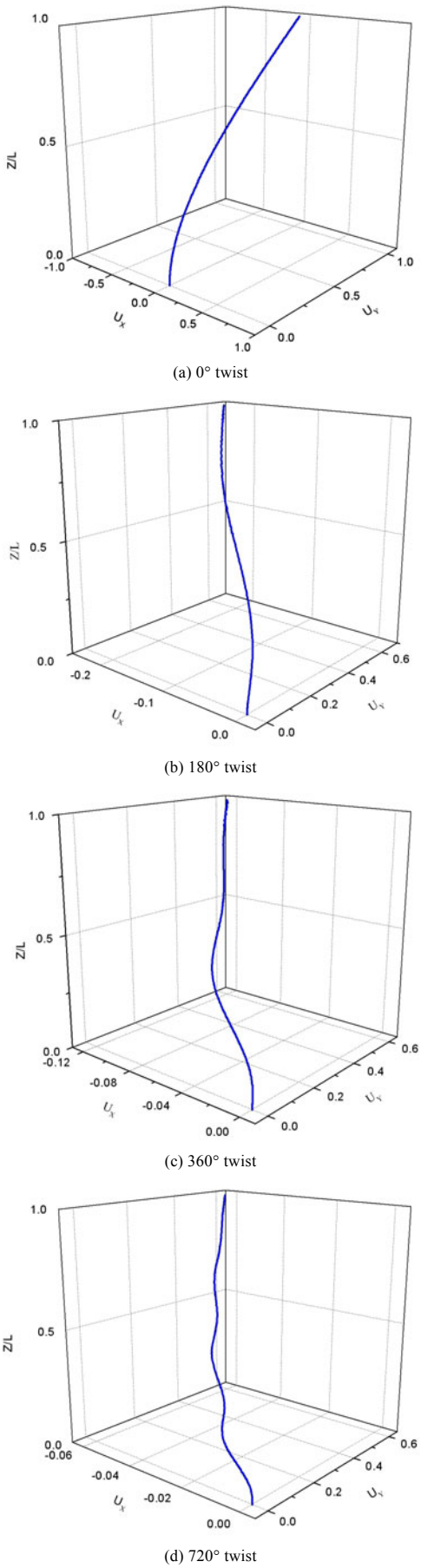


Fig. 2 Deformation plots of centroidal axis for clamped-free twisted Timoshenko beams of L/h = 100, h/b = 0.25 subjected to unit tip force

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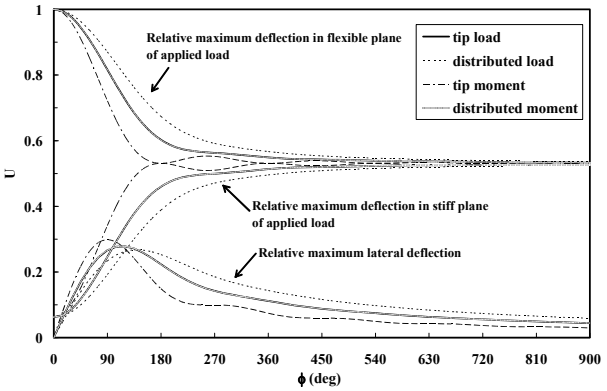


Fig. 3 Relative displacements versus twisted angles for clamped-free twisted Timoshenko beams of L/h = 100, h/b = 0.25 subjected to various loadings

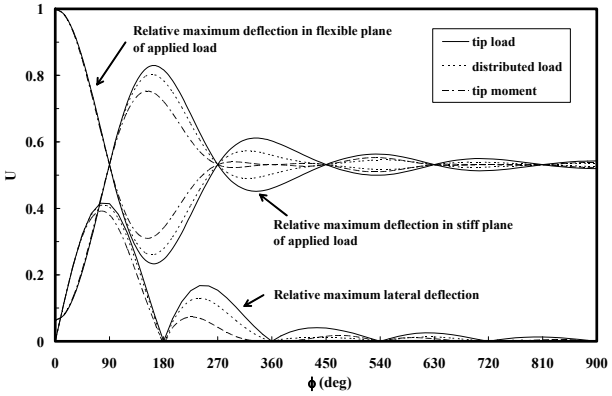
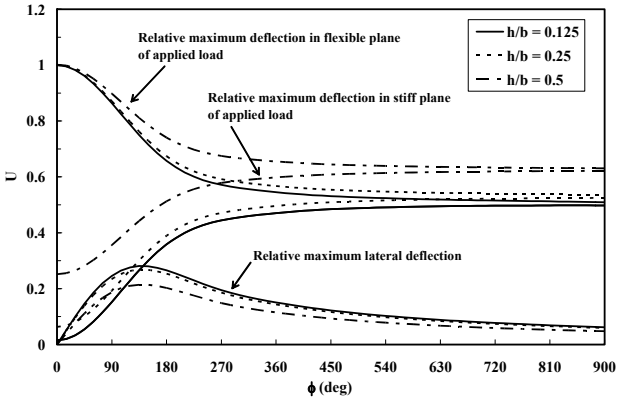


Fig. 4 Relative displacements versus twisted angles for simply-supported twisted Timoshenko beams of L/h = 100, h/b = 0.25 subjected to various loadings

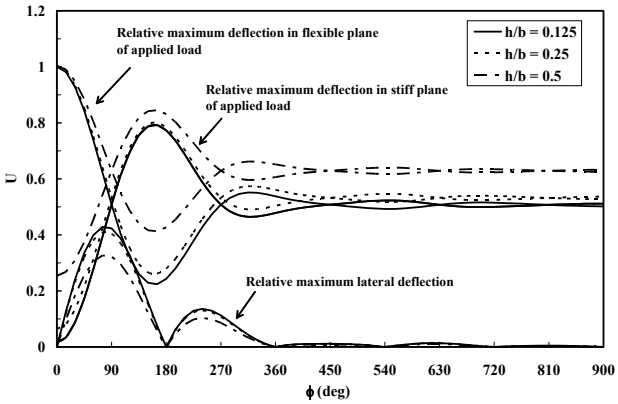
deflection becomes minimal. Due to the coupling effect of pretwist, there exists a lateral deflection in addition to the deflection in the direction of the applied load. The lateral deflection increases with the twist angle up to 120° for the beams under tip force and distributed moment, and up to 135° for the beam under distributed force, and then decreases gradually. For the beam subjected to tip moment, the lateral deflection increases with the twist angle up to 90° and thereafter varies up and down and reduces eventually. The investigated twisted beams under tip (or distributed) force and clamped-free restrained condition can be used to study the static bending of pretwisted blades.

The plots of the relative displacement at the center of the simply-supported beams with L/h = 100 and h/b = 0.25 under different loadings are presented in Fig. 4. The variation of the displacement versus the twist angle is quite different from that of the cantilever twisted beam. With the increase in twist angle, the two relative maximum deflections oscillate and intersect to each other and form local maxima and minima at typical twist angles. As can be seen, this phenomenon varies with the loading applied. Unlike the cantilever twisted beam, the bending stiffness of the simply-supported twisted beam increases and decreases alternatively with the increasing pretwist. The relative maximum deflection might not be affected by the orientation of the applied loading at certain twist angles where the two deflection curves intersect. However, for a typical twist angle and greater the difference between the two maximum deflections is reduced with the increasing twist angle so the influence of the direction of applied loads will become minor. As the twist angle is increased, the relative maximum lateral deflection oscillates in such a way that the amplitude of the oscillations diminishes with the increase in twist angle.

The effect of the thickness ratio h/b on the relative displacements of the twisted beams with L/b = 12 under distributed force is presented in Figs. 5(a) and 5(b). With the increasing thickness ratio, the relative maximum deflection is increased and the maximum lateral deflection is decreased regardless of the twist angle. Similar behavior can also be observed for other loading cases. Figures 6(a) ~ 6(d) show the influence of the



(a) clamped-free

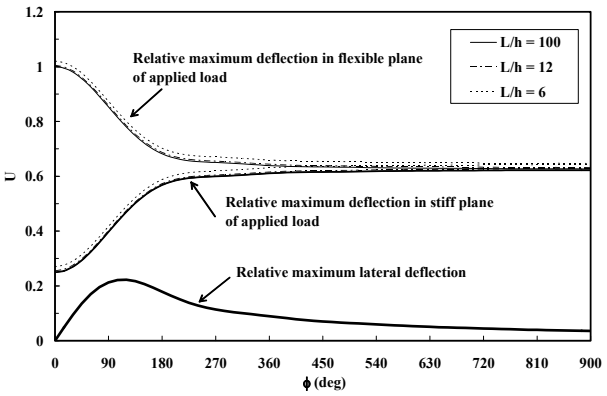


(b) simply-supported

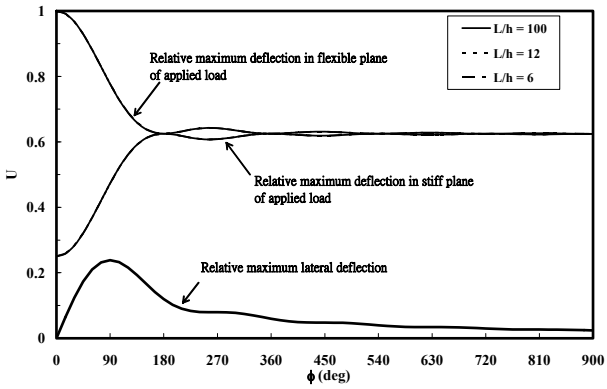
Fig. 5 Relative displacements versus thickness-to-width ratios for twisted Timoshenko beams of L/b = 12 subjected to uniformly distributed forces

length-to-thickness ratio (L/h) on the deflections of the twisted beams with h/b = 0.5. The relative maximum deflection increases with the decreasing length-to-thickness ratio when the twisted beam is under the concentrated transverse force. The reason for this difference arises from the effect of shear deformation, which makes the beam less stiff. The effect of shear deformation on the deflections is much significant for the short twisted beam with simply-supported constraint.

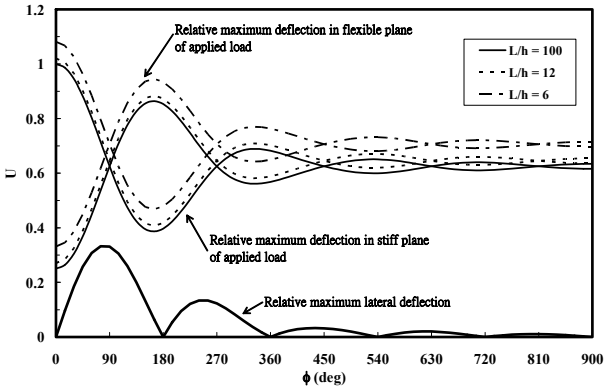
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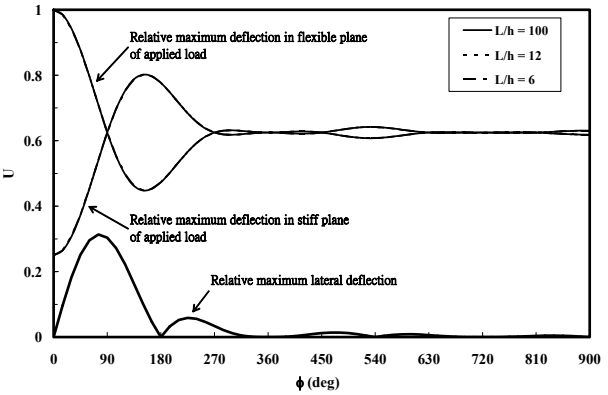
(a) clamped-free, tip force



(b) clamped-free, tip moment



(c) simply-supported, tip force

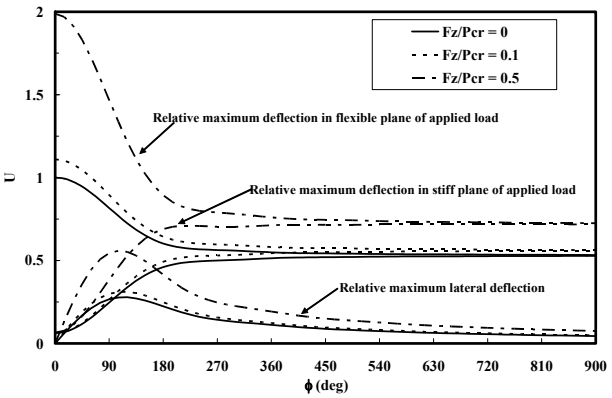


(d) simply-supported, tip moment

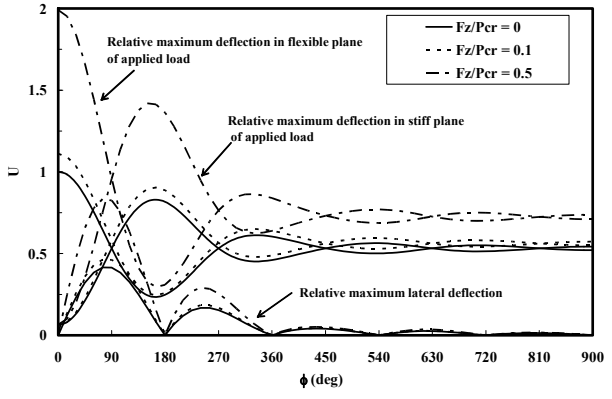
Fig. 6 Relative displacements versus length-to-thickness ratios for twisted Timoshenko beams of $h/b = 0.5$

Likewise, the displacements for twisted beams under a distributed force have the same tendency. On contrast, the relative maximum deflection is not affected by the length-to-thickness ratio for the twisted beam subject to the moment. However, the relative maximum lateral displacement remains unchanged regardless of the length-to-thickness ratio for each loading case.

The influence of axial compressive force on the deflections of the cantilevered and simply-supported beam under combined axial and transverse forces is given in Figs. 7(a) and 7(b). Three values of axial load parameter, $F_Z / P_{cr} = 0$, 0.1, 0.5, are considered. The critical buckling load P_{cr} is $0.25\pi^2 EI_\xi / L^2$ and $\pi^2 EI_\xi / L^2$ for the corresponding untwisted Euler-Bernoulli beam with clamped-free and pinned-pinned end condition, respectively. Both the relative maximum deflection and lateral deflection are increased for all twisted angles as the compressive axial load is increased. This is due to the fact that the contribution of the compressive axial load tends to lessen the system stiffness of the twisted beam in both directions. The cantilevered twisted beam subjected to combined transverse and axial loads at the free end is able to model the bending behaviors of fluted cutters, such as drill bits and end-milling cutters.



(a) clamped-free



(b) simply-supported

Fig. 7 Relative displacements versus axial load parameters for twisted Timoshenko beams of $L/b = 12$ and $h/b = 0.25$ under simultaneous axial compression and tip load

https://doi.org/10.1017/jmech.2012.23 Published online by Cambridge University Press

4. CONCLUDING REMARKS

The present Timoshenko beam model can be used to analyze the static bending characteristics of twisted beam structures, such as the blades and the twisted drill bits. The finite element results demonstrate the influence of the twist angle, shear deformation, axial force, transverse loadings and boundary conditions on the bending behaviors of twisted beams. Based on the results discussed earlier, several conclusions can be summarized as follows.

1. For cantilever twisted beam under the tip force, uniformly distributed force or couple, the increasing pretwist always increases the resistance to bending in the flexible-plane direction but decreases the resistance to bending in the stiff-plane direction.
2. With the increase in twist angle, bending stiffness of the simply-supported twisted beam under the end moments, concentrated or uniformly distributed force, decreases and increases alternatively along the flexible-plane direction but has the opposite tendency along the stiff-plane direction.
3. The shear deformation or length-to-thickness ratio L/h will merely affect the bending properties of the short twisted beam subjected to single concentrated or distributed force. The relative maximum deflection is increased as the decreasing length-to-thickness ratio, but the relative maximum lateral deflection remains unchanged.
4. The increasing compressive axial force tends to reduce the resistance to bending of the twisted beam.

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(Manuscript received September 20, 2010,
accepted for publication February 9, 2011.)

https://doi.org/10.1017/jmech.2012.23 Published online by Cambridge University Press