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A steady vortex ring close to Hill's spherical vortex

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Abstract. The existence of a steady vortex ring close to Hill's spherical vortex is established, and an approximate description of its boundary is given. The vorticity in the ring is proportional to the distance from the axis of symmetry. The core propagates steadily in an unbounded fluid at rest at infinity. The boundary of the vortex ring is close to an interior stream surface of Hill's vortex.

CONTENTS

	PAGE
1. Introduction	253
2. Formulation of the problem	255
2·1. The basic non-linear integral equation	255
2·2. The operator equation $u-\mathscr{K}_\epsilon u = f_\epsilon+\mathscr{N}_\epsilon u$	256
3. The main results	259
4. The linear equation $u-\mathscr{K}_\epsilon u = f$	260
4·1. The approximation of $\mathscr{K}_\epsilon$ by $\mathscr{K}_0$	260
4·2. The existence of the inverse operator $(\mathscr{I}-\mathscr{K}_\epsilon)^{-1}$	266
5. The non-linear terms	268
6. Proof of the main results	274
Appendix A. The interior streamline ∂H_ϵ of Hill's vortex	277
B. The Green's function $G(r, \hat{r}, z-\hat{z})$ and derivatives	278
C. Bounds for an integral of G	280

1. *Introduction.* By a steady vortex ring we mean an axi-symmetric distribution of azimuthal vorticity that moves without change of form, and at constant speed, through unbounded fluid at rest at infinity, the fluid being inviscid and of uniform density. Alternatively, if we take axes fixed with respect to the ring, the flow is steady, consisting of a uniform axial stream and the flow induced by the vorticity. With this latter viewpoint, which we adopt henceforth, the ratio ω/r of vorticity to cylindrical radius must be constant on each streamline.

The simplest and best-known example of a steady ring is Hills' spherical vortex(1), in which ω/r is constant throughout the spherical core. (The core of a vortex ring is the region to which the vorticity is confined.) It has long been thought that certain rings of small cross-section, having approximately toroidal cores throughout which ω/r is

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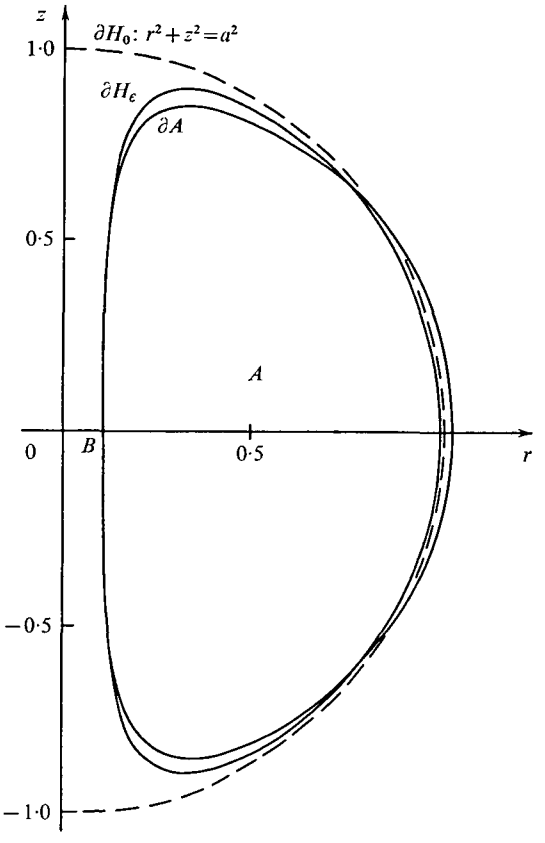


Fig. 1. The region A, with boundary ∂A, depicts the core cross-section of the desired vortex ring. The curve ∂H_0, on which $\psi_H = 0$, marks the boundary of the core of Hill's vortex; and the curve ∂H_ϵ, on which $\psi_H = \epsilon^2\{1-(10/\Omega a^4)\,\epsilon^2\}$, is an interior streamline of that vortex.

again constant, are also steady; and Maruhn(2) and Fraenkel(3) have proved results of this kind (actually for arbitrary distributions of ω/r over the core streamlines). In a further paper Fraenkel (4) has given a detailed description of rings with small cross-section and constant ω/r. An obvious conjecture is then that, with suitably normalised variables, Hill's vortex is the 'largest' member, and the rings described by Fraenkel(4) are 'small' members of a one-parameter family of steady vortex rings characterized by uniformity of ω/r in each core. However, Hill's vortex differs from all other members of the conjectured family in that its core is spherical, while the other cores are toroidal in the sense of being topologically equivalent to a torus. (Henceforth, the word 'toroidal' will be used in this sense.)

In the present paper we find support for this conjecture by proving the existence of a steady vortex ring (Fig. 1) that is toroidal, is characterized by a small positive parameter ϵ, and tends to Hill's vortex as $\epsilon \to 0$. The basic difficulty in such a perturbation of Hill's vortex is clearly the topological difference between the configurations for $\epsilon = 0$ and $\epsilon > 0$; that is, in finding the shape of the central hole of the perturbed core.

The essentials of our method are as follows. We seek a function describing the core boundary as the solution of a non-linear integral equation. As a first approximation

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to the desired boundary we consider not the spherical boundary of Hill's vortex, but a certain toroidal stream surface of that vortex, represented by the curve ∂H_ϵ in Fig. 1. The dependent variable q, which measures the distance from the known curve ∂H_ϵ to the desired curve ∂A, is then subjected to a transformation which allows us to decompose the integral operator into a dominant linear one and 'smaller' non-linear one, to invert the linear operator, and to complete the proof by successive approximations. (Such a decomposition does not succeed for more obvious dependent variables, like q itself, the properties of which are less uniform than those of the transformed variable that we shall adopt.) Our existence proof is thus constructive, and yields an explicit approximate description of the core boundary. Once this boundary is known, the flow field can be determined, in principle, by certain integrations.

2. *Formulation of the problem.* 2.1. *The basic integral equation.* We start from the equations of continuity and vorticity for the steady, axi-symmetric flow, with no swirl, of an inviscid fluid with uniform density filling all space. The continuity equation $\operatorname{div} \mathbf{v} = 0$ implies the existence of a vector potential $(0, \psi/r, 0)$ such that

$$\mathbf{v} = \operatorname{curl}(0, \psi/r, 0) = \left(-\frac{1}{r}\frac{\partial \psi}{\partial z},\ 0,\ \frac{1}{r}\frac{\partial \psi}{\partial r}\right), \tag{2·1}$$

where $\psi(r, z)$ is the Stokes stream function, $\mathbf{v}$ the fluid velocity, and all components are with respect to cylindrical coordinates (r, λ, z). The vorticity equation,

$$\mathbf{v} \cdot \operatorname{grad} \omega/r = 0, \tag{2·2}$$

is satisfied trivially by the particular distribution $\psi = \Omega r$, $\Omega =$ constant, of vorticity that we take in the core of the vortex ring. The flow outside the core is irrotational. We call Ω the vorticity constant, and denote the meridional cross-section of the core by A (Fig. 1). Finally, we take axes fixed in the ring.

The problem can now be stated: given the free stream velocity W, the vorticity constant Ω, and a positive parameter ϵ, find the stream function ψ and the boundary ∂A of the core cross-section A such that

$$\left\{\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right\}\psi(r, z) = \begin{cases} -\Omega r^2 & \text{inside } \partial A \\ 0 & \text{outside } \partial A \end{cases}, \tag{2·3a}$$

(where we have taken the curl of (2·1)), and such that

$$\psi \text{ and } \operatorname{grad} \psi \text{ are continuous across } \partial A, \tag{2·3b}$$

$$\psi = \text{const.} = \epsilon^2 \quad \text{on} \quad \partial A, \tag{2·3c}$$

$$\psi + \tfrac{1}{2} W r^2 \to 0 \quad \text{as} \quad r^2 + z^2 \to \infty. \tag{2·3d}$$

Equations (2·3b, c) correspond to no flow of fluid through the boundary ∂A and to continuity of the tangential velocity across ∂A.

We observe that if ∂A is assumed known, then ψ is given by

$$\psi(r, z) = -\tfrac{1}{2} W r^2 + \frac{\Omega}{2\pi}\iint_A G(r, \hat{r}, z - \hat{z})\, d\hat{r}\, d\hat{z}, \tag{2·4}$$

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where
$$G(r,\hat{r},z-\hat{z}) = \tfrac{1}{2}r\hat{r}^2\int_{-\pi}^{\pi}\frac{\cos\hat{\lambda}\,d\hat{\lambda}}{\{r^2+\hat{r}^2-2r\hat{r}\cos\hat{\lambda}+(z-\hat{z})^2\}^{\frac{1}{2}}}, \tag{2·5}$$

and where (2·4) is based upon the fundamental solution $|\mathbf{x}-\hat{\mathbf{x}}|^{-1}$ of the Poisson equation satisfied by the vector potential. The kernel G is the streamfunction at $\mathbf{x} = (r,z)$ of a singular vortex circle at $\hat{\mathbf{x}} = (\hat{r},\hat{z})$ having vorticity of delta-function type with circulation $2\pi\hat{r}$.

Once ∂A is known, (2·4) represents the solution of the problem (2·3). Accordingly, we combine (2·3c) and (2·4) to obtain

$$\epsilon^2 = -\tfrac{1}{2}Wr^2+\frac{\Omega}{2\pi}\iint_A G(r,\hat{r},z-\hat{z})\,d\hat{r}\,d\hat{z} \quad \text{for} \quad (r,z)\in\partial A, \tag{2·6}$$

and the problem is reduced to solving this non-linear equation for the desired boundary ∂A.

When $\epsilon = 0$ equation (2·6) is satisfied by the curve

$$\partial A = \partial H_0 \equiv \{(r,z)| \quad \text{either} \quad r^2+z^2 = a^2 \quad \text{or} \quad r = 0,\ |z| < a\},$$

where $a^2 = 15W/2\Omega$. This, of course, represents the boundary of Hill's spherical vortex, whose core cross-section we denote by H_0. The corresponding streamfunction for Hill's spherical vortex is

$$\psi_H = \begin{cases} \dfrac{\Omega}{10}r^2(a^2-r^2-z^2) & (r^2+z^2 \leqslant a^2), \\ -\dfrac{\Omega}{15}r^2\left(a^2-\dfrac{a^5}{(r^2+z^2)^{\frac{3}{2}}}\right) & (r^2+z^2 \geqslant a^2). \end{cases} \tag{2·7}$$

We now seek, for small positive ϵ, smooth curves ∂A satisfying (2·6) and close to the boundary ∂H_0, as in Fig. 1.

2·2. *The operator equation.* We first approximate the desired curve ∂A not by ∂H_0 but rather by an interior streamline ∂H_ϵ (bounding an interior domain H_ϵ) of Hill's vortex. This curve is defined by

$$\partial H_\epsilon\colon \psi_H \equiv \frac{\Omega}{10}r^2(a^2-r^2-z^2) = \epsilon^2\left(1-\frac{10}{\Omega a^4}\epsilon^2\right). \tag{2·8}$$

The constants are chosen to suit the following non-dimensionalization. We refer all lengths to the spherical vortex radius a, given by

$$W = \tfrac{2}{15}\Omega a^2, \tag{2·9}$$

and introduce non-dimensional variables $\bar{r},\bar{z},\bar{\epsilon}$, etc., by

$$r = a\bar{r}, \quad z = a\bar{z}, \quad \epsilon^2 = \frac{\Omega a^4}{10}\bar{\epsilon}^2, \quad \text{etc.} \tag{2·10}$$

Equations (2·6) and (2·8) become, on dropping the bars,

$$\epsilon^2 = -\tfrac{2}{3}r^2+\frac{5}{\pi}\iint_A G(r_1\bar{r}_1,z-\hat{z})\,d\hat{r}\,d\hat{z}, \quad \text{for} \quad (r,z)\in\partial A. \tag{2·11}$$

and
$$\partial H_\epsilon\colon \psi_H \equiv r^2(1-r^2-z^2) = \epsilon^2(1-\epsilon^2). \tag{2·12}$$

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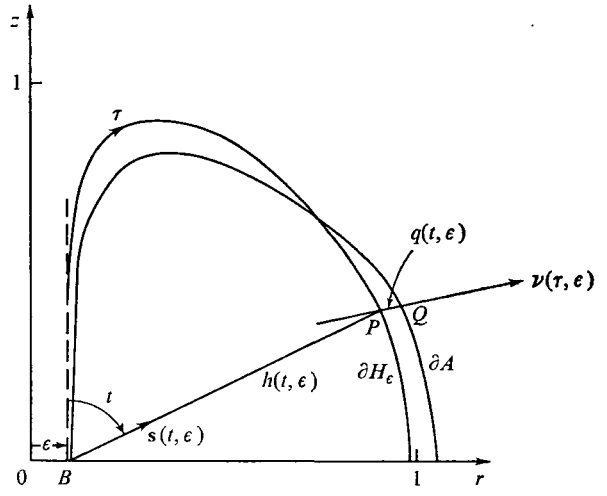


Fig. 2. The known curve ∂H_ϵ: $s = h(t, \epsilon)$ approximates the unknown curve ∂A; which is given by the distance q from ∂H_ϵ to ∂A along the normal $\boldsymbol{\nu}$ to the curve ∂H_ϵ. The arc length τ is measured from B: $r = \epsilon, z = 0$ to P (clockwise) along $\partial H\epsilon$. The problem is symmetric about $t = \frac{1}{2}\pi$ (or $z = 0$).

Note that in terms of spherical polar coordinates ρ, θ ($r = \rho \sin\theta, z = \rho\cos\theta$) we have

$$\partial H_\epsilon\colon \rho = \{\tfrac{1}{2} + \tfrac{1}{2}(1 - 4\epsilon^2(1-\epsilon^2)/\sin^2\theta)^{\frac{1}{2}}\}^{\frac{1}{2}} = 1 - \frac{\epsilon^2(1-\epsilon^2)}{2\sin^2\theta} + O\left(\frac{\epsilon^4}{\theta^4}\right) \quad \text{for} \quad \frac{\epsilon}{\theta} \to 0,$$

and the non-uniformity of this expansion near $\theta = 0$ will be reflected in much of the work that follows.

To describe ∂A in terms of the known curve ∂H_ϵ we proceed as follows. Let τ denote arc length from the point B: $r = \epsilon$, $z = 0$ measured clockwise along ∂H_ϵ, and let $\boldsymbol{\nu}(\tau, \epsilon)$ denote the unit outward normal to ∂H_ϵ (Fig. 2). We define q to be the signed distance from the ∂H_ϵ to ∂A along the normal $\boldsymbol{\nu}$, assuming that such a (one-valued) function $q(\tau, \epsilon)$ exists and that the normals ν through distinct points of ∂H_ϵ do not intersect in the region between ∂H_ϵ and ∂A. More precisely: writing $R(\tau, \epsilon)$ for the local radius of curvature of the ∂H_ϵ, we assume that $-q(\tau, \epsilon) < R(\tau, \epsilon)$ for all τ and for ϵ sufficiently small. This assumption will be verified *a posteriori*.

We now introduce a new independent variable t so that our functions will be defined on the *same* interval $0 \leqslant t \leqslant \pi$ for *all* parameter values $\epsilon \to 0$. Thus (s, t) coordinates are defined by

$$r = \epsilon + s\sin t \quad \text{and} \quad z = s\cos t. \tag{2·13}$$

The equation of the known curve ∂H_ϵ is now $s = h(t, \epsilon)$, where the function $h(t, \epsilon)$ is defined (implicitly) by (2·12) and (2·13), (see also Appendix A); the signed distance q between the curves ∂A and ∂H_ϵ is written $q(t, \epsilon)$. Let $\mathbf{s}(t, \epsilon)$ be the unit vector in the direction s increasing. Then we can write, (Fig. 2),

$$\partial A\colon \mathbf{BQ} = h(t, \epsilon)\,\mathbf{s}(t, \epsilon) + q(t, \epsilon)\,\boldsymbol{\nu}(t, \epsilon) = h\mathbf{s} + q\boldsymbol{\nu}, \tag{2·14}$$

where $\boldsymbol{\nu}(\tau, \epsilon)$ is relabelled $\boldsymbol{\nu}(t, \epsilon)$ after using $\tau(t, \epsilon)$.

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Next we seek a dependent variable $u(t,\epsilon)$ which is more uniformly behaved in t, as $\epsilon \to 0$, than is q. The transformation is

$$\left.\begin{aligned} u &= gq, \\ \text{where} \qquad g(t,\epsilon) &= -\frac{1}{r}\frac{\partial}{\partial \nu}\frac{5}{\pi}\left\{-\tfrac{2}{3}r^2 + \iint_{H_\epsilon} G\, d\hat{r}\, d\hat{z}\right\} \quad \text{for} \quad (r,z)\in \partial H_\epsilon, \end{aligned}\right\} \tag{2·15}$$

and where $\partial/\partial\nu$ denotes the normal derivative to ∂H_ϵ. This choice of dependent variable is suggested by formal linearization of (2·11) for small q, and enables us to split the non-linear equation (2·11) into a dominant linear part and a smaller non-linear part.

In fact, we shall now cast (2·11) into the form

$$u - \mathscr{K}_\epsilon u = f_\epsilon + \mathscr{N}_\epsilon u, \tag{2·16}$$

where $\mathscr{K}_\epsilon$ is a linear integral operator, f_ϵ a known function, and $\mathscr{N}_\epsilon$ a non-linear operator – all to be defined presently. First, we split the integral in (2·11) as follows:

$$\iint_A = \iint_{H_\epsilon} + \iint_{A - H_\epsilon} \equiv \iint_{H_\epsilon} + \iint_{A \setminus H_\epsilon} - \iint_{H_\epsilon \setminus A} \tag{2·17}$$

where (as usual) $X \setminus Y$ denotes the set of points in X not belonging to Y. Next, with $|_{\partial A}$ denoting values when $(r,z)\in\partial A$, we isolate terms linear in q (or u) by writing

$$\left\{-\tfrac{2}{3}r^2 + \frac{5}{\pi}\iint_{H_\epsilon} G\right\}\Big|_{\partial A} = \left\{-\tfrac{2}{3}r^2 + \frac{5}{\pi}\iint_{H_\epsilon} G\right\}\Big|_{\partial H_\epsilon} + q\frac{\partial}{\partial \nu}\left\{-\tfrac{2}{3}r^2 + \frac{5}{\pi}\iint_{H_\epsilon} G\right\}\Big|_{\partial H_\epsilon} + \mathscr{R}_\epsilon q, \tag{2·18}$$

$$\left.\begin{aligned} \text{and} \qquad \frac{5}{\pi}\iint_{A - H_\epsilon} G(r,\hat{r},z-\hat{z})\Big|_{\partial A}\, d\hat{r}\, d\hat{z} &= \frac{5}{\pi}\int_0^\pi \Gamma_\epsilon(t,\hat{t})\,\hat{q}\frac{d\hat{r}}{d\hat{t}}.d\hat{t} + \mathscr{S}_\epsilon q, \\ \text{where} \qquad \Gamma_\epsilon(t,\hat{t}) &= G(r,\hat{r},z-\hat{z}) \quad \text{for} \quad (r,z) \quad \text{and} \quad (\hat{r},\hat{z})\in\partial H_\epsilon \end{aligned}\right\} \tag{2·19}$$

(The notation $\hat{q} = q(\hat{t},\epsilon)$ is used consistently.) These equations *define* the non-linear operators $\mathscr{R}_\epsilon$ and $\mathscr{S}_\epsilon$. Dividing by r (where $r = \epsilon + h.\sin t$ on ∂H_ϵ), we now define

$$f_\epsilon \equiv f(t,\epsilon) = \left\{-\tfrac{2}{3}r + \frac{5}{\pi r}\iint_{H_\epsilon} G - \frac{\epsilon^2}{r}\right\}\Big|_{\partial H_\epsilon}, \tag{2·20}$$

and note that the third term in (2·18) becomes $-u(t,\epsilon)$ because of (2·15). We further define

$$\left.\begin{aligned} \mathscr{K}_\epsilon u &= \int_0^\pi K_\epsilon(t,\hat{t})\,\hat{u}\, d\hat{t}, \\ \text{where} \qquad K_\epsilon(t,\hat{t}) &= \frac{5}{\pi}\frac{\Gamma_\epsilon(t,\hat{t})\, d\hat{r}/d\hat{t}}{\{\epsilon + h(t,\epsilon)\sin t\}\, g(\hat{t},\epsilon)}, \end{aligned}\right\} \tag{2·21}$$

$$\text{and} \qquad \mathscr{N}_\epsilon u = \frac{(\mathscr{R}_\epsilon q)(t) + (\mathscr{S}_\epsilon q)(t)}{\epsilon + h(t,\epsilon)\sin t}. \tag{2·22}$$

With these definitions (2·11) and (2·16) are equivalent.

The interpretation of the operator equation (2·16) is as follows. The basic flow consists of the free stream, and of vorticity Ωr in H_ϵ (only), and has the (non-dimensional) stream function

$$\Psi(r,z) = -\tfrac{2}{3}r^2 + \frac{5}{\pi}\iint_{H_\epsilon} G.$$

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To this we add the stream function

$$\tilde{\psi}(r,z) = \frac{5}{\pi}\int_{\partial H_\epsilon} G\hat{q}\,d\hat{r}$$

of a vortex sheet (or 'single layer') along ∂H_ϵ, of unknown strength Ωq; this represents the additional vorticity between ∂H_ϵ and ∂A. With this notation, we have

$$-\frac{1}{r}\frac{\partial \Psi}{\partial \nu}\bigg|_{\partial H_\epsilon} q = gq \equiv u, \quad \frac{1}{r}\tilde{\psi}|_{\partial H_\epsilon} = \mathscr{K}_\epsilon u, \quad \frac{1}{r}(\Psi - \epsilon^2)|_{\partial H_\epsilon} = f_\epsilon,$$

and the operator equation becomes

$$\frac{\epsilon^2}{r}\bigg|_{\partial H_\epsilon} = \frac{1}{r}\left(\Psi + \frac{\partial \Psi}{\partial \nu} q + \tilde{\psi}\right)\bigg|_{\partial H_\epsilon} + \text{non-linear terms},$$

in accordance with (2·6). The right-hand side is thus $(1/r)\,|_{\partial H_\epsilon} \cdot \psi|_{\partial A}$ and represents the prescribed flux in a meridional plane between the axis $r = 0$ and the desired curve ∂A.

An equivalent formulation of the linearized problem is

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\tilde{\psi} = 0 \quad \text{for} \quad (r,z) \notin \partial H_\epsilon,$$

with

$$\frac{g}{10r^2}\left[\frac{\partial \tilde{\psi}}{\partial \nu}\right]_{\nu=0_-}^{\nu=0_+} + \frac{\tilde{\psi}}{r} = -f_\epsilon \quad \text{on} \quad \partial H_\epsilon.$$

Here the signs in the boundary condition are *not* appropriate to an energy inequality, and therefore we are forced to use more elaborate means in section 4 to prove that $u - \mathscr{K}_\epsilon u = 0$ implies $u = 0$.

After defining an appropriate function space $\mathscr{U}$, we shall use the contraction mapping principle to establish existence and uniqueness (in a certain small ball of $\mathscr{U}$) of the solution u of equation (2·16). This is equivalent to the following iterative scheme:

$$\left.\begin{aligned} u_1 - \mathscr{K}_\epsilon u_1 &= f_\epsilon, \\ u_n - \mathscr{K}_\epsilon u_n &= f_\epsilon + \mathscr{N}_\epsilon u_{n-1} \quad \text{for} \quad n \geqslant 2. \end{aligned}\right\} \tag{2·23}$$

This scheme also yields approximations to u, asymptotic for $\epsilon \to 0$, if we expand f_ϵ asymptotically and iterate to sufficient order.

3. *The main results*

THEOREM 3·1. *Given the free-stream velocity W, the vorticity constant Ω, and a sufficiently small positive parameter ϵ, there exists a steady vortex ring whose stream function $\psi(r,z)$ and core boundary ∂A satisfy equations* (2·3). *The core boundary is (non-uniformly) approximated by the interior streamline ∂H_ϵ of Hill's spherical vortex (specified by the same W, Ω). Further, a first approximation to $q(t,\epsilon)$, the distance of ∂A from ∂H_ϵ along the normal to ∂H_ϵ, is given by equations* (6·9–10).

Proof. See Theorem 6·2.

This vortex ring is the only ring having a core boundary which asymptotes, as $\epsilon \to 0$, to the curve ∂H_ϵ within the error $O(\epsilon^2/g(t,\epsilon))$.

We note that no regular perturbation of the spherical surface of Hill's vortex,

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Table 1

T	$q(T,\epsilon)\,\epsilon^{-2}$	T	$q(T,\epsilon)\,\epsilon^{-2}$	$T\epsilon^{-\frac{2}{3}}$	$q(T,\epsilon)\,\epsilon^{-\frac{4}{3}}$
$\frac{\pi}{60}\times 1$	$-28{\cdot}0$	$\frac{\pi}{60}\times 16$	$0{\cdot}866$	$0{\cdot}25$	$-0{\cdot}001$
2	$-14{\cdot}72$	17	$1{\cdot}140$	$0{\cdot}50$	$-0{\cdot}019$
3	$-9{\cdot}94$	18	$1{\cdot}392$	$0{\cdot}75$	$-0{\cdot}069$
4	$-7{\cdot}370$	19	$1{\cdot}618$	$1{\cdot}00$	$-0{\cdot}101$
5	$-5{\cdot}688$	20	$1{\cdot}822$	$1{\cdot}25$	$-0{\cdot}094$
6	$-4{\cdot}449$	21	$2{\cdot}007$	$1{\cdot}50$	$-0{\cdot}076$
7	$-3{\cdot}496$	22	$2{\cdot}169$	$1{\cdot}75$	$-0{\cdot}059$
8	$-2{\cdot}716$	23	$2{\cdot}311$	$2{\cdot}00$	$-0{\cdot}047$
9	$-2{\cdot}052$	24	$2{\cdot}436$	$3{\cdot}00$	$-0{\cdot}021$
10	$-1{\cdot}485$	25	$2{\cdot}539$	$4{\cdot}00$	$-0{\cdot}012$
11	$-0{\cdot}983$	26	$2{\cdot}624$	$5{\cdot}00$	$-0{\cdot}008$
12	$-0{\cdot}533$	27	$2{\cdot}690$	$6{\cdot}00$	$-0{\cdot}005$
13	$-0{\cdot}132$	28	$2{\cdot}736$	$7{\cdot}00$	$-0{\cdot}004$
14	$0{\cdot}232$	29	$2{\cdot}764$	$8{\cdot}00$	$-0{\cdot}003$
$\frac{15\pi}{60}=\frac{\pi}{4}$	$-0{\cdot}565$	$\frac{30\pi}{60}=\frac{1}{2}\pi$	$2{\cdot}774$	$9{\cdot}00$	$-0{\cdot}002$
				$10{\cdot}00$	$-0{\cdot}002$

(aside from the trivial translations parallel to the z-axis), leads to a steady vortex satisfying (2·3) with $\epsilon = 0$.

To draw approximations to the curves ∂A we use, for $r \succ \epsilon^{\frac{2}{3}}$†, the asymptotic formula (6·10) for q, and ∂H_ϵ; while for $r \prec \epsilon^{\frac{2}{3}}$ we use ∂H_ϵ alone. When $r \asymp \epsilon^{\frac{2}{3}}$ we use a numerical approximation in (6·9). Thus we form table 1 for $(r,z) \in \partial A$, and for parameter values $\epsilon = 0{\cdot}05$ and $0{\cdot}1$. These curves are shown in Figs. 9 and 1 respectively.

4. *The linear equation* $u - \mathscr{K}_\epsilon u = f$. 4·1. *The approximation of* $\mathscr{K}_\epsilon$ *by* $\mathscr{K}_0$.

First, we introduce the Banach space $\mathscr{U}$ appropriate to the solution u of our problem. Denoting (as usual) by $C[0,\pi]$ the Banach space of real-valued continuous functions $u(t)$ defined on the interval $0 \leqslant t \leqslant \pi$ with the norm $\|u\| = \max_{0\leqslant t\leqslant\pi} |u(t)|$, we define the closed subspace $\mathscr{U}$ by

$$\mathscr{U} = \{u \,|\, u \in C[0,\pi],\ u(t) = u(\pi - t)\}. \tag{4·1}$$

We now proceed to prove the existence of the bounded linear inverse operator $(\mathscr{I} - \mathscr{K}_\epsilon)^{-1}$ by introducing the limiting form $\mathscr{K}_0$ of $\mathscr{K}_\epsilon$ and showing (*a*) that $\|\mathscr{K}_\epsilon - \mathscr{K}_0\|$ is small, (*b*) that $(\mathscr{I} - \mathscr{K}_0)^{-1}$ exists. Here $\mathscr{I}$ is the identity operator $\mathscr{I}u \equiv u$. To this end, we first note the inequalities (proved in Appendix B)

$$0 \leqslant G(r,\hat{r},z-\hat{z}) \leqslant \frac{2r^2\hat{r}^3}{\sigma^3}, \tag{4·2a}$$

$$0 \leqslant G(r,\hat{r},z-\hat{z}) \leqslant r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}} \sin^{-1}\frac{\sqrt{(2r\hat{r})}}{\sigma}, \tag{4·2b}$$

where

$$\sigma^2 = (r-\hat{r})^2 + (z-\hat{z})^2;$$

† Here we use the order notation of Hardy (9). If $f(\epsilon) = o\{g(\epsilon)\}$ for $\epsilon \downarrow 0$, we write $f \prec g$ (or $g \succ f$); if $f = O(g)$, we write $f \leqslant g$; if $f = O(g)$ and $g = O(f)$, we write $f \asymp g$; and if $\lim f/g$ exists and is not zero, we write $f \underset{\sim}{\asymp} g$.

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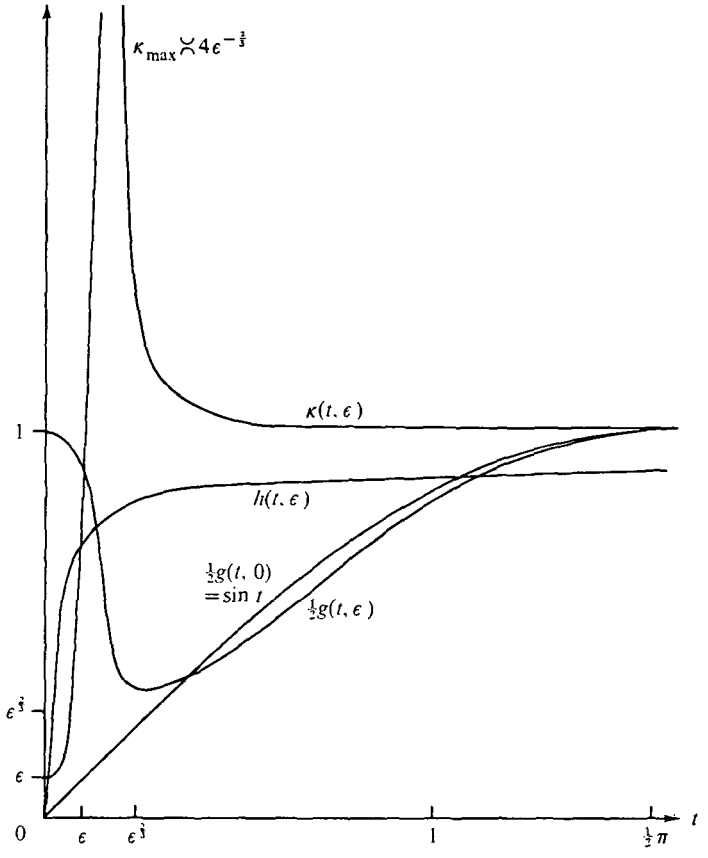


Fig. 3. The curve ∂H_ϵ, given by $s = h(t, \epsilon)$ which has a familiar boundary-layer structure for $t \to 0$, has curvature $\kappa(t, \epsilon)$, whose singular behaviour for $t \asymp \epsilon^{\frac{2}{3}}$ exhibits the difficulties of the problem there. The function $g(t, \epsilon)$, which relates the physical and mathematical variables by $q = u/g$, tends to $g(t, 0)$ as $\epsilon \to 0$ *only for* $t \succ \epsilon^{\frac{2}{3}}$.

the former is useful for $r \to 0$ or $\hat{r} \to 0$ with $\sigma \neq 0$, and the latter corresponds to the logarithmic singularity of G for $\sigma \to 0$.

We also observe that for $\epsilon = 0$ the coordinates (s, t) defined by (2·13) are simply spherical polars, which we write (ρ, θ). Accordingly, writing $\Gamma_0(t, \hat{t}) \equiv \Gamma_0(\theta, \hat{\theta})$ for the restriction of G to $\rho = \hat{\rho} = 1$, and noting that in (2·21) we have for $\epsilon = 0$, $\sin t \neq 0$ and $\sin \hat{t} \neq 0$.

$$\left.\frac{d\hat{\tau}}{d\hat{t}}\right|_{\epsilon=0} = 1, \quad h(t, 0) = 1, \quad g(\hat{t}, 0) = 2 \sin \hat{t}, \tag{4·3}$$

we define the operator $\mathscr{K}_0$ in terms of the kernel

$$K_0(t, \hat{t}) = \begin{cases} \dfrac{5}{2\pi} \dfrac{\Gamma_0(t, \hat{t})}{\sin t \sin \hat{t}} & \text{on} \quad (0, \pi) \times (0, \pi), \quad t \neq \hat{t}, \\ 0 \qquad \text{for} \quad t = 0, \pi \quad \text{or} \quad \hat{t} = 0, \pi. \end{cases} \tag{4·4}$$

Then (4·2*a*) shows that K_0 is continuous on $[0, \pi] \times [0, \pi]$ except on the diagonal $t = \hat{t}$, (where it has a logarithmic singularity, as suggested by (4·2*b*)).

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The proof that $\|\mathscr{K}_\epsilon - \mathscr{K}_0\|$ is small is based on explicit approximations to $h(t,\epsilon)$, $g(t,\epsilon)$ and on a detailed application of the inequalities (4·2) and their extensions (4·11) to the integral defining $(\mathscr{K}_\epsilon - \mathscr{K}_0)u$. The invertibility of $\mathscr{I} - \mathscr{K}_0$ follows from the classical representation of G (and hence K_0) in spherical polar coordinates. We consider first the function $g(t,\epsilon)$ which relates the physical and mathematical variables, q and u, of our problem. We shall restrict ϵ to intervals $(0,\epsilon_i]$, $(i = 1,2,\ldots)$ and c, C are to be generic symbols for numbers that are then independent of $t, \hat{t}, \epsilon$.

LEMMA 4·1. *The function $g(t,\epsilon)$ is in $\mathscr{U}$ and*

$$g(t,\epsilon) = 2\sqrt{\{\sin^2\theta(1-2\rho^2)^2 + \cos^2\theta(1-\rho^2)^2\}}|_{\partial H_\epsilon}\{1+O(\epsilon^{\frac{2}{3}})\} \quad (0 \leqslant t \leqslant \pi), \tag{4·5}$$

where (ρ,θ) are spherical coordinates.

Proof. Collecting the information about ψ_H in (2·6)–(2·13), we write

$$\psi_H = -\tfrac{2}{3}r^2 + \frac{5}{\pi}\iint_{H_0} G(r,\hat{r},z-\hat{z})\,d\hat{r}\,d\hat{z} = \rho^2\sin^2\theta\,(1-\rho^2) \quad (\rho \leqslant 1). \tag{4·6}$$

We insert this in the definition (2·15) of $g(t,\epsilon)$ and obtain

$$g(t,\epsilon) = -\frac{1}{r}\frac{\partial\psi_H}{\partial\nu}\bigg|_{\partial H_\epsilon} + \frac{5}{\pi}\iint_{H_0-H_\epsilon}\frac{1}{r}\frac{\partial G}{\partial\nu}\bigg|_{\partial H_\epsilon} d\hat{r}\,d\hat{z}. \tag{4·7}$$

Now explicit calculation shows that

$$-\frac{1}{r}\frac{\partial\psi_H}{\partial\nu}\bigg|_{\partial H_\epsilon} = \frac{1}{r}|\nabla\psi_H|_{\partial H_\epsilon} = 2\sqrt{\{\sin^2\theta\,(1-2\rho^2)^2 + \cos^2\theta(1-\rho^2)^2\}}|_{\partial H_\epsilon} \geqslant 3\epsilon^{\frac{2}{3}},$$

and a natural extension of the work in Appendix C proves that

$$\iint_{H_0-H_\epsilon}\frac{1}{r}\left|\frac{\partial G}{\partial\nu}\right|\bigg|_{\partial H_\epsilon} d\hat{r}\,d\hat{z} \leqslant C\epsilon^{\frac{4}{3}}, \tag{4·8}$$

and hence we have (4·5).

The symmetry $g(t,\epsilon) = g(\pi - t,\epsilon)$, is clear from (2·15) on noting the symmetries of ∂H_ϵ and $G(r,\hat{r},z-\hat{z})$. Classical arguments from potential theory show that $g \in C[0,\pi]$, and hence $g \in \mathscr{U}$.

The function $g(t,\epsilon)$ is shown in Fig. 3.

LEMMA 4·2. *The linear operator $\mathscr{K}_0$ uniformly approximates $\mathscr{K}_\epsilon$ in $\mathscr{U}$; that is,*

$$\|\mathscr{K}_\epsilon - \mathscr{K}_0\| \leqslant C\epsilon^{\frac{2}{3}}\ln(1/\epsilon) \quad \text{for} \quad \epsilon \leqslant \tfrac{1}{12}. \tag{4·9}$$

Proof. (*a*) By symmetry it suffices to consider $0 \leqslant t \leqslant \frac{1}{2}\pi$.

We prove the result

$$\int_0^\pi |K_\epsilon(t,\hat{t}) - K_0(t,\hat{t})|\,d\hat{t} \leqslant C\epsilon^{\frac{2}{3}}\ln(1/\epsilon), \quad \forall t \in [0,\pi] \quad (\epsilon \leqslant \tfrac{1}{12}), \tag{4·10}$$

from which (4·9) follows immediately on noting that

$$|(\mathscr{K}_\epsilon - \mathscr{K}_0)u| \leqslant \int_0^\pi |K_\epsilon(t,\hat{t}) - K_0(t,\hat{t})|\,|\hat{u}|d\hat{t} \leqslant \|u\|\int_0^\pi |K_\epsilon(t,\hat{t}) - K_0(t,\hat{t})|\,d\hat{t}.$$

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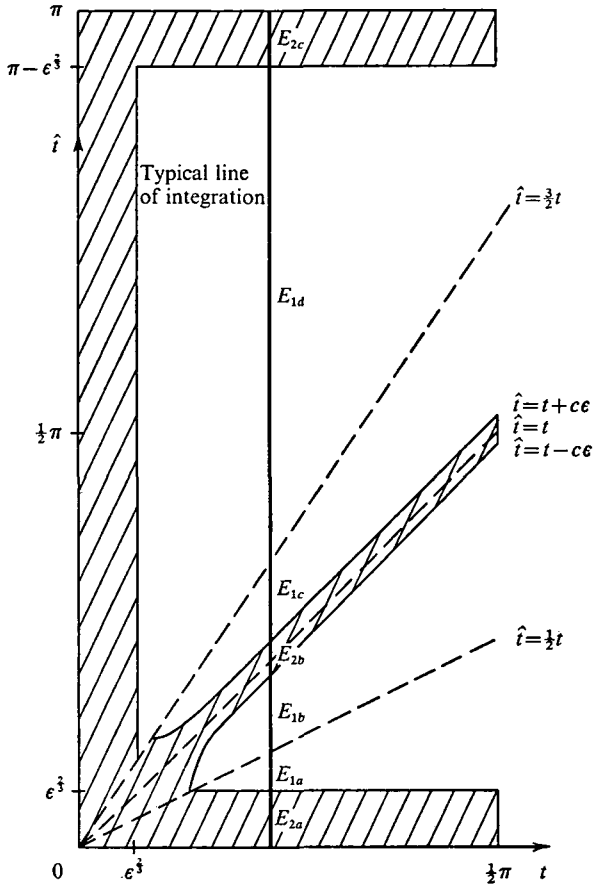


Fig. 4. The line of integration $0 \leqslant \hat{t} \leqslant \pi$, for a typical value $\epsilon^{\frac{2}{3}} \leqslant t \leqslant \frac{1}{2}\pi$, passes through the sections E_1, where K_ϵ is approximated by K_0, and the remainder E_2, which has small total length. For $0 \leqslant t \leqslant \epsilon^{\frac{2}{3}}$ the kernels K_ϵ and K_0 are both small.

The proof of (4·10) is difficult because of the non-uniform behaviour of the integrand in the domain $0 \leqslant t \leqslant \frac{1}{2}\pi$, $0 \leqslant \hat{t} \leqslant \pi$.

We prove the result (4·10) for $4\epsilon^{\frac{2}{3}} \leqslant t \leqslant \epsilon^{\frac{1}{2}}$, where $\epsilon \leqslant \epsilon_1$; for other values of $t \in [\epsilon^{\frac{2}{3}}, \frac{1}{2}\pi]$ equation (4·10) follows similiarly and more easily. The basic idea is to divide the line of integration $0 \leqslant \hat{t} \leqslant \pi$ into a section E_1, where $K_\epsilon(t,\hat{t})$ is indeed approximated by $K_0(t,\hat{t})$, and a section E_2 (the complement of E_1) which has small total length (Fig. 4). For $t \in [0, \epsilon^{\frac{2}{3}}]$ the calculation is simpler.

(*b*) Until the contrary is stated we fix t in the interval $[4\epsilon^{\frac{2}{3}}, \epsilon^{\frac{1}{2}}]$. We desire as large a section E_1 as possible, and trial suggests the definition

$$E_1 = \left\{\hat{t} \,\middle|\, |\hat{t}-t| \geqslant \frac{6\epsilon^2}{\sin^2 t} \quad \text{and} \quad \epsilon^{\frac{2}{3}} \leqslant \hat{t} \leqslant \pi - \epsilon^{\frac{2}{3}}\right\}. \tag{4·11}$$

In Appendix B we bound, by equations (B 7 *a*, *b*), the effect of perturbing the variables $r, z, \hat{r}, \hat{z}$ of the function G, (which itself is bounded in (4·2)). Now $\Gamma_0(t,\hat{t}) = G$ with $r = \sin t$, $z = \cos t$, $\hat{r} = \sin\hat{t}$, $\hat{z} = \cos\hat{t}$ and then

$$\sigma = \sqrt{\{(\sin t - \sin\hat{t})^2 + (\cos t - \cos\hat{t})^2\}} = 2\sin\tfrac{1}{2}|t-\hat{t}|.$$

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With the notation of Appendix B we have $\Gamma_\epsilon(t,\hat{t}) = G$ with $\bar{r} = \epsilon + h(t,\epsilon)\sin t$, $\bar{z} = h(t,\epsilon)\cos t$, etc.; also (A 2) implies that

$$1 - h(t,\epsilon) - \epsilon\sin t - \epsilon^2\cos^2 t/2 \sim \epsilon^2/2\sin^2 t + O\left(\frac{\epsilon^4}{t^4}\right) \quad \text{for} \quad \frac{\epsilon}{t} \to 0.$$

Accordingly, for $0 < \epsilon < \frac{1}{12}$ and $t, \hat{t} \geqslant \epsilon^{\frac{3}{4}}$, we have

$$\xi \equiv \sqrt{\{(\bar{r}-r)^2 + (\bar{z}-z)^2\}} \leqslant \epsilon + \epsilon^2/\sin^2 t,$$

$$\hat{\xi} \equiv \sqrt{\{(\hat{\bar{r}}-\hat{r})^2 + (\hat{\bar{z}}-\hat{z})^2\}} \leqslant \epsilon + \epsilon^2/\sin^2 \hat{t}.$$

Let $\hat{t} \in E_1$ and $0 < \epsilon < \frac{1}{12}$. We wish to use the bounds (B 7a, b) of Appendix B and must check that condition (B 8) is satisfied. In fact, the definition (4·11) of E_1 implies, after some calculation, that (for $\hat{t} \in E_1$ and $0 < \epsilon < \frac{1}{12}$)

$$\sigma = 2\sin\tfrac{1}{2}|t-\hat{t}| \geqslant \mu\{\epsilon^2/\sin^2 t + \epsilon^2/\sin^2\hat{t} + 2\epsilon\} \quad (\mu > 1),$$

and this inequality is illustrated in Fig. 5. Thus $\sigma \geqslant \mu\{\xi + \hat{\xi}\}$ with $\mu > 1$, and (B 8) holds. Retaining only the dominant terms of (B 7a, b) we have

$$|\Gamma_\epsilon(t,\hat{t}) - \Gamma_0(t,\hat{t})| \leqslant C\frac{r^2\hat{r}^3}{\sigma^3}\cdot\frac{\xi+\hat{\xi}}{\sigma},$$

$$\leqslant C\frac{\sin^2 t.\sin^3\hat{t}}{\sigma^4}\left\{\frac{\epsilon^2}{\sin^2 t} + \frac{\epsilon^2}{\sin^2\hat{t}} + 2\epsilon\right\}, \tag{4·12a}$$

and

$$|\Gamma_\epsilon(t,\hat{t}) - \Gamma_0(t,\hat{t})| \leqslant Cr^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}\frac{\xi+\hat{\xi}}{\sigma},$$

$$\leqslant \frac{C\sin^{\frac{1}{2}} t.\sin^{\frac{3}{2}}\hat{t}}{\sigma}\left\{\frac{\epsilon^2}{\sin^2 t} + \frac{\epsilon^2}{\sin^2\hat{t}} + 2\epsilon\right\}, \tag{4·12b}$$

(for $\hat{t} \in E_1$ and $\epsilon < \frac{1}{12}$). Now

$$\int_{E_1}[K_\epsilon - K_0]\,d\hat{t} = \frac{5}{\pi}\left\{\int_{E_1}\Gamma_\epsilon(t,\hat{t})\left[\frac{d\hat{r}/d\hat{t}}{\{\epsilon + h.\sin t\}\hat{g}} - \frac{1}{2\sin t.\sin\hat{t}}\right]d\hat{t} + \int_{E_1}\frac{\Gamma_\epsilon(t,\hat{t}) - \Gamma_0(t,\hat{t})}{2\sin t.\sin\hat{t}}\,d\hat{t}\right\}. \tag{4·13}$$

We further divide E_1 into sets E_{1a}, E_{1d} (Fig. 4) where we use (4·2a) for Γ_ϵ and (4·12a) for $\Gamma_\epsilon - \Gamma_0$ because $\sigma \geqslant c\sqrt{(r\hat{r})}$ there, and sets E_{1b}, E_{1c} where we use (4·2b) for Γ_ϵ and (4·12b) for $\Gamma_\epsilon - \Gamma_0$ because $\sigma \leqslant c\sqrt{(r\hat{r})}$. We observe that the points $\hat{t} = \frac{1}{2}t$, $\hat{t} = \frac{3}{2}t$ do divide E_1 appropriately because, for $\hat{t} \leqslant \frac{1}{2}t$ or $\hat{t} \geqslant \frac{3}{2}t$,

$$\frac{\sigma}{\sqrt{(r\hat{r})}} = \frac{2\sin\frac{1}{2}|\hat{t}-t|}{\sqrt{(\sin t.\sin\hat{t})}} \geqslant \frac{2\sin\frac{1}{4}t}{\sqrt{(\sin t.\sin\frac{3}{2}t)}} \geqslant c > 0.$$

Thus we define

$$\left.\begin{aligned} E_{1a}&: \epsilon^{\frac{3}{4}} \leqslant \hat{t} \leqslant \tfrac{1}{2}t, \quad E_{1b}: \tfrac{1}{2}t \leqslant \hat{t} \leqslant t - \frac{6\epsilon^2}{\sin^2 t}, \\ E_{1c}&: t + \frac{6\epsilon^2}{\sin^2 t} \leqslant \hat{t} \leqslant \tfrac{3}{2}t, \quad E_{1d}: \tfrac{3}{2}t \leqslant \hat{t} \leqslant \pi - \epsilon^{\frac{3}{4}}. \end{aligned}\right\} \tag{4·14}$$

In Appendix A we prove that

$$\left|\frac{d\hat{r}/d\hat{t}}{\{\epsilon + h\sin t\}\hat{g}} - \frac{1}{2\sin t.\sin\hat{t}}\right| \leqslant \frac{C\epsilon^{\frac{3}{4}}}{\sin t.\sin\hat{t}} + \frac{2\epsilon}{\sin t.\sin\hat{t}} + \frac{2\epsilon}{\sin^2 t.\sin\hat{t}} \quad (t \in E_1); \tag{4·15}$$

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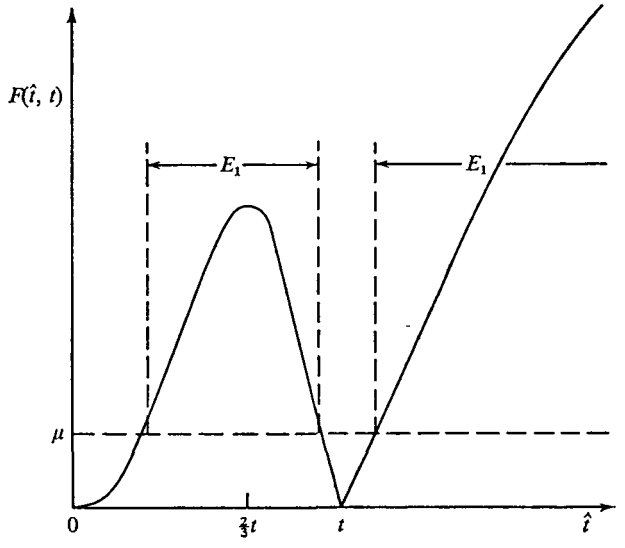


Fig. 5. The function

$$F(\hat{t}, t) = 2 \sin \tfrac{1}{2}|\hat{t}-t| \Big/ \left\{\frac{\epsilon^2}{\sin^2 \hat{t}} + \frac{\epsilon^2}{\sin^2 t} + 2\epsilon\right\}.$$

The set E_1 is defined so that $F(\hat{t}, t) \geqslant \mu > 1$ for $\hat{t} \in E_1$.

from this, and the bounds on Γ_ϵ implied by the inequalities (4·2a, b), it is proved (by estimates similar to those in (c) below) that

$$\int_{E_1} \Gamma_\epsilon(t, \hat{t}) \left| \frac{d\hat{\tau}/d\hat{t}}{\{\epsilon + h \sin t\}\hat{g}} - \frac{1}{2 \sin t . \sin \hat{t}} \right| . d\hat{t} \leqslant C\epsilon^{\frac{3}{2}}. \tag{4·16}$$

We illustrate the bounding of the remaining integral in (4·13) in some detail for the typical, but crucial, segments E_{1b} and E_{1d}. (We use the convention that the integral vanishes if the upper limit is less than the lower limit.) Using (4·12b) for $\Gamma_\epsilon - \Gamma_0$ in this integral in (4·13), and noting that $\sigma \geqslant c|\hat{t}-t|$, we see that

$$\int_{E_{1b}} C \sqrt{\left\{\frac{\sin \hat{t}}{\sin t}\right\}} \left\{\frac{\epsilon^2}{\sin^2 t} + \frac{\epsilon^2}{\sin^2 \hat{t}} + 2\epsilon\right\} d\hat{t} \leqslant \int_{E_{1b}} \frac{C\epsilon^2}{\hat{t}^2|\hat{t}-t|} d\hat{t} \leqslant C\epsilon^{\frac{3}{2}}. \tag{4·17}$$

Next, (4·12a) for $\Gamma_\epsilon - \Gamma_0$ in E_{1d} yields

$$\int_{E_{1d}} \frac{C \sin t \sin^2 \hat{t}}{\sigma^4} \left\{\frac{\epsilon^2}{\sin^2 t} + \frac{\epsilon^2}{\sin^2 \hat{t}} + 2\epsilon\right\} d\hat{t} \leqslant \int_{E_{1d}} C \frac{\epsilon^2}{\sin^2 t} \frac{\hat{t}^2 d\hat{t}}{|\hat{t}-t|^4} \leqslant C\epsilon^{\frac{3}{2}}. \tag{4·18}$$

Thus grouping (4·16) and results like (4·17), (4·18), we bound (4·13).

(c) We now consider the rest of (4·10)

$$\int_{E_2} |K_\epsilon - K_0| \, d\hat{t} \leqslant \int_{E_2} \{K_\epsilon + K_0\} d\hat{t}. \tag{4·19}$$

We split the section E_2, as before, into E_{2a}, E_{2c} where we use (4·2a), and E_{2b} where we use (4·2b) for Γ_ϵ, Γ_0. We define

$$E_{2a}\colon 0 \leqslant \hat{t} \leqslant \epsilon^{\frac{3}{2}}, \quad E_{2b}\colon t - \frac{6\epsilon^2}{\sin^2 t} \leqslant \hat{t} \leqslant t + \frac{6\epsilon^2}{\sin^2 t}, \quad E_{2c}\colon \pi - \epsilon^{\frac{3}{2}} \leqslant \hat{t} \leqslant \pi,$$

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and observe these are appropriate (trivially from Fig. 4). We bound typical integrals that make up (4·19) as follows. Using (4·2a) for Γ_0 when $\hat{t} \in E_{2a}$, and noting again $\sigma \geqslant c|\hat{t}-t|$, we see that

$$\begin{aligned}\int_{E_{2a}} K_0(t,\hat{t})\,d\hat{t} &\leqslant \int_0^{\epsilon^{\frac{1}{2}}} \frac{5}{\pi}\frac{\sin t.\sin^2\hat{t}}{2\sin^3\frac{1}{2}|\hat{t}-t|}\,d\hat{t},\\ &\leqslant \int_0^{\epsilon^{\frac{1}{2}}} C\frac{t\hat{t}^2\,d\hat{t}}{|\hat{t}-t|^3},\\ &\leqslant \frac{C}{t^2}\int_0^{\epsilon^{\frac{1}{2}}} \hat{t}^2 d\hat{t} \leqslant C\epsilon^{\frac{3}{2}}. \end{aligned} \tag{4·20}$$

Using (4·2b) for Γ_0 when $\hat{t} \in E_{2b}$, we have

$$\begin{aligned}\int_{E_{2b}} K_0(t,\hat{t})\,d\hat{t} &\leqslant \int_{E_{2b}} C\sqrt{\left\{\frac{\sin\hat{t}}{\sin t}\right\}}\ln\left(\frac{ct}{\sigma}\right).d\hat{t},\\ &\leqslant C\int_0^{6\epsilon^{\frac{1}{2}}/\sin^2 t} \ln\left(\frac{ct}{\zeta}\right)d\zeta \quad (\zeta = |\hat{t}-t|),\\ &\leqslant C\epsilon^{\frac{3}{2}}. \end{aligned} \tag{4·21}$$

When K_ϵ replaces K_0 in these calculations similar results ensue. More effort has to be made in determining the behaviour of $(d\hat{\tau}/d\hat{t})/[\{\epsilon + h\sin t\}g]$ for $\hat{t} \to 0$; Appendix A shows that $d\hat{\tau}/d\hat{t}$ becomes large, but $\hat{g}^{-1}$ does not, so the product is not worse than $1/2\sin\hat{t}$, the form for $\epsilon = 0$. Hence calculations like (4·20) and (4·21) lead to bounds for (4·19), and combined with the results from (b) for (2·13), we finally obtain the bounds (4·10).

(d) The same pattern of calculations as in (b) and (c) leads to the required result (4·10) for $\epsilon^{\frac{1}{2}} \leqslant t \leqslant \frac{1}{2}\pi$, when we replace (4·11) by

$$E_1 = \{\hat{t}\,|\,|\hat{t}-t| \geqslant 4\epsilon \quad \text{and} \quad \epsilon^{\frac{3}{2}} \leqslant \hat{t} \leqslant \pi - \epsilon^{\frac{3}{2}}\}. \tag{4·11*}$$

For $0 \leqslant t \leqslant 4\epsilon^{\frac{3}{2}}$ we need consider only E_2-type calculations. Thus we use the simpler partitioning

$$\int_0^\pi |K_\epsilon - K_0|\,d\hat{t} \leqslant \int_0^{\frac{1}{2}t} + \int_{\frac{1}{2}t}^{\frac{3}{2}t} + \int_{\frac{3}{2}t}^{\pi} \{K_\epsilon + K_0\}\,d\hat{t}, \tag{4·22}$$

and apply (4·2a), (4·2b) appropriately to bound Γ_ϵ and Γ_0. A series of calculations as in (4·20), (4·21) leads to the result (4·10); we show only one typical case (again using (4·2a) and noting that $\sigma \geqslant c|\hat{t}-t|$),

$$\begin{aligned}\int_{\frac{3}{2}t}^{\pi} K_0(t,\hat{t})\,d\hat{t} &\leqslant \int_{\frac{3}{2}t}^{\pi} \frac{C\sin t\sin^2\hat{t}}{|\hat{t}-t|^3}\,d\hat{t},\\ &\leqslant Ct\ln(1/t) \leqslant C\epsilon^{\frac{3}{2}}\ln(1/\epsilon). \end{aligned} \tag{4·23}$$

Again we note that when K_ϵ replaces K_0, it is $d\hat{\tau}/d\hat{t}$, not $\hat{g}^{-1}$, that becomes large as $t \to 0$; however, the same bound $C\epsilon^{\frac{3}{2}}\ln 1/\epsilon$ follows. Hence (4·10) is established, and we observe the $\epsilon < \frac{1}{12}$ is not critical for the result.

4·2. *Existence of the inverse operator* $(\mathscr{I} - \mathscr{K}_\epsilon)^{-1}$. We now proceed to show the existence of the bounded linear inverse operator $(\mathscr{I} - \mathscr{K}_\epsilon)^{-1}$ by showing the existence of the approximating operator $(\mathscr{I} - \mathscr{K}_0)^{-1}$.

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LEMMA 4·3. *The linear integral operators $\mathscr{K}_0$, defined by* (4·4) *and*

$$\mathscr{K}_0 u = \int_0^\pi K_0(t,\hat{t})\,\hat{u}\,d\hat{t},$$

and $\mathscr{K}_\epsilon$, defined by (2·21), *map $\mathscr{U}$ into itself and are completely continuous.*

Proof. We treat the symmetry property of the linear operator $\mathscr{K}_0$ first.

$$\begin{aligned}(\mathscr{K}_0 u)(\pi-t) &= \frac{5}{2\pi}\int_0^\pi \frac{\Gamma_0(\pi-t,\hat{t})}{\sin(\pi-t)\sin\hat{t}}u(\hat{t})\,d\hat{t}\\ &= \frac{5}{2\pi}\int_0^\pi \frac{\Gamma_0(t,\pi-\hat{t})}{\sin t.\sin(\pi-\hat{t})}u(\pi-\hat{t})\,d\hat{t}\\ &= (\mathscr{K}_0 u)(t), \end{aligned} \tag{4·24}$$

where $\Gamma_0(\pi-t,\hat{t}) = \Gamma_0(t,\pi-\hat{t})$ because $G(r,\hat{r},z-\hat{z}) = G(r,\hat{r},\hat{z}-z)$. Lemma 4·1 and the properties of ∂H_ϵ show similarly that $(\mathscr{K}_\epsilon u)(\pi-t) = (\mathscr{K}_\epsilon u)(t)$. Since the kernels K_0, K_ϵ are not only quadratically summable, but also satisfy

$$\int_0^\pi \{K(t,\hat{t})\}^2\,d\hat{t} \leqslant M, \quad M \text{ independent of } t,$$

where K is discontinuous only on a continuous line $\hat{t} = t$ in the basic square

$$[0,\pi]\times[0,\pi],$$

standard theory ((5), p. 88) shows that $\mathscr{K}_\epsilon u \in C[0,\pi]$, (and hence $\mathscr{K}_\epsilon u \in \mathscr{U}$), and that $\mathscr{K}_\epsilon$ is completely continuous for $0 \leqslant \epsilon \leqslant \frac{1}{12}$.

LEMMA 4·4. *The operator $\mathscr{I}-\mathscr{K}_0$ has only the zero vector in its null space in $\mathscr{U}$, that is*

$$u\in\mathscr{U} \quad \textit{and} \quad u-\mathscr{K}_0 u \equiv (\mathscr{I}-\mathscr{K}_0)u = 0 \quad \textit{imply} \quad u = 0.$$

Proof. We use the following series representations for G, (derived from the expansion in spherical polars of the fundamental solution of Laplace's equation), divergent only at the point $\{\hat{\theta}=\theta, \hat{\rho}=\rho\}$,

$$G(r,\hat{r},z-\hat{z}) = \pi\rho\sin\theta.\,\hat{\rho}^2\sin^2\hat{\theta}\sum_{n=1}^{\infty}\frac{P_n^1(\cos\theta)\,P_n^1(\cos\hat{\theta})}{n(n+1)}\begin{cases}\rho^n/\hat{\rho}^{n+1} & \text{if} \quad \rho\leqslant\hat{\rho}\\ \hat{\rho}^n/\rho^{n+1} & \text{if} \quad \hat{\rho}\leqslant\rho\end{cases}, \tag{4·25}$$

to write $K_0(t,\hat{t})$ as (where $\rho=\hat{\rho}=1$),

$$K_0(t,\hat{t}) = \tfrac{5}{2}\sin\hat{t}\sum_{n=1}^{\infty}\frac{P_n^1(\cos t)\,P_n^1(\cos\hat{t})}{n(n+1)} \quad (t\neq\hat{t}). \tag{4·26}$$

Here P_n^1 is the associated Legendre function of the first kind, satisfying the orthogonality condition

$$\int_{-1}^{+1} P_n^1(\cos\hat{t})\,P_n^1(\cos\hat{t})\,d\cos\hat{t} = \delta_{nm}\frac{2n(n+1)}{(2n+1)}. \tag{4·27}$$

Changing the independent variable to $\cos t \equiv \mu$, and writing

$$(\mathscr{K}_0 v)(\mu) = \int_{-1}^{+1}\left\{\frac{5}{2}\sum_{n=1}^{\infty}\frac{P_n^1(\mu)\,P_n^1(\mu)}{n(n+1)}\right\}\hat{v}\,d\hat{\mu},$$

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we now consider the operator $\mathscr{I}-\mathscr{K}_0$ in the Hilbert space $L_2(-1,1)$ of square integrable functions $v(\mu)$. Then standard theory ((7), § 136) shows that

$$\mathscr{K}_0 v_n = \lambda_n v_n, \quad \|v_n\| = 1,$$

implies $$v_n = \sqrt{\left(\frac{2n+1}{2n(n+1)}\right)} P_n^1(\mu) \quad \text{and} \quad \lambda_n = \frac{2n+1}{5} \quad (n = 1, 2, \ldots).$$

Hence, the only non-trivial solutions of $v = \mathscr{K}_0 v$ are those for $\lambda_n = 1$, i.e. $n = 2$, and are multiples of $P_2^1(\cos t) = \sin t \cos t$. Observing that $\mathscr{U} \subset L_2$, by transformation of variables $t \to \cos t \equiv \mu$ we see that only multiples of $\sin t \cos t \in C[0, \pi]$ are possible non-trivial solutions of $u = \mathscr{K}_0 u$ in $\mathscr{U}$. But $P_2^1(\cos t) \neq P_2^1(\cos(\pi - \hat{t})$ and hence $\sin t \cos t \notin \mathscr{U}$. Thus $u = \mathscr{K}_0 u$ has only the trivial solution zero in $\mathscr{U}$.

Remark. The solution $\sin t \cos t$ is in $C[0, \pi]$, and represents a translation of the spherical vortex core parallel to the axis of symmetry. Thus we could have defined $\mathscr{U}$ as the subspace of $C[0, \pi]$ orthogonal to $\sin t \cos t$ and still have achieved our aim of inverting $\mathscr{I} - \mathscr{K}_\epsilon$.

LEMMA 4·5. *The linear equation $u - \mathscr{K}_0 u = f$, for $f \in \mathscr{U}$, has the unique solution $u = (\mathscr{I} - \mathscr{K}_0)^{-1} f \in \mathscr{U}$, where $(\mathscr{I} - \mathscr{K}_0)^{-1}$ is a bounded linear operator.*

Proof. The Riesz theory for compact operators mapping a Banach space into itself, with Lemmas 4·3 and 4·4, proves the result.(see, for example, section 135 of (7)).

THEOREM 4·1. *The linear equation $u - \mathscr{K}_\epsilon u = f$, for $f \in \mathscr{U}$, has the unique solution*

$$u = (\mathscr{I} - \mathscr{K}_\epsilon)^{-1} f \in \mathscr{U},$$

where $(\mathscr{I} - \mathscr{K}_\epsilon)^{-1}$ is a bounded linear operator, for $0 \leqslant \epsilon \leqslant \epsilon_2$.

Proof. The result is proved from Lemmas 4·2 and 4·5 by a standard theorem of functional analysis (see, for example, chapter 4 of (8)).

5. *The non-linear terms.* This section is devoted to proving the following.

THEOREM 5·1. *Define the ball $\mathscr{B}(M\epsilon^2) = \{u | u \in \mathscr{U} \text{ and } \|u\| \leqslant M\epsilon^2\}$.*

The non-linear operator $\mathscr{N}_\epsilon$ maps $\mathscr{B}$ into $\mathscr{U}$, and satisfies the relations, for $0 < \epsilon \leqslant \epsilon_3$ and $0 \leqslant t \leqslant \pi$,

$$|\mathscr{N}_\epsilon u| \leqslant C_2 \|u\| \{\epsilon^{-\frac{4}{3}} \|u\| + \epsilon\}, \tag{5·1}$$

$$|\mathscr{N}_\epsilon u_1 - \mathscr{N}_\epsilon u_2| \leqslant C_3 \|u_1 - u_2\| \{\epsilon^{-\frac{4}{3}} (\|u_1\| + \|u_2\|) + \epsilon\}, \tag{5·2}$$

where $$u = g.q \in \mathscr{B}(M\epsilon^2).$$

Proof. The non-linear operator $\mathscr{N}_\epsilon$ is defined by (2·22), (2·18) and (2·19). We first cast the non-linear terms into a more manageable form. We use the relation (4·6) to simplify the various terms in (2·18), which becomes

$$\left\{\psi_H - \frac{5}{\pi}\iint_{H_0 - H_\epsilon} G\right\}\Big|_{\partial A} = \left\{\psi_H - \frac{5}{\pi}\iint_{H_0 - H_\epsilon} G\right\}\Big|_{\partial H_\epsilon} + q\frac{\partial}{\partial v}\left\{\psi_H - \frac{5}{\pi}\iint_{H_0 - H_\epsilon} G\right\}\Big|_{\partial H_\epsilon} + \mathscr{R}_\epsilon q. \tag{5·3}$$

Thus, on regrouping terms, (5·3) becomes

$$\mathscr{R}_\epsilon q = \left\{\psi_H|_{\partial A} - \psi_H|_{\partial H_\epsilon} - q\frac{\partial \psi_H}{\partial \nu}\Big|_{\partial H_\epsilon}\right\} - \frac{5}{\pi}\iint_{H_0 - H_\epsilon}\left\{G|_{\partial A} - G|_{\partial H_\epsilon} - q\frac{\partial G}{\partial \nu}\Big|_{\partial H_\epsilon}\right\}. \tag{5·4}$$

Consider (2·19), and note that, under the transformation $(r, z) \to (\nu, \tau)$, where ν, τ

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measure distance normal and tangential to ∂H_ϵ, we have $d\hat{r}\,d\hat{z} = (1+\kappa(\hat{r},\epsilon)\,\hat{v})\,d\hat{v}\,d\hat{t}$, where κ denotes the curvature of ∂H_ϵ. Then

$$\begin{aligned}\frac{5}{\pi}\int_0^\pi \Gamma_\epsilon(t,\hat{t})\,\hat{q}\frac{d\hat{r}}{d\hat{t}}\,d\hat{t} &= \frac{5}{\pi}\oint_{\partial H_\epsilon}\Gamma_\epsilon(t,\hat{t})\,d\hat{r}\int_0^{\hat{q}}d\hat{v}\\ &= \frac{5}{\pi}\oint_{\partial H_\epsilon}d\hat{r}\int_0^{\hat{q}}\Gamma_\epsilon(t,\hat{t})\{1+\hat{\kappa}\hat{v}-\hat{\kappa}\hat{v}\}\,d\hat{v},\\ &= \frac{5}{\pi}\iint_{A-H_\epsilon}\Gamma_\epsilon(t,\hat{t})\,d\hat{r}\,d\hat{z}-\frac{5}{\pi}\int_0^\pi\Gamma_\epsilon(t,\hat{t})\frac{d\hat{r}}{d\hat{t}}\frac{\hat{\kappa}\hat{q}^2}{2}d\hat{t}. \qquad (5{\cdot}5)\end{aligned}$$

Hence (2·19) becomes

$$\mathscr{S}_\epsilon q = \frac{5}{\pi}\iint_{A-H_\epsilon}\{G|_{\partial A}-\Gamma_\epsilon(t,\hat{t})\}+\frac{5}{\pi}\int_0^\pi\Gamma_\epsilon(t,\hat{t})\frac{d\hat{r}}{d\hat{t}}\frac{\hat{\kappa}\hat{q}^2}{2}d\hat{t}. \qquad (5{\cdot}6)$$

Thus, on dividing (5·4) and (5·6) through by $r|_{\partial H_\epsilon} = \epsilon + h\sin t$, we have the non-linear terms $\mathscr{N}_\epsilon u$ defined by (2·22).

The various terms in (5·4) and (5·6) are clearly symmetric, that is,

$$(\mathscr{N}_\epsilon u)(t) = (\mathscr{N}_\epsilon u)(\pi - t) \quad \text{for} \quad u\in\mathscr{U},$$

because of (4·22) and the symmetry of ψ_H and ∂H_ϵ. Therefore, if we can show for $u\in\mathscr{B}$, that $\mathscr{N}_\epsilon u\in C[0,\pi]$, then it will follow that $\mathscr{N}_\epsilon$ maps $\mathscr{B}$ into $\mathscr{U}$. This continuity with respect to t is obvious for the terms of (5·4) involving ψ_H on noting that $\epsilon + h\sin t \geqslant \epsilon$. The continuity with respect to t of the integrals follows from the usual potential-theory arguments (6). That is, G has only a logarithmic singularity at $(r,z) = (\hat{r},\hat{z})$, see (4·2), and the boundaries of the domains $H_0 - H_\epsilon$ and $A - H_\epsilon$ are continuous. For, since $u\in\mathscr{B}$, we have (i) $q = u/g\in\mathscr{U}$, and (ii) ∂A does not intersect the axis $\hat{r} = 0$ for ϵ sufficiently small, because Lemma 4·1 implies that $|q| \leqslant C\epsilon^{\frac{1}{3}}$. With regard to $\partial G/\partial\nu$ on ∂H_ϵ we note that ∂H_ϵ is analytic; and this also ensures continuity of the line integral in (5·6).

The proof that the various terms in (5·4) and (5·6), when divided by $\epsilon + h\sin t$, satisfy the inequalities (5·1) and (5·2) is involved because of the non-uniform behaviour of the integrands in both t and $\hat{t}$. However, it is the simpler terms involving ψ_H in (5·4) that are critical and force the coefficient $\epsilon^{-\frac{1}{3}}$ in (5·1), (5·2); the integral terms in (5·4) and (5·6) contribute only the corresponding coefficient $\epsilon^{-\frac{2}{3}}\ln 1/\epsilon$. We proceed to consider the non-linear terms in stages, tackling the simplest first.

(*a*) Using the explicit form for ψ_H in (4·6), and defining α as the angle between the z-axis and $\boldsymbol{\nu}$ (Fig. 2), we write the first part of (5·4) as

$$\begin{aligned}\psi_H|_{\partial A}-\psi_H|_{\partial H_\epsilon}-q\frac{\partial\psi_H}{\partial\nu}\bigg|_{\partial H_\epsilon} &= q^2\{\sin^2\alpha(1-r^2-z^2)-r^2(1+4\sin^2\alpha)-2rz\,2\sin 2\alpha\\ &\qquad -q\sin^2\alpha\,[2r(1+\sin\alpha)+2z\cos\alpha+q]\}\\ &\equiv q^2 R\ (\text{say}),\end{aligned}$$

where $(r,z)\in\partial H_\epsilon$ and $(r+q\sin\alpha, z+q\cos\alpha)\in\partial A$. We divide through by $r|_{\partial H_\epsilon}$, substitute $q^2 = u^2/g^2$, and note that $u^2 \leqslant \|u\|^2$. Then we see that these terms satisfy (5·1) because, with $|q| \leqslant M\epsilon^{\frac{1}{3}}$, $\forall u\in\mathscr{B}$,

$$\max_{0\leqslant t\leqslant\pi}\left|\frac{R}{g^2 r}\right| = \left|\frac{R}{g^2 r}\right|_{t=c\epsilon^{\frac{1}{3}}} \leqslant \frac{\epsilon^{-\frac{1}{3}}}{2}, \quad \text{for} \quad \epsilon\leqslant\epsilon_3\leqslant M^{-\frac{3}{2}}.$$

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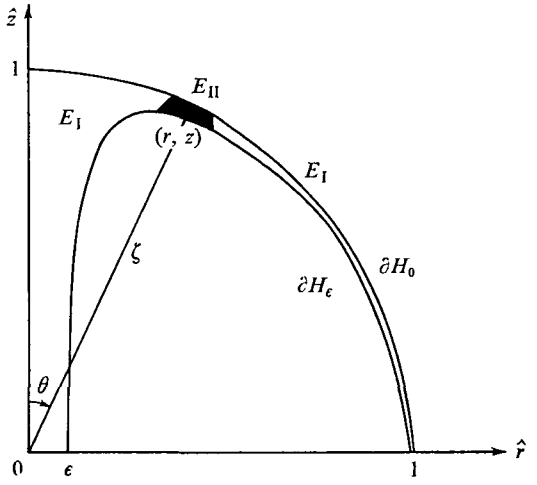


Fig. 6. The domain $H_0 - H_\epsilon$ enclosed by ∂H_0 and ∂H_ϵ is split into the domain E_I, where an inequality can be derived for $\{G|_{\partial A} - G|_{\partial H_\epsilon} - q\,(\partial G/\partial \nu)|_{\partial H_\epsilon}\}$, and the small complement E_{II}, of area $O(q^2)$, by the curve $\sigma = 2|q|$.

To prove the result (5·2), we perform a similar calculation on

$$\psi_H|_{\partial A_1} - \psi_H|_{\partial A_2} - (q_1 - q_2)\frac{\partial \psi_H}{\partial \nu}\bigg|_{\partial H_\epsilon},$$

where $$(r + q_i \sin\alpha,\ z + q_i \cos\alpha) \in \partial A_i, \quad i = 1, 2.$$

(*b*) We now bound the integral in (5·4). The basic idea is to split the domain of integration $H_0 - H_\epsilon$ into two sets E_I and E_{II} (Fig. 6) such that, for a given $t \in [0, \frac{1}{2}\pi]$, the integrand can be bounded by an expression $O(q^2)$ in E_I, while the contribution of E_{II} is small, either because its area is small (for $t \geqslant \epsilon^{\frac{2}{3}}$) or because each term of the integrand is small there (for $t < \epsilon^{\frac{2}{3}}$). Thus we define

$$E_I = \{(\hat{r}, \hat{z}) | (\hat{r}, \hat{z}) \in H_0 - H_\epsilon, \sigma \geqslant 2q\}, \quad E_{II} = H_0 - H_\epsilon - E_I, \tag{5·7}$$

where $\sigma^2 = (r - \hat{r})^2 + (z - \hat{z})^2$ and $(r, z) \in \partial H_\epsilon$. We note that if $(\hat{r}, \hat{z}) \in E_I$ then condition (B 8) of Appendix B is satisfied, and hence equations (B 9*a*, *b*) imply

$$\left| G|_{\partial A} - G|_{\partial H_\epsilon} - q\frac{\partial G}{\partial \nu}\bigg|_{\partial H_\epsilon} \right| \leqslant Cq^2 \frac{r^2 \hat{r}^3}{\sigma^3}\left\{\frac{1}{\sigma^2} + \frac{1}{r^2}\right\}, \tag{5·8a}$$

and

$$\left| G|_{\partial A} - G|_{\partial H_\epsilon} - q\frac{\partial G}{\partial \nu}\bigg|_{\partial H_\epsilon} \right| \leqslant Cq^2 . r^{\frac{1}{2}} \hat{r}^{\frac{3}{2}}\left\{\frac{4}{\sigma^2} + \frac{1}{r^2}\ln\left(\frac{c\sqrt{(r\hat{r})}}{\sigma}\right)\right\}, \tag{5·8b}$$

where, on the right-hand side, $(r, z) \in \partial H_\epsilon$ and $(\hat{r}, \hat{z}) \in E_I$. To apply (5·8) we divide E_I further into E_{Ia} where $\sigma^2 \geqslant r\hat{r}$, and E_{Ib} where $\sigma^2 \leqslant r\hat{r}$. For definiteness we take $\epsilon^{\frac{2}{3}} \leqslant t \leqslant \frac{1}{2}\pi$, which actually represents the most critical case. Then the lines $\hat{\theta} = \frac{3}{2}\theta$, $\hat{\theta} = \frac{1}{2}\theta$ divide E_I appropriately, since, if $\hat{\theta} \geqslant \frac{3}{2}\theta$, or $\hat{\theta} \leqslant \frac{1}{2}\theta$, we have

$$\frac{\sigma^2}{4r\hat{r}} = \frac{(\rho - \hat{\rho})^2 + 4\rho\hat{\rho}\sin^2(|\theta - \hat{\theta}|/2)}{4\rho\hat{\rho}\sin\theta . \sin\hat{\theta}} \geqslant c\frac{|\theta - \hat{\theta}|^2}{\theta\hat{\theta}} = c\{\sqrt{(\theta/\hat{\theta})} - \sqrt{(\hat{\theta}/\theta)}\}^2 \geqslant c > 0. \tag{5·9}$$

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We split the integral in (5·4) as follows:

$$\left|\iint_{H_0-H_\epsilon}\left\{G|_{\partial A}-G|_{\partial H_\epsilon}-q\frac{\partial G}{\partial \nu}\Big|_{\partial H_\epsilon}\right\}\right| \leqslant \iint_{E_{\mathrm{I}}}\left|G|_{\partial A}-G|_{\partial H_\epsilon}-q\frac{\partial G}{\partial \nu}\Big|_{\partial H_\epsilon}\right| + \iint_{E_{\mathrm{II}}}\left\{G|_{\partial A}+G|_{\partial H_\epsilon}+\left|q\frac{\partial G}{\partial \nu}\right|\Big|_{\partial H_\epsilon}\right\}. \quad (5\cdot10)$$

We now bound the most critical parts of the right-hand side. First, we include the domains of integration $E_{\mathrm{I}a}$, $E_{\mathrm{I}b}$, E_{II} in more manageable subdomains in the following manner:

$$\iint_{E_{\mathrm{I}a}}\left|\left|d\hat{r}\,d\hat{z}\right| \leqslant \int_0^{3\epsilon} d\hat{r}\int_{-\infty}^{\infty}\left|\left|d\hat{z}+\int_{3\epsilon}^{\frac{1}{2}\theta} d\hat{\theta}\int_{1-\hat{\eta}}^{1}\right|\right|\hat{\rho}\,d\hat{\rho}+\int_{\frac{3}{2}\theta}^{\pi-3\epsilon} d\hat{\theta}\int_{1-\hat{\eta}}^{1}\right|\left|\hat{\rho}\,d\hat{\rho}, \quad (5\cdot11)$$

where

$$\eta = \epsilon^2/2\sin^2\theta,$$

$$\iint_{E_{\mathrm{I}b}}\left|\left|d\hat{r}\,d\hat{z}\right| \leqslant \int_{\frac{1}{2}\theta}^{\theta-\sqrt{(2)}\,|q|} d\hat{\theta}\int_{1-\hat{\eta}}^{1}\left|\left|\hat{\rho}\,d\hat{\rho}+\int_{\theta+\sqrt{(2)}\,|q|}^{\frac{3}{2}\theta} d\hat{\theta}\int_{1-\hat{\eta}}^{1}\right|\right|\hat{\rho}\,d\hat{\rho} + \int_{\theta-\sqrt{(2)}\,|q|}^{\theta+\sqrt{(2)}\,|q|} d\hat{\theta}\int_{1-\hat{\eta}}^{1}\left|\left|\hat{\rho}\,d\hat{\rho}, \quad (5\cdot12)\right.\right.$$

$$\iint_{E_{\mathrm{II}}}\left|\left|d\hat{r}\,d\hat{z}\right| \leqslant \int_0^{2\pi} dv\int_0^{2|q|}\left|\left|\sigma\,d\sigma, \quad \text{with} \quad v=\tan^{-1}\frac{z-\hat{z}}{r-\hat{r}}. \right.\right. \quad (5\cdot13)$$

We are now in a position to make some detailed calculations. Using (5·8a) we obtain for the middle term of (5·11), noting $\sigma^2 \geqslant 4\rho\hat{\rho}\sin^2(\frac{1}{2}|\hat{\theta}-\theta|)$,

$$\begin{aligned}\int_{3\epsilon}^{\frac{1}{2}\theta} d\hat{\theta}\int_{1-\hat{\eta}}^{1}\left|G|_{\partial A}-G|_{\partial H_\epsilon}-q\frac{\partial G}{\partial \nu}\Big|_{\partial H_\epsilon}\right|\hat{\rho}\,d\hat{\rho} &\leqslant Cq^2\int_{3\epsilon}^{\frac{1}{2}\theta} d\hat{\theta}\int_{1-\hat{\eta}}^{1}\frac{r^2\hat{r}^3}{\sigma^5}\hat{\rho}\,d\hat{\rho},\\ &\leqslant Cq^2\int_{3\epsilon}^{\frac{1}{2}\theta} d\hat{\theta}\frac{\hat{\theta}^3}{\theta^3}\int_{1-\hat{\eta}}^{1} d\hat{\rho}, \quad \text{since} \quad \sigma \geqslant c\theta,\\ &\leqslant Cq^2\frac{\epsilon^2}{\theta},\\ &\leqslant C\|u\|^2\frac{\epsilon^2}{g^2\theta},\\ &\leqslant C\|u\|^2\epsilon^{-\frac{2}{3}}r\Big|_{\partial H_\epsilon}, \end{aligned} \quad (5\cdot14)$$

where we use $u = gq$.

For the first term of (5·12) we note that

$$\begin{aligned}\int_{1-\hat{\eta}}^{1}\frac{\hat{\rho}\,d\hat{\rho}}{\sigma^2} &\leqslant \int_{1-\hat{\eta}}^{1}\frac{d\hat{\rho}}{(\hat{\rho}-\rho)^2+c\sin^2\frac{1}{2}|\hat{\theta}-\theta|},\\ &\leqslant \frac{c}{|\hat{\theta}-\theta|}\left[\tan^{-1}\frac{\hat{\rho}-\rho}{c|\hat{\theta}-\theta|}\right]_{1-\hat{\eta}}^{1},\\ &\leqslant \frac{c}{|\hat{\theta}-\theta|}, \end{aligned}$$

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and, using (5·8b), we obtain

$$\begin{aligned}
&\int_{\frac{1}{2}\theta}^{\theta-\sqrt{(2)}\,|q|} d\hat{\theta} \int_{1-\hat{\gamma}}^{1} \left| G|_{\partial A} - G|_{\partial H_\epsilon} - q \frac{\partial G}{\partial \nu}\Big|_{\partial H_\epsilon} \right| \hat{\rho}\, d\hat{\rho} \\
&\quad \leqslant Cq^2 \int_{\frac{1}{2}\theta}^{\theta-\sqrt{(2)}\,|q|} d\hat{\theta} \int_{1-\hat{\gamma}}^{1} r^{\frac{1}{2}} \hat{r}^{\frac{3}{2}} \left\{ \frac{1}{\sigma^2} + \frac{\ln\left(c\sqrt{(r\hat{r})}/\sigma\right)}{r^2} \right\} \hat{\rho}\, d\hat{\rho}, \\
&\quad \leqslant Cq^2 \int_{\frac{1}{2}\theta}^{\theta-\sqrt{(2)}\,|q|} \left\{ \frac{1}{|\hat{\theta}-\theta|} + \frac{\epsilon^2}{\theta^2} \frac{\ln|\hat{\theta}-\theta|^{-1}}{\hat{\theta}^2} \right\} d\hat{\theta}, \\
&\quad \leqslant Cq^2 \ln\left(|q|^{-1}\right) \{1 + \epsilon^2 \theta^{-3}\}, \\
&\quad \leqslant C \|u\|^2 \ln\left(\|u\|^{-1}\right) \epsilon^{-\frac{2}{3}} r|_{\partial H_\epsilon}. \qquad (5{\cdot}15)
\end{aligned}$$

For (5·13) we use (4·2b) for G, its extension from Appendix B for $\partial G/\partial \nu$, and note that

$$\int_0^{2\pi} d\nu \int_0^{2|q|} \left\{ \ln\frac{1}{\sigma}, \frac{1}{\sigma} \right\} r\, d\sigma \leqslant C\{q^2 \ln|q|^{-1}, |q|\}, \qquad (5{\cdot}15a)$$

to obtain

$$\begin{aligned}
\int_0^{2\pi} d\nu \int_0^{2|q|} \left\{ G|_{\partial A} + G|_{\partial H_\epsilon} + \left| q \frac{\partial G}{\partial \nu}\Big|_{\partial H_\epsilon} \right| \right\} \sigma\, d\sigma &\leqslant \int_0^{2\pi} d\nu \int_0^{2|q|} C \left\{ r^{\frac{1}{2}} \hat{r}^{\frac{3}{2}} \sinh^{-1} \frac{\sqrt{(2r\hat{r})}}{\sigma} \right. \\
&\quad \left. + |q| \left(\frac{\hat{r}^{\frac{3}{2}}}{r^{\frac{1}{2}}} \sinh^{-1} \frac{\sqrt{(2r\hat{r})}}{\sigma} + \frac{r^{\frac{1}{2}} r^{\frac{3}{2}}}{\sigma} \right) \right\} \sigma\, d\sigma \\
&\leqslant Cq^2 \left\{ \theta^2 \ln\left(q|^{-1} + \frac{|q|}{\theta} \ln|q|^{-1} + \theta^2 \right) \right\}, \\
&\leqslant C\|u\|^2 \ln\left(\|u\|^{-1}\right) \epsilon^{-\frac{2}{3}} r|_{\partial H_\epsilon}. \qquad (5{\cdot}16)
\end{aligned}$$

The range $0 \leqslant t \leqslant \epsilon^{\frac{2}{3}}$ leads similarly, and more simply to results like (5·14–5·16).

We need only group these results (5·14), (5·15), (5·16) and their analogues to show that, for $0 \leqslant t \leqslant \pi$,

$$\begin{aligned}
\frac{5}{\pi r}\Big|_{\partial H_\epsilon} \iint_{H_0 - H_\epsilon} \left\{ G|_{\partial A} - G|_{\partial H_\epsilon} - q \frac{\partial G}{\partial \nu}\Big|_{\partial H_\epsilon} \right\} d\hat{r}\, d\hat{z} &\leqslant C\|u\| \{ \|u\| \epsilon^{-\frac{2}{3}} \ln(1/\epsilon) + \epsilon^{-\frac{2}{3}} \|u\| \ln\left(\|u\|^{-1}\right) \}, \\
&\leqslant C\|u\| \{ \|u\| \epsilon^{-\frac{2}{3}} \ln(1/\epsilon) + \epsilon \}, \qquad (5{\cdot}17)
\end{aligned}$$

which is the desired bound.

To prove (5·2) we repeat the above calculation for

$$\frac{5}{\pi r}\Big|_{\partial H_\epsilon} \iint_{H_0 - H_\epsilon} \left\{ G|_{\partial A_1} - G|_{\partial A_2} - (q_1 - q_2) \frac{\partial G}{\partial \nu}\Big|_{\partial H_\epsilon} \right\} d\hat{r}\, d\hat{z}.$$

We replace the $|q|^2$ in the above work by $\{|q_1| + |q_2|\}\, |q_1 - q_2|$ in E_{I}, and E_{II} has radius $\{|q_1| + |q_2|\}$, not $2|q|$. The only essential difference in achieving the desired analogue of (5·17) lies in the analogue of (5·16). We perform this bounding only in a circle of radius $|q_1 - q_2|$ about $(\frac{1}{2}(r_1 + r_2), \frac{1}{2}(z_1 + z_2))$, where $(r_i, z_i) \in \partial A_i\, (i = 1, 2)$, while in the rest of E_{II} we obtain a bound for $G|_{\partial A_1} - G|_{\partial A_2}$ from appendix (B 8), (B 9).

(c) Finally, we establish the bounds (5·1, 2) for the terms of (5·6). The line integral in (5·6) may be written

$$r|_{\partial H_\epsilon} \int_0^{\pi} K_\epsilon(t, \hat{t}) \frac{\hat{\kappa}}{2\hat{g}} \hat{u}^2\, d\hat{t},$$

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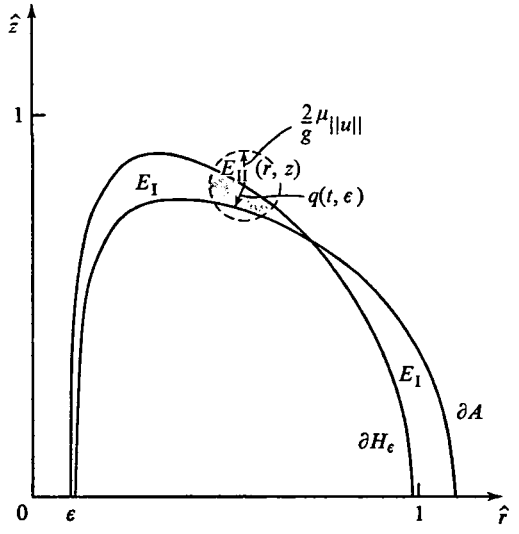


Fig. 7. The domain $A - H_\epsilon$ is divided into E_I, where an inequality holds for $\{G|_{\partial A} - \Gamma_\epsilon(t, \hat{t})\}$, and E_{II} with small area of $O(\|u\|^2)$.

and bounded by estimates like (4·20), (4·21). The behaviour of κ is determined by the methods of Appendix A and is shown in Fig. 3.

The bounding of the double integral of (5·6), namely

$$\oint_{\partial H_\epsilon} d\hat{\tau} \int_0^{\hat{q}} |G|_{\partial A} - \Gamma_\epsilon(t, \hat{t})| (1 + \hat{\kappa}\hat{v})\, d\hat{v}, \tag{5·18}$$

follows the method of the previous section. We note that $|\hat{\kappa}\hat{v}| \leqslant c\epsilon^{\frac{2}{3}}$. Again we seek a domain E_I where we can bound the integrand. Thus we define (Fig. 7)

$$E_I = \left\{(\hat{r}, \hat{z})|\ (\hat{r}, \hat{z}) \in A - H_\epsilon \quad \text{and} \quad \sigma \geqslant \frac{4}{g}\|u\|\right\}, \quad E_{II} = A - H_\epsilon - E_I.$$

We prove in Appendix B that

$$|G|_{\partial A} - \Gamma_\epsilon(t, \hat{t})| \leqslant 3\frac{r^2\hat{r}^3}{\sigma^3}\{|q|\,[\sigma^{-1} + r^{-1}] + |\hat{v}|\,[\sigma^{-1} + \hat{r}^{-1}]\}, \tag{5·19a}$$

and

$$|G|_{\partial A} - \Gamma_\epsilon(t, \hat{t})| \leqslant r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}\left\{|q|\left[4\sigma^{-1} + \frac{1 + \sinh^{-1}(\sqrt{(2r\hat{r})}/\sigma)}{r}\right] + |\hat{v}|\left[4\sigma^{-1} + \frac{1 + \sinh^{-1}(\sqrt{(2r\hat{r})}/\sigma)}{\hat{r}}\right]\right\}, \tag{5·19b}$$

where, on the right-hand sides (r, z), $(\hat{r}, \hat{z}) \in \partial H_\epsilon$ and $(\hat{r}, \hat{z}) \in E_I$ as well.

We take, for definiteness, $3\epsilon \leqslant t \leqslant \frac{1}{2}\pi$.

Again we split E_I into E_{Ia} and E_{Ib} by the lines $\hat{\theta} = \frac{1}{2}\theta$, $\frac{3}{2}\theta$ so that we can apply (5·19a) in E_{Ia}, where $\sigma^2 \geqslant r\hat{r}$, and (5·19$b$) in E_{Ib}, where $\sigma^2 \leqslant r\hat{r}$.

Then we bound the integral (5·18) with a series of calculations like the following.

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Using (5·19a) in the part of E_{Ia} where $\hat{\theta} \geqslant \frac{3}{2}\theta$, we have, since $\sigma \geqslant cr$,

$$\begin{aligned}\int_{\hat{\theta}=\frac{3}{2}\theta}^{\hat{t}=\pi} d\hat{\tau} \int_0^{\hat{q}} \{G|_{\partial A} - \Gamma_\epsilon(t,\hat{t})\}\, d\hat{v} &\leqslant \int_{\hat{\theta}=\frac{3}{2}\theta}^{\hat{t}=\pi} d\hat{\tau} \int_0^{|\hat{q}|} C \frac{r^2\hat{r}^3}{\sigma^3}\left\{\frac{|q|}{r}+\frac{|\hat{v}|}{\hat{r}}\right\} d\hat{v},\\ &\leqslant Cr|_{\partial H_\epsilon}.\|u\|^2 \int_{\hat{\theta}=\frac{3}{2}\theta}^{\hat{t}=\pi}\left\{\frac{\hat{r}^3}{g\hat{g}\sigma^3}+\frac{r\hat{r}^2}{\hat{g}^2\sigma^3}\right\} d\hat{\tau},\\ &\leqslant C\|u\|^2.r|_{\partial H_\epsilon}\{\theta \ln(|\theta|^{-1})+\theta\},\\ &\leqslant C\|u\|^2.r|_{\partial H_\epsilon}.\epsilon^{+\frac{2}{3}} \ln(1/\epsilon). \qquad (5\cdot20)\end{aligned}$$

We use (5·19b) in the part of E_{Ib} where $\theta < \hat{\theta} \leqslant \frac{3}{2}\theta$:

$$\begin{aligned}&\int_{\hat{\theta}=\theta+(4/g)\|u\|}^{\hat{\theta}=\frac{3}{2}\theta} d\hat{\tau} \int_0^{\hat{q}} \{G|_{\partial A} - \Gamma_\epsilon(t,\hat{t})\}\, d\hat{v}\\ &\quad\leqslant \int_{\hat{\theta}=\theta+(4/g)\|u\|}^{\hat{\theta}=\frac{3}{2}\theta} d\hat{\tau} \int_0^{|\hat{q}|} Cr^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}\left\{\frac{|q|+|\hat{v}|}{\sigma}+\left(\frac{|q|}{r}+\frac{|\hat{v}|}{\hat{r}}\right)\sinh^{-1}\frac{\sqrt{(2r\hat{r})}}{\sigma}\right\} d\hat{v},\\ &\quad\leqslant \int_{\hat{\theta}=\theta+(4/g)\|u\|}^{\hat{\theta}=\frac{3}{2}\theta} Cr^2\frac{\|u\|^2}{g^2}\left\{\frac{1}{\sigma}+\frac{1}{r}\ln\left(\frac{\sqrt{(2r\hat{r})}}{\sigma}\right)\right\} d\hat{\tau},\\ &\quad\leqslant C\|u\|^2 r|_{\partial H_\epsilon}\int_{\hat{t}=t+(4/g)\|u\|}^{\hat{t}=\frac{3}{2}t} \frac{r}{g^2}\left\{\frac{1}{|\hat{t}-t|}+\frac{1}{r}\ln\left(\frac{ct}{|\hat{t}-t|}\right)\right\} d\hat{\tau},\\ &\quad\leqslant C\|u\|\, r|_{\partial H_\epsilon}\{\|u\|\,\epsilon^{-\frac{2}{3}}\ln(1/\epsilon)+\epsilon^{-\frac{2}{3}}\|u\|\ln(\|u\|^{-1})\},\\ &\quad\leqslant C\|u\|\,\{\|u\|\,\epsilon^{-\frac{2}{3}}\ln(1/\epsilon)+\epsilon\} r|_{\partial H_\epsilon}. \qquad (5\cdot21)\end{aligned}$$

Finally, we use (4·2b) in E_{II}:

$$\begin{aligned}\int_{E_{II}} \{G|_{\partial A}+\Gamma_\epsilon(t,\hat{t})\}\, d\hat{r}\, d\hat{z} &\leqslant \iint_{\sigma\leqslant(4/g)\|u\|} Cr^2 \ln\left(\frac{cr}{\sigma}\right) d\hat{r}\, d\hat{z},\\ &\leqslant Cr|_{\partial H_\epsilon}\frac{r}{g^2}\|u\|^2\ln(\|u\|^{-1}), \quad \text{using} \quad (5\cdot16a),\\ &\leqslant C\|u\|r|_{\partial H_\epsilon}\{\|u\|\,\epsilon^{-\frac{2}{3}}\ln(1/\epsilon)+\epsilon\}. \qquad (5\cdot22)\end{aligned}$$

When $0 \leqslant t \leqslant 3\epsilon$ we replace the lines $\hat{\theta} = \frac{1}{2}\theta, \frac{3}{2}\theta$ by lines $\hat{r} = r \pm \epsilon, \hat{z} = z \pm \epsilon$ to divide E_I appropriately into E_{Ia}, where $\sigma^2 \geqslant r\hat{r}$, and E_{Ib}, where $\sigma^2 \leqslant r\hat{r}$.

We need only group the results (5·20, 21, 22) and their analogues to establish (5·1). To show (5·2) we repeat the above calculations on

$$\oint d\hat{\tau} \int_0^{\hat{q}_1} \{G|_{\partial A_1} - G|_{\partial A_2}\}\, d\hat{v} + \oint d\hat{\tau} \int_{\hat{q}_2}^{\hat{q}_1} \{G|_{\partial A_2} - \Gamma_\epsilon(t,\hat{t})\}\, d\hat{v}.$$

The second integral is analogous to (5·18) on replacing $\hat{q}$ by $\hat{q}_1 - \hat{q}_2$; the first integral is bounded as before with the role of q taken by $q_1 - q_2$ (so that E_I is defined for $\sigma \geqslant (4/g)\|u_1 - u_2\|$, and so on). Hence we show (5·2).

6. *Proof of the main results.* Before proving the main results we consider some properties of the forcing function $f(t,\epsilon) \equiv f_\epsilon$, defined by (2·20).

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LEMMA 6·1. *The function f_ϵ is in $\mathscr{U}$, and satisfies the bounds*

$$-C\epsilon^2 \leqslant f(t,\epsilon) \leqslant -\epsilon^2\{\epsilon + h(t,\epsilon)\sin t\} \quad (C > 0). \tag{6·1}$$

Proof. Using (4·6) in (2·20) we write

$$f(t,\epsilon) = -\frac{5}{\pi}\iint_{H_0-H_\epsilon} \frac{1}{r} G|_{\partial H_\epsilon}\, d\hat{r}\, d\hat{z} - \frac{\epsilon^4}{r}\bigg|_{\partial H_\epsilon} \quad (0 \leqslant t \leqslant \pi). \tag{6·2}$$

The symmetry $f(t,\epsilon) = f(\pi - t, \epsilon)$ follows from arguments like (4·24).

The inequalities (4·2a, b) for G, with standard potential-theory ideas, show that $f(t,\epsilon) \in C[0,\pi]$. Hence $f_\epsilon \in U$.

We prove in Appendix C that

$$\epsilon^2 r|_{\partial H_\epsilon} \leqslant \iint_{H_0-H_\epsilon} \frac{1}{r} G|_{\partial H_\epsilon}\, d\hat{r}\, d\hat{z} \leqslant C\epsilon^2,$$

and thus (6·1) follows.

We now prove the main result by the contraction mapping principle applied to the closed ball $\mathscr{B}(M\epsilon^2)$, defined in Theorem 5·1. M is specified in the proof.

THEOREM 6·1. *For $0 \leqslant \epsilon \leqslant \epsilon_*$ and ϵ_* sufficiently small there exists a solution*

$$u \in \mathscr{B}(M\epsilon^2),$$

which is unique in $\mathscr{B}$, of the non-linear operator equation

$$u - \mathscr{K}_\epsilon u = f_\epsilon + \mathscr{N}_\epsilon u, \tag{6·3}$$

in the space $\mathscr{U}$. Here M is independent of ϵ.

Proof. We write (6·3) in the form

$$u = (\mathscr{I} - \mathscr{K}_\epsilon)^{-1}\{f_\epsilon + \mathscr{N}_\epsilon u\} \equiv \mathscr{A}_\epsilon u,$$

and this defines the non-linear operator $\mathscr{A}_\epsilon$ for $\epsilon \leqslant \epsilon_2$ (Theorem 4·1). To show that $\mathscr{A}_\epsilon$ is a contraction mapping of the ball $\mathscr{B}(M\epsilon^2)$, we must prove that

$$\mathscr{A}_\epsilon u \in \mathscr{B}(M\epsilon^2) \quad \forall u \in \mathscr{B}(M\epsilon^2), \tag{6·4}$$

and

$$\|\mathscr{A}_\epsilon u_1 - \mathscr{A}_\epsilon u_2\| \leqslant \mu\|u_1 - u_2\| \quad (\mu < 1), \quad \forall\, u_1, u_2 \in \mathscr{B}(M\epsilon^2). \tag{6·5}$$

For $u \in \mathscr{B}$, we have $f_\epsilon, \mathscr{N}_\epsilon u \in \mathscr{U}$, and hence $\mathscr{A}_\epsilon u \in \mathscr{U}$, using Lemma 6·1, and Theorems 4·1 and 5·1. Then, for $u, u_1, u_2 \in \mathscr{B}(M\epsilon^2)$, we have

$$\begin{aligned} \|\mathscr{A}_\epsilon u\| &\leqslant \|(\mathscr{I} - \mathscr{K}_\epsilon)^{-1}\|\,\{\|f_\epsilon\| + \|\mathscr{N}_\epsilon u\|\} \\ &\leqslant c\{C_1\epsilon^2 + c_2\|u\|(\epsilon^{-\frac{4}{3}}\|u\| + \epsilon)\} \\ &\leqslant c\epsilon^2\{C_1 + C_2 M(M\epsilon^{\frac{2}{3}} + \epsilon)\}, \end{aligned} \tag{6·6}$$

and

$$\begin{aligned} \|\mathscr{A}_\epsilon u_1 - \mathscr{A}_\epsilon u_2\| &\leqslant \|(\mathscr{I} - \mathscr{K}_\epsilon)^{-1}\|\,.\,\|\mathscr{N}_\epsilon u_1 - \mathscr{N}_\epsilon u_2\| \\ &\leqslant c\{\epsilon^{-\frac{4}{3}}\|u_1\| + \epsilon^{-\frac{4}{3}}\|u_2\| + \epsilon\}\|u_1 - u_2\| \\ &\leqslant c\{M\epsilon^{\frac{2}{3}} + \epsilon\}\|u_1 - u_2\|, \end{aligned} \tag{6·7}$$

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where $0 \leqslant \epsilon \leqslant \epsilon_3$, for $\epsilon_3 \leqslant M$ in $\{\epsilon_1, \epsilon_2\}$. We now choose $M = 2cC_1$, and then choose ϵ_* so small that

$$cC_1\{1 + 2cC_2(2cC_1\epsilon_*^{\frac{2}{3}} + \epsilon_*)\} \leqslant 2cC_1,$$

and

$$c\{2cC_1\epsilon_*^{\frac{2}{3}} + \epsilon_*\} \leqslant \mu < 1.$$

Then (6·4) and (6·5) hold, $\mathscr{A}_\epsilon$ is a contraction mapping of the ball $\mathscr{B}(M\epsilon^2)$ into itself, and hence there is a unique fixed point $u \in \mathscr{B}(M\epsilon^2)$ such that $u = \mathscr{A}_\epsilon u$. Thus the theorem follows.

We now show that the solution u of (6·3) leads to a solution $\{\psi, \partial A\}$ of the original problem (2·3).

THEOREM 6·2. *Let u, ϵ_* be as in Theorem* 6·1. *For $0 < \epsilon \leqslant \epsilon_*$ the core boundary ∂A, described by the normal distance $q = u/g$ from the curve ∂H_ϵ, is a continuous curve. The streamfunction ψ, given by* (2·4), *is analytic inside and outside ∂A; and $\{\psi, \partial A\}$ is a solution of the original problem* (2·3). *Moreover, ∂A is the only such curve within a distance of $M\epsilon^2/g$ of ∂H_ϵ.*

Proof. We note a basic result of potential theory for integrals of the type (2·4) from Courant and Hilbert ((10), §4·1, 2 and §4·7).

The continuous curve ∂A, given by the solution u of (6·3) and formulae (2·14) and (2·15), defines the bounded, open, connected set A. Then $\psi(r, z)$, defined by (2·4), is continuously differentiable everywhere ($r \geqslant 0$, $-\infty < z < +\infty$), and satisfies equation (2·3a).

We observe that equation (2·3c) is merely (6·3) rewritten in the original variables and rearranged, while (2·3d) follows from the properties of G. Hence $\{\psi, \partial A\}$ is a solution of (2·3). The streamfunction is analytic inside and outside ∂A because the vorticity distribution is analytic inside ∂A (see Courant and Hilbert (10)). The uniqueness of the solution of (6·3) in $\mathscr{B}(M\epsilon^2)$ leads to the mild uniqueness result for ∂A.

Consideration of $q(t, \epsilon) = [u(t, \epsilon)]/[g(t, \epsilon)]$ shows that the curve ∂H_ϵ approximates the boundary ∂A within an error of at most $O(\epsilon^{\frac{4}{3}})$. We can improve our approximate description of ∂A by finding the first term $u_1(t, \epsilon) = O(\epsilon^2)$ in an expansion of $u(t, \epsilon)$, asymptotic as $\epsilon \to 0$. Thus we solve the linear equation

$$u_1 - \mathscr{K}_0 u_1 = f_\epsilon \tag{6·8}$$

for u_1, (because of Theorems 4·1, 5·1 and 6·1). This is not straightforward because of the difficulty of expanding f_ϵ asymptotically. We finally obtain, see (11), for

$$\epsilon^{\frac{2}{3}} \prec t_1 \prec \epsilon^{\frac{2}{3}} \prec t_2 \prec 1,$$

$$u_1(t, \epsilon) = \begin{cases} -\dfrac{5\epsilon^2}{2} \displaystyle\sum_{n=0}^{\infty} \frac{(4n+3)P^1_{2n+1}(\cos t)}{(2n-1)(2n+1)(2n+2)}, & \text{when} \quad t_2 \leqslant t \leqslant \tfrac{1}{2}\pi, \\ -5\epsilon^2 \displaystyle\int_0^\infty d\hat{r} \int_0^{1/2\hat{r}^2} d\hat{z}\, \hat{r}^2 \int_0^\infty \exp\left[-k\left|\frac{\epsilon^{\frac{4}{3}}}{2t^2} - \hat{z}\right|\right] J_1(kt/\epsilon^{\frac{2}{3}})\, J_1(k\hat{r})\, dk, & (t \succsim \epsilon^{\frac{2}{3}}), \\ 0 \quad \text{when} \quad 0 \leqslant t \leqslant t_1. \end{cases} \tag{6·9}$$

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Thus we have the first term in an asymptotic expansion for $q(t,\epsilon)$ away from the axis $r=0$:

$$q(t,\epsilon) = -\frac{5\epsilon^2}{4}\sum_{n=0}^{\infty}\frac{(4n+3)\,(dP_{2n+1}(\mu)/d\mu)}{(2n-1)\,(2n+1)\,(2n+2)}\{1+o(1)\} \quad (t_2 \leqslant t \leqslant \tfrac{1}{2}\pi). \tag{6·10}$$

Where $r = O(\epsilon^{\frac{2}{3}})$ we need numerical methods to find approximate descriptions of the curve ∂A; but for $r \leqslant \epsilon^{\frac{2}{3}}$ we know that $q = O(r^5/\epsilon^2)$.

Finally we note that the representation that was assumed in section 2 is in fact consistent with the results. For (*a*) q is a continuous function of t on $[0,\pi]$, and (*b*) we have $|q| \leqslant C\epsilon^{\frac{4}{3}}$ and $R(\tau,\epsilon) > \frac{1}{5}\epsilon^{\frac{2}{3}}$ (where R denotes the radius of curvature of ∂H_ϵ), so that the condition $-q(\tau,\epsilon) < R(\tau,\epsilon)$ is indeed satisfied for ϵ sufficiently small.

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APPENDIX A

The interior streamline ∂H_ϵ of Hill's vortex

We express the interior streamline ∂H_ϵ of Hill's vortex, defined by (2·12)

$$\psi_H = r^2(1-r^2-z^2) = \rho^2\sin^2\theta(1-\rho^2) = \epsilon^2(1-\epsilon^2),$$

((ρ,θ) being the usual spherical coordinates), in the form

$$z = \pm\left\{1-r^2-\frac{\epsilon^2(1-\epsilon^2)}{r^2}\right\}^{\frac{1}{2}}; \tag{A 1}$$

and thus calculate properties of the curve, like the arc length and curvature, as functions of r from elementary formulae. It is convenient to use the (ρ,θ) form of (A 1) in some calculations, e.g. the curvature in terms of θ is given by

$$\kappa(\theta,\epsilon) = \rho\sin\theta\{(1-\rho^2)\,(1-2\rho^2)-\sin^2\theta\,(3-8\rho^2+6\rho^4)\} \times\{(1-\rho^2)^2-\rho^2\sin^2\theta\,(2-3\rho^2)\}^{-\frac{3}{2}}\big|_{\partial H_\epsilon},$$

However, we need these properties of ∂H_ϵ evaluated as functions of t. Equation (2·13) enables us to do this provided we know $s = h(t,\epsilon)$, the equation of ∂H_ϵ in (s,t) coordinates. We use the method of matched asymptotic expansions, p. 228 of (**12**), to write, for $0 \leqslant t \leqslant \frac{1}{2}\pi$ and $0 < \epsilon \leqslant \frac{1}{2}$,

$$\left.\begin{aligned} s = h(t,\epsilon) = \gamma\left(\frac{t}{\epsilon}\right)+\epsilon\{t-\sin t\}+\epsilon^2\left\{\gamma_2\left(\frac{t}{\epsilon}\right)+\frac{1}{2t^2}-\frac{1}{2\sin^2 t}+\frac{2}{3}-\frac{\cos^2 t}{2}\right\}+O(\epsilon^3), \\ \text{where } \gamma(t/\epsilon) \text{ is determined by} \\ \gamma(\gamma^2-1)\left(\frac{t}{\epsilon}\right)^2+2(\gamma^2-1)\left(\frac{t}{\epsilon}\right)+\gamma = 0, \\ \text{and } \gamma_2(t/\epsilon) \text{ by} \\ \gamma_2\left(\frac{t}{\epsilon}\right) = -\tfrac{1}{2}\gamma\left\{4-\frac{1}{3}\left(\frac{t}{\epsilon}\right)^2+\frac{16}{3}\left(\frac{t}{\epsilon}\right)\gamma+\frac{7}{3}\left(\frac{t}{\epsilon}\right)^2\gamma^2\right\}\left\{1+\gamma^2+\left(\frac{t}{\epsilon}\right)\gamma^3\right\}^{-1}. \end{aligned}\right\} \tag{A 2}$$

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The error bound $O(\epsilon^3)$ is shown by direct substitution of (A 2) in (2·12) with (2·13). We can now evaluate approximately the arc length $\tau(t,\epsilon)$, and $d\tau/dt = \{h^2+(dh/dt)^2\}^{\frac{1}{2}}$, (used in the linear operator $\mathscr{K}_\epsilon$), the curvature $\kappa(t,\epsilon)$ (Fig. 3), and other quantities used in the body of the work. We note that from (2·13) we have, away from the axis,

$$\theta = t+O(\epsilon) \quad \text{and} \quad \frac{d\theta}{dt} = 1+O(\epsilon) \quad \text{for} \quad 2\epsilon \leqslant t \leqslant \pi-2\epsilon \tag{A 3}$$

holding on the curve ∂H_ϵ.

The only result that we shall show is the following:

$$\left|\frac{d\hat{r}/d\hat{t}}{\{\epsilon+h\sin t\}\hat{g}} - \frac{1}{2\sin t\sin\hat{t}}\right| \leqslant \frac{2\epsilon}{\sin t\sin\hat{t}}\left\{\frac{1}{\sin t}+\frac{1}{\sin\hat{t}}\right\}+\frac{C\epsilon^{\frac{2}{3}}}{\sin t\sin\hat{t}}, \quad \text{when} \quad \epsilon^{\frac{2}{3}} \leqslant t,$$

which is used in determining the behaviour of the integrand $K_\epsilon(t,\hat{t})$. We write the left hand side of (A 4) as follows

$$\frac{d\hat{r}/d\hat{t}}{\hat{g}}\left\{\frac{\sin t-\epsilon-h\sin t}{\sin t\,(\epsilon+h\sin t)}\right\}+\frac{1}{\sin t}\left\{\left(\frac{d\hat{r}/d\hat{t}}{\hat{g}}-\frac{(d\hat{r}/d\hat{\theta})\,r}{|\nabla\psi_H|}\bigg|_{\partial H_\epsilon}\right)+\left(\frac{(d\hat{r}/d\hat{\theta})\,r}{|\nabla\psi_H|}\bigg|_{\partial H_\epsilon}-\frac{1}{2\sin\hat{t}}\right)\right\}$$

and proceed to bound each term.

First, we use (4·7, 8) to write

$$g(\hat{t},\epsilon) = \frac{1}{\hat{r}}\,|\nabla\hat{\psi}_H|_{\partial H_\epsilon}.\{1+O(\epsilon^{\frac{2}{3}})\}.$$

Next, denoting ∂H_ϵ by $\rho = \rho(\theta,\epsilon)$, we have, on differentiating $\psi_H|_{\partial H_\epsilon} = \text{const.}$ with respect to θ, that

$$\frac{\partial\rho}{\partial\theta}\bigg|_{\partial H_\epsilon} = -\frac{\partial\psi_H}{\partial\theta}\bigg/\frac{\partial\psi_H}{\partial\rho}\bigg|_{\partial H_\epsilon},$$

and so

$$\frac{\partial\hat{r}}{d\hat{\theta}} = \left\{\hat{\rho}^2+\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\right)^2\right\}^{\frac{1}{2}} = \hat{\rho}\,|\nabla\hat{\psi}_H|\bigg/\left|\frac{\partial\hat{\psi}_H}{\partial\hat{\rho}}\right| \quad \text{for} \quad (\hat{\rho},\hat{\theta})\in\partial H_\epsilon.$$

Finally, using (A 2), (A 3) with more detail, the above work, and the result

$$\frac{\partial\hat{\psi}_H}{\partial\hat{\rho}}\bigg|_{\partial H_\epsilon} = 2\hat{\rho}\sin^2\hat{\theta}\,(1-2\hat{\rho}^2)|_{\partial H_\epsilon},$$

we obtain the inequality (A 4).

APPENDIX B

Bounds involving the Green's function

We establish here bounds on the Green's function G, defined in (2·5), and its derivatives, and then consider the effect of perturbing the variables of G. Defining variables σ, a by

$$\sigma^2 = (r-\hat{r})^2+(z-\hat{z})^2, \quad a^2 = \frac{\sigma^2}{4r\hat{r}}, \tag{B 1}$$

we write the Green's function in terms of $f(a)$, defined in the following

$$G = r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}\int_0^1 \frac{(1-2v^2)}{\sqrt{(a^2+v^2)}}\,\frac{dv}{\sqrt{(1-v^2)}} \equiv r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}f(a), \quad \text{where} \quad v = \sin\tfrac{1}{2}\hat{\lambda}. \tag{B 2}$$

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Consideration of the sign of the integrand over the range of integration leads to the following (Fig. 10):

$$O \leqslant f(a) \leqslant \int_0^{2^{-\frac{1}{2}}} \frac{dv}{\sqrt{(a^2+v^2)}} = \sinh^{-1}\frac{1}{a\sqrt{2}}, \quad \text{where} \quad 0 \leqslant a < \infty. \tag{B 3}$$

On noting that $0 \geqslant (1+v^2/a^2)^{-\frac{1}{2}} - 1 \geqslant -v^2/2a^2$, and that

$$\int_0^1 \frac{1-2v^2}{\sqrt{(1-v^2)}}\,dv = 0,$$

we have

$$\begin{aligned} f(a) &= \frac{1}{a}\int_0^1 \frac{1-2v^2}{\sqrt{(1-v^2)}}\left(1+\frac{v^2}{a^2}\right)^{-\frac{1}{2}} dv - \frac{1}{a}\int_0^1 \frac{1-2v^2}{\sqrt{(1-v^2)}}\,dv, \\ &\leqslant \frac{1}{a}\int_{2^{-\frac{1}{2}}}^1 \frac{1-2v^2}{\sqrt{(1-v^2)}}\left(-\frac{v^2}{2a^2}\right)dv < \frac{1}{4a^3}. \end{aligned} \tag{B 4}$$

Thus we have the bounds for G:

$$0 \leqslant G(r_1, \hat{r}, z-\hat{z}) \leqslant \begin{cases} \dfrac{r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}}{4a^3}, & \text{(B 5}a\text{)} \\[2ex] r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}\sinh^{-1}\dfrac{1}{a\sqrt{2}}. & \text{(B 5}b\text{)} \end{cases}$$

We differentiate G, to obtain formulae like

$$\frac{\partial G}{\partial r} = \tfrac{1}{2}r^{-\frac{1}{2}}\hat{r}^{\frac{3}{2}} f(a) + r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}} f'(a)\frac{\partial a}{\partial r} \quad \text{and} \quad \frac{\partial a}{\partial r} = a\left\{\frac{r-\hat{r}}{\sigma^2} - \frac{1}{2r}\right\},$$

and bound $f'(a)$ as we bounded $f(a)$ in (B 3) and (B 4). When we put these results together there follow formulae like

$$0 \leqslant \left|\frac{\partial G}{\partial r}\right| \leqslant \left\{ \begin{array}{ll} \dfrac{r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}}{4a^3}\left[\dfrac{1}{2r} + 3\left(\dfrac{|r-\hat{r}|}{\sigma^2} + \dfrac{1}{2r}\right)\right], & \text{(B 6}a\text{)} \\[2ex] r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}\left[\dfrac{1}{2r}\sinh^{-1}\dfrac{1}{a\sqrt{2}} + \dfrac{|r-\hat{r}|}{\sigma^2} + \dfrac{1}{2r}\right], & \text{(B 6}b\text{)} \end{array} \right\}$$

We can now estimate the effect of perturbing the variables of G. Defining $\bar{G}$, the perturbed G, and $\boldsymbol{\xi}$ by

$$\bar{G} = G(r+\xi_1, \hat{r}+\hat{\xi}_1, z+\xi_2-\hat{z}-\hat{\xi}_2) \quad \text{and} \quad \boldsymbol{\xi} = (\xi_1, \xi_2) \quad \text{with} \quad \xi = |\boldsymbol{\xi}|,$$

and using the notation $\bar{\sigma}^2 = (r+\xi_1-\hat{r}-\hat{\xi}_1)^2 + (z+\xi_2-\hat{z}-\hat{\xi}_2)^2$, etc., we establish the following results:

$$|\bar{G}-G| \leqslant \left\{ \begin{array}{ll} C\dfrac{r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}}{a^3}\left[\dfrac{|\xi_2|}{r} + \dfrac{|\hat{\xi}_2|}{\hat{r}} + 3\,\dfrac{\xi+\hat{\xi}}{\sigma}\right], & \text{(B 7}a\text{)} \\[2ex] Cr^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}\left[\left(\dfrac{|\xi_1|}{r} + \dfrac{|\hat{\xi}_1|}{\hat{r}}\right)\left(1+\sinh^{-1}\dfrac{1}{a\sqrt{2}}\right) + 4\,\dfrac{\xi+\hat{\xi}}{\sigma}\right], & \text{(B 7}b\text{)} \end{array} \right\}$$

provided that

$$|\xi_1|/r, \quad |\xi_1|/\hat{r}, \quad (\xi+\hat{\xi})/\sigma \leqslant 1/\mu < 1. \tag{B 8}$$

(C depends on μ.) The condition (B 8) implies that

$$1-\frac{1}{\mu} \leqslant \frac{\bar{r}}{r} \leqslant 1+\frac{1}{\mu}, \quad \text{etc.}, \quad \frac{(u-1)^2}{(\mu+1)^2} \leqslant \frac{\bar{a}^2}{a^2} \leqslant \frac{(\mu+1)^2}{(\mu-1)^2},$$

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and thus (B 6), used in the relation

$$\bar{G}-G=\int_{\mathfrak{x},\,\hat{\mathfrak{z}}}^{\mathfrak{x}+\xi,\,\hat{\mathfrak{z}}+\hat{\xi}}\left\{\frac{\partial G}{\partial r}dr+\frac{\partial G}{\partial \hat{r}}d\hat{r}+\frac{\partial G}{\partial z}dz+\frac{\partial G}{\partial \hat{z}}d\hat{z}\right\},$$

yield the result (B 7).

By a series of steps similar to those that led to (B 6) and (B 7), though more involved, we prove, under condition (B 8), that

$$\left|G(r+\xi_1,\,\hat{r},\,z+\xi_2-\hat{z})-G(r,\,\hat{r},\,z-\hat{z})-\xi\frac{\partial G}{\partial \xi}\right| \leqslant \begin{cases} C\xi^2\dfrac{r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}}{a^3}\left[\dfrac{4}{\sigma^2}+\dfrac{1}{r^2}\right], & \text{(B 9}a\text{)} \\ C\xi^2 r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}\left[\dfrac{4}{\sigma^2}+\dfrac{1}{r^2}\left(1+\sinh^{-1}\dfrac{1}{a\sqrt{2}}\right)\right], & \text{(B 9}b\text{)} \end{cases}$$

where $\partial/\partial\xi$ denotes differentiation in the direction of $\boldsymbol{\xi}$.

APPENDIX C

An integral of G

We proceed to obtain the following bounds for the integral of G over the domain H_0-H_ϵ (defined by (2·8));

$$\epsilon^2 r|_{\partial H_\epsilon} \leqslant \iint_{H_0-H_\epsilon}\frac{1}{r}G|_{\partial H_\epsilon}\,d\hat{r}\,d\hat{z} \leqslant C\epsilon^2 \quad \text{for} \quad 0\leqslant t\leqslant \pi. \tag{C 1}$$

For $0\leqslant t\leqslant 3\epsilon$ we improve the upper bound from $C\epsilon^2$ to $C\epsilon^3$; and for $3\epsilon\leqslant t\leqslant \epsilon^{\frac{2}{3}}$ we improve it to $Cr^3|_{\partial H_\epsilon}$.

We perform the calculations in some detail as a model for the work in section 5.

Since $1/r.G$ is non-negative, we simplify the domain of integration by including it in a more conveniently shaped domain; that is,

$$\left\{\frac{1}{r}G\geqslant 0,\quad S\subseteq T\right\} \text{ implies } \iint_S\frac{1}{r}G\leqslant\iint_T\frac{1}{r}G. \tag{C 2}$$

Hence, we define, with $\eta\equiv\epsilon^2/2\sin^2\theta$,

$$T=\{(\hat{r},\hat{z})|\,0\leqslant\hat{r}\leqslant 3\epsilon,\ -\infty<\hat{z}<\infty\},$$

and

$$U=\{(\hat{\rho},\hat{\theta})|\ 3\epsilon\leqslant\hat{\theta}\leqslant\pi-3\epsilon,\quad 1-\hat{\eta}\leqslant\hat{\rho}\leqslant 1\},$$

and note that $H_0-H_\epsilon\subseteq T+U$.

We can evaluate exactly the integral over T.:

$$\iint_T\frac{1}{r}G\,d\hat{r}\,d\hat{z}=\int_0^{3\epsilon}d\hat{r}\int_{-\infty}^{\infty}d\hat{z}\int_0^{\infty}\pi\hat{r}^2\,J_1(kr)\,J_1(k\hat{r})\exp(-k|z-\hat{z}|)dk$$
$$=\begin{cases}\pi(3\epsilon)^4/4r & \text{for}\quad r\geqslant 3\epsilon,\\ \pi r(18\epsilon^2-r^2)/4 & \text{for}\quad 0\leqslant r\leqslant 3\epsilon,\end{cases} \tag{C 3}$$

using a standard integral from (13), p. 47 on the cylindrical coordinate form of (2·5).

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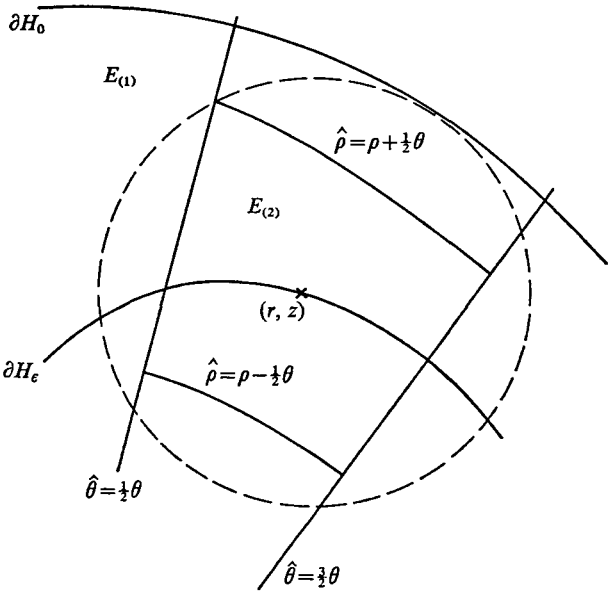


Fig. 8. The integral of G over the domain H_0-H_ϵ is bounded by (i) bounding G itself, and (ii) splitting the domain H_0-H_ϵ into subdomains $E_{(1)}$ and $E_{(2)}$ by the line $a \asymp 1$. The figure shows how $E_{(1)}$ naturally splits into pieces which can be integrated over straightforwardly.

Thus we have only to bound

$$\iint_U \frac{1}{r} G|_{\partial H_\epsilon} \hat{\rho} d\hat{\rho} d\hat{\theta} \quad \text{for} \quad 0 \leqslant t \leqslant \tfrac{1}{2}\pi, \tag{C4}$$

where we use the symmetry of the integral about $t = \frac{1}{2}\pi$. The exact result (using (4·4) and (4.6)),

$$\int_0^\pi d\hat{\theta} \int_{1-\alpha}^1 \frac{1}{r} G \hat{\rho} d\hat{\rho} = \frac{2\pi r}{3} \alpha \left(1 - \frac{\alpha}{2}\right) \quad \text{for} \quad \rho \leqslant 1-\alpha, 0 \leqslant t \leqslant \pi, \tag{C5}$$

shows that

$$\iint_{H_0-H_\epsilon} \frac{1}{r} G|_{\partial H_\epsilon} d\hat{r} d\hat{z} \geqslant \epsilon^2 r|_{\partial H_\epsilon} \quad \text{for} \quad 0 \leqslant t \leqslant \pi \quad 0 < \epsilon \leqslant \tfrac{1}{10}.$$

However it does not lead to an upper bound that is sharp enough.

Hence, we bound (C4) by using (B 5) to bound the integrand. The calculation is rather tedious because of the dependence of the integrand on (ρ, θ); it is necessary to consider the integral (C 4) separately for $0 \leqslant t \leqslant 2\epsilon$, $2\epsilon \leqslant t \leqslant \epsilon^{\frac{2}{3}}$, and $\epsilon^{\frac{2}{3}} \leqslant t \leqslant \frac{1}{2}\pi$. We go into detail only for the range $2\epsilon \leqslant t \leqslant \epsilon^{\frac{2}{3}}$. Our first task is to split the domain U into the sets $E_{(1)}$ and $E_{(2)}$ by the line $a \asymp 0(1)$, (figure 8), so that $a \geqslant 0(1)$ in $E_{(1)}$ where the a^{-3} term of (B 5) is appropriate, and $a \leqslant 0(1)$ in $E_{(2)}$ where the $\sinh^{-1} 1/a\sqrt{2}$ terms of (B 5) is appropriate. We define

$$E_{(2)} = \{(\hat{\rho}, \hat{\theta}) | \tfrac{1}{2}\theta \leqslant \hat{\theta} \leqslant \tfrac{3}{2}\theta, \quad \rho - \tfrac{1}{2}\theta \leqslant \hat{\rho} \leqslant \rho + \tfrac{1}{2}\theta\}, \tag{C6}$$

and

$$E_{(1)} = U - E_{(2)}.$$

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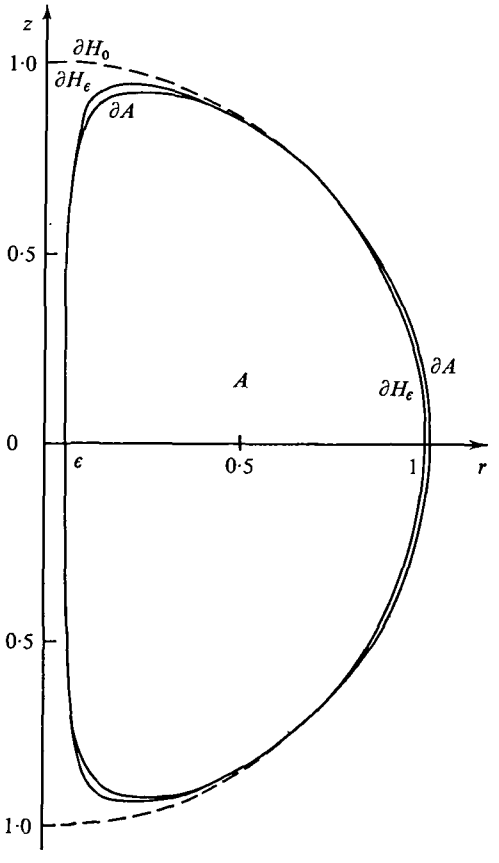


Fig. 9. The curve ∂A, for $\epsilon = 0{\cdot}05$ (in non-dimensional units), and its first approximation ∂H_ϵ.

We show this is a satisfactory definition by considering $(\hat{\rho}, \hat{\theta}) \in E_{(1)}$ and noting, if $\hat{\theta} \leqslant \frac{1}{2}\theta$ or $\hat{\theta} \geqslant \frac{3}{2}\theta$,

$$a^2 = \frac{\sigma^2}{4r\hat{r}} \geqslant \frac{\sin^2 \frac{1}{2}|\theta - \hat{\theta}|}{\sin\theta \,.\, \sin\hat{\theta}} \geqslant c\left\{\sqrt{\frac{\hat{\theta}}{\theta}} - \sqrt{\frac{\theta}{\hat{\theta}}}\right\}^2 \geqslant c > 0,$$

while if $\frac{1}{2}\theta \leqslant \hat{\theta} \leqslant \frac{3}{2}\theta$ and $|\hat{\rho} - \rho| \geqslant \frac{1}{2}\theta$,

$$a^2 \geqslant \frac{(\rho - \hat{\rho})^2}{4\hat{\rho} \,.\, \rho \sin\hat{\theta} \,.\, \sin\theta} \geqslant \frac{(\frac{1}{2}\theta)^2}{6\theta^2} = \frac{1}{24}.$$

Finally, we bound the following integrals. We use the $\sinh^{-1}$ alternative of (B 5) on the integrand of (C 4) over the domain in (C 6) to give

$$\begin{aligned}
\iint_{E_{(1)}} \frac{1}{r} G|_{\partial H_\epsilon} \hat{\rho}\, d\hat{\rho}\, d\hat{\theta} &\leqslant \int_{\frac{1}{2}\theta}^{\frac{3}{2}\theta} d\hat{\theta} \int_{\rho - \frac{1}{2}\theta}^{\rho + \frac{1}{2}\theta} \frac{\hat{r}^{\frac{3}{2}}}{r^{\frac{1}{2}}} \sinh^{-1} \frac{1}{a\sqrt{2}} \hat{\rho}\, d\hat{\rho}, \\
&\leqslant 3r \int_{\frac{1}{2}\theta}^{\frac{3}{2}\theta} d\hat{\theta} \int_{\rho - \frac{1}{2}\theta}^{\rho + \frac{1}{2}\theta} \ln\left(\frac{c\theta}{\sigma}\right) \hat{\rho}\, d\hat{\rho}, \\
&\leqslant c\theta^3 \{\ln\theta^{-1} + c + \ln\theta\} \leqslant c\theta^3. \qquad \text{(C 7)}
\end{aligned}$$

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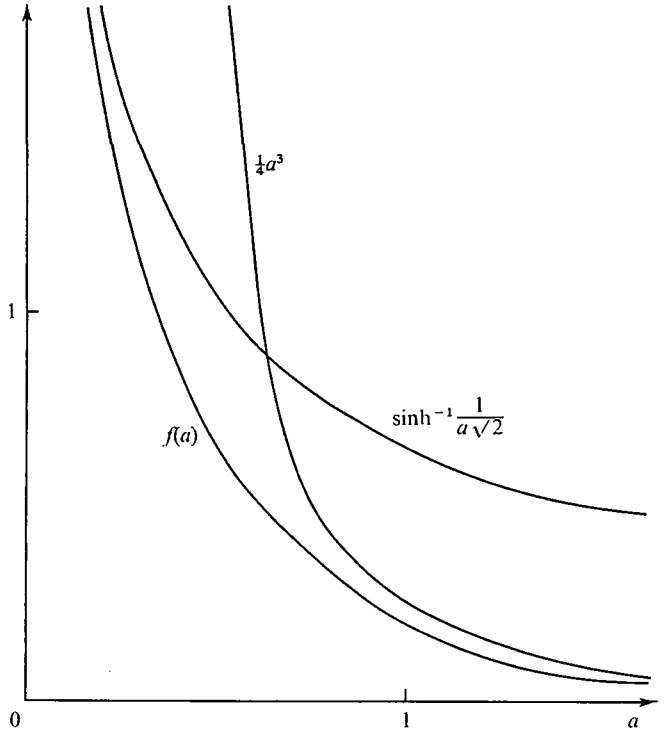


Fig. 10. The function $f(a)$, defined by $G = r^{\frac{1}{2}}\hat{r}^{\frac{3}{2}}$. $f(a)$, is bounded by the functions $1/(4a^3)$ and $\sinh^{-1}$. $1/a\sqrt{2}$, and in fact it can be shown that $f(a) \sim \ln 1/a$ as $a \to 0$, and that $f(a) \sim$ const. a^{-3} as $a \to \infty$.

We now use the a^{-3} alternative of (B 5) over the domain $E_{(1)}$, which we split as follows.

$$E_{(1)} = \{3\epsilon \leqslant \hat{\theta} \leqslant \tfrac{1}{2}\theta, 1-\hat{\eta} \leqslant \hat{\rho} \leqslant 1\} + \{\tfrac{1}{2}\theta \leqslant \hat{\theta} \leqslant \tfrac{3}{2}\theta,\ \rho + \tfrac{1}{2}\theta \leqslant \hat{\rho} \leqslant 1\}$$
$$+ \{\tfrac{3}{2}\theta \leqslant \hat{\theta} \leqslant \pi - 3\epsilon,\ 1-\hat{\eta} \leqslant \hat{\rho} \leqslant 1\} + \{\tfrac{1}{2}\theta \leqslant \hat{\theta} \leqslant \tfrac{3}{2}\theta, 1-\hat{\eta} \leqslant \hat{\rho} \leqslant \rho - \tfrac{1}{2}\theta\}.$$

Then we have, on noting $\frac{17}{18} \leqslant \rho, \hat{\rho} \leqslant 1$, that

$$\int_{\frac{3}{2}\theta}^{\pi-3\epsilon} d\hat{\theta} \int_{1-\hat{\eta}}^{1} \frac{r\hat{r}^3}{\sigma^3}\hat{\rho}\, d\hat{\rho} \leqslant \int_{\frac{3}{2}\theta}^{\pi-3\epsilon} d\hat{\theta} \int_{1-\hat{\eta}}^{1} \frac{\theta\hat{\theta}^3 d\hat{\rho}}{\{(\hat{\rho}-\rho)^2 + c(\hat{\theta}-\theta)^2\}^{\frac{3}{2}}}$$
$$\leqslant \int_{\frac{3}{2}\theta}^{e^2\theta^{-1}} d\hat{\theta} \int_{1-\hat{\eta}}^{1} \frac{\theta\hat{\theta}^3 d\hat{\rho}}{(\hat{\rho}-\rho)^3} + \int_{e^2\theta^{-1}}^{\pi-3\epsilon} d\hat{\theta} \int_{1-\hat{\eta}}^{1} C\frac{\theta\hat{\theta}^3 d\hat{\rho}}{(\hat{\theta}-\theta)^3}$$
$$\leqslant \int_{\frac{3}{2}\theta}^{e^2\theta^{-1}} \frac{C\theta\hat{\theta}^3}{\eta^3}\hat{\eta}\, d\hat{\theta} + \int_{e^2\theta^{-1}}^{\pi-3\epsilon} \frac{C\theta\hat{\theta}^3}{(\hat{\theta}-\theta)^3}\hat{\eta}\, d\hat{\theta}$$
$$\leqslant C\theta^3. \tag{C 8}$$

We use $\sigma^2 = (\hat{\rho}-\rho)^2 + 4\hat{\rho}\rho\sin^2\frac{1}{2}(\hat{\theta}-\theta) \geqslant (\hat{\rho}-\rho)^2$ in showing

$$\int_{\frac{1}{2}\theta}^{\frac{3}{2}\theta} d\hat{\theta} \int_{\rho+\frac{1}{2}\theta}^{1} \frac{r\hat{r}^3}{\sigma^2}\hat{\rho}\, d\hat{\rho} \leqslant \int_{\frac{1}{2}\theta}^{\frac{3}{2}\theta} d\hat{\theta} \int_{\rho+\frac{1}{2}\theta}^{1} \theta\hat{\theta}^3 \frac{d\hat{\rho}}{(\hat{\rho}-\rho)^3},$$
$$\leqslant C.\theta.\theta^4 \frac{1}{(\frac{1}{2}\theta)^2} = 4C\theta^3. \tag{C 9}$$

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Finally, we show that

$$\int_{3\epsilon}^{\frac{1}{2}\theta} d\hat{\theta} \int_{1-\hat{\eta}}^{1} \frac{r\hat{r}^3}{\sigma^3} \hat{\rho}\, d\hat{\rho} \leqslant \int_{3\epsilon}^{\frac{1}{2}\theta} d\hat{\theta} \int_{1-\hat{\eta}}^{1} C\theta\hat{\theta}^3 \frac{d\hat{\rho}}{\{(\hat{\rho}-\rho)^2 + c(\hat{\theta}-\theta)^2\}^{\frac{3}{2}}},$$

$$\leqslant \int_{3\epsilon}^{\frac{1}{2}\theta} C\theta\hat{\theta}^3 d\theta \left[\frac{\hat{\rho}-\rho}{(\hat{\rho}-\rho)^2 + c(\hat{\theta}-\theta)^2} \right]_{1-\hat{\eta}}^{1},$$

$$\leqslant C\theta \int_{3\epsilon}^{\frac{1}{2}\theta} \frac{\hat{\theta}^3 d\hat{\theta}}{(\hat{\theta}-\theta)^2},$$

$$\leqslant C\theta^3. \tag{C 10}$$

Grouping the results (C 7–10), we have the bound (C 1) as stated for $3\epsilon \leqslant t \leqslant \epsilon^{\frac{2}{3}}$. When $\epsilon^{\frac{2}{3}} \leqslant t \leqslant \frac{1}{2}\pi$, similar calculations produce (6·1). For $0 \leqslant t \leqslant 2\epsilon$ it is easier to use (C 3) over the area of integration $\{0 \leqslant \hat{r} \leqslant 4\epsilon,\ -\infty < z < \infty\}$, and note that $\sigma^2 \geqslant (\hat{\rho}-\rho)^2 \geqslant c > 0$ holds for the remainder of the domain of $E_{(1)}$, to produce the bound $C\epsilon^3$.

Again, using a similar method and results like (B 6), we bound the integral of $\partial G/\partial\nu|_{\partial H_\epsilon}$ over the area $H_0 - H_\epsilon$:

$$\iint_{H_0-H_\epsilon} \left| \frac{\partial G}{\partial \nu} \right|_{\partial H_\epsilon} d\hat{r}\, d\hat{z} \leqslant C\epsilon^2 \quad (0 \leqslant t \leqslant \pi). \tag{C 11}$$

It is possible to improve the bound for $0 \leqslant t \leqslant \epsilon^{\frac{2}{3}}$ from $C\epsilon^2$ to $C\epsilon^{\frac{4}{3}} r|_{\partial H_\epsilon}$.

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