Ergod. Th. & Dynam. Sys. (2019), **39**, 2827–2854 © *Cambridge University Press, 2018*
doi:10.1017/etds.2017.136

Sturmian words and Cantor sets arising from unique expansions over ternary alphabets

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(*Received* 2 *June* 2016 *and accepted in revised form* 13 *November* 2017)

Abstract. Over a finite alphabet A of real numbers, unique expansions in base β are considered. A real number G_A called the generalized golden ratio is a critical point of a situation of unique expansions. If $\beta < G_A$, then there are only trivial unique expansions in base β, while there are non-trivial unique expansions in base β whenever $\beta > G_A$. Komornik, Lai and Pedicini [Generalized golden ratios of ternary alphabets. *J. Eur. Math. Soc.* **13**(4) (2011), 1113–1146] investigated the case where A consists of three real numbers, and demonstrated that Sturmian words curiously emerge out of the generalized golden ratio. The present paper focuses on Sturmian words under this context. For a given alphabet $A = \{a_1, a_2, a_3\}$ with $a_1 < a_2 < a_3$, we give a complete characterization of the corresponding Sturmian words effectively and algorithmically, which reveals interesting structures behind the generalized golden ratios.

1. *Introduction*

Since their introduction by Rényi [**22**], expansions in non-integer bases have been studied in diverse contexts, including but not limited to ergodic theory, number theory, theoretical computer science, etc. The most characteristic feature of non-integer bases is that they allow multiple representations. Among others, Parry [**21**] explored the lexicographically greatest expansions, which are called, in the literature, *greedy expansions* or *β-expansions*. On the other hand, Erdös, Joó and Komornik [**9**] concentrated on this multiplicity nature of expansions in non-integer bases. They proved that if $1 < \beta < (1 + \sqrt{5})/2$ then every $x \in (0, 1/(\beta - 1))$ has a continuum of expansions in base β over the alphabet $\{0, 1\}$. Daróczy and Kátai [**8**] found that the golden ratio $(1 + \sqrt{5})/2$ is the best possible upper bound, i.e. if $(1 + \sqrt{5})/2 < \beta \leq 2$ then some $x \in (0, 1/(\beta - 1))$ have unique expansions in base β over $\{0, 1\}$. Recently, Komornik, Lai and Pedicini [**12**] considered unique expansions over a ternary alphabet $A = \{a_1, a_2, a_3\}$ with $a_1 < a_2 < a_3$. In a fixed base, the uniqueness of

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expansion is not affected if we replace the alphabet A by

$$A' := \{a_1, a_1 - a_2 + a_3, a_3\},$$
$$c + A := \{c + a_1, c + a_2, c + a_3\}, \quad \text{or}$$
$$cA := \{ca_1, ca_2, ca_3\}$$

for any non-zero real c [**12**, Lemma 2.9]. Hence it suffices to consider a ternary alphabet $A_m := \{0, 1, m\}$, where

$$m := \max\left\{\frac{a_3 - a_1}{a_2 - a_1}, \frac{a_3 - a_1}{a_3 - a_2}\right\} \geq 2.$$

For a given $m \geq 2$, a real number $G_m (= p_m) \geq 2$ was decided, which is called a *critical base* or a *generalized golden ratio* for A_m.

THEOREM A. [**12**] *Let $m \geq 2$ be a real number. Then there continuously exists p_m such that:*

(a) $2 \leq p_m \leq P_m := 1 + \sqrt{m/(m-1)}$;
(b) *if $\beta < p_m$, then unique expansions over A_m in base β are only 0^∞ and m^∞, the trivial unique expansions;*
(c) *if $\beta > p_m$, then there are non-trivial unique expansions over A_m in base β;*
(d) *the set $C := \{m \geq 2 : p_m = P_m\}$ is a Cantor set, i.e. a perfect set that is dense nowhere.*

From behind the generalized golden ratio, Komornik *et al* exhibited that, curiously enough, Sturmian words emerge in terms of admissible sequences. A sequence or infinite word $d = (d_i)_{i=1}^\infty \in \{0, 1\}^{\mathbb{N}}$ is *admissible* if

$$0d_2d_3 \cdots \leq (d_{n+i})_{i=1}^\infty \leq d_1d_2d_3 \cdots \tag{1}$$

for every $n \geq 0$. Here and throughout the paper, inequalities on words are lexicographic, and we will not distinguish between sequences and infinite words. Komornik *et al* observed that Sturmian words are involved in admissible sequences. The present paper gives the entire picture of how their characterization makes Sturmian words out of admissible sequences.

We write σ for the shift map on infinite words. Suppose that $d = (d_i)_{i=1}^\infty \in \{0, 1\}^{\mathbb{N}}$ is an admissible sequence which is not 1^∞. If d is either aperiodic or not purely periodic, then we set $d' := \sigma(d) = (d_{1+i})_{i=1}^\infty$. On the other hand, if d is purely periodic, then we put $d' := \sigma(\hat{d})$, where $\hat{d}$ is the smallest admissible sequence greater than d. Let us denote by μ, the map on $\{0, 1\}^{\mathbb{N}}$ such that $0 \mapsto 1$ and $1 \mapsto m$. Given an admissible sequence $d \neq 1^\infty$, let $\delta = (\delta_i)_{i=1}^\infty := \mu(d)$ and $\delta' = (\delta'_i)_{i=1}^\infty := \mu(d')$, and define the values $p'_m, p''_m > 1$ by series equations

$$\sum_{i=1}^{\infty} \frac{\delta_i}{(p'_m)^i} = m - 1 \tag{2}$$

and

$$\sum_{i=1}^{\infty} \frac{m - \delta'_i}{(p''_m)^i} = 1, \tag{3}$$

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respectively. As functions of m, the values P_m, p'_m, p''_m behave as the next theorem outlines.

THEOREM B. [**12**] *Let $d \neq 1^\infty$ be an admissible sequence.*

(a) *The function $m \mapsto p'_m - P_m$ is strictly decreasing and continuous in $(1, \infty)$, and has a unique zero m_d.*

(b) *The function $m \mapsto p''_m - P_m$ is strictly increasing and continuous in $(1, \infty)$, and has a unique zero M_d.*

(c) *The function $m \mapsto p'_m - p''_m$ is strictly decreasing and continuous in $(1, \infty)$, and has a unique zero μ_d.*

For a given $m \geq 2$, the procedure to determine the generalized golden ratio for A_m is as follows.

THEOREM C. [**12**] *Let $m \geq 2$ be a real number. Then there exists the lexicographically greatest admissible sequence $d \neq 1^\infty$ such that*

$$\sum_{i=1}^{\infty} \frac{\delta_i}{P_m^i} \leq m - 1. \tag{4}$$

Moreover, $m_d \leq m < M_d$ and

$$p_m := \max\{p'_m, p''_m\} = \begin{cases} p'_m & \text{if } m_d \leq m \leq \mu_d, \\ p''_m & \text{otherwise} \end{cases}$$

is the critical base for A_m.

In §§2 and 3, we review some preliminaries on β-expansions and Sturmian words. Another characterization of admissible sequences is also established in §3. This will be easier to handle for our purpose. Section 4 inaugurates an effective algorithm to determine the admissible sequence satisfying (4). This was possible only after characterizing admissible sequences in a more familiar manner, which enables us to utilize existing theory on Sturmian words: characteristic, central, and skew words, standard factorizations, palindromic closures, derivations, and so on. Beginning with trivial but enlightening cases, §5 describes the process to obtain m_d, M_d, and μ_d for a given admissible sequence d. Finally, the critical base p_m for the alphabet A_m is also concretized here. Generically, only algebraic operations will be involved in doing this. Here, the algebraic operations mean finding the zeros of polynomials. Cantor sets $C := \{m \geq 2 : p_m = P_m\}$ and $\mathcal{C} := \{p_m : p_m = P_m, m \geq 2\}$ are focused on in §6 to some degree of detail. We specify their sizes in terms of Hausdorff dimension. Their arithmetic properties are also pursued from a folklore on transcendence. In the last section, we present a brief summary with an implemented computer program and some illuminating figures.

2. *β-expansions*

Let $\mathbb{N}$ be the set of non-negative integers. As usual, $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions respectively, and $\{\cdot\}$ the fractional part. For a real $\beta > 1$, the *β-expansions* are greedy expansions in base β over the alphabet

$$\mathcal{A}_\beta := \{0, 1, \ldots, \lceil \beta \rceil - 1\}.$$

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β-expansions can be realized by symbolic dynamics of the *β-transformation* $T_\beta : [0, 1] \to [0, 1)$ defined by

$$T_\beta(x) = \beta x \mod 1.$$

More precisely, the β-expansion $d_\beta(x) := (x_i)_{i\geq 1}$ of $x \in [0, 1]$ is given by

$$x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor \quad \text{for } i \geq 1.$$

Other important expansions in this paper are quasi-greedy expansions. The *quasi-greedy expansion* in base β over $\mathcal{A}_\beta$ is the lexicographically greatest infinite expansion, i.e. non-zero letters appear infinitely often in the expansion. In [**14**], it was shown that the quasi-greedy expansion can also be implemented by a map $\overline{T}_\beta : [0, 1] \to (0, 1]$ defined by

$$\overline{T}_\beta(x) := \langle \beta x \rangle,$$

where

$$\langle t \rangle := \begin{cases} \{t\} & \text{if } t \notin \mathbb{Z}, \\ 1 & \text{if } t \in \mathbb{Z}. \end{cases}$$

In other words, the quasi-greedy expansion $d^*_\beta(x) = (x_i)_{i\geq 1}$ of $x \in [0, 1]$ is determined by

$$x_i = \lceil \beta \overline{T}_\beta^{i-1}(x) \rceil - 1 \quad \text{for } i \geq 1.$$

We need the next theorem from [**21**].

PROPOSITION 2.1. [**21**]

(a) *Let $\mathcal{A} \subset \mathbb{N}$ be a finite alphabet, and $s \in \mathcal{A}^{\mathbb{N}}$. Then $s = d_\beta(1)$ for some $\beta > 1$ if and only if*

$$\sigma^n(s) < s \quad \textit{for every } n \geq 1.$$

(b) *Given $\beta > 1$, let $s \in \mathcal{A}_\beta^{\mathbb{N}}$. Then $s = d_\beta(x)$ for some $x \in [0, 1)$ if and only if*

$$\sigma^n(s) < d^*_\beta(1) \quad \textit{for every } n \geq 0.$$

The present paper considers the greedy or quasi-greedy expansion in base $\beta := P_m = 1 + \sqrt{m/(m-1)}$. For $m \geq 2$, one observes that

$$2 < P_m \leq 1 + \sqrt{2}$$

with the equality if and only if $m = 2$. The next lemma is for future reference.

LEMMA 2.2. *Given $m \geq 2$, let $\beta = P_m$. The β-expansion $d_\beta(1) \in \{0, 1, 2\}^{\mathbb{N}}$ of 1 is given by*

$$d_\beta(1) = \begin{cases} 210^\infty & \textit{if } m = 2, \\ 20\cdots & \textit{otherwise}. \end{cases}$$

Proof. We proceed along with the β-transformation. Since $2 < \beta < 3$, the greedy expansion of 1 begins with 2. Next, one has

$$\beta T_\beta(1) = \beta(\beta - 2) = \left(1 + \sqrt{\frac{m}{m-1}}\right)\left(-1 + \sqrt{\frac{m}{m-1}}\right) = \frac{1}{m-1} \leq 1,$$

where the equality holds if and only if $m = 2$. □

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3. *Sturmian words*

For a finite alphabet $\mathcal{A}$, we mean by $\mathcal{A}^*$ the set of finite words over $\mathcal{A}$, whereas $\mathcal{A}^n$ is the set of words with length n. So, $\mathcal{A}^* = \bigcup_{n\geq 0} \mathcal{A}^n$. The word with length 0 is the *empty word* ε that serves as the identity in $\mathcal{A}^*$ under concatenation. If w is a word finite or infinite, then $\mathrm{alph}(w)$ denotes the set of letters appearing in w. For a letter $a \in \mathcal{A}$ and a word $w \in \mathcal{A}^*$, we write $|w|_a$ for the number of occurrences of a in w, while $|w|$ is the length of w. Suppose $w = a_1 a_2 \cdots a_n \in \mathcal{A}^*$. Then $\tilde{w} := a_n a_{n-1} \cdots a_1$ means the *reversal* of w. In particular, if $\tilde{w} = w$, then w is termed a *palindrome*. As usual, the topology given in $\mathcal{A}^{\mathbb{N}}$ is a product topology while $\mathcal{A}$ has a discrete one.

Given reals $\alpha \geq 0$ and $0 \leq \rho \leq 1$, we define two arithmetic functions $s_{\alpha,\rho}, s'_{\alpha,\rho} : \mathbb{N} \to \mathbb{N}$ by

$$s_{\alpha,\rho}(n) := \lfloor \alpha(n+1) + \rho \rfloor - \lfloor \alpha n + \rho \rfloor,$$
$$s'_{\alpha,\rho}(n) := \lceil \alpha(n+1) + \rho \rceil - \lceil \alpha n + \rho \rceil.$$

Then the infinite words

$$s_{\alpha,\rho} := s_{\alpha,\rho}(0)s_{\alpha,\rho}(1)\cdots \quad \text{and} \quad s'_{\alpha,\rho} := s'_{\alpha,\rho}(0)s'_{\alpha,\rho}(1)\cdots$$

are called *lower* and *upper mechanical words* with *slope* α and *intercept* ρ, respectively. One observes that $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are binary infinite words unless α is an integer, i.e. for a non-integer α, the set of letters appearing in $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are

$$\mathrm{alph}(s_{\alpha,\rho}) = \mathrm{alph}(s'_{\alpha,\rho}) = \{\lceil \alpha \rceil - 1, \lceil \alpha \rceil\}.$$

If $\alpha \in \mathbb{N}$, then $s_{\alpha,\rho} = s'_{\alpha,\rho} = \alpha^\infty$ are unary words. Mechanical words are known to be recurrent [**18**]. An infinite word x is said to be *recurrent* if any finite factor of x has an infinite number of occurrences in x. We readily note that $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are aperiodic if and only if α is irrational. For a rational α, on the other hand, both $s_{\alpha,\rho}$ and $s'_{\alpha,\rho}$ are purely periodic. Mechanical words with irrational slopes are called *Sturmian words*.

Let w be a finite or infinite word such that $\mathrm{alph}(w) = \{a, b\}$. We say that w is a *balanced* word if $||u|_b - |v|_b| \leq 1$ whenever both u and v are finite factors of w with $|u| = |v|$. Another characterization of Sturmian words is due to Morse and Hedlund [**20**]: Sturmian words are aperiodic balanced words. Morse and Hedlund also proved that an infinite balanced word is either a mechanical word or a skew word. A *skew word* is a non-recurrent but ultimately periodic infinite word. Let $\{x, y\} = \{a, b\}$, and define morphisms ψ_a and ψ_b by

$$\psi_a(a) = a, \quad \psi_b(a) = ba,$$
$$\psi_a(b) = ab, \quad \psi_b(b) = b.$$

Then skew words are suffixes of $\Psi(x^l y x^\infty)$, $l \in \mathbb{N}$ but not of $\Psi(x^\infty)$, where Ψ is a composition of any finite number of morphisms ψ_a and ψ_b. Suffixes of $\Psi(x^\infty)$ are mechanical words of rational slopes.

Mechanical words with zero intercept ($\rho = 0$) are of special interest. Let $\alpha \geq 0$ and $a + 1 = b = \lceil \alpha \rceil$. If α is irrational, then $s_{\alpha,0}$ and $s'_{\alpha,0}$ have a suffix c_α in common:

$$s_{\alpha,0} = ac_\alpha, \quad s'_{\alpha,0} = bc_\alpha.$$

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This c_α is called the *characteristic word* of slope α. On the other hand, if $\alpha = p/q$ with $\gcd(p, q) = 1$, then $s_{\alpha,0}$ and $s'_{\alpha,0}$ are purely periodic and their shortest periodic words are of the form

$$s_{\alpha,0} = (az_{p,q}b)^\infty, \quad s'_{\alpha,0} = (bz_{p,q}a)^\infty,$$

where the common factor $z_{p,q}$ is a palindrome called the *central word*. See, e.g., [**18**]. Not every palindrome can be a central word. Central words obey the next rule.

PROPOSITION 3.1. [**18**] *Let $\mathcal{P}$ be the set of palindromes over $\{a, b\}$. A word $w \in \{a, b\}^*$ is central if and only if*

$$w \in \{a\}^* \cup \{b\}^* \cup (\mathcal{P} \cap \mathcal{P}ab\mathcal{P}).$$

Let $p/q \in (0, 1)$ be a rational with $\gcd(p, q) = 1$. Along with the above proposition, the word $1z_{p,q}0$ can be written as $1z_{p,q}0 = (1z_{p',q'}0)(1z_{p'',q''}0)$ for some rationals p'/q', $p''/q'' \in [0, 1]$ with $p' + p'' = p$ and $q' + q'' = q$. This is called a *standard factorization*.

PROPOSITION 3.2. [**5**] *Let $p/q \in (0, 1)$ be a rational with $\gcd(p, q) = 1$. The word $1z_{p,q}0$ has a unique standard factorization*

$$1z_{p,q}0 = (1z_{p',q'}0)(1z_{p'',q''}0).$$

Furthermore, the integers $q' = |1z_{p',q'}0|$ and $q'' = |1z_{p'',q''}0|$ satisfy

$$q'(q - p) \equiv 1 \pmod{q}$$

and

$$q''p \equiv 1 \pmod{q}.$$

Given a finite word w, we denote by w^+ the unique shortest palindrome having w as a prefix. For example, $(abbab)^+ = abbabba$. The palindrome w^+ is called the *(right) palindromic closure* of w. Suppose that a finite word $w = uv$, where v is the longest palindromic suffix of w. One immediately sees that $w^+ = w\tilde{u}$.

Over a binary alphabet $\mathcal{A} = \{a, b\}$, a function $\mathrm{Pal} : \mathcal{A}^* \to \mathcal{A}^*$ is recursively defined as follows:

(i) $\mathrm{Pal}(\varepsilon) := \varepsilon$;

(ii) if $w = vx$ for some $x \in \{a, b\}$, then $\mathrm{Pal}(w) := (\mathrm{Pal}(v)x)^+$.

We need another characterization of central words.

PROPOSITION 3.3. [**3**] *A word $u \in \{a, b\}^*$ is central if and only if $u = \mathrm{Pal}(v)$ for some $v \in \{a, b\}^*$.*

Given a central word u, the next proposition makes it possible to find the word v such that $u = \mathrm{Pal}(v)$. This will play an essential role in finding the admissible sequence satisfying Theorem C.

PROPOSITION 3.4. [**3**] *The function $\mathrm{Pal} : \{a, b\}^* \to \{a, b\}^*$ is injective. Let $\{\varepsilon = \mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n, \mathfrak{p}_{n+1} = \mathrm{Pal}(w)\}$ be the set of palindromic prefixes of $\mathrm{Pal}(w)$ such that $0 = |\mathfrak{p}_1| < |\mathfrak{p}_2| < \cdots < |\mathfrak{p}_n| < |\mathrm{Pal}(w)|$. And let x_i be the letter appearing just after $\mathfrak{p}_i$ in $\mathrm{Pal}(w)$. Then $w = x_1x_2 \cdots x_n$.*

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All mechanical words of rational slope α can be realized by shifting $s'_{\alpha,0}$. We say that two finite words w and w' are *cyclically conjugate* if there exist words u and v such that $w = uv$ and $w' = vu$.

PROPOSITION 3.5. [**3**] *Let $0 \le \rho \le 1$ and let $\alpha = p/q \ge 0$ be a rational number with $b = a + 1 = \lceil \alpha \rceil$. Any of the finite words $s_{\alpha,\rho}(0)s_{\alpha,\rho}(1)\cdots s_{\alpha,\rho}(q-1)$ and $s'_{\alpha,\rho} := s'_{\alpha,\rho}(0)s'_{\alpha,\rho}(1)\cdots s'_{\alpha,\rho}(q-1)$ are cyclically conjugate to $bz_{p,q}a$ and $az_{p,q}b$.*

The order between mechanical words is rather well known.

PROPOSITION 3.6. [**5**] *Let $\alpha > 0$ and $0 < \rho < 1$ be real numbers. Then*

$$s_{\alpha,0} \le s_{\alpha,\rho} \le s'_{\alpha,\rho} \le s'_{\alpha,0}.$$

If α is irrational, then the first and the last inequalities are strict.

We should pay special attention to a particular type of skew word. With intercept $\rho = 0$ fixed, the right limit of an upper mechanical word gives us this type of skew word.

PROPOSITION 3.7. *Let $\alpha_0 = p/q \in [0, 1)$ be rational with $\gcd(p, q) = 1$. Then*

$$\lim_{\alpha \searrow p/q} s'_{\alpha,0} = 1(z_{p,q}10)^\infty.$$

Proof. See [**15**, Lemma 2.3]. □

Let E be the unique automorphism on $\{0, 1\}^*$ which sends each letter 0 to 1, and 1 to 0. This map naturally extends to $\{0, 1\}^{\mathbb{N}}$. We also write $E(w) =: \overline{w}$. The symmetry of mechanical words is revealed in the following form.

PROPOSITION 3.8. [**18**]

(a) *If $\alpha \in [0, 1]$ is irrational, then*

$$\overline{c_\alpha} = c_{1-\alpha}.$$

(b) *If $p/q \in [0, 1]$ is rational with $\gcd(p, q) = 1$, then*

$$\overline{z_{p,q}} = z_{q-p,q}.$$

Given a sequence $h = (h_i)_{i=1}^\infty$ of positive integers, the finite words $S_h(j, 1)$, $S_h(j, 0) \in \{0, 1\}^*$, $j = 0, 1, 2, \ldots$, were presented in [**12**] by the following recursive formulae:

$$\begin{aligned} S_h(0, 1) &:= 1 \quad \text{and} \quad S_h(0, 0) := 0, \\ S_h(j, 1) &:= S_h(j-1, 1)^{h_j} S_h(j-1, 0), \\ S_h(j, 0) &:= S_h(j-1, 1)^{h_j - 1} S_h(j-1, 0). \end{aligned} \tag{5}$$

And admissible sequences were illustrated in terms of these words.

THEOREM D. [**12**] *A sequence $d \in \{0, 1\}^{\mathbb{N}}$ is admissible if and only if one of the following conditions holds:*

- *$d = 0^\infty$;*
- *there exist some $N \in \mathbb{N}$ and a finite sequence $h = (h_1, h_2, \ldots, h_N)$ of positive integers such that $d = S_h(N, 1)^\infty$;*
- *there exists an infinite sequence $h = (h_i)_{i=1}^\infty$ of positive integers such that $d = \lim_{N\to\infty} S_h(N, 1)$.*

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Here, $\lim_{N\to\infty} S_h(N, 1)$ makes sense because $S_h(N, 1)$ begins with $S_h(N-1, 1)$ for every $N \geq 1$. Komornik *et al* observed that an infinite word s is a characteristic word c_α for some irrational $\alpha \in (0, 1)$ if and only if $1s = \lim_{N\to\infty} S_h(N, 1)$ for some infinite sequence $h = (h_i)_{i=1}^{\infty}$ such that $h_i \geq 2$ for infinitely many i. Now we make this characterization more user-friendly, which will then be employed throughout the paper.

An infinite word $s \in \{a, b\}^{\mathbb{N}}$ is said to be *allowable* if $0 \leq a = b - 1$ and s begins with the letter b. So an allowable word s may be written in the form: $s = b^{n_1}ab^{n_2}ab^{n_3}\cdots$, $n_i \geq 0$. We define the *derivation* D on allowable words by

$$D(s) = D(b^{n_1}ab^{n_2}ab^{n_3}\cdots) := n_1n_2\cdots .$$

If a word $n_1n_2\cdots$ is also allowable, then we say that s is *derivable*. By convention, we let a^∞, b^∞, and ∞ be both allowable and derivable, and express

$$D(a^\infty) = D(b^\infty) = \infty \quad \text{and} \quad D(\infty) = \infty.$$

Note that the other fixed point of D is 10^∞: $D(10^\infty) = 10^\infty$. Suppose that an infinite word $s \in \{a, b\}^{\mathbb{N}}$ is derivable indefinitely many times, i.e. $D^n(s)$ is allowable for every $n \geq 0$, and let h_i be the first letter of $D^i(s)$ for $i \geq 1$. If $h_i < \infty$ for every $i \geq 1$, then we associate, with s, a sequence $h = (h_i)_{i=1}^{\infty}$. Associated with s is a finite sequence $h = (h_1, h_2, \ldots, h_N)$ if $h_N < \infty$ and $h_{N+1} = \infty$. We obtain another characterization of admissible sequences owing to [**13**]. This concrete characterization makes it possible to produce the algorithm of the present paper.

THEOREM 3.9. *For an infinite word $d \in \{0, 1\}^{\mathbb{N}}$, the following are equivalent.*

(a) *d is admissible.*

(b) *One of the following conditions holds:*
- *$d = 0^\infty$;*
- *there exist some $N \in \mathbb{N}$ and a finite sequence $h = (h_1, h_2, \ldots, h_N)$ of positive integers such that $d = S_h(N, 1)^\infty$;*
- *there exists an infinite sequence $h = (h_i)_{i=1}^{\infty}$ of positive integers such that $d = \lim_{N\to\infty} S_h(N, 1)$.*

(c) *$D^n(d)$ is allowable for every $n \geq 0$.*

(d) *One of the following conditions holds:*
- *$d = (1z_{p,q}0)^\infty$ for some rational $p/q \in [0, 1]$;*
- *$d = 1z_{p,q}1(0z_{p,q}1)^\infty = 1(z_{p,q}10)^\infty$ for some rational $p/q \in [0, 1)$;*
- *$d = 1c_\alpha$ for some irrational $\alpha \in (0, 1)$.*

Remark 3.10. If p/q is either $0/1$ or $1/1$, then $(1z_{p,q}0)^\infty$ should read $(1z_{0,1}0)^\infty = 0^\infty$ and $(1z_{1,1}0)^\infty = 1^\infty$, while $1(z_{0,1}10)^\infty = 10^\infty$.

Proof of Theorem 3.9. (b) $\Rightarrow$ (c). By our convention, $d = 0^\infty$ is derivable indefinitely many times. Suppose that $d = S_h(N, 1)^\infty$ for some finite sequence $h = (h_1, h_2, \ldots, h_N)$ of positive integers. We claim that d is derivable indefinitely with the associated sequence h. Indeed, $d = S_h(N, 1)^\infty$ is an infinite word over the alphabet $\{S_h(0, 0) = 0, S_h(0, 1) = 1\}$, and begins with 1. Hence d is allowable, and $D(d)$ is an infinite word over the alphabet $\{h_1 - 1, h_1\}$. In other words, we regard d as an infinite word over the alphabet

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$\{S_h(1, 0), S_h(1, 1)\}$, but the word $S_h(1, 0)$ is collapsed into a letter $h_1 - 1$ and $S_h(1, 1)$ into h_1. Since $D(d)$ begins with h_1, it is also allowable. Hence, $D^2(d)$ is an infinite word over the alphabet $\{h_2 - 1, h_2\}$, i.e. d is regarded as an infinite word over the alphabet $\{S_h(2, 0), S_h(2, 1)\}$ with the word $S_h(2, 0)$ collapsed into a letter $h_2 - 1$ and with $S_h(2, 1)$ into h_2. Continuing this process, we finally attain $D^N(d) = h_N^\infty$ and $D^k(d) = \infty$ for any $k \geq N + 1$.

The case of $d = \lim_{N\to\infty} S_h(N, 1)$ for some $h = (h_i)_{i=1}^\infty$ has the same proof, but the process continues indefinitely.

(c) $\Rightarrow$ (d). This implication follows from [**13**, Theorem 3.6].

(d) $\Rightarrow$ (a). The admissibility of $d = (1z_{p,q}0)^\infty$ and $d = 1c_\alpha$ follows from Proposition 3.6. As for the admissibility of $d = 1(z_{p,q}10)^\infty$, the inequality

$$(0z_{p,q}1)^\infty \leq \sigma^n(1(z_{p,q}10)^\infty) \leq 1(z_{p,q}10)^\infty \quad \text{for any } n \geq 0$$

is a consequence of Proposition 3.5. □

We state two easy corollaries, which will be used implicitly throughout the paper.

COROLLARY 3.11. *Every admissible sequence d is lexicographically greater than or equal to its proper suffixes. In particular, if $d = 1c_\alpha$ for some irrational $\alpha \in (0, 1)$, then the inequality is strict.*

Proof. The claim follows from Proposition 3.6. □

COROLLARY 3.12. *Suppose that an admissible sequence d is purely periodic, i.e. $d = (1z_{p,q}0)^\infty$ for some rational $p/q \in [0, 1]$. Let $\hat{d}$ be the smallest admissible sequence greater than d with $d' := \sigma(\hat{d})$. Then*

$$\hat{d} = 1(z_{p,q}10)^\infty \quad \text{and} \quad d' = (z_{p,q}10)^\infty.$$

Proof. Since any upper mechanical word $s'_{\alpha,0}$ with $\alpha > p/q$ is greater than d, Proposition 3.7 proves the claim. □

Remark 3.13. Assume $d = \lim_{n\to\infty} S_h(n, 1)$ for some $h = (h_i)_{i=1}^\infty$ where $h_N > 1$ and $h_{N+1} = h_{N+2} = \cdots = 1$. Then $D^N(d) = h_N(h_N - 1)^\infty$, and thus $D^k(d) = 10^\infty$ for all $k \geq N + 1$. In the case, $d = 1(z_{p,q}10)^\infty$ for some rational $p/q \in [0, 1)$.

Example 1. Let $h = (2, 1, 3)$. Then $d := S_h(3, 1)^\infty = (11010110101101010)^\infty$. One verifies

$$D(d) = (2121211)^\infty, \quad D^2(d) = (1110)^\infty, \quad D^3(d) = 3^\infty,$$

and $D^k(d) = \infty$ for $k \geq 4$. The first letters of $D(d)$, $D^2(d)$, $D^3(d)$ coincide with h.

Let an admissible sequence d be purely or ultimately periodic. Without appeal to the recursive formulae (5), the inverse derivation constructs d from h.

Example 2. Suppose $h = (3, 2, 1)$. Then one can revive d as

$$\begin{aligned} 1^\infty &\mapsto (21)^\infty \mapsto (33232)^\infty \\ &\mapsto (111011101101110110)^\infty = (1z_{13,18}0)^\infty = S_h(3, 1)^\infty = d. \end{aligned}$$

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If $h = (2, 3, 1, 1, 1, \ldots)$, then d is also revitalized as

$$\begin{aligned} 10^\infty &\mapsto 32^\infty \mapsto 2221(221)^\infty \\ &\mapsto 11011011010(11011010)^\infty = 1(10110110)^\infty \\ &= 1(z_{5,8}10)^\infty = \lim_{N\to\infty} S_h(N, 1) = d. \end{aligned}$$

4. *Admissible sequences*

In this section, the admissible sequences described in Theorem C are determined.

Suppose that $d = (d_i)_{i=1}^\infty \in \{0, 1\}^{\mathbb{N}}$ is admissible. We regard d as a quasi-greedy expansion of some x in base β. Then the admissibility (1) means that the $\overline{T}_\beta(x)$-orbit of x is contained in an interval with length $1/\beta$. A detailed study on such sequences was developed for the case $1 < \beta < 2$ in [**1**], and later extended to the general case $\beta > 1$ in [**14**]. For the present paper, we need the case $\beta = P_m = 1 + \sqrt{m/(m-1)} \in (2, 1 + \sqrt{2}]$. If so, then the quasi-greedy expansion of 1 begins with 2: $d_\beta^*(1) = 2 \cdots$. Let $e = (e_i)_{i=1}$ be an infinite word over the alphabet $\{0, 1, 2\}$ with $e \le d_\beta^*(1)$. The next two propositions determine an infinite word $s \in \{0, 1\}^{\mathbb{N}}$ for which

$$(e_1 - 1)e_2e_3 \cdots \le \sigma^n(s) \le e_1e_2e_3 \cdots \quad \text{for all } n \ge 0. \tag{6}$$

The propositions come from [**14**], but are translated in accordance with our framework.

PROPOSITION 4.1. [**14**] *Given $m \ge 2$, set $\beta := P_m$. Suppose that an infinite word $e = (e_i)_{i=1}^\infty \in \{0, 1, 2\}^{\mathbb{N}}$ is lexicographically less than or equal to $d_\beta^*(1)$:*

(a) *If $e \le 10^\infty$, then $s'_{0,0} = 0^\infty$ is the unique infinite word satisfying (6).*

(b) *If $e \ge 1^\infty$, then $s'_{1,0} = 1^\infty$ is the unique infinite word satisfying (6).*

(c) *If $e_1e_2e_3 \cdots = s'_{\alpha,0}$ for some irrational $\alpha \in [0, 1]$, then the set of infinite words satisfying (6) is*

$$\overline{\{\sigma^n(s'_{\alpha,0}) : n \ge 0\}},$$

the shift orbit closure of $s'_{\alpha,0}$.

(d) *If $(e_1 - 1)e_2e_3 \cdots = s_{\alpha,0}$ or $e_1e_2e_3 \cdots = s'_{\alpha,0}$ for some rational $\alpha \in [0, 1]$, then the set of recurrent infinite words satisfying (6) is*

$$\{\sigma^n(s'_{\alpha,0}) : n \ge 0\},$$

the shift orbit of $s'_{\alpha,0}$.

Remark 4.2. Note that our attention in (d) above is confined to recurrent infinite words. There are non-recurrent infinite words satisfying (6). They are skew words. See [**14**].

The other cases not covered in Proposition 4.1 are stated in the next one.

PROPOSITION 4.3. [**14**] *In addition to the hypothesis in Proposition 4.1, assume that the infinite word e satisfies none of the above four cases. Hence we have $e_1 = 1$. Let u be the longest central prefix of $\sigma(e) = e_2e_3 \cdots$ over $\{0, 1\}$.*

(a) *Suppose $u = 0^k$ for some $k \ge 1$, and let $e = 10^k ve'$ where $v \in \{0, 1, 2\}^{k+1}$ and $e' \in \{0, 1, 2\}^{\mathbb{N}}$.*

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(i) *If* $0^{k+1} < v < 10^k$, *then the set of infinite words satisfying (6) is*

$$\{\sigma^n((10^{k+1})^\infty) : n \geq 0\},$$

the shift orbit of a mechanical word of slope $1/(k+2)$.

(ii) *If* $v > 10^k$, *then the set of infinite words satisfying (6) is*

$$\{\sigma^n((10^k)^\infty) : n \geq 0\},$$

the shift orbit of a mechanical word of slope $1/(k+1)$.

(b) *Suppose* $u = 1^k$ *for some* $k \geq 1$, *and let* $e = 1^{k+1}ve'$ *where* $v \in \{0, 1, 2\}^{k+1}$ *and* $e' \in \{0, 1, 2\}^{\mathbb{N}}$.

(i) *If* $v > 1^{k+1}$, *then e falls into Case (b) of Proposition 4.1.*

(ii) *If* $01^k < v < 1^{k+1}$, *then the set of infinite words satisfying (6) is*

$$\{\sigma^n((1^{k+1}0)^\infty) : n \geq 0\},$$

the shift orbit of a mechanical word of slope $(k+1)/(k+2)$.

(iii) *If* $v < 01^k$, *then the set of infinite words satisfying (6) is*

$$\{\sigma^n((1^k0)^\infty) : n \geq 0\},$$

the shift orbit of a mechanical word of slope $k/(k+1)$.

(c) *Suppose that* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are the unique pair of central words satisfying* $u = \mathfrak{p}01\mathfrak{q} = \mathfrak{q}10\mathfrak{p}$. *Let* $e = 1uxyve'$, *where* $x, y \in \{0, 1, 2\}$, $v \in \{0, 1, 2\}^{|u|+2}$ *and* $e' \in \{0, 1, 2\}^{\mathbb{N}}$.

(i) *Either if* $xy = 01$ *and* $v > u01$, *or if* $xy = 10$ *and* $v < u10$, *then the set of infinite words satisfying (6) is*

$$\{\sigma^n((1u0)^\infty) : n \geq 0\},$$

the shift orbit of a mechanical word of slope $(|u|_1 + 1)/(|u| + 2)$.

(ii) *Either if* $xy = 00$, *or if* $xy = 01$ *and* $v < u01$, *then the set of infinite words satisfying (6) is*

$$\{\sigma^n((1\mathfrak{q}0)^\infty) : n \geq 0\},$$

the shift orbit of a mechanical word of slope $(|\mathfrak{q}|_1 + 1)/(|\mathfrak{q}| + 2)$.

(iii) *Either if* $xy \geq 11$, *or if* $xy = 10$ *and* $v > u10$, *then the set of infinite words satisfying (6) is*

$$\{\sigma^n((1\mathfrak{p}0)^\infty) : n \geq 0\},$$

the shift orbit of a mechanical word of slope $(|\mathfrak{p}|_1 + 1)/(|\mathfrak{p}| + 2)$.

(iv) *If* $01 < xy < 10$, *then the set of infinite words satisfying (6) is*

$$\{\sigma^n((1u0)^\infty) : n \geq 0\},$$

the shift orbit of a mechanical word of slope $(|u|_1 + 1)/(|u| + 2)$.

One notes that, wherever v is involved in the above, equality never holds because u is the longest central prefix of $e_2e_3 \cdots$ over $\{0, 1\}$.

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LEMMA 4.4. *Given $m \geq 2$, put $\beta := P_m$, and let d and δ be the sequences guaranteed in Theorem C. Then d satisfies*

$$1 - \frac{1}{\sqrt{m(m-1)}} - \frac{\sqrt{m-1}}{\sqrt{m}+\sqrt{m-1}} \leq \overline{T}_\beta^n\left(\sum_{i=1}^{\infty} \frac{d_i}{\beta^i}\right) \leq 1 - \frac{1}{\sqrt{m(m-1)}}. \tag{7}$$

for every $n \geq 0$. Moreover, d is a unique admissible sequence satisfying (7), unless the greedy expansion of $1 - 1/\sqrt{m(m-1)}$ in base β is

$$d_\beta\left(1 - \frac{1}{\sqrt{m(m-1)}}\right) = 1(z_{p,q}10)^\infty$$

for some rational $p/q \in [0, 1)$. In this case, only two admissible sequences $1(z_{p,q}10)^\infty$ and $(1z_{p,q}0)^\infty$ satisfy (7). In particular, if $d_\beta(1 - 1/\sqrt{m(m-1)}) = 10^\infty$, then only two admissible sequences 10^∞ and 0^∞ satisfy (7). See Remark 3.10.

Proof. From (4) it follows that

$$\sum_{i=1}^{\infty} \frac{\delta_i - 1}{P_m^i} \leq m - 1 - \frac{1}{P_m - 1} = m - 1 - \frac{\sqrt{m-1}}{\sqrt{m}}.$$

Divided by $m - 1$, this inequality is derived as

$$\sum_{i=1}^{\infty} \frac{d_i}{P_m^i} \leq 1 - \frac{1}{\sqrt{m(m-1)}}. \tag{8}$$

The lower bound in (7) is originated from $1 - 1/\sqrt{m(m-1)} - 1/P_m$. Propositions 4.1 and 4.3 together with Theorem 3.9 guarantee a unique admissible sequence $\check{d}$ satisfying (7), unless

$$d_\beta\left(1 - \frac{1}{\sqrt{m(m-1)}}\right) = (1z_{p,q}0)^\infty \tag{9}$$

or

$$d_\beta\left(1 - \frac{1}{\sqrt{m(m-1)}} - \frac{\sqrt{m-1}}{\sqrt{m}+\sqrt{m-1}}\right) = (0z_{p,q}1)^\infty \tag{10}$$

for some rational $p/q \in [0, 1)$. Since d should be the lexicographically greatest admissible sequence satisfying (8), we find $\check{d} = d$. Indeed, if $\check{d} \lneqq d$, then it contradicts the uniqueness of recurrent words in Propositions 4.1 and 4.3.

Proofs for non-uniqueness in the case of (9) and (10) now only remain. Suppose first that (9) holds. According to Theorem 3.9, a possible admissible sequence $s \neq (1z_{p,q}0)^\infty$ satisfying

$$0z_{p,q}0(1z_{p,q}0)^\infty = 0(z_{p,q}01)^\infty \leq \sigma^n(s) \leq (1z_{p,q}0)^\infty \quad \text{for all } n \geq 0 \tag{11}$$

must be $s = 1z_{p',q'}1(0z_{p',q'}1)^\infty = 1(z_{p',q'}10)^\infty$ for some rational $p'/q' \in [0, 1)$. Clearly, p'/q' cannot be equal to p/q because the above inequality (11) is violated. Even if $p'/q' \neq p/q$, then another recurrent word $(1z_{p',q'}0)^\infty$ which is some suffix of s, fulfills the inequality (11). This fact contradicts the uniqueness of recurrent words in Propositions 4.1 and 4.3. Therefore, no admissible sequence other than $(1z_{p,q}0)^\infty$ satisfies (11).

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Next, suppose that (10) holds. Then a possible admissible sequence $s \neq (1z_{p,q}0)^\infty$ satisfying

$$(0z_{p,q}1)^\infty \leq \sigma^n(s) \leq 1z_{p,q}1(0z_{p,q}1)^\infty = 1(z_{p,q}10)^\infty \quad \text{for all } n \geq 0 \tag{12}$$

is $s = 1z_{p',q'}1(0z_{p',q'}1)^\infty = 1(z_{p',q'}10)^\infty$ for some rational $p'/q' \in [0, 1)$. As in the previous case, $p'/q' \neq p/q$ cannot be true. But the skew word $1(z_{p,q}10)^\infty$ itself satisfies (12). □

For the sake of convenience and for coherence with (7), equations (2) and (3) are restated in terms of the admissible sequence d.

LEMMA 4.5. *Let d, d', δ, δ', p'_m, and p''_m be defined as in (2) and (3). Then*

$$\sum_{i=1}^{\infty} \frac{d_i}{(p'_m)^i} = 1 - \frac{1}{(m-1)(p'_m - 1)}, \tag{13}$$

$$\sum_{i=1}^{\infty} \frac{d'_i}{(p''_m)^i} = \frac{1}{p''_m - 1} - \frac{1}{m-1} \quad \text{or} \quad \sum_{i=1}^{\infty} \frac{\overline{d'_i}}{(p''_m)^i} = \frac{1}{m-1}, \tag{14}$$

where the overline in $\overline{d'_i}$ is the morphism mentioned in Proposition 3.8.

Proof. We formulate (3) as

$$\sum_{i=1}^{\infty} \frac{\delta'_i}{(p''_m)^i} = \frac{m}{p''_m - 1} - 1 \;\Rightarrow\; \sum_{i=1}^{\infty} \frac{\delta'_i - 1}{(p''_m)^i} = \frac{m-1}{p''_m - 1} - 1,$$

which is followed by

$$\sum_{i=1}^{\infty} \frac{d'_i}{(p''_m)^i} = \frac{1}{p''_m - 1} - \frac{1}{m-1} \;\Rightarrow\; \sum_{i=1}^{\infty} \frac{\overline{d'_i}}{(p''_m)^i} = \frac{1}{m-1}.$$

The equality for p'_m has a similar proof. □

A language-theoretic version of Lemma 4.4 is more convenient to grasp how critical bases work intertwined with mechanical words and β-expansions. The next theorem together with Propositions 4.1 and 4.3 will play a key role in what follows.

THEOREM 4.6. *Given $m \geq 2$, let $d = (d_i)_{i=1}^\infty \in \{0, 1\}^{\mathbb{N}}$ be given as in Theorem C. For $\beta = P_m$, suppose that the β-expansion of 1 is $d_\beta(1) = (e_i)_{i=1}^\infty \in \{0, 1, 2\}^{\mathbb{N}}$. Then*

$$e_2e_3e_4 \cdots \leq \sigma^n(\overline{d}) \leq (e_2 + 1)e_3e_4 \cdots \tag{15}$$

for all $n \geq 0$.

Remark 4.7. The above theorem verifies that the admissible sequence $d = 1^\infty$ never occurs in Theorem C. If $d = 1^\infty$ does satisfy the condition of Theorem C, then by Proposition 4.1, $(e_2 + 1)e_3e_4 \cdots$ should be lexicographically less than or equal to 10^∞. In other words, $d_\beta(1) = 20^\infty$. But we have seen that this phenomenon cannot happen in the proof of Lemma 2.2.

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Proof of Theorem 4.6. From $(\beta-1)^2 = m/(m-1)$, we have $m = (\beta-1)^2/((\beta-1)^2 - 1)$ in hand. Thus, one derives

$$\sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \le 1 - \frac{1}{\sqrt{m(m-1)}} = 1 - \frac{(\beta-1)^2-1}{\beta-1} = 2 - \beta + \frac{1}{\beta-1},$$

and hence,

$$2 - \beta + \frac{1}{\beta-1} - \frac{1}{\beta} \le \sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \le 2 - \beta + \frac{1}{\beta-1}. \tag{16}$$

Noting $e_1 = 2$ from Lemma 2.2, we deduce that

$$\sum_{i=1}^{\infty} \frac{d_i}{\beta^i} \le 2 - \beta\left(\frac{2}{\beta} + \frac{e_2}{\beta^2} + \frac{e_3}{\beta^3} + \cdots\right) + \left(\frac{1}{\beta} + \frac{1}{\beta^2} + \frac{1}{\beta^3} + \cdots\right),$$

which is turned into

$$\frac{e_2}{\beta} + \frac{e_3}{\beta^2} + \frac{e_4}{\beta^3} + \cdots \le \sum_{i=1}^{\infty} \frac{\overline{d_i}}{\beta^i}.$$

On the other hand, the left inequality of (16) gives

$$\sum_{i=1}^{\infty} \frac{\overline{d_i}}{\beta^i} \le \frac{e_2+1}{\beta} + \frac{e_3}{\beta^2} + \frac{e_4}{\beta^3} + \cdots.$$

Since $e_2e_3e_4\cdots$ is a shift of $d_\beta(1)$ and since $\overline{d}$ is an infinite word over $\{0, 1\}$, both of them are β-expansions of some numbers in $[0, 1]$. Lemma 2.2 tells us that

$$(e_2+1)e_3e_4\cdots = \begin{cases} 20^\infty < d_\beta(1) = 210^\infty & \text{if } m = 2, \\ 1e_3e_4\cdots < d_\beta(1) = 20e_3e_4\cdots & \text{otherwise.} \end{cases}$$

In either case, $(e_2+1)e_3e_4\cdots$ is a β-expansion of some number in $[0, 1]$. Therefore, the inequalities between real numbers are compatible with lexicographical ones. □

Remark 4.8. Resting upon Propositions 4.1 and 4.3, the previous theorem determines $\overline{d}$, and hence d by Proposition 3.8. If the sequence satisfying (15) is $\overline{d} = 0c_\alpha$ for some irrational $\alpha \in [0, 1]$, then $d = 1c_{1-\alpha}$, whereas one obtains $d = (1z_{q-p,q}0)^\infty$ from $\overline{d} = (0z_{p,q}1)^\infty$. We need to pay special attention only when $d_\beta(1) = 20(z_{p,q}01)^\infty$ for some rational $p/q \in [0, 1]$. In that case, both $0(z_{p,q}01)^\infty$ and $(0z_{p,q}1)^\infty$ satisfy the inequality (15). But we should pick up the lexicographically smaller one. Hence, $d = 1(z_{q-p,q}10)^\infty$.

The relation between mechanical words and unique expansions over ternary alphabets was earlier investigated in [**16**]. Unfortunately, however, the paper neglected the skew word case in [**16**, Theorem 2.1] upon which the whole context of the paper heavily depends. So the results in it should be carefully translated. For instance, let β be such that $d_\beta(1) = 20(z_{2,3}01)^\infty$, or in other words, the unique real root >1 of the polynomial $x^5 - 2x^4 - 2x^2 + 2x - 1 = 0$. If $m := 1 + 1/\beta(\beta-2)$, then [**16**, Theorem 3.1] is no more true by Theorem 5.5 below.

Furthermore, compared to [**16**] and also to [**12**], the present paper effectively and algorithmically determines the central words for a given m. This process turns out to be far from being trivial as illustrated in the next examples.

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Example 3. Let $m = 6$ and $\beta := P_6 = 1 + \sqrt{\frac{6}{5}}$. Let us find the admissible sequence d for P_6 in Theorem C. One calculates

$$d_\beta(1) = 20001111010\cdots .$$

By Theorem 4.6, we have

$$0s := 0001111010\cdots \le \sigma^n(\overline{d}) \le 1001111010\cdots =: 1s$$

for all $n \ge 0$. To determine the longest central prefix $u \in \{0, 1\}^*$ of s, we have to consult Proposition 3.4 as follows.

$$\begin{aligned} x_1 = 0 &\implies \mathrm{Pal}(0) = 0 : \text{prefix of } s \\ x_2 = 0 &\implies \mathrm{Pal}(00) = (\mathrm{Pal}(0)0)^+ = 00 : \text{prefix of } s \\ x_3 = 1 &\implies \mathrm{Pal}(001) = (\mathrm{Pal}(00)1)^+ = 00100 \ne 00111. \end{aligned}$$

One has $u = 00$, and thus applies Proposition 4.3(a) with $u = 00$ and $v = 111$. Since $v = 111 > 100$, we finally obtain $\overline{d} = (001)^\infty$, and $d = (110)^\infty$.

Example 4. The case of $m = 5$ turns out to be subtle enough. Let $\beta := P_5 = 1 + \frac{\sqrt{5}}{2}$. A computation shows that

$$d_\beta(1) = 2001001001020002000012000100\cdots.$$

According to Theorem 4.6, the admissible sequence d in Theorem C satisfies

$$0s \le \sigma^n(\overline{d}) \le 1010010010200020001200010 0\cdots =: 1s$$

for all $n \ge 0$. The following describes the procedure to find the longest central prefix $u \in \{0, 1\}^*$ of s:

$$\begin{aligned} x_1 = 0 &\implies \mathrm{Pal}(0) = 0 : \text{prefix of } s \\ x_2 = 1 &\implies \mathrm{Pal}(01) = (\mathrm{Pal}(0)1)^+ = 010 : \text{prefix of } s \\ x_3 = 0 &\implies \mathrm{Pal}(010) = (\mathrm{Pal}(01)0)^+ = 010010 : \text{prefix of } s \\ x_4 = 0 &\implies \mathrm{Pal}(0100) = (\mathrm{Pal}(010)0)^+ = 010010010 : \text{prefix of } s. \end{aligned}$$

But the last word is followed by 2 in s. So we have

$$u = 010010010.$$

By Proposition 3.2, the standard factorization of $1u0 = 1z_{4,11}0$ is

$$1u0 = 1z_{4,11}0 = (1z_{3,8}0)(1z_{1,3}0) = (10100100)(100).$$

Now Proposition 4.3(c) fits our case with

$$u = 010010010, \quad \mathfrak{p} = 010010, \quad \mathfrak{q} = 0, \quad xy = 20, \quad v = 00200012000.$$

Since $xy = 20 \ge 11$, we establish

$$\overline{d} = (0\mathfrak{p}1)^\infty = (00100101)^\infty, \quad \text{and} \quad d = (11011010)^\infty.$$

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5. *Generalized golden ratios*

Before a deeper plunge into generic critical bases, we first deal with trivial cases. The next theorem is a consequence of [**12**], but its proof demonstrates how useful and powerful Theorem 4.6 is. The rest of the paper will appeal to Theorem 4.6 consistently and implicitly.

THEOREM 5.1. *Let $m \geq 2$ and $A_m := \{0, 1, m\}$. If $2 \leq m \leq m_0$ where $m_0 \approx 2.3247$ is the unique real zero of $x^3 - 3x^2 + 2x - 1$, then the critical base p_m for A_m is $p_m = m$.*

Komornik *et al* [**12**] recognized that $m_0 - 1$ is the smallest Pisot number, i.e. the unique real zero of $x^3 - x - 1$. We note that m_0 is also a Pisot number. A *Pisot number* is an algebraic integer greater than 1 whose Galois conjugates lie inside the unit circle.

Proof. Let $\beta = P_m$. In view of Theorem 4.6, Proposition 4.1 shows that $d = 0^\infty$ if and only if $e_2e_3e_4 \cdots \geq 01^\infty$, which is equivalent to

$$\beta - 2 \geq \frac{1}{\beta^2} + \frac{1}{\beta^3} + \frac{1}{\beta^4} + \cdots = \frac{1}{\beta(\beta - 1)}.$$

Since

$$\beta(\beta - 1)(\beta - 2) = \frac{\sqrt{m}}{(m - 1)\sqrt{m - 1}} \geq 1,$$

this inequality is simplified as

$$m^3 - 3m^2 + 2m - 1 \leq 0.$$

Thus, $2 \leq m \leq m_0$. Recalling Lemma 4.5, we find

$$1 - \frac{1}{(m - 1)(p'_m - 1)} = 0 = \frac{1}{p''_m - 1} - \frac{1}{m - 1},$$

then $p'_m = m/(m - 1)$ and $p''_m = m$. We conclude $p_m = \max\{p'_m, p''_m\} = m$. □

Given an admissible sequence $d \neq 1^\infty$, let $m_d \leq \mu_d \leq M_d$ be the numbers produced in Theorem B. By the manner of their constructions, the following hold.

- If $m = m_d$, then $p_m = p'_m = P_m = 1 + \sqrt{m/(m - 1)} \geq p''_m$.
- If $m = \mu_d$, then $p_m = p'_m = p''_m \leq P_m = 1 + \sqrt{m/(m - 1)}$. Further, $p_m = P_m$ if and only if d is not purely periodic.
- If $m = M_d$, then $p_m = p''_m = P_m = 1 + \sqrt{m/(m - 1)} \geq p'_m$.

In each case we now compute the greedy expansion of 1 in base p_m, i.e. the p_m-expansion $d_{p_m}(1)$ of 1. This gives us much information on critical bases.

THEOREM 5.2. *Given an admissible sequence $d \neq 1^\infty$, let $m = m_d$ and $\beta'_d := p_m = p'_m = P_m$. Then the β'_d-expansion $d_{\beta'_d}(1)$ of 1 is given by*

$$d_{\beta'_d}(1) = 2\overline{d}$$

$$= \begin{cases} 2\overline{1c_\alpha} = 20c_{1-\alpha} & \text{if } d = 1c_\alpha \text{ for some } \alpha \in [0, 1] \setminus \mathbb{Q}, \\ 2\overline{1(z_{p,q}10)^\infty} = 20(z_{q-p,q}01)^\infty & \text{if } d = 1(z_{p,q}10)^\infty \\ & \text{for some } p/q \in [0, 1) \cap \mathbb{Q}, \\ 2\overline{(1z_{p,q}0)^\infty} = 2(0z_{q-p,q}1)^\infty & \text{if } d = (1z_{p,q}0)^\infty \\ & \text{for some } p/q \in (0, 1) \cap \mathbb{Q}. \end{cases}$$

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Proof. Since $\beta'_d = p'_m = P_m$, the right inequality in (16) becomes indeed an equality in this case. We hence have $d_{\beta'_d}(1) = 2\overline{d}$. □

Given $\beta > 1$, we introduce a function $(\cdot)_\beta$ which maps a sequence $a_1a_2a_3\cdots$ of numbers to a real number $\sum_{i=1}^{\infty} a_i/\beta^i$ as long as the series converges.

THEOREM 5.3. *Given an admissible sequence $d \neq 1^\infty$, let $m = M_d$ and $\beta''_d := p_m = p''_m = P_m$. Then the β''_d-expansion $d_{\beta''_d}(1)$ of 1 is given by*

$$d_{\beta''_d}(1) = 20\overline{d'} = \begin{cases} 20\overline{c_\alpha} = 20c_{1-\alpha} & \text{if } d = 1c_\alpha \\ & \text{for some } \alpha \in [0,1]\setminus\mathbb{Q}, \\ 20\overline{(z_{p,q}10)^\infty} = 20(z_{q-p,q}01)^\infty & \text{if } d = 1(z_{p,q}10)^\infty \\ & \text{for some } p/q \in [0,1)\cap\mathbb{Q}, \\ 20\overline{(z_{p,q}10)^\infty} = 20(z_{q-p,q}01)^\infty & \text{if } d = (1z_{p,q}0)^\infty \\ & \text{for some } p/q \in [0,1)\cap\mathbb{Q}. \end{cases}$$

Note that the purely periodic case and the skew word case yield the same result.

Proof. On the one hand, $\beta''_d = P_m$ shows $m = (\beta''_d - 1)^2/((\beta''_d - 1)^2 - 1)$, and then

$$\frac{1}{m-1} = \beta''_d(\beta''_d - 2).$$

On the other hand, $\beta''_d = p''_m$ together with (14) of Lemma 4.5 guarantees

$$(\overline{d'_1}\,\overline{d'_2}\,\overline{d'_3}\cdots)_{\beta''_d} = \frac{1}{m-1}.$$

Gathering these, one infers

$$\beta''_d - 2 = (0\overline{d'_1}\,\overline{d'_2}\,\overline{d'_3}\cdots)_{\beta''_d} \implies 1 = (20\overline{d'_1}\,\overline{d'_2}\,\overline{d'_3}\cdots)_{\beta''_d}.$$

Since $\sigma^n(20\overline{d'}) < 20\overline{d'}$ for all $n \geq 1$, Proposition 2.1 proves $d_{\beta''_d}(1) = 20\overline{d'}$. □

The last and involved case remains.

THEOREM 5.4. *Given an admissible sequence $d \neq 1^\infty$, let $m = \mu_d$ and $\beta_d := p_m = p'_m = p''_m$. Then the β_d-expansion $d_{\beta_d}(1)$ of 1 is given by*

$$d_{\beta_d}(1) = 2\overline{d} = \begin{cases} 2\overline{1c_\alpha} = 20c_{1-\alpha} & \text{if } d = 1c_\alpha \\ & \text{for some } \alpha \in [0,1]\setminus\mathbb{Q}, \\ 2\overline{1(z_{p,q}10)^\infty} = 20(z_{q-p,q}01)^\infty & \text{if } d = 1(z_{p,q}10)^\infty \\ & \text{for some } p/q \in [0,1)\cap\mathbb{Q}, \end{cases}$$

and

$$d_{\beta_d}(1) = 20\overline{z_{p,q}} = 20z_{q-p,q} \quad \text{if } d = (1z_{p,q}0)^\infty \text{ for some } p/q \in [0,1)\cap\mathbb{Q}.$$

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Proof. Since $\beta_d = p'_m = p''_m$, it follows from (13) and (14) that

$$(\beta_d - 1) - (\beta_d - 1)(d_1 d_2 d_3 \cdots)_{\beta_d} = \frac{1}{m-1} = (\overline{d'_1}\,\overline{d'_2}\,\overline{d'_3}\cdots)_{\beta_d}.$$

One derives

$$\begin{aligned}\beta_d - 1 &= (\beta_d - 1)(d_1 d_2 d_3 \cdots)_{\beta_d} + (\overline{d'_1}\,\overline{d'_2}\,\overline{d'_3}\cdots)_{\beta_d} \\ &= d_1 + (d_2 d_3 d_4 \cdots)_{\beta_d} - (d_1 d_2 d_3 \cdots)_{\beta_d} + (\overline{d'_1}\,\overline{d'_2}\,\overline{d'_3}\cdots)_{\beta_d}.\end{aligned}$$

Since $d_1 = 1$, we have

$$\beta_d = 2 + (d_2 d_3 d_4 \cdots)_{\beta_d} - (d_1 d_2 d_3 \cdots)_{\beta_d} + (\overline{d'_1}\,\overline{d'_2}\,\overline{d'_3}\cdots)_{\beta_d},$$

which is, in turn, equivalent to

$$1 = (2 d_2 d_3 d_4 \cdots)_{\beta_d} - (0 d_1 d_2 d_3 \cdots)_{\beta_d} + (0\overline{d'_1}\,\overline{d'_2}\,\overline{d'_3}\cdots)_{\beta_d}. \tag{17}$$

Suppose first that $d = 1c_\alpha$ for some irrational $\alpha \in [0, 1]$. Then equation (17) becomes

$$1 = (2c_\alpha)_{\beta_d} - (0d)_{\beta_d} + (0\overline{c_\alpha})_{\beta_d} = (21^\infty)_{\beta_d} - (0d)_{\beta_d} = (2\overline{d})_{\beta_d},$$

which shows that $d_{\beta_d}(1) = 2\overline{1c_\alpha} = 20c_{1-\alpha}$.

If $d = 1(z_{p,q}10)^\infty$ for some rational $p/q \in [0, 1]$, then

$$\begin{aligned}1 &= (2(z_{p,q}10)^\infty)_{\beta_d} - (0d)_{\beta_d} + (0\overline{(z_{p,q}10)^\infty})_{\beta_d} \\ &= (21^\infty)_{\beta_d} - (0d)_{\beta_d} = (2\overline{d})_{\beta_d}.\end{aligned}$$

Hence, we obtain $d_{\beta_d}(1) = 2\overline{1(z_{p,q}10)^\infty} = 20(z_{q-p,q}01)^\infty$.

Assume finally $d = (1z_{p,q}0)^\infty$ for some rational $p/q \in [0, 1]$. Then equation (17) is rewritten as

$$1 = (2(z_{p,q}01)^\infty)_{\beta_d} - (0(1z_{p,q}0)^\infty)_{\beta_d} + (0\overline{(z_{p,q}10)^\infty})_{\beta_d}.$$

If $p/q = 1/2$, then $d = (10)^\infty$ is plugged into the above equation to get

$$\begin{aligned}1 &= (2(01)^\infty)_{\beta_d} - (0(10)^\infty)_{\beta_d} + (0\overline{(10)^\infty})_{\beta_d} \\ &= ((20)^\infty)_{\beta_d} - ((01)^\infty)_{\beta_d} = ((2(-1))^\infty)_{\beta_d} = (20^\infty)_{\beta_d},\end{aligned}$$

where the last equality follows from an observation $(00(2(-1))^\infty)_{\beta_d} = (010^\infty)_{\beta_d}$. We have $d_{\beta_d}(1) = 2$, i.e. $\beta_d = 2$.

Let $p/q = 1/3$, and so $d = (100)^\infty$. Consequently, one derives

$$\begin{aligned}1 &= (2(001)^\infty)_{\beta_d} - (0(100)^\infty)_{\beta_d} + (0\overline{(010)^\infty})_{\beta_d} \\ &= (2(102)^\infty)_{\beta_d} - (0(100)^\infty)_{\beta_d} = ((200)^\infty)_{\beta_d} = (2010^\infty)_{\beta_d},\end{aligned}$$

because $(000(200)^\infty)_{\beta_d} = (0010^\infty)_{\beta_d}$. Thus, we have $d_{\beta_d}(1) = 201$.

The proof for $p/q = 2/3$ is similar to the above ones. The remaining proof is divided into two cases.

Suppose $p/q < 1/2$ with $q \geq 4$. Then $z_{p,q} = 0u0$ for some words $u \in \{0, 1\}^{q-4}$. We rewrite equation (17) as

$$\begin{aligned}1 &= (2(0u001)^\infty)_{\beta_d} - (0(10u00)^\infty)_{\beta_d} + (0\overline{(0u010)^\infty})_{\beta_d} \\ &= (2(1^{q-2}02)^\infty)_{\beta_d} - (0(10u00)^\infty)_{\beta_d} = (2(01\overline{u}02)^\infty)_{\beta_d} \\ &= ((201\overline{u}0)^\infty)_{\beta_d} = (201\overline{u}10^\infty)_{\beta_d} = (20\overline{z_{p,q}}0^\infty)_{\beta_d}.\end{aligned}$$

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Therefore, we have $d_{\beta_d}(1) = 20\overline{z_{p,q}} = 20z_{q-p,q}$.

Let $p/q > 1/2$ with $q \geq 4$. Then $z_{p,q} = 1v1$ for some words $v \in \{0, 1\}^{q-4}$. Equation (17) is evolved as

$$\begin{aligned} 1 &= (2(1v101)^\infty)_{\beta_d} - (0(11v10)^\infty)_{\beta_d} + (0\overline{(1v110)^\infty})_{\beta_d} \\ &= (2(1^{q-2}02)^\infty)_{\beta_d} - (0(11v10)^\infty)_{\beta_d} = (2(00\overline{v}(-1)2)^\infty)_{\beta_d} \\ &= ((200\overline{v}(-1))^\infty)_{\beta_d} = (200\overline{v}00^\infty)_{\beta_d} = (20\overline{z_{p,q}}0^\infty)_{\beta_d}. \end{aligned}$$

We also establish $d_{\beta_d}(1) = 20\overline{z_{p,q}} = 20z_{q-p,q}$, but this expression terminates with some consecutive zeros with length ≥ 1. □

The virtue of the previous three theorems is that they enable us to compute m_d, μ_d, and M_d for a given admissible sequence $d \neq 1^\infty$. Except the case where d is a Sturmian word of irrational slope, only algebraic operations are required in performing the computation. The next two theorems give us an algorithm to find m_d, μ_d, and M_d.

THEOREM 5.5. *Let $\alpha \in [0, 1]$ be irrational, and let $p/q \in [0, 1)$ a rational with* $\gcd(p, q) = 1$. *Assume that one of the following holds:*

- *$d = 1c_\alpha$, and β is the unique real root >1 of an equation $d_\beta(1) = 20c_{1-\alpha}$;*
- *$d = 1(z_{p,q}10)^\infty$, and β is the unique real root >1 of an equation $d_\beta(1) = 20(z_{q-p,q}01)^\infty$.*

Then

$$m_d = \mu_d = M_d = 1 + \frac{1}{\beta(\beta-2)} = \frac{(\beta-1)^2}{\beta(\beta-2)}.$$

Proof. Suppose either $m = m_d$ or $m = M_d$. Then we have $\beta = P_m$, which proves that

$$m = 1 + \frac{1}{\beta(\beta-2)}.$$

If $m = \mu_d$, then our β satisfies $1 = (20\overline{d_2}\,\overline{d_3}\cdots)_\beta$, which implies that

$$\beta(\beta-2) = (\overline{d_2}\,\overline{d_3}\,\overline{d_4}\cdots)_\beta = (\overline{d_1'}\,\overline{d_2'}\,\overline{d_3'}\cdots)_\beta = \frac{1}{m-1}.$$ □

For a word $w = a_1a_2\cdots a_n$, we mean by $\overrightarrow{w}$ the vector $(a_1, a_2, \ldots, a_n)$. If $d_\beta(1) = e_1\cdots e_n(e_{n+1}\cdots e_{n+p})^\infty$, then β is the unique real zero >1 of a polynomial

$$\left(x^{n+p} - \sum_{i=1}^{n+p} e_i x^{n+p-i}\right) - \left(x^n - \sum_{i=1}^{n} e_i x^{n-i}\right).$$

This computation will be frequently used below.

THEOREM 5.6. *Let $d = (1z_{p,q}0)^\infty$ for some rational $p/q \in [0, 1)$ with* $\gcd(p, q) = 1$.

(a) *The value m_d is given by*

$$m_d = 1 + \frac{1}{\beta'(\beta'-2)} = \frac{(\beta'-1)^2}{\beta'(\beta'-2)},$$

where β' is the unique real zero >1 of a polynomial

$$x^{q+1} - 2x^q - \overrightarrow{z_{q-p,q}} \cdot (x^{q-2}, x^{q-3}, \ldots, x) - x + 1.$$

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(b) *The value M_d is given by*

$$M_d = 1 + \frac{1}{\beta''(\beta''-2)} = \frac{(\beta''-1)^2}{\beta''(\beta''-2)},$$

where β'' is the unique real zero >1 of a polynomial

$$x^{q+2} - 2x^{q+1} - \overrightarrow{z_{q-p,q}} \cdot (x^{q-1}, x^{q-2}, \ldots, x^2) - x^2 + 2x - 1.$$

(c) *The value μ_d is given by*

$$\mu_d = 1 + \frac{\beta^q - 1}{\beta^{q+2} - 2\beta^{q+1} + 1} = \frac{\beta^q(\beta-1)^2}{\beta^{q+2} - 2\beta^{q+1} + 1},$$

where β is the unique real zero >1 of a polynomial

$$x^q - 2x^{q-1} - \overrightarrow{z_{q-p,q}} \cdot (x^{q-3}, x^{q-4}, \ldots, 1).$$

Note that if $p/q = 0$ then m_d does not exist in $[2, \infty)$, while $\mu_d = 2$ and $M_d = m_0 \approx 2.3247$.

Proof. If $m = m_d$ (respectively $m = M_d$), then (a) (respectively (b)) follows from $\beta' = P_m$ (respectively $\beta'' = P_m$) and from $d_{\beta'}(1) = 2(0z_{q-p,q}1)^\infty$ (respectively $d_{\beta''}(1) = 20(z_{q-p,q}01)^\infty$) as in Theorem 5.5. We prove (c).

Let $m = \mu_d$. Then the unique real root $\beta > 1$ of an equation $d_\beta(1) = 20z_{q-p,q}$ satisfies

$$\frac{1}{m-1} = (\overline{d_1'}\,\overline{d_2'}\,\overline{d_3'} \cdots)_\beta = ((z_{q-p,q}01)^\infty)_\beta.$$

On the other hand, from $1 = (20z_{q-p,q}0^\infty)_\beta$ it follows that

$$\beta(\beta-2) = (z_{q-p,q}0^\infty)_\beta \implies \beta(\beta-2) + \frac{1}{\beta^q} = (z_{q-p,q}010^\infty)_\beta.$$

Therefore, we have

$$\frac{1}{m-1} = ((z_{q-p,q}01)^\infty)_\beta = \frac{\beta^q}{\beta^q - 1}\left(\beta(\beta-2) + \frac{1}{\beta^q}\right). \qquad \square$$

Recall that if $d \neq 1^\infty$ is a purely periodic admissible sequence then $\hat{d}$ denotes the smallest admissible sequence greater than d. The next corollary is an easy consequence of Theorems 5.5 and 5.6, but worthy of note. This is a new proof of [**12**, Lemma 5.13(a)].

COROLLARY 5.7. *Let $d \neq 1^\infty$ be a purely periodic admissible sequence. Then*

$$M_d = m_{\hat{d}} = \mu_{\hat{d}} = M_{\hat{d}}.$$

We are now in a position to find the critical base p_m for a given $m \geq 2$. We will see in the next section that for a generic m, the corresponding admissible sequence d is purely periodic. If d is not purely periodic, then Theorem 5.5 already produced the critical base p_m. The remaining case is the purely periodic one, where the whole job is reduced to finding a zero of a polynomial.

THEOREM 5.8. *Let $m \geq 2$ and $d = (1z_{p,q}0)^\infty$ for some rational $p/q \in [0, 1)$ with* $\gcd(p, q) = 1$.

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(a) *If $m_d \le m \le \mu_d$, then the critical base p_m is the unique real zero >1 of a polynomial*

$$(m-1)x^q - mx^{q-1} - ((m-1)\overrightarrow{z_{p,q}} + \overrightarrow{1^{q-2}}) \cdot (x^{q-2}, x^{q-3}, \dots, x) - m.$$

(b) *If $\mu_d \le m \le M_d$, then the critical base p_m is the unique real zero >1 of a polynomial*

$$x^q - (m-1)\overrightarrow{z_{q-p,q}} \cdot (x^{q-1}, x^{q-2}, \dots, x^2) - m.$$

Proof. (a) Let $m_d \le m \le \mu_d$, and hence set

$$\beta' := p_m = \max\{p'_m, p''_m\} = p'_m.$$

Then Lemma 4.5 says that

$$\begin{aligned} 1 - \frac{1}{(m-1)(\beta'-1)} &= (d_1 d_2 d_3 \cdots)_{\beta'} \\ &= \left(\frac{d_1}{\beta'} + \frac{d_2}{(\beta')^2} + \cdots + \frac{d_q}{(\beta')^q}\right)\left(1 + \frac{1}{(\beta')^q} + \frac{1}{(\beta')^{2q}} + \cdots\right) \\ &= \left(\frac{d_1}{\beta'} + \frac{d_2}{(\beta')^2} + \cdots + \frac{d_q}{(\beta')^q}\right)\frac{(\beta')^q}{(\beta')^q - 1}. \end{aligned}$$

The simplification of this equation provides our claim.

(b) If $\mu_d \le m \le M_d$, then the critical base p_m satisfies

$$\beta'' := p_m = \max\{p'_m, p''_m\} = p''_m.$$

Accordingly, we find that

$$\begin{aligned} \frac{1}{m-1} &= (\overline{d'_1}\,\overline{d'_2}\,\overline{d'_3}\cdots)_{\beta''} \\ &= \left(\frac{\overline{d'_1}}{\beta''} + \frac{\overline{d'_2}}{(\beta'')^2} + \cdots + \frac{\overline{d'_q}}{(\beta'')^q}\right)\left(1 + \frac{1}{(\beta'')^q} + \frac{1}{(\beta'')^{2q}} + \cdots\right) \\ &= \left(\frac{\overline{d'_1}}{\beta''} + \frac{\overline{d'_2}}{(\beta'')^2} + \cdots + \frac{\overline{d'_q}}{(\beta'')^q}\right)\frac{(\beta'')^q}{(\beta'')^q - 1}. \end{aligned}$$

□

Komornik *et al* [**12**] proved the critical base $p_m = 2$ if and only if $m = 2^q$ for some positive integer q. This was done by observing that an admissible sequence d satisfies $d = d'$ if and only if $d = (1^{q-1}0)^\infty = (1z_{q-1,q}0)^\infty$. Theorems 5.5 and 5.6 apply to the general case, but in particular make it possible to extend the result of [**12**].

COROLLARY 5.9. *Let $d = (1^{q-1}0)^\infty = (1z_{q-1,q}0)^\infty$ for some positive integer q. Then $\mu_d = 2^q$, and the following hold.*

(a) *$m_d = (\beta')^{q-1}(\beta'-1)$, where β' is the unique real zero >1 of a polynomial*

$$x^{q+1} - 2x^q - x + 1.$$

(b) *$M_d = (\beta'')^q$, where β'' is the unique real zero >1 of a polynomial*

$$x^{q+2} - 2x^{q+1} - x^2 + 2x - 1.$$

(c) *If $m_d \le m \le 2^q = \mu_d$, then the critical base p_m is the unique real zero >1 of a polynomial*

$$(m-1)x^q - mx^{q-1} - mx^{q-2} - \cdots - mx - m.$$

(d) *If $\mu_d = 2^q \le m \le M_d$, then the critical base p_m is given by*

$$p_m = m^{1/q}.$$

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Remark 5.10.

(i) If $m = \mu_d = 2^q$ for some positive integer, then one immediately notes that $m^{1/q} = 2$, and that 2 is the unique real zero >1 of

$$(2^q - 1)x^q - 2^q x^{q-1} - 2^q x^{q-2} - \cdots - 2^q x - 2^q.$$

That is to say, both (c) and (d) are consistent with the result of [**12**].

(ii) The above theorem also extends [**2**, Theorem 4]. Baker and Steiner proved that if

$$2^q \le m \le \left(1 + \sqrt{\frac{m}{m-1}}\right)^q$$

for some positive integer q then $p_m = m^{1/q}$. In fact, their result is equivalent to (d). Putting $x = m^{1/q}$, one can transform the equality $m = (1 + \sqrt{m/(m-1)})^q$ into the polynomial equation $x^{q+2} - 2x^{q+1} - x^2 + 2x - 1 = 0$.

Example 5. Let $d = (110)^\infty$.

- $\beta' \approx 2.11769$ is the unique real zero >1 of $x^4 - 2x^3 - x + 1$. So we have
$$m_d = (\beta')^2(\beta' - 1) \approx 5.01239.$$
- $\beta'' \approx 2.06239$ is the unique real zero >1 of $x^5 - 2x^4 - x^2 + 2x - 1$. So we have
$$M_d = (\beta'')^3 \approx 8.77222.$$
- If $8 \le m \le M_d \approx 8.77222$, then $p_m = m^{1/3}$.
- If $5.01239 \approx m_d \le m \le 8$, then p_m is the unique real zero >1 of
$$(m-1)x^3 - mx^2 - mx - m.$$
For example, $p_6 \approx 2.06341$ and $p_7 \approx 2.02647$.

It is natural to consider the case $d = (10^{q-1})^\infty = (1z_{1,q}0)^\infty$ that is symmetric to Corollary 5.9. Unfortunately, however, the situation is not so simple as in (d) of Corollary 5.9.

COROLLARY 5.11. *Let $d = (10^{q-1})^\infty = (1z_{1,q}0)^\infty$ for some positive integer q. The following hold.*

(a) $m_d = (\beta' - 1)^2/(\beta'(\beta' - 2))$, *where β' is the unique real zero >1 of a polynomial*
$$x^{q+1} - 2x^q - x^{q-2} - x^{q-3} - \cdots - x^2 - 2x + 1.$$

(b) $\mu_d = \beta^q(\beta - 1)^2/(\beta^{q+2} - 2\beta^{q+1} + 1)$, *where β is the unique real zero >1 of a polynomial*
$$x^q - 2x^{q-1} - x^{q-3} - x^{q-4} - \cdots - 1.$$

(c) $M_d = (\beta'' - 1)^2/(\beta''(\beta'' - 2))$, *where β'' is the unique real zero >1 of a polynomial*
$$x^{q+2} - 2x^{q+1} - x^{q-1} - x^{q-2} - \cdots - x^3 - 2x^2 + 2x - 1.$$

(d) *If $m_d \le m \le \mu_d$, then the critical base p_m is the unique real zero >1 of a polynomial*
$$(m-1)x^q - mx^{q-1} - x^{q-2} - x^{q-3} - \cdots - x - m.$$

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(e) *If $\mu_d \le m \le M_d$, then the critical base p_m is the unique real zero >1 of a polynomial*

$$x^q - (m-1)x^{q-1} - (m-1)x^{q-2} - \cdots - (m-1)x^2 - m.$$

Example 6. Let $d = (100)^\infty$.

- $\beta' \approx 2.29663$ is the unique real zero >1 of $x^4 - 2x^3 - 2x + 1$. Thus,
$$m_d = \frac{(\beta'-1)^2}{\beta'(\beta'-2)} \approx 2.46789.$$
- $\beta \approx 2.20557$ is the unique real zero >1 of $x^3 - 2x^2 - 1$. Thus,
$$\mu_d = \frac{\beta^3(\beta-1)^2}{\beta^5 - 2\beta^4 + 1} \approx 2.65897.$$
- $\beta'' \approx 2.25717$ is the unique real zero >1 of $x^5 - 2x^4 - 2x^2 + 2x - 1$. Thus,
$$M_d = \frac{(\beta''-1)^2}{\beta''(\beta''-2)} \approx 2.72275.$$
- The critical base for an alphabet $\{0, 2, 5\}$ is approximately 2.27977 which is the unique real zero >1 of $1.5x^3 - 2.5x^2 - x - 2.5$.
- The critical base for an alphabet $\{0, 10, 27\}$ is approximately 2.23872 which is the unique real zero >1 of $x^3 - 1.7x^2 - 2.7$.

6. *Cantor sets*

We consider a set

$$C := \{m \ge 2 : p_m = P_m\}.$$

This set was proved to be a Cantor set, i.e. a perfect set that is dense nowhere [**12**]. In this section, we determine its size. The present paper was, in fact, motivated by trying to compute its Hausdorff dimension.

Instead of the set C, we manipulate a slightly modified set

$$\mathcal{C} := \{p_m : p_m = P_m, m \ge 2\} \subset (2, m_0],$$

where $m_0 - 1 \approx 1.3247$ is the smallest Pisot number. See Theorem 5.1. Let us define a function $f : (2, m_0] \to \mathbb{R}$ by

$$f(x) := \frac{(x-1)^2}{x(x-2)} = 1 + \frac{1}{x(x-2)},$$

which is the inverse of the function $m \mapsto 1 + \sqrt{m/(m-1)}$. We choose any strictly decreasing sequence $m_0 = a_0 > a_1 > a_2 > \cdots > 2$ tending to 2, and set

$$\mathcal{C}_i := \mathcal{C} \cap (a_i, a_{i-1}] \quad \text{and} \quad C_i := f(\mathcal{C}_i)$$

for $i \ge 1$. Then $\mathcal{C} = \bigcup_{i\ge1} \mathcal{C}_i$ and $C = \bigcup_{i\ge1} C_i$. Since f is bi-Lipschitz on each interval $(a_i, a_{i-1}]$, we have $\dim_H \mathcal{C}_i = \dim_H C_i$. Therefore, $\dim_H \mathcal{C} = 0$ is followed by $\dim_H C = 0$. We prove below $\dim_H \mathcal{C} = 0$, more precisely $\dim_H \widetilde{\mathcal{C}} = 0$, where

$$\widetilde{\mathcal{C}} := \{\beta : d_\beta(1) = 2s_{\alpha,0}, \alpha \in (0, 1)\}.$$

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This is justified by Theorems 5.2 and 5.3. The cardinality of a set $(\mathcal{C} \setminus \widetilde{\mathcal{C}}) \cup (\widetilde{\mathcal{C}} \setminus \mathcal{C})$ is at most countable.

The number of prefixes of $d_\beta(1)$ with length $k+1$ for all $\beta \in \widetilde{\mathcal{C}}$ is $\sum_{i=1}^{k} \varphi(i)$, where φ is the Euler totient function. See the proof of [**18**, Theorem 2.2.36]. The asymptotic of this number is given by

$$n_k := \sum_{i=1}^{k} \varphi(i) = \frac{3k^2}{\pi^2} + O(k \log k).$$

See, e.g., [**11**].

THEOREM 6.1. $\dim_H \widetilde{\mathcal{C}} = 0$ *and hence* $\dim_H C = 0$ *hold.*

Proof. We follow the proof of [**6**, Lemma 4]. At each kth step, $\widetilde{\mathcal{C}}$ can be covered by n_k intervals of length less than $\delta_k := m_0^2/2^{k+2}$. We have

$$\dim_H \widetilde{\mathcal{C}} \le \lim_{k\to\infty} \frac{\log n_k}{-\log \delta_k} \le \lim_{k\to\infty} \frac{\log(3k^2/\pi^2 + O(k \log k))}{-2 \log m_0 + (k+2) \log 2} = 0. \qquad \square$$

After the author had obtained the whole work in this paper, he recognized in [**2**] the same result as Theorem 6.1. The current specific proof leads us to consider a sharper *dimension function* than that of the usual Hausdorff dimension. See [**10**, §2.5].

Remark 6.2. For a continuous and increasing function $h : \mathbb{R}^+ \to \mathbb{R}^+$ and a set $S \subset \mathbb{R}$, we define

$$\mathcal{H}^h_\delta(S) := \inf \left\{ \sum_{i=1}^{\infty} h(\operatorname{diam} U_i) : S \subset \bigcup_{i=1}^{\infty} U_i, \operatorname{diam} U_i \le \delta \right\}.$$

Taking $\mathcal{H}^h(S) := \lim_{\delta \searrow 0} \mathcal{H}^h_\delta(S)$, one obtains a measure. Under this language, Theorem 6.1 has proved that $\mathcal{H}^h(C) = \mathcal{H}^h(\mathcal{C}) = 0$ whenever $h(t) = t^s$ with $s > 0$. Now suppose $h(t) := (-\log t)^{-2-\varepsilon}$ for any given $\varepsilon > 0$. Then $\mathcal{H}^h$ is a much thinner measure than the s-dimensional Hausdorff measure for any $s > 0$. The following inequality implies that we can say more about the sizes of our Cantor sets than 'being of Hausdorff dimension zero':

$$\mathcal{H}^h(\widetilde{\mathcal{C}}) \le \lim_{k\to\infty} n_k h(\delta_k) \le \lim_{k\to\infty} \frac{3k^2/\pi^2 + O(k \log k)}{(-2 \log m_0 + (k+2) \log 2)^{2+\varepsilon}} = 0.$$

The Cantor sets C and $\mathcal{C}$ are also distinguished in view of arithmetic. They consist only of very special algebraic numbers and even more special transcendental numbers. We need a classical result on transcendence.

PROPOSITION 6.3. [**19**] *Let a function f be defined by*

$$f(\alpha, z) = \sum_{n=1}^{\infty} \lfloor \alpha n \rfloor z^n,$$

where α is real and z is complex with $|z| < 1$. Then $f(\alpha, \gamma)$ is a transcendental number, provided α is irrational and γ is a non-zero algebraic number with $|\gamma| < 1$.

An algebraic integer $\beta > 1$ is called a *Perron number* if its Galois conjugates other than β have moduli less than β.

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THEOREM 6.4. *Let* $m \in C = \{m \geq 2 : p_m = P_m\}$.

(a) *The critical base* p_m *is either a transcendental number or a Perron unit.*

(b) *The number* $(m-1)^{-1}$ *is either a transcendental number or an algebraic integer.*

(c) *In particular,* m *can be neither an integer nor a rational number.*

Proof. (a) Let $\beta := p_m = P_m$, and suppose that $d_\beta(1) = 20c_\alpha$ for some irrational $\alpha \in [0, 1]$. Then one finds that

$$\begin{aligned}\beta - 2 &= \sum_{n=0}^{\infty} \frac{\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor}{\beta^{n+1}} = \sum_{n=1}^{\infty} \frac{\lfloor \alpha n \rfloor}{\beta^n} - \frac{1}{\beta} \sum_{n=1}^{\infty} \frac{\lfloor \alpha n \rfloor}{\beta^n} \\ &= \left(1 - \frac{1}{\beta}\right) f(\alpha, \beta^{-1}).\end{aligned}$$

Owing to Proposition 6.3, a contradiction is yielded if we assume that β is an algebraic number.

Let the β-expansion $d_\beta(1)$ of 1 be $20(z_{q-p,q}01)^\infty$ or $2(0z_{q-p,q}1)^\infty$ for some rational $p/q \in [0, 1)$. Then β is a Perron number [**4**, **17**]. In addition, β is the unique real zero >1 of a polynomial

$$x^{q+2} - 2x^{q+1} - \overrightarrow{z_{q-p,q}} \cdot (x^{q-1}, x^{q-2}, \ldots, x^2) - x^2 + 2x - 1,$$

or

$$x^{q+1} - 2x^q - \overrightarrow{z_{q-p,q}} \cdot (x^{q-2}, x^{q-3}, \ldots, x) - x + 1,$$

respectively.

(b) Since $(m-1)^{-1} = \beta(\beta - 2)$, it is a transcendental number or an algebraic integer.

(c) From $\beta = 1 + \sqrt{m/(m-1)}$, it follows that β is also a zero of a polynomial

$$(m-1)x^2 - 2(m-1)x - 1. \tag{18}$$

The smallest element in C is $m_0 \approx 2.3247$. For integers $m \geq 3$, the above polynomial is irreducible over $\mathbb{Q}$. Let $m = p/q > 2$ be a non-integer rational with $\gcd(p, q) = 1$. Then $(m-1)^{-1} = q/(p-q)$ is an algebraic integer if and only if $p - q = 1$. Therefore, we have $m = 1 + 1/q < 2$, a contradiction. □

7. *Summary*

In summary, we portray the procedure to find the critical base p_m.

For a given $m \geq 2$, we set $\beta = P_m = 1 + \sqrt{m/(m-1)}$, and calculate the β-expansion $d_\beta(1)$ of 1. Next, Theorem 4.6 with the aid of Propositions 4.1 and 4.3 determines the admissible sequence d satisfying Theorem C. By Theorem 6.1, we encounter a purely periodic admissible sequence d for a generic m. In the next step, the values m_d, μ_d, and M_d are computed according to Theorems 5.5 and 5.6. In case d is not purely periodic, Theorem 5.5 also determines the critical base p_m. Otherwise, if d is purely periodic, then we appeal to Theorem 5.8 for the critical base p_m. This is the case, in particular, if $(m-1)^{-1}$ is an algebraic number that is not an algebraic integer ($\because$ Theorem 6.4).

The series of our theorems is quite programmable. They allow a short computer program to evaluate the critical base p_m. A Mathematica program, for instance, is available at the following website.

www.math.jnu.ac.kr/doyong/paper/golden3.nb

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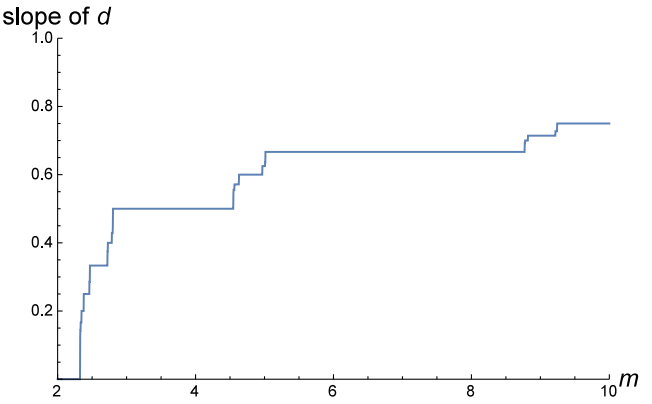


FIGURE 1. Slope of admissible sequence d.

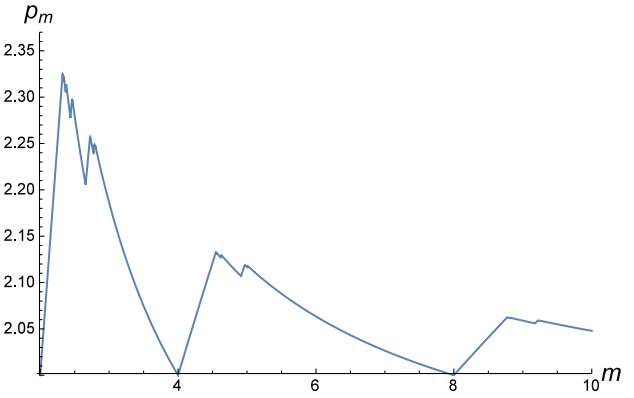


FIGURE 2. Critical base p_m.

The program computes the critical base p_m unless m belongs to the Cantor set $C = \{m \geq 2 : p_m = P_m\}$. This constraint on m is far from being a critical issue. Every number inside the computer is rational, and no rational number is contained in the Cantor set C. Moreover, p_m is a continuous function of m.

As a benefit of the computer program, we obtain some illustrating figures. Figure 1 depicts the slope of the admissible sequence d satisfying Theorem C. Note that the slope is also a continuous function of m. It is reminiscent of the function studied by Bullett and Sentenac [**7**]. Figure 2 is the graph of p_m as a function of m.

The whole context of the paper proves the following.

THEOREM 7.1. *The function $m \mapsto p_m$ from $[2, \infty)$ into $\mathbb{R}$ sends algebraic numbers to algebraic ones, and transcendental numbers to transcendental ones.*

Proof. For a given $m \geq 2$, suppose first that its associated admissible sequence d is either $d = 1c_\alpha$ for some irrational $\alpha \in [0, 1]$, or $d = 1(z_{p,q}10)^\infty$ for some rational $p/q \in [0, 1)$.

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Then Theorem 5.5 guarantees that

$$m = \frac{(p_m - 1)^2}{p_m(p_m - 2)}.$$

Next, let $d = (1z_{p,q}0)^\infty$ be the associated admissible sequence for some rational $p/q \in [0, 1)$. If m is an algebraic number, then p_m is a zero of the polynomials with algebraic coefficients by Theorem 5.8. Suppose that m is transcendental but that p_m is algebraic. If $m_d \le m \le \mu_d$, then p_m is the unique real zero >1 of a polynomial

$$\begin{aligned} &m(x^q - x^{q-1} - \overrightarrow{z_{p,q}} \cdot (x^{q-2}, x^{q-3}, \ldots, x) - 1) \\ &\quad - (x^q - \overrightarrow{z_{p,q}} \cdot (x^{q-2}, x^{q-3}, \ldots, x) + x^{q-2} + x^{q-3} + \cdots + x) = 0, \end{aligned}$$

and hence, of two polynomials

$$x^q - x^{q-1} - \overrightarrow{z_{p,q}} \cdot (x^{q-2}, x^{q-3}, \ldots, x) - 1 = 0,$$

and

$$x^q - \overrightarrow{z_{p,q}} \cdot (x^{q-2}, x^{q-3}, \ldots, x) + x^{q-2} + x^{q-3} + \cdots + x = 0.$$

This forces p_m to be a zero of a cyclotomic polynomial $x^{q-1} + x^{q-2} + \cdots + x + 1 = 0$. For the case of $\mu_d \le m \le M_d$, the proof is similar. □

Acknowledgements. The referee's comments, which made the paper more readable and brought [**16**] to the author's attention, are gratefully acknowledged. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (grant number: NRF-2016R1D1A1B02010219).

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