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SOJOURN FUNCTIONALS OF TIME-DEPENDENT χ^2-RANDOM FIELDS ON TWO-POINT HOMOGENEOUS SPACES

ALESSIA CAPONERA,* *LUISS Guido Carli*
MAURIZIA ROSSI,** *Università di Milano-Bicocca*
MARÍA DOLORES RUIZ MEDINA,*** *Universidad de Granada*

Abstract

We investigate geometric properties of invariant spatio-temporal random fields $X\colon \mathbb{M}^d \times \mathbb{R} \to \mathbb{R}$ defined on a compact two-point homogeneous space $\mathbb{M}^d$ in any dimension $d \geq 2$, and evolving over time. In particular, we focus on chi-squared-distributed random fields, and study the large-time behavior (as $T \to +\infty$) of the average on $[0,T]$ of the volume of the excursion set on the manifold, i.e. of $\{X(\cdot, t) \geq u\}$ (for any $u > 0$). The Fourier components of X may have short or long memory in time, i.e. integrable or non-integrable temporal covariance functions. Our argument follows the approach developed in Marinucci et al. (2021) and allows us to extend their results for invariant spatio-temporal Gaussian fields on the two-dimensional unit sphere to the case of chi-squared distributed fields on two-point homogeneous spaces in any dimension. We find that both the asymptotic variance and limiting distribution, as $T \to +\infty$, of the average empirical volume turn out to be non-universal, depending on the memory parameters of the field X.

Keywords: Chi-square random fields; central and non-central limit theorems; Wiener chaos; two-point homogeneous spaces

2010 Mathematics Subject Classification: Primary 60G60
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1. Introduction

In this note we investigate geometric properties of invariant spatio-temporal random fields $X\colon \mathbb{M}^d \times \mathbb{R} \to \mathbb{R}$, $(x, t) \mapsto X(x, t)$ defined on compact two-point homogeneous spaces $\mathbb{M}^d$ in any dimension $d \geq 2$, and evolving over time. A key example is $\mathbb{M}^d = \mathbb{S}^d$, the d-dimensional sphere with the round metric.

In recent decades the theory of spherical random fields, or more generally random fields on Riemannian manifolds, has received growing interest, also in view of possible applications to cosmology, brain imaging, machine learning, and climate science, to cite a few. As for the latter, it is indeed natural to model the Earth surface temperature as a random field on $\mathbb{S}^2$ evolving over time.

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* Email address: acaponera@luiss.it
** Email address: maurizia.rossi@unimib.it
*** Email address: mruiz@ugr.es

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1.1. Previous work on the sphere

In [2] the authors, relying on this probabilistic model, are able to detect Earth surface temperature anomalies via analysis of climate data collected from 1949 to 2020. On the other hand, in [23] geometric properties of invariant *Gaussian* random fields Z on $\mathbb{S}^2 \times \mathbb{R}$ are investigated (see also [14]). (In this manuscript, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ while $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$.) The Fourier components of Z, say $\{Z_\ell\}_{\ell \in \mathbb{N}_0}$, satisfying $\Delta_{\mathbb{S}^2} Z_\ell(\cdot, t) + \ell(\ell+1) Z_\ell(\cdot, t) = 0$ (where $\Delta_{\mathbb{S}^2}$ is the Laplace–Beltrami operator on the sphere) are allowed to have short or long memory in time (independently of one another), i.e. an integrable or non-integrable temporal covariance function. The parametric model is related to functions that at infinity behave like t^{-2} (or any other integrable power) in the short-memory case, and $t^{-\beta_\ell}$ in the long-memory case, where $\beta_\ell \in (0, 1)$. In order to make the notation more compact, the short-memory case is associated with $\beta_\ell = 1$, representing an integrable power law at infinity. For details, see (10) and Assumption 1.

In particular, in [23] the authors focus on the large-time (as $T \to +\infty$) fluctuations of the average empirical area

$$\mathcal{M}_T^Z(u) := \int_{[0,T]} \int_{\mathbb{S}^2} (\mathbf{1}_{\{Z(x,t) \geq u\}} - \mathbb{P}(Z(x, t) \geq u)) \, \mathrm{d}x \, \mathrm{d}t, \tag{1}$$

where $u \in \mathbb{R}$ is the threshold at which we 'cut' the field. Note that in (1) the spatial domain is fixed.

Both first- and second-order fluctuations of $\mathcal{M}_T^Z(u)$ depend on a subtle interplay between the parameters of the model, in particular between

$$\beta_{\ell^*} := \min_{\ell \in \mathbb{N}} \beta_\ell \tag{2}$$

(which the authors of [23] assume exists), the value of β_0, and the threshold u. The core of their proofs relies on the decomposition of (1) into so-called Wiener chaoses: basically, when Z has zero mean and unit variance, $\mathcal{M}_T^Z(u)$ can be written as an *orthogonal* series in $L^2(\Omega)$ of the form

$$\mathcal{M}_T^Z(u) = \sum_{q=1}^{+\infty} \mathcal{M}_T^Z(u)[q] = \sum_{q=1}^{+\infty} \int_{[0,T]} \left(\frac{\phi(u) H_{q-1}(u)}{q!} \int_{\mathbb{S}^2} H_q(Z(x, t)) \, \mathrm{d}x \right) \mathrm{d}t, \tag{3}$$

where $\{H_q\}_{q \in \mathbb{N}_0}$ represents the family of Hermite polynomials, and ϕ the standard Gaussian density. This expansion is satisfied by any $L^2(\Omega)$-functional of a Gaussian field, and it is based on the fact that Hermite polynomials form a complete orthogonal basis for the space of square-integrable functions on the real line with respect to the Gaussian measure.

The quantity $\beta_0 \in (0, 1]$ is the memory parameter of the temporal part of $\mathcal{M}_T^Z(u)[1]$ in (3), i.e. of the stochastic process $\mathbb{R} \ni t \mapsto \phi(u) \int_{\mathbb{S}^2} Z(x, t) \, \mathrm{d}x$. If $\beta_0 = 1$, then the asymptotic variance of $\mathcal{M}_T^Z(u)[1]$ is of order T, otherwise it is of order $T^{2-\beta_0}$. Of course, $\mathcal{M}_T^Z(u)[1]$ is Gaussian for every $T > 0$.

As for the second chaotic component, it is worth noticing that it vanishes if and only if $u = 0$ (indeed, $H_1(u) = u$); for $u \neq 0$, roughly speaking, the dependence property of its integrand stochastic process relies on the interplay between β_{ℓ^*} and β_0: if $2\beta_{\ell^*} < \min(1, 2\beta_0)$ then the asymptotic variance of $\mathcal{M}_T^Z(u)[2]$ is of order $T^{2-2\beta_{\ell^*}}$, while if both $2\beta_{\ell^*}$ and $2\beta_0$ are > 1 it is of order T. In the former case, the limiting distribution of $\mathcal{M}_T^Z(u)[2]$ after standardization

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is non-Gaussian, in the latter case a central limit theorem holds. The analysis of higher-order chaotic components of (3) is somewhat analogous.

In particular, in the 'short-memory case' (that corresponds to $\beta_0 = 1$ and $\beta_{\ell^*} = \min_{\ell\in\mathbb{N}} \beta_\ell$ such that $\beta_{\ell^*} > \frac{1}{2}$ for $u \neq 0$ or $\beta_{\ell^*} > \frac{1}{3}$ for $u = 0$) the statistic in (1), once divided by $\sqrt{T}$, shows Gaussian fluctuations. Indeed, each non-null chaotic component in (3) has an asymptotic variance of order T (so the whole variance series is of order T), and once divided by $\sqrt{T}$, converges to a Gaussian random variable (r.v.) via an application of the fourth moment theorem (hence, thanks to a Breuer–Major argument, for $\mathcal{M}_T^Z(u)/\sqrt{T}$ a central limit theorem holds [26, Theorem 6.3.1]).

In the 'long-memory case', in particular when $2\beta_{\ell^*} < \min(1, \beta_0)$, for $u \neq 0$ the statistics $\mathcal{M}_T^Z(u)$ and $\mathcal{M}_T^Z(u)[2]$ are asymptotically equivalent in $L^2(\Omega)$, and $\mathcal{M}_T^Z(u)[2]$ converges in distribution towards a linear combination of independent Rosenblatt r.v.s, once divided by $T^{1-\beta_{\ell^*}}$ (hence (1) does not have Gaussian fluctuations). This non-central limit theorem follows from standard results for Hermite rank-two functionals of Gaussian fields in the presence of long memory [5, 31].

1.2. Our main contribution

The aim of the present note is to take a first step towards possible natural generalizations in two directions of the results of [23] mentioned in Section 1.1: the type of randomness of the field and the underlying manifold. Indeed, as briefly anticipated, we want to study the large-time behavior of the statistic $\mathcal{M}_T^X(u)$ defined as in (1) with the spherical random field Z replaced by an invariant time-dependent chi-squared distributed process X on a two-point homogeneous space $\mathbb{M}^d$ in any dimension $d \geq 2$ (see Definition 1). Of course, we consider only the case of positive thresholds, i.e. $u > 0$. (For $u = 0$, it is natural to study the time behavior of the set $\{X(\cdot, t) = 0\}$. We leave this as a topic for future research – see [24] for fluctuations of the level lines of a spatio-temporal Gaussian field on the 2-sphere.)

To be more precise, the marginal law of $X = X_k$ is a chi-square distribution with $k \in \mathbb{N}$ degrees of freedom. In order to define X_k, it suffices to consider k independent and identically distributed (i.i.d.) copies $Z_1, \ldots, Z_k$ of an invariant Gaussian random field Z on $\mathbb{M}^d \times \mathbb{R}$ (generalizing the definition for $\mathbb{S}^2 \times \mathbb{R}$) with standard Gaussian marginals, and take the sum of their squares, $X_k(x, t) = Z_1(x, t)^2 + \cdots + Z_k(x, t)^2$, $(x, t) \in \mathbb{M}^d \times \mathbb{R}$. Our main results mirror those obtained for the statistic in (1) in the spherical case. It is worth stressing that although our proofs closely follow the strategy developed in [23], there are a few differences and we obtain some novel results that may be of independent interest. First of all, it is possible to obtain a chaotic expansion similar to (3) for $\mathcal{M}_T^{X_k}(u)$, actually the distribution of the underlying noise $(Z_1, \ldots, Z_k)$ is Gaussian. Due to parity of the involved functions, terms of odd order q vanish, and this expansion can hence be rewritten in terms of *Laguerre polynomials*. We find along the way an explicit formula for chaotic coefficients of the function $\mathbf{1}_{\{\chi^2(k)\geq u\}}$ for $u > 0$, a result that may have some independent interest. (Here and in what follows, $\chi^2(k)$ denotes a chi-square r.v. with k degrees of freedom.)

Let us define $\beta_{\ell^*} := \min_{\ell\in\mathbb{N}_0} \beta_\ell$ – see also (2). As for the asymptotic variance and limiting distribution of $\mathcal{M}_T^{X_k}(u)$, for $\beta_{\ell^*} > \frac{1}{2}$ it is of order T and a central limit theorem holds for every $u > 0$, while for $\beta_{\ell^*} < \frac{1}{2}$ it is of order $T^{2-2\beta_{\ell^*}}$ and convergence to a linear combination of independent Rosenblatt r.v.s holds (still for every $u > 0$).

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1.3. Brief overview of the literature

The existing literature is rich in results on sojourn times of random fields, and more generally on the geometry of their excursion sets. As for chi-square random fields on manifolds, it is worth mentioning as a possible application in astrophysics in the analysis of the cosmic microwave background (CMB) data. Indeed, the CMB polarization intensity is modeled as a spin-2 spherical random field [21, Chapter 12] and, as noted in [3], the *expected* Lipschitz–Killing curvatures of corresponding excursion sets coincide with those of a chi-square random field on the sphere. In this regard, recently in [25] critical points of chi-square random fields on manifolds have been investigated. In the Euclidean setting the distribution of sojourn functionals for spatio-temporal Gaussian random fields with long memory has been studied in [14], while in [15] the case of nonlinear functionals of Gamma-correlated random fields with long memory (and no time dependence) has been treated by means of Laguerre chaos. See also [20] for limit theorems in the case of subordinated Gaussian random fields, still with long-range dependence. The paper [12] contains recent developments on this subject; indeed, the authors investigate limit theorems for p-domain functionals of stationary Gaussian fields. Moreover, in [4] the authors show that under a very general framework there is an interesting relationship between tail asymptotics of sojourn times and that of the supremum of the random field. To conclude, let us recall that the decomposition structure of (time-varying) Gaussian random fields on compact two-point homogeneous spaces was studied, e.g., in [19]; see also [18] for regularity properties of their sample paths. Of course, this list is by no means complete.

2. Background and notation

Let $\mathbb{M}^d$ be a *two-point homogeneous* compact Riemannian manifold with dimension $d \geq 2$, and denote by ρ its geodesic distance. Here we assume that $\rho(x, y) \in [0, \pi]$ for every $x, y \in \mathbb{M}^d$, and consider also a *normalized* Riemannian measure, denoted by $\mathrm{d}v$, so that $\int_{\mathbb{M}^d} \mathrm{d}v = 1$. Denoting by G the group of isometries of $\mathbb{M}^d$, for any set of four points $x_1, y_1, x_2, y_2 \in \mathbb{M}^d$ with $\rho(x_1, y_1) = \rho(x_2, y_2)$, there exists $\varphi \in G$ such that $\varphi(x_1) = x_2$ and $\varphi(y_1) = y_2$. As pointed out in [32], $\mathbb{M}^d$ belongs to one of the following categories: the unit (hyper)sphere $\mathbb{S}^d$, $d = 2, 3, \ldots$; the real projective spaces $\mathbb{P}^d(\mathbb{R})$, $d = 2, 3, \ldots$; the complex projective spaces $\mathbb{P}^d(\mathbb{C})$, $d = 4, 6, \ldots$; the quaternionic projective spaces $\mathbb{P}^d(\mathbb{H})$, $d = 8, 12, \ldots$; and the Cayley projective plane $\mathbb{P}^d(\mathrm{Cay})$, $d = 16$.

2.1. Fourier analysis for two-point homogeneous spaces

Let $\Delta_{\mathbb{M}^d}$ be the Laplace–Beltrami operator on $\mathbb{M}^d$ (see, e.g., [6, 7, 19]). It is well known that the spectrum of $\Delta_{\mathbb{M}^d}$ is purely discrete, the eigenvalues being $\lambda_\ell = \lambda_\ell(\mathbb{M}^d) = -\ell(\ell + \alpha + \beta + 1)$, $\ell \in \Lambda$, where

$$\Lambda = \Lambda(\mathbb{M}^d) = \begin{cases} \{n \in \mathbb{N}_0 : n \text{ even}\} & \text{if } \mathbb{M}^d = \mathbb{P}^d(\mathbb{R}), \\ \mathbb{N}_0 & \text{if } \mathbb{M}^d \neq \mathbb{P}^d(\mathbb{R}), \end{cases} \tag{4}$$

and the parameters α, β are reported in Table 1. Now consider $L^2(\mathbb{M}^d)$, the space of real-valued, square-integrable functions on $\mathbb{M}^d$ with respect to the normalized Riemannian measure $\mathrm{d}v$, endowed with the standard inner product $\langle \cdot, \cdot \rangle_{L^2(\mathbb{M}^d)}$.

(Recall the definition of the set Λ in (4).) For every $\ell \in \Lambda$, eigenfunctions corresponding to the same eigenvalue λ_ℓ form a finite-dimensional vector space, denoted by $\mathcal{Y}_\ell = \mathcal{Y}_\ell(\mathbb{M}^d)$, with dimension

$$\dim(\mathcal{Y}_\ell) = \frac{(2\ell + \alpha + \beta + 1)\Gamma(\beta + 1)\Gamma(\ell + \alpha + \beta + 1)\Gamma(\ell + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(\alpha + \beta + 2)\Gamma(\ell + 1)\Gamma(\ell + \beta + 1)}.$$

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TABLE 1. Eigenvalue parameters α and β.

$\mathbb{M}^d$	$\mathbb{S}^d$	$\mathbb{P}^d(\mathbb{R})$	$\mathbb{P}^d(\mathbb{C})$	$\mathbb{P}^d(\mathbb{H})$	$\mathbb{P}^d(\mathrm{Cay})$
Dimension	$d=1, 2, \dots$	$d=2, 3, \dots$	$d=4, 6, \dots$	$d=8, 12, \dots$	$d=16$
α	$(d-2)/2$	$(d-2)/2$	$(d-2)/2$	$(d-2)/2$	7
β	$(d-2)/2$	$(d-2)/2$	0	1	3

It is well known that $L^2(\mathbb{M}^d)=\bigoplus_{\ell\in\Lambda}\mathcal{Y}_\ell$, where $\bigoplus$ stands for the orthogonal sum, in L^2, of vector spaces. Moreover, given a real-valued orthonormal basis of $\mathcal{Y}_\ell$, say $\{Y_{\ell,m}, m=1,\dots,\dim(\mathcal{Y}_\ell)\}$, any function $f\in L^2(\mathbb{M}^d)$ can be written as a series, converging in L^2, of the form

$$f=\sum_{\ell\in\Lambda}\sum_{m=1}^{\dim(\mathcal{Y}_\ell)}\langle f, Y_{\ell,m}\rangle_{L^2(\mathbb{M}^d)}Y_{\ell,m}.$$

From [6, Theorem 3.2] an *addition formula* holds: for any $\ell\in\Lambda$,

$$\frac{1}{\dim(\mathcal{Y}_\ell)}\sum_{m=1}^{\dim(\mathcal{Y}_\ell)}Y_{\ell,m}(x)Y_{\ell,m}(y)=\frac{P_\ell^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))}{P_\ell^{(\alpha,\beta)}(1)}, \tag{5}$$

where $\varepsilon=\frac{1}{2}$ if $\mathbb{M}^d=\mathbb{P}^d(\mathbb{R})$ and $\varepsilon=1$ otherwise. The functions $P_\ell^{(\alpha,\beta)}\colon[-1,1]\to\mathbb{R}$, $\ell\in\mathbb{N}_0$, are the well-known *Jacobi polynomials* [29]. Notably, when $\mathbb{M}^d=\mathbb{S}^2$ (the unit two-dimensional sphere with the round metric), they reduce to the *Legendre polynomials* P_ℓ, $\ell\in\mathbb{N}_0$. Recall that

$$P_\ell^{(\alpha,\beta)}(1)=\binom{\ell+\alpha}{\ell},$$

and that $|P_\ell^{(\alpha,\beta)}(z)|\le P_\ell^{(\alpha,\beta)}(1)$, $z\in[-1,1]$.

In general, the reason for the 'exceptional' behavior of the real projective spaces is discussed, for instance, in [10] (see also [11] and the references therein).

2.2. Spatio-temporal invariant random fields

Consider a mean-square continuous isotropic and stationary centered Gaussian field Z on $\mathbb{M}^d\times\mathbb{R}$, that is, a measurable map $Z\colon\Omega\times\mathbb{M}^d\times\mathbb{R}\to\mathbb{R}$, where $(\Omega,\mathcal{F},\mathbb{P})$ is some probability space, such that:

- $(Z(x_1,t_1),\dots,Z(x_n,t_n))$ is Gaussian for every n and every $x_1,\dots,x_n\in\mathbb{M}^d, t_1,\dots,t_n\in\mathbb{R}$;
- $\mathbb{E}[Z(x,t)]=0$ for every $x\in\mathbb{M}^d, t\in\mathbb{R}$;
- there exists a function $\Sigma\colon[0,\pi]\times\mathbb{R}\to\mathbb{R}$ such that, for every $x,y\in\mathbb{M}^d$, $t,s\in\mathbb{R}$, $\mathbb{E}[Z(x,t)Z(y,s)]=\Sigma(\rho(x,y),t-s)$ – see also [19, Definition 1];
- Σ is a continuous function.

It is well known [19, Theorem 3/Corollary 1] that, under these assumptions, Σ admits the spectral decomposition

$$\Sigma(\theta,\tau)=\sum_{\ell\in\Lambda}\kappa_\ell C_\ell(\tau)P_\ell^{(\alpha,\beta)}(\cos\varepsilon\theta),\qquad\tau\in\mathbb{R},\theta\in[0,\pi], \tag{6}$$

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where $\kappa_\ell = \dim(\mathcal{Y}_\ell)/P_\ell^{(\alpha,\beta)}(1)$, $\varepsilon = \frac{1}{2}$ if $\mathbb{M}^d = \mathbb{P}^d(\mathbb{R})$ and $\varepsilon = 1$ otherwise; the series on the right-hand side (r.h.s.) of (6) converges uniformly on $[0,\pi]\times\mathbb{R}$, and $C_\ell\colon \mathbb{R}\to\mathbb{R}$, $\ell\in\Lambda$, are semi-positive-definite functions. Note that the uniform convergence of the series in (6) is *equivalent* to the condition $\sum_{\ell\in\Lambda}\dim(\mathcal{Y}_\ell)C_\ell(0) < +\infty$.

Throughout this work we assume that the field Z has *unit variance*:

$$\Sigma(0,0) = \sum_{\ell\in\Lambda}\dim(\mathcal{Y}_\ell)C_\ell(0) = 1. \tag{7}$$

Definition 1. Let $Z_1, Z_2, \dots, Z_k$ be $k\in\mathbb{N}$ i.i.d. copies of an isotropic and stationary centered unit-variance Gaussian field Z on $\mathbb{M}^d\times\mathbb{R}$ For $k\in\mathbb{N}$, we define the spatio-temporal $\chi^2(k)$-random field X_k as $X_k(x,t) := Z_1^2(x,t) + \cdots + Z_k^2(x,t)$, $x\in\mathbb{M}^d$, $t\in\mathbb{R}$.

Note that the marginal distribution of X_k is a chi-square probability law with k degrees of freedom. Hence $\mathbb{E}[X_k(x,t)] = k$ and, moreover,

$$\mathrm{Cov}\,(X_k(x,t), X_k(y,s)) = 2k\,\Sigma(\rho(x,y), t-s)^2. \tag{8}$$

Indeed,

$$\begin{aligned}\mathrm{Cov}\,(X_k(x,t), X_k(y,s)) = k\,\mathrm{Cov}\,(Z(x,t)^2, Z(y,s)^2) &= k\,\mathbb{E}[H_2(Z(x,t))H_2(Z(y,s))]\\ &= 2k\,\mathrm{Cov}(Z(x,t), Z(y,s))^2,\end{aligned}$$

where $H_2(r) = r^2 - 1$, $r\in\mathbb{R}$, is the second Hermite polynomial.

Note that the covariance function of invariant chi-square random fields always takes positive values; see (8).

Remark 1. In this regard, we mention an application in percolation theory that is attracting growing interest: the positivity of the covariance kernel of a random field replaces the assumption of 'positive association' in the study of percolation properties of an invariant bond percolation process on the square lattice. In particular, the results in [9] that are valid for the latter can be extended at least to the case of isotropic random fields on $\mathbb{R}^2$ whose covariance function is positive, in order to investigate whether or not their excursion sets have an unbounded component.

2.2.1. *Short and long memory* Let Z be a mean-square continuous isotropic and stationary centered Gaussian field Z on $\mathbb{M}^d\times\mathbb{R}$. From (6), it is straightforward to prove that Z admits the spectral decomposition

$$Z(x,t) = \sum_{\ell\in\Lambda}\sum_{m=1}^{\dim(\mathcal{Y}_\ell)} a_{\ell,m}(t)Y_{\ell,m}(x), \tag{9}$$

where $a_{\ell,m}$, $\ell\in\Lambda$, $m = 1,\dots,\dim(\mathcal{Y}_\ell)$ are independent stationary (centered) Gaussian stochastic processes, indexed by $\mathbb{R}$, such that $\mathbb{E}[a_{\ell,m}(t)a_{\ell',m'}(s)] = \delta_\ell^{\ell'}\delta_m^{m'}C_\ell(t-s)$. Note that the convergence of the series in (9) is in $L^2(\Omega\times\mathbb{M}^d\times[-T,T])$ for any $T>0$, and that

$$\sup_{\tau\in\mathbb{R}}\sum_{\ell\in\Lambda}\dim(\mathcal{Y}_\ell)\,|\,C_\ell(\tau)| \le \sum_{\ell\in\Lambda}\dim(\mathcal{Y}_\ell)C_\ell(0) = 1.$$

Of course, from now on we can restrict ourselves to $\tilde{\Lambda} := \{\ell\in\Lambda\colon C_\ell(0)\neq 0\}$.

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Similarly to the spatio-temporal model on $\mathbb{S}^2$ defined in [23], we allow the stochastic processes $a_{\ell,m}$, $\ell \in \tilde{\Lambda}$, $m = 1, \dots, \dim(\mathcal{Y}_\ell)$ to have short- or long-range dependence. Let us define the family of functions g_β, $\beta \in (0, 1]$, as follows: for $\tau \in \mathbb{R}$,

$$g_\beta(\tau) := \begin{cases} (1 + |\tau|)^{-\beta}, & \beta \in (0, 1), \\ (1 + |\tau|)^{-2}, & \beta = 1. \end{cases} \tag{10}$$

Assumption 1. *The covariance functions C_ℓ, $\ell \in \tilde{\Lambda}$, can be written as $C_\ell(\tau) = G_\ell(\tau) \cdot g_{\beta_\ell}$, $\tau \in \mathbb{R}$, for some parameters $\beta_\ell \in (0, 1]$, $\ell \in \tilde{\Lambda}$, where*

$$\sup_{\ell \in \tilde{\Lambda}} \left| \frac{G_\ell(\tau)}{G_\ell(0)} - 1 \right| = o(1), \quad \tau \to +\infty.$$

Note that $G_\ell(0) = C_\ell(0)$. From now on we assume that Assumption 1 holds. If $\beta_\ell = 1$ (resp. $\beta_\ell < 1$), the processes $a_{\ell,m}$, $m = 1, \dots, \dim(\mathcal{Y}_\ell)$, have short (resp. long) memory; indeed, their common covariance function C_ℓ is integrable on $\mathbb{R}$ (resp. $\int_{\mathbb{R}} |C_\ell| = +\infty$) – see (10).

Remark 2. In (10), the assumption $\beta \in (0, 1]$ can be relaxed. To be more precise, in principle we may consider covariance functions that behave like any power at infinity, i.e. for $\ell \in \tilde{\Lambda}$ we may assume C_ℓ to be of the form $G_\ell(\tau) \cdot (1 + |\tau|)^{-\beta_\ell}$, $\tau \in \mathbb{R}$, for some $\beta_\ell > 0$. Indeed, we expect the nature of the results for the average empirical volume in (11) to be analogous even though the discussion is more tangled. In particular, in the 'short-memory regime' (see Section 4) that basically corresponds to any chaotic component temporal process

$$t \mapsto \sum_{\substack{n_1, \dots, n_k \in \mathbb{N}_0 \\ n_1 + \cdots + n_k = q}} \frac{\alpha_{n_1, \dots, n_k}(u)}{n_1! \cdots n_k!} \int_{\mathbb{M}^d} \prod_{j=1}^{k} H_{n_j}(Z_j(x, t)) \, \mathrm{d}\nu(x)$$

(see Lemma 2) having an integrable covariance function, we would need to improve some technical lemmas in [23]. Indeed, to simplify some of their arguments, the authors of [23] only consider a quadratic power law in the integrable case, while actually the same results are expected for any other integrable power (and not necessarily independent of ℓ).

Assumption 2. *There exists $\beta_{\ell^*} := \min_{\ell \in \tilde{\Lambda}} \beta_\ell$.*

From now on we assume that Assumption 2 holds. In particular, $\beta_{\ell^*} \in (0, 1]$. (Note that the definition of β_{ℓ^*} in Assumption 2 is slightly different than the one given in [23] for the spherical case; actually, in that paper the minimum is taken over $\ell \in \tilde{\Lambda}(\mathbb{S}^2) \setminus \{0\}$, while here we consider $\ell \in \tilde{\Lambda}(\mathbb{M}^d)$. The reason relies on the fact that in our case the first chaotic component $\mathcal{M}_T^{X_k}(u)[1]$ vanishes for every $u > 0$, in contrast with the spherical Gaussian case where the integrand process in $\mathcal{M}_T^{Z}(u)[1]$ has memory parameter β_0.)

3. The sojourn functional

For $u > 0$, let us consider the following geometric functional of the spatio-temporal $\chi^2(k)$-random field X_k on $\mathbb{M}^d \times \mathbb{R}$ (as in Definition 1): for any $T > 0$,

$$\mathcal{M}_T(u) = \mathcal{M}_T^{X_k}(u) := \int_{[0,T]} \int_{\mathbb{M}^d} \left(\mathbf{1}_{\{X_k(x,t) \geq u\}} - \mathbb{P}(\chi^2(k) \geq u) \right) \mathrm{d}\nu(x) \, \mathrm{d}t, \tag{11}$$

where $\mathbb{P}(\chi^2(k) \geq u)$ is the tail distribution of a chi-square r.v. with k degrees of freedom. Note that $\mathbb{E}[\mathcal{M}_T(u)] = 0$. This geometric functional can be interpreted as the temporal average,

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over $[0, T]$, of the volume of the excursion set on the manifold, evaluated at a fixed positive threshold u. We will refer to (11) as the average empirical volume.

A key aspect in obtaining our results is that the random variable $\mathcal{M}_T(u)$, being bounded, is a square-integrable functional of the underlying Gaussian field $(Z_1, Z_2, \dots, Z_k)$, and hence it admits a Wiener–Itô chaos expansion [26, Section 2.2] (see also [13]) of the form

$$\mathcal{M}_T(u) = \sum_{q=1}^{+\infty} \mathcal{M}_T(u)[q], \tag{12}$$

where the series converges in $L^2(\Omega)$, and $\mathcal{M}_T(u)[q]$, $\mathcal{M}_T(u)[q']$ are orthogonal in $L^2(\Omega)$ whenever $q \neq q'$ (see Lemmas 1 and 2). Specifically, for $q \geq 1$,

$$\mathcal{M}_T(u)[q] = \sum_{\substack{n_1,\dots,n_k \in \mathbb{N}_0 \\ n_1+\cdots+n_k=q}} \frac{\alpha_{n_1,\dots,n_k}(u)}{n_1! \cdots n_k!} \int_{[0,T]} \int_{\mathbb{M}^d} \prod_{j=1}^{k} H_{n_j}(Z_j(x,t)) \, \mathrm{d}\nu(x) \, \mathrm{d}t,$$

where, for $N_1, \dots, N_k$ i.i.d. standard Gaussians,

$$\alpha_{n_1,\dots,n_k}(u) = \mathbb{E}\left[\mathbf{1}_{\{\sum_{j=1}^k N_j^2 \geq u\}} \prod_{j=1}^{k} H_{n_j}(N_j) \right] \tag{13}$$

(note that the coefficients in (13) are *symmetric*). Here, H_q, $q \in \mathbb{N}_0$, again denotes the qth Hermite polynomial [26, Section 1.4], which can be defined as follows: $H_0 \equiv 1$; for $q \geq 1$, $H_q(r) = (-1)^q \phi^{-1}(r) \mathrm{d}^q \phi(r) / \mathrm{d}r^q$, $r \in \mathbb{R}$, ϕ denoting the standard Gaussian density. Recall that $\{H_q,\ q \in \mathbb{N}_0\}$ form an orthogonal basis for the space of square-integrable real-valued functions with respect to the Gaussian measure $\phi(r)\,\mathrm{d}r$, denoted by $L^2(\mathbb{R}, \phi(r)\,\mathrm{d}r)$. Moreover, the set of tensor products $\left\{ \otimes_{j=1}^k H_{n_j},\ n_1, \dots, n_k \in \mathbb{N}_0 \right\}$ provides an orthogonal basis for the space of square-integrable real-valued functions with respect to $\otimes_{j=1}^k \phi(r_j)\,\mathrm{d}r_j$.

Lemma 1. *Let $k \in \mathbb{N}$ and $n_1, \dots, n_k \in \mathbb{N}_0$. If there exists $j \in \{1, 2, \dots, k\}$ such that n_j is an odd number, then $\alpha_{n_1,\dots,n_k}(u) = 0$.*

Proof. For $k = 1$ it is straightforward due to parity of Hermite polynomials. For $k \geq 2$ we can assume without loss of generality that n_1 is *odd*; then, since

$$\begin{aligned} \alpha_{n_1,\dots,n_k}(u) &= \mathbb{E}\left[\mathbb{E}\left[1_{\sum_{j=1}^k N_j^2 \geq u} \prod_{j=1}^{k} H_{n_j}(N_j) \mid N_2, \dots, N_k \right] \right] \\ &= \mathbb{E}\left[\prod_{j=2}^{k} H_{n_j}(N_j) \mathbb{E}\left[\mathbf{1}_{N_1^2 + \sum_{j=2}^k x_j^2 \geq u} H_{n_1}(N_1) \right]_{x_2 = N_2, \dots, x_k = N_k} \right] \end{aligned}$$

and $\mathbb{E}[1_{N_1^2 \geq r} H_{n_1}(N_1)] = 0$ for every $r \in \mathbb{R}$, we deduce that $\alpha_{n_1,\dots,n_k}(u) = 0$. □

Thus (12) can be rewritten as follows.

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Lemma 2. *The chaotic expansion of $\mathcal{M}_T(u)$ in (11) is*

$$\mathcal{M}_T(u) = \sum_{q=1}^{+\infty} \mathcal{M}_T(u)[2q] = \sum_{q=1}^{+\infty} \sum_{n_1+\cdots+n_k=q} \frac{\alpha_{2n_1,\ldots,2n_k}(u)}{(2n_1)!\cdots(2n_k)!} \int_{[0,T]} \int_{\mathbb{M}^d} \prod_{j=1}^{k} H_{2n_j}(Z_j(x,t))\,\mathrm{d}\nu(x)\,\mathrm{d}t,$$

where the series is orthogonal and converges in $L^2(\Omega)$.

Proof. The proof of Lemma 2 closely follows the proof of [23, Lemma 4.1]. Let $N_1, \ldots, N_k$ be i.i.d. standard Gaussians. Since $\mathbf{1}_{\{\sum_{j=1}^k N_j^2 \geq u\}}$ is square-integrable with respect to the underlying Gaussian measure induced by the random vector $(N_1, N_2, \ldots, N_k)$, applying Lemma 1 gives the following orthogonal expansion in $L^2(\Omega)$:

$$\mathbf{1}_{\{\sum_{j=1}^k N_j^2 \geq u\}} = \sum_{q=0}^{+\infty} \sum_{n_1+\cdots+n_k=q} \frac{\alpha_{2n_1,\ldots,2n_k}(u)}{(2n_1)!\cdots(2n_k)!} \prod_{j=1}^{k} H_{2n_j}(N_j).$$

Note that the component corresponding to $q=0$ is exactly $\mathbb{P}(\chi^2(k) \geq u)$, and that, for fixed $x \in \mathbb{M}^d$ and $t \in \mathbb{R}$, $Z_1(x,t), \ldots, Z_k(x,t)$ are i.i.d. standard Gaussians. From our expansion, the Jensen inequality and the Fubini–Tonelli theorem, we have

$$\mathbb{E}\left[\left(\mathcal{M}_T(u) - \sum_{q=1}^{Q} \sum_{n_1+\cdots+n_k=q} \frac{\alpha_{2n_1,\ldots,2n_k}(u)}{(2n_1)!\cdots(2n_k)!} \int_0^T \int_{\mathbb{M}^d} \prod_{j=1}^{k} H_{2n_j}(Z_j(x,t))\,\mathrm{d}\nu(x)\,\mathrm{d}t\right)^2\right]$$
$$\leq T^2 \mathbb{E}\left[\left(\mathbf{1}_{\{\sum_{j=1}^k N_j^2 \geq u\}} - \sum_{q=0}^{Q} \sum_{n_1+\cdots+n_k=q} \frac{\alpha_{2n_1,\ldots,2n_k}(u)}{(2n_1)!\cdots(2n_k)!} \prod_{j=1}^{k} H_{2n_j}(N_j)\right)^2\right] \to 0$$

as $Q \to \infty$, and orthogonality follows from [26, Proposition 2.2.1]. □

The chaotic expansion in Lemma 2 can be rewritten in terms of Laguerre polynomials thanks to [1, (22.5.40), (22.5.18)].

Note that, by standard properties of Hermite polynomials [26, Section 1.4],

$$\mathrm{Var}\left(1_{\chi^2(k)\geq u}\right) = \mathbb{P}\left(\chi^2(k) \geq u\right)\left(1 - \mathbb{P}(\chi^2(k) \geq u)\right) = \sum_{q=1}^{+\infty} \sum_{n_1+\cdots+n_k=q} \frac{(\alpha_{2n_1,\ldots,2n_k}(u))^2}{(2n_1)!\cdots(2n_k)!}, \quad (14)$$

thus the series on the r.h.s. of (14) converges.

Lemma 3. *Let $u > 0$, $q \in \mathbb{N}$, and $n_1, \ldots, n_k \in \mathbb{N}_0$ be such that $n_1 + \cdots + n_k = q$. Then, for $k=1$, $\alpha_{2n_1}(u) = 2\phi(\sqrt{u})H_{2n_1-1}(\sqrt{u})$, where ϕ is the standard Gaussian density, while for $k \geq 2$,*

$$\alpha_{2n_1,\ldots,2n_k}(u) = \sum_{0\leq r_1\leq n_1,\ldots,0\leq r_k\leq n_k} \frac{(-1)^{r_1+\cdots+r_k+1}}{2^{n_1}\cdots 2^{n_k}} \frac{(2n_1)!\cdots(2n_k)!}{r_1!(n_1-r_1)!\cdots r_k!(n_k-r_k)!} \times \frac{\gamma(k/2+n_1-r_1+\cdots+n_k-r_k, u/2)}{\Gamma(k/2+n_1-r_1+\cdots+n_k-r_k)},$$

where γ is the lower incomplete Gamma function [1, Section 6.5]*, and Γ is the (standard) Gamma function.*

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Lemma 3 immediately entails that $\alpha_2(u) = 2\phi(\sqrt{u})\sqrt{u} > 0$ for every $u > 0$. Actually, we are able to show that $\alpha_{2n_1,2n_2,\dots,2n_k}(u) > 0$ for $n_1 = 1$, $n_2 = \cdots = n_k = 0$, for all $k \geq 1$ and $u > 0$, i.e.,

$$\alpha_{2,0,\dots,0}(u) > 0 \quad \text{for all } u > 0. \tag{15}$$

Lemma 3 is new, its proof is technical and hence we collect it in Appendix A, together with the proof of (15).

4. Main results

We are interested in the large-time behavior (as $T \to \infty$) of the average empirical volume (11) at any positive threshold. We will observe a phase transition for both asymptotic variance and limiting distribution depending on the memory parameters of the field, in particular under the 'long-memory regime', i.e. for $2\beta_{\ell^*} < 1$ (where β_{ℓ^*} is defined as in Assumption 2) we will observe a non-Gaussian behavior for the second-order fluctuations of $\mathcal{M}_T(u)$ as $T \to +\infty$, in contrast with the central limit theorem that we will prove in the case of 'short memory', i.e. for $2\beta_{\ell^*} > 1$.

In order to state our main results, we first need some more notation.

Definition 2. The r.v. X_β has the standard Rosenblatt distribution (see, e.g., [30] and also [5, 31]) with parameter $\beta \in \left(0, \frac{1}{2}\right)$ if it can be written as

$$X_\beta = a(\beta) \int_{(\mathbb{R}^2)'} \frac{e^{i(\lambda_1+\lambda_2)} - 1}{i(\lambda_1 + \lambda_2)} \frac{W(d\lambda_1)W(d\lambda_2)}{|\lambda_1\lambda_2|^{(1-\beta)/2}}, \tag{16}$$

where W is complex Hermitian Gaussian measure with Lebesgue control measure satisfying conditions (i)–(iv) in [27, Definition 9.2.1], and the double stochastic integral in (16) is defined in the Itô sense excluding the diagonals ($(\mathbb{R}^2)'$ represents the set $\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_1 \neq \lambda_2\}$). Moreover,

$$a(\beta) := \frac{\sigma(\beta)}{2\,\Gamma(\beta)\,\sin((1-\beta)\pi/2)}, \qquad \sigma(\beta) := \sqrt{\frac{1}{2}(1-2\beta)(1-\beta)}.$$

Following [23], we say the random vector V satisfies a composite Rosenblatt distribution of degree $N \in \mathbb{N}$ with parameter β and coefficients $c_1, \dots, c_N \in \mathbb{R} \setminus \{0\}$ if

$$V = V_N(c_1, \dots, c_N; \beta) \overset{d}{=} \sum_{k=1}^{N} c_k X_{k;\beta}, \tag{17}$$

where $\{X_{k;\beta}\}_{k=1,\dots,N}$ is a collection of i.i.d. standard Rosenblatt r.v.s (16) of parameter β.

The Rosenblatt distribution was first introduced in [30] and has already appeared in the context of spherical isotropic Gaussian random fields as the exact distribution of the correlogram [17].

Some comments are in order. The r.h.s. of (16) provides the spectral representation of the marginal, at time $t = 1$, of the Hermite process of order two, with self-similar parameter $H = 1 - \beta$. The Hermite process has stationary increments, and has the same covariance function as fractional Brownian motion (see [27, Section 9.5]), in particular with our parametrization $\mathbb{E}[X_\beta] = 0$, $\mathrm{Var}(X_\beta) = 1$ (see also [31, Theorem 6.3]). Moreover, since X_β belongs to a second-order Wiener chaos, being a double stochastic integral in the Itô sense, it is non-Gaussian and, thanks to [26, Theorem 2.10.1], has a density with respect to the Lebesgue measure.

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Remark 3. The characteristic function Ξ_V of $V = V_N(c_1, \dots, c_N; \beta)$ in (17) is

$$\Xi_V(\theta) = \prod_{k=1}^{N} \xi_\beta(c_k\theta), \qquad \xi_\beta(\theta) = \exp\left(\frac{1}{2}\sum_{j=2}^{+\infty} (2\mathrm{i}\theta\sigma(\beta))^j \frac{a_j}{j}\right),$$

where ξ_β is the characteristic function of X_β in (16), the series is only convergent near the origin, and

$$a_j := \int_{[0,1]^j} |x_1 - x_2|^{-\beta} |x_2 - x_3|^{-\beta} \cdots |x_{j-1} - x_j|^{-\beta} |x_j - x_1|^{-\beta} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \cdots \, \mathrm{d}x_j.$$

Note that when $\beta \to 0^+$ (resp. $\beta \to \frac{1}{2}^-$) then ξ_β approaches the characteristic function of $H_2(Z)/\sqrt{2} = (Z^2 - 1)/\sqrt{2}$ (resp. Z), where $Z \sim \mathcal{N}(0, 1)$ is a standard Gaussian random variable. Many more characterizations of the Rosenblatt distribution have been given in the literature; for instance, its infinite divisibility properties and Lévy–Khinchin representation are discussed in [16] and the references therein.

Theorem 1. *For $2\beta_{\ell^*} < 1$, as $T \to +\infty$,*

$$\lim_{T\to+\infty} \frac{\mathrm{Var}(\mathcal{M}_T(u))}{T^{2-2\beta_{\ell^*}}} = k\frac{(\alpha_{2,0,\dots,0}(u))^2}{2(2-2\beta_{\ell^*})(1-2\beta_{\ell^*})} \sum_{\ell\in\mathcal{I}^*} \dim(\mathcal{Y}_\ell)(C_\ell(0))^2 =: v_{\beta_{\ell^*}}(u). \tag{18}$$

Assume moreover that the set

$$\mathcal{I}^* := \{\ell \in \tilde{\Lambda} : \beta_\ell = \beta_{\ell^*}\} \tag{19}$$

is finite, i.e. the number of dominant eigenspaces where the projected process displays the largest dependence range is finite. Then

$$\frac{\mathcal{M}_T(u)}{\sqrt{\mathrm{Var}(\mathcal{M}_T(u))}} \xrightarrow{\mathrm{d}} \mathcal{R}(u), \tag{20}$$

where

$$\mathcal{R}(u) \overset{\mathrm{d}}{=} \frac{\alpha_{2,0,\dots,0}(u)}{2!} \sum_{j=1}^{k} \sum_{\ell_j\in\mathcal{I}^*} \frac{C_{\ell_j}(0)}{a(\beta_{\ell^*})\sqrt{v_{\beta_{\ell^*}}(u)}} \sum_{m_j=1}^{\dim(\mathcal{Y}_{\ell_j})} X_{m_j;\beta_{\ell^*}}$$

is a composite Rosenblatt r.v. with parameter β_{ℓ^*} and degree $\sum_{j=1}^{k} \dim(\mathcal{Y}_{\ell_j})$ (see Definition 2).

Note that the r.h.s. of (18) is (strictly) positive and finite, indeed $\alpha_{2,0,\dots,0}(u) > 0$ from (15), $\beta_{\ell^*} \in \left(0, \frac{1}{2}\right)$, and $0 < \sum_{\ell\in\mathcal{I}^*} \dim(\mathcal{Y}_\ell) C_\ell(0)^2 < +\infty$ (actually, from (7) we deduce that $C_\ell(0) \le 1$ for all $\ell \in \Lambda$).

Theorem 2. *For $2\beta_{\ell^*} > 1$, as $T \to +\infty$,*

$$\lim_{T\to+\infty} \frac{\mathrm{Var}(\mathcal{M}_T(u))}{T} = \sum_{q=1}^{+\infty} s_{2q}^2, \tag{21}$$

where

$$s_2^2 = k\frac{(\alpha_{2,0,\dots,0}(u))^2}{2!} \sum_{\ell\in\tilde{\Lambda}} \dim(\mathcal{Y}_\ell) \int_{\mathbb{R}} (C_\ell(\tau))^2 \, \mathrm{d}\tau \tag{22}$$

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and, for $q \geq 2$,

$$s_{2q}^2 := \sum_{n_1+\cdots+n_k=q} \frac{(\alpha_{2n_1,\ldots,2n_k}(u))^2}{(2n_1)!\cdots(2n_k)!} \sum_{\ell_1,\ldots,\ell_{2q}\in\tilde{\Lambda}} \kappa_{\ell_1}\cdots\kappa_{\ell_{2q}} \int_{\mathbb{R}} C_{\ell_1}(\tau)\cdots C_{\ell_{2q}}(\tau)\,\mathrm{d}\tau \times \int_{(\mathbb{M}^d)^2} P_{\ell_1}^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))\cdots P_{\ell_{2q}}^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y). \quad (23)$$

Moreover,

$$\frac{\mathcal{M}_T(u)}{\sqrt{\mathrm{Var}(\mathcal{M}_T(u))}} \xrightarrow{\mathrm{d}} Z,$$

where $Z \sim \mathcal{N}(0,1)$ *is a standard Gaussian r.v.*

As for the asymptotic variance (21), note that $0 < \sum_{q=1}^{+\infty} s_{2q}^2 < +\infty$. Positivity is due to $s_2^2 > 0$ (see (22) and (15)), while the convergence of the series follows from the inequality

$$\int_{(\mathbb{M}^d)^2} \left|P_{\ell_1}^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))\cdots P_{\ell_{2q}}^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))\right| \mathrm{d}\nu(x)\,\mathrm{d}\nu(y) \leq P_{\ell_1}^{(\alpha,\beta)}(1)\cdots P_{\ell_{2q}}^{(\alpha,\beta)}(1)$$

and (7). Indeed,

$$s_{2q}^2 \leq \sum_{n_1+\cdots+n_k=q} \frac{(\alpha_{2n_1,\ldots,2n_k}(u))^2}{(2n_1)!\cdots(2n_k)!} \sum_{\ell_1,\ell_2\in\tilde{\Lambda}} \dim(\mathcal{Y}_{\ell_1})\dim(\mathcal{Y}_{\ell_2}) \int_{\mathbb{R}} |C_{\ell_1}(\tau)||C_{\ell_2}(\tau)|\,\mathrm{d}\tau,$$

and the double sum over $\tilde{\Lambda}$ is finite thanks to Assumption 1 (recall that $2\beta_{\ell^*} > 1$) and (7). We then have

$$\sum_{q=2}^{+\infty} s_{2q}^2 \leq \sum_{\ell_1,\ell_2\in\tilde{\Lambda}} \dim(\mathcal{Y}_{\ell_1})\dim(\mathcal{Y}_{\ell_2}) \int_{\mathbb{R}} |C_{\ell_1}(\tau)||C_{\ell_2}(\tau)|\,\mathrm{d}\tau \sum_{q=2}^{+\infty} \sum_{n_1+\cdots+n_k=q} \frac{(\alpha_{2n_1,\ldots,2n_k}(u))^2}{(2n_1)!\cdots(2n_k)!},$$

where the sum over q is finite given (14).

4.1. Outline of the proofs

Our results in Theorems 1 and 2 rely on the non-universal asymptotic behavior of chaotic components in (12). Their proofs are fully inspired by [23].

Proposition 1. *Let* $q \geq 1$. *For* $2q\beta_\ell^* < 1$,

$$\lim_{T\to+\infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[2q])}{T^{2-2q\beta_{\ell^*}}} = \sum_{n_1+\cdots+n_k=q} \frac{(\alpha_{2n_1,\ldots,2n_k}(u))^2}{(2n_1)!\cdots(2n_k)!} \times \sum_{\ell_1,\ldots,\ell_{2q}\in\mathcal{I}^*}^{+\infty} \kappa_{\ell_1}\cdots\kappa_{\ell_{2q}} \frac{2C_{\ell_1}(0)\cdots C_{\ell_{2q}}(0)}{(1-2q\beta_{\ell^*})(2-2q\beta_{\ell^*})} \times \int_{(\mathbb{M}^d)^2} P_{\ell_1}^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))\cdots P_{\ell_{2q}}^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y);$$

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for $2q\beta_\ell^* = 1$,

$$\lim_{T\to+\infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[2q])}{T\log T} = \sum_{n_1+\cdots+n_k=q} \frac{(\alpha_{2n_1,\ldots,2n_k}(u))^2}{(2n_1)!\cdots(2n_k)!} \times \sum_{\ell_1,\ldots,\ell_{2q}\in\mathcal{I}^*}^{+\infty} \kappa_{\ell_1}\cdots\kappa_{\ell_{2q}} 2C_{\ell_1}(0)\cdots C_{\ell_{2q}}(0) \times \int_{(\mathbb{M}^d)^2} P_{\ell_1}^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))\cdots P_{\ell_{2q}}^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y);$$

and finally, for $2q\beta_\ell^* > 1$,

$$\lim_{T\to+\infty} \frac{\mathrm{Var}(\mathcal{M}_T(u)[2q])}{T} = s_{2q}^2,$$

where s_{2q}^2 *is as in (22) for* $q = 1$ *and (23) for* $q \geq 2$.

The proof of Proposition 1 is postponed to Section 5.1. Proposition 1 implies the following.

Corollary 1. *For* $2\beta_{\ell^*} < 1$, *as* $T \to +\infty$,

$$\frac{\mathcal{M}_T(u)}{\sqrt{\mathrm{Var}(\mathcal{M}_T(u))}} = \frac{\mathcal{M}_T(u)[2]}{\sqrt{\mathrm{Var}(\mathcal{M}_T(u)[2])}} + o_{\mathbb{P}}(1), \tag{24}$$

where $o_{\mathbb{P}}(1)$ *denotes a sequence of r.v.s converging to zero in probability.*

Proposition 2. *For* $2\beta_{\ell^*} < 1$, *as* $T \to +\infty$,

$$\frac{\mathcal{M}_T(u)[2]}{\sqrt{\mathrm{Var}(\mathcal{M}_T(u)[2])}} \xrightarrow{\mathrm{d}} \mathcal{R}(u),$$

where $\mathcal{R}(u)$ *is as in (20).*

The proof of Proposition 2 is based on non-central limit theorems for Hermite transforms of long-memory stochastic processes [5]. Proposition 2 together with (24) allow us to prove Theorem 1.

From Proposition 1, for $2\beta_{\ell^*} > 1$, the order of magnitude of $\mathrm{Var}(\mathcal{M}_T(u)[2q])$ is T for every $q \geq 1$ (since $2q\beta_{\ell^*} > 1$ for every $q \geq 1$).

Proposition 3. *For any* $q \geq 1$ *such that* $2q\beta_{\ell^*} > 1$,

$$\frac{\mathcal{M}_T(u)[2q]}{\sqrt{\mathrm{Var}(\mathcal{M}_T(u)[2q])}} \xrightarrow{\mathrm{d}} Z,$$

where $Z \sim \mathcal{N}(0,1)$.

The proof of Theorem 2 is then an application of the central limit theorem à la Breuer–Major via chaotic expansion [26, Theorem 6.3.1], whose key intermediate step is indeed Proposition 3.

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5. Proofs of the main results

5.1. Asymptotic variance

Let us study the variance of each chaotic component in (12). By independence and equidistribution of the fields $Z_1, \dots, Z_k$, and standard properties of Hermite polynomials [26, Section 1.4], we can write, for any $q \geq 1$,

$$\begin{aligned}
\mathrm{Var}(\mathcal{M}_T(u)[2q]) &= \sum_{\substack{n_1+\cdots+n_k=q \\ n_1'+\cdots+n_k'=q}} \frac{\alpha_{2n_1,\dots,2n_k}(u)}{(2n_1)!\cdots(2n_k)!} \frac{\alpha_{2n_1',\dots,2n_k'}(u)}{(2n_1')!\cdots(2n_k')!} \\
&\quad \times \int_{[0,T]^2} \int_{(\mathbb{M}^d)^2} \prod_{j=1}^{k} \mathbb{E}\big[H_{2n_j}(Z_j(x,t))H_{2n_j'}(Z_j(y,s))\big]\, \mathrm{d}\nu(x)\,\mathrm{d}\nu(y)\,\mathrm{d}t\,\mathrm{d}s \\
&= \sum_{n_1+\cdots+n_k=q} \frac{(\alpha_{2n_1,\dots,2n_k}(u))^2}{(2n_1)!\cdots(2n_k)!} \\
&\quad \times \int_{[0,T]^2} \int_{(\mathbb{M}^d)^2} (\Sigma(\rho(x,y),t-s))^{2q}\, \mathrm{d}\nu(x)\,\mathrm{d}\nu(y)\,\mathrm{d}t\,\mathrm{d}s \\
&= \sum_{n_1+\cdots+n_k=q} \frac{(\alpha_{2n_1,\dots,2n_k}(u))^2}{(2n_1)!\cdots(2n_k)!} \\
&\quad \times \sum_{\ell_1,\dots,\ell_{2q}\in\tilde{\Lambda}}^{+\infty} \kappa_{\ell_1}\cdots\kappa_{\ell_{2q}} \int_{[0,T]^2} C_{\ell_1}(t-s)\cdots C_{\ell_{2q}}(t-s)\,\mathrm{d}t\,\mathrm{d}s \\
&\quad \times \int_{(\mathbb{M}^d)^2} P_{\ell_1}^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))\cdots P_{\ell_{2q}}^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y). \qquad (25)
\end{aligned}$$

In particular, for $q=1$, thanks to the addition formula (5) and the *symmetry* of chaotic coefficients,

$$\begin{aligned}
\mathrm{Var}(\mathcal{M}_T(u)[2]) &= k\frac{(\alpha_{2,0,\dots,0}(u))^2}{2!} \sum_{\ell\in\tilde{\Lambda}} \kappa_\ell^2 \frac{(P_\ell^{(\alpha,\beta)}(1))^2}{\dim(\mathcal{Y}_\ell)} \int_{[0,T]^2} (C_\ell(t-s))^2\,\mathrm{d}t\,\mathrm{d}s \\
&= k\frac{(\alpha_{2,0,\dots,0}(u))^2}{2!} \sum_{\ell\in\tilde{\Lambda}} \dim(\mathcal{Y}_\ell) \int_{[0,T]^2} (C_\ell(t-s))^2\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}$$

For notational convenience, let us set, for $\ell\in\tilde{\Lambda}$, $K_\ell(T) := \int_{[0,T]^2} (C_\ell(t-s))^2\,\mathrm{d}t\,\mathrm{d}s$ and, for $q\geq 2$, $\ell_1,\dots,\ell_{2q}\in\tilde{\Lambda}$, $K_{\ell_1,\dots,\ell_{2q}}(T) := \int_{[0,T]^2} C_{\ell_1}(t-s)\cdots C_{\ell_{2q}}(t-s)\,\mathrm{d}t\,\mathrm{d}s$.

Proof of Proposition 1. From [23, Lemma 4.4], for $\ell\in\tilde{\Lambda}$, if $2\beta_\ell < 1$,

$$\lim_{T\to+\infty} \frac{K_\ell(T)}{T^{2-2\beta_\ell}} = \frac{2(C_\ell(0))^2}{(1-2\beta_\ell))(2-2\beta_\ell)};$$

for $2\beta_\ell = 1$,

$$\lim_{T\to+\infty} \frac{K_\ell(T)}{T\log T} = 2(C_\ell(0))^2;$$

finally, for $2\beta_\ell > 1$,

$$\lim_{T\to+\infty} \frac{K_\ell(T)}{T} = \int_{\mathbb{R}} (C_\ell(\tau))^2\,\mathrm{d}\tau.$$

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From [23, Lemma 4.11], for every $q \geq 2$, for $\beta_{\ell_1} + \cdots + \beta_{\ell_{2q}} < 1$,

$$\lim_{T\to+\infty} \frac{K_{\ell_1,\ldots,\ell_{2q}}(T)}{T^{2-(\beta_{\ell_1}+\cdots+\beta_{\ell_{2q}})}} = \frac{2C_{\ell_1}(0)\cdots C_{\ell_{2q}}(0)}{(1-(\beta_{\ell_1}+\cdots+\beta_{\ell_{2q}}))(2-(\beta_{\ell_1}+\cdots+\beta_{\ell_{2q}}))};$$

for $\beta_{\ell_1} + \cdots + \beta_{\ell_{2q}} = 1$,

$$\lim_{T\to+\infty} \frac{K_{\ell_1,\ldots,\ell_{2q}}(T)}{T\log T} = 2C_{\ell_1}(0)\cdots C_{\ell_{2q}}(0);$$

finally, for $\beta_{\ell_1} + \cdots + \beta_{\ell_{2q}} > 1$,

$$\lim_{T\to+\infty} \frac{K_{\ell_1,\ldots,\ell_{2q}}(T)}{T} = \int_{\mathbb{R}} C_{\ell_1}(\tau)\cdots C_{\ell_{2q}}(\tau)\,\mathrm{d}\tau.$$

Now, consider the series in (25) divided respectively by $T^{2-2q\beta_{\ell^*}}$ if $2q\beta_{\ell^*} < 1$, by $T\log T$ if $2q\beta_{\ell^*} = 1$, and by T if $2q\beta_{\ell^*} > 1$. Thanks to [23, Lemmas 4.7–4.10], [23, Lemmas 4.13–4.17], and the fact that

$$\begin{aligned}
&\sum_{\ell_1,\ldots,\ell_{2q}\in\tilde{\Lambda}} \kappa_{\ell_1}\cdots\kappa_{\ell_{2q}} C_{\ell_1}(0)\cdots C_{\ell_{2q}}(0)\\
&\quad\times \int_{(\mathbb{M}^d)^2} \left|P_{\ell_1}^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))\cdots P_{\ell_{2q}}^{(\alpha,\beta)}(\cos\varepsilon\rho(x,y))\right| \mathrm{d}\nu(x)\,\mathrm{d}\nu(y)\\
&\qquad\leq \sum_{\ell_1,\ldots,\ell_{2q}\in\tilde{\Lambda}} \kappa_{\ell_1}\cdots\kappa_{\ell_{2q}} C_{\ell_1}(0)\cdots C_{\ell_{2q}}(0) P_{\ell_1}^{(\alpha,\beta)}(1)\cdots P_{\ell_{2q}}^{(\alpha,\beta)}(1)\\
&\qquad= \sum_{\ell_1,\ldots,\ell_{2q}\in\tilde{\Lambda}} \dim(\mathcal{Y}_{\ell_1})\cdots\dim(\mathcal{Y}_{\ell_{2q}}) C_{\ell_1}(0)\cdots C_{\ell_{2q}}(0)\\
&\qquad= \left(\sum_{\ell\in\tilde{\Lambda}} \dim(\mathcal{Y}_\ell) C_\ell(0)\right)^{2q} = 1
\end{aligned}$$

by (7), we can apply the dominated convergence theorem as in the proofs of [23, Propositions 4.5 and 4.12], thus concluding the proof of our Proposition 1. □

5.2. Limiting distribution

Proof of Proposition 2. Due to symmetry of chaotic coefficients in (13), and the definitions in Assumption 2 and (19),

$$\begin{aligned}
\frac{\mathcal{M}_T(u)[2]}{T^{1-\beta_{\ell^*}}} &= \frac{1}{T^{1-\beta_{\ell^*}}}\frac{\alpha_{2,0,\ldots,0}(u)}{2!}\sum_{j=1}^{k}\int_0^T\int_{\mathbb{M}^d} H_2(Z_j(x,t))\,\mathrm{d}x\,\mathrm{d}t\\
&= \frac{1}{T^{1-\beta_{\ell^*}}}\frac{\alpha_{2,0,\ldots,0}(u)}{2!}\sum_{j=1}^{k}\sum_{\ell_j\in\mathcal{I}^*} C_{\ell_j}(0)\sum_{m_j=1}^{\dim(\mathcal{Y}_{\ell_j})}\int_0^T H_2\left(\frac{a_{\ell_j,m_j}^{(j)}(t)}{\sqrt{C_{\ell_j}(0)}}\right)\mathrm{d}t + o_{\mathbb{P}}(1),
\end{aligned} \tag{26}$$

where $a_{\ell_j,m_j}^{(j)}$, $\ell_j\in\tilde{\Lambda}$, $m_j = 1,\ldots,\dim(\mathcal{Y}_{\ell_j})$ are k i.i.d. copies of $a_{\ell,m}$, $\ell\in\tilde{\Lambda}$, $m = 1,\ldots,\dim(\mathcal{Y}_\ell)$.

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From (26) and [5, Theorem 1] we have

$$\begin{aligned}\frac{\mathcal{M}_T(u)[2]}{T^{1-\beta_{\ell^*}}} &= \frac{1}{T^{1-\beta_{\ell^*}}}\frac{\alpha_{2,0,\dots,0}(u)}{2!}\sum_{j=1}^{k}\sum_{\ell_j\in\mathcal{I}^*}C_{\ell_j}(0)\sum_{m_j=1}^{\dim(\mathcal{Y}_{\ell_j})}\int_0^T H_2\left(\frac{a^{(j)}_{\ell_j,m_j}(t)}{\sqrt{C_{\ell_j}(0)}}\right)\mathrm{d}t + o_{\mathbb{P}}(1)\\ &= \frac{\alpha_{2,0,\dots,0}(u)}{2!}\sum_{j=1}^{k}\sum_{\ell_j\in\mathcal{I}^*}\frac{C_{\ell_j}(0)}{a(\beta_{\ell^*})}\sum_{m_j=1}^{\dim(\mathcal{Y}_{\ell_j})}\frac{a(\beta_{\ell^*})}{T^{1-\beta_{\ell^*}}}\int_0^T H_2\left(\frac{a^{(j)}_{\ell_j,m_j}(t)}{\sqrt{C_{\ell_j}(0)}}\right)\mathrm{d}t + o_{\mathbb{P}}(1)\\ &\xrightarrow{\mathrm{d}} \frac{\alpha_{2,0,\dots,0}(u)}{2!}\sum_{j=1}^{k}\sum_{\ell_j\in\mathcal{I}^*}\frac{C_{\ell_j}(0)}{a(\beta_{\ell^*})}\sum_{m_j=1}^{\dim(\mathcal{Y}_{\ell_j})}X_{m_j;\beta_{\ell^*}},\end{aligned}$$

where $X_{m_j;\beta_{\ell^*}}$, $j=1,\dots,k$, $m_j=1,\dots,\dim(\mathcal{Y}_{\ell_j})$, are i.i.d. copies of a Rosenblatt r.v. with parameter β_{ℓ^*} (recall Definition 2). □

The proof of Proposition 3 is similar to the proof of [23, Proposition 5.3], hence we omit the details.

Appendix A. Technical proofs

Proof of Lemma 3. For $k=1$, since $N_1 =^{\mathrm{d}} -N_1$ it is easy to check that, for all $u>0$,

$$\alpha_{2n_1}(u) = 2\mathbb{E}\big[\mathbf{1}_{N_1\geq\sqrt{u}}H_{2n_1}(N_1)\big] = 2\int_{\sqrt{u}}^{+\infty}H_{2n_1}(t)\phi(t)\,\mathrm{d}t = 2\phi(\sqrt{u})H_{2n_1-1}(\sqrt{u}),$$

where ϕ still denotes the standard Gaussian density (in the last equality, it is enough to use the definition of Hermite polynomials).

The approach followed is inspired by [22]. Recall the generating function of Hermite polynomials [26, Proposition 1.4.2],

$$\mathrm{e}^{tr-(t^2/2)} = \sum_{q=0}^{+\infty}H_q(r)\frac{t^q}{q!}\quad \text{in } L^2(\mathbb{R},\phi(r)\,\mathrm{d}r), \tag{27}$$

and, for $k\geq 2$, consider the integral

$$\begin{aligned}&\frac{1}{(2\pi)^{k/2}}\int_{\mathbb{R}^k}\mathbf{1}_{\{x_1^2+\cdots+x_k^2\geq u\}}\mathrm{e}^{\lambda_1x_1-(\lambda_1^2/2)}\cdots\mathrm{e}^{\lambda_kx_k-(\lambda_k^2/2)}\mathrm{e}^{-(x_1^2+\cdots+x_k^2)/2}\,\mathrm{d}x_1\,\cdots\,\mathrm{d}x_k\\ &\qquad= \frac{1}{(2\pi)^{k/2}}\int_{\mathbb{R}^k}\mathbf{1}_{\{x_1^2+\cdots+x_k^2\geq u\}}\mathrm{e}^{-((x_1-\lambda_1)^2+\cdots+(x_k-\lambda_k)^2)/2}\,\mathrm{d}x_1\,\cdots\,\mathrm{d}x_k.\end{aligned}$$

This integral coincides with $\mathbb{P}(X_1^2+\cdots+X_k^2\geq u)$, where $X_1,\dots,X_k$ are independent Gaussian r.v.s with variance one and mean $\mathbb{E}[X_i]=\lambda_i$, $i=1,\dots,k$. Note that $X_1^2+\cdots+X_k^2\sim\chi^2(k,\lambda_1^2+\cdots+\lambda_k^2)$ is a non-central chi-square r.v. with k degrees of freedom and non-centrality parameter $\lambda_1^2+\cdots+\lambda_k^2$.

It is known that

$$\mathbb{P}(\chi^2(k,\lambda_1^2+\cdots+\lambda_k^2)\geq u) = Q_{k/2}\left(\sqrt{\lambda_1^2+\cdots+\lambda_k^2},\sqrt{u}\right), \tag{28}$$

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where Q is the so-called $(k/2)$th-order generalized Marcum Q-function (see, e.g., [28]). The canonical representation of the νth-order generalized Marcum function Q_ν is

$$Q_\nu(a,b) = 1 - \mathrm{e}^{-a^2/2} \sum_{l=0}^{\infty} \frac{1}{l!} \frac{\gamma(\nu+l, \frac{b^2}{2})}{\Gamma(\nu+l)} \left(\frac{a^2}{2}\right)^l \tag{29}$$

(for $\nu, a > 0$ and $b \geq 0$), where γ is the lower incomplete Gamma function, and Γ denotes the (standard) Gamma function. Hence, from (28) and (29) we can write

$$\begin{aligned}
&1 - \mathbb{P}(\chi^2(k, \lambda_1^2 + \cdots + \lambda_k^2) \geq u) \\
&= \mathrm{e}^{-\lambda_1^2+\cdots+\lambda_k^2/2} \sum_{l=0}^{+\infty} \frac{1}{l!} \frac{\gamma(k/2+l, u/2)}{\Gamma(k/2+l)} \left(\frac{\lambda_1^2 + \cdots + \lambda_k^2}{2}\right)^l \\
&= \mathrm{e}^{-\lambda_1^2+\cdots+\lambda_k^2/2} \sum_{l=0}^{+\infty} \frac{1}{l!} \frac{\gamma(k/2+l, u/2)}{\Gamma(k/2+l)} \sum_{m_1+\cdots+m_k=l} \frac{1}{2^l} \binom{l}{m_1, \ldots, m_k} \lambda_1^{2m_1} \cdots \lambda_k^{2m_k} \\
&= \sum_{l, r_1, \ldots, r_k=0}^{+\infty} \frac{(-1)^{r_1+\cdots+r_k}}{2^{r_1} \cdots 2^{r_k} r_1! \cdots r_k!} \frac{1}{l!} \frac{\gamma(k/2+l, u/2)}{\Gamma(k/2+l)} \sum_{m_1+\cdots+m_k=l} \frac{1}{2^l} \binom{l}{m_1, \ldots, m_k} \\
&\quad \times (2m_1+2r_1)! \cdots (2m_k+2r_k)! \frac{\lambda_1^{2m_1+2r_1}}{(2m_1+2r_1)!} \cdots \frac{\lambda_k^{2m_k+2r_k}}{(2m_k+2r_k)!}.
\end{aligned}$$

Note that the series is absolutely summable. Performing the change of variable $2t_1 = 2m_1 + 2r_1$, $2t_k = 2m_k + 2r_k$, we get

$$\begin{aligned}
&\sum_{l, r_1, \ldots, r_k=0}^{+\infty} \frac{(-1)^{r_1+\cdots+r_k}}{2^{r_1} \cdots 2^{r_k} r_1! \cdots r_k!} \frac{1}{l!} \frac{\gamma(k/2+l, u/2)}{\Gamma(k/2+l)} \sum_{m_1+\cdots+m_k=l} \frac{1}{2^l} \binom{l}{m_1, \ldots, m_k} \\
&\quad \times (2m_1+2r_1)! \cdots (2m_k+2r_k)! \frac{\lambda_1^{2m_1+2r_1}}{(2m_1+2r_1)!} \cdots \frac{\lambda_k^{2m_k+2r_k}}{(2m_k+2r_k)!} \\
&\qquad = \sum_{t_1, \ldots, t_k=0}^{+\infty} (2t_1)! \cdots (2t_k)! \sum_{r_1 \leq t_1, \ldots, r_k \leq t_k} \frac{(-1)^{r_1+\cdots+r_k}}{2^{r_1} \cdots 2^{r_k} r_1! \cdots r_k!} \\
&\qquad\quad \times \frac{1}{(t_1 - r_1 + \cdots + t_k - r_k)!} \frac{\gamma(k/2 + t_1 - r_1 + \cdots + t_k - r_k, u/2)}{\Gamma(k/2 + t_1 - r_1 + \cdots + t_k - r_k)} \\
&\qquad\quad \times \frac{1}{2^{t_1 - r_1 + \cdots + t_k - r_k}} \binom{t_1 - r_1 + \cdots + t_k - r_k}{t_1 - r_1, \ldots, t_k - r_k} \frac{\lambda_1^{2t_1}}{(2t_1)!} \cdots \frac{\lambda_k^{2t_k}}{(2t_k)!}.
\end{aligned}$$

On the other hand, due to (27) we have the following (we can exchange integrals and summation by applying Fubini's theorem thanks to standard estimates for Hermite

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polynomials [8, (30)]):

$$\begin{aligned}
&\frac{1}{(2\pi)^{k/2}}\int_{\mathbb{R}^k}\mathbf{1}_{x_1^2+\cdots+x_k^2\geq u}\sum_{t_1=0}^{+\infty}H_{t_1}(x_1)\frac{\lambda_1^{t_1}}{t_1!}\cdots\sum_{t_k=0}^{+\infty}H_{t_k}(x_k)\frac{\lambda_k^{t_k}}{t_k!}\mathrm{e}^{-(x_1^2+\cdots+x_k^2)/2}\,\mathrm{d}x_1\,\cdots\,\mathrm{d}x_k\\
&=\sum_{t_1,\ldots,t_k=0}^{+\infty}\underbrace{\frac{1}{(2\pi)^{k/2}}\int_{\mathbb{R}^k}\mathbf{1}_{x_1^2+\cdots+x_k^2\geq u}H_{t_1}(x_1)\cdots H_{t_k}(x_k)\mathrm{e}^{-(x_1^2+\cdots+x_k^2)/2}\,\mathrm{d}x_1\,\cdots\,\mathrm{d}x_k}_{=\alpha_{t_1,\ldots,t_k}(u)}\frac{\lambda_1^{t_1}}{t_1!}\cdots\frac{\lambda_k^{t_k}}{t_k!}\\
&=\sum_{t_1,\ldots,t_k=0}^{+\infty}\alpha_{2t_1,\ldots,2t_k}(u)\frac{\lambda_1^{2t_1}}{(2t_1)!}\cdots\frac{\lambda_k^{2t_k}}{(2t_k)!},
\end{aligned}$$

where the last two equalities follow from (13) and Lemma 1 respectively. Hence, necessarily,

$$\alpha_{(0,\ldots,0)}(u)=1-\frac{\gamma(k/2,u/2)}{\Gamma(k/2)},$$

and more generally,

$$\begin{aligned}
\alpha_{2t_1,\ldots,2t_k}(u)&=(2t_1)!\cdots(2t_k)!\sum_{r_1\leq t_1,\ldots,r_k\leq t_k}\frac{(-1)^{r_1+\cdots+r_k+1}}{2^{r_1}\cdots 2^{r_k}r_1!\cdots r_k!}\\
&\quad\times\frac{1}{(t_1-r_1+\cdots+t_k-r_k)!}\frac{\gamma(k/2+t_1-r_1+\cdots+t_k-r_k,u/2)}{\Gamma(k/2+t_1-r_1+\cdots+t_k-r_k)}\\
&\quad\times\frac{1}{2^{t_1-r_1+\cdots+t_k-r_k}}\binom{t_1-r_1+\cdots+t_k-r_k}{t_1-r_1,\ldots,t_k-r_k}.
\end{aligned}$$

For instance, for $t_1=q$, $t_2=\cdots=t_k=0$ we have

$$\begin{aligned}
\alpha_{(2q,0,\ldots,0)}(u)&=(2q)!\sum_{r_1\leq q}\frac{(-1)^{r_1+1}}{2^{r_1}r_1!}\frac{1}{2^{q-r_1}(q-r_1)!}\frac{\gamma(k/2+q-r_1,u/2)}{\Gamma(k/2+q-r_1)}\\
&=(2q)!\sum_{r_1\leq q}\frac{(-1)^{r_1+1}}{q!2^q}\binom{q}{r_1}\frac{\gamma(k/2+q-r_1,u/2)}{\Gamma(k/2+q-r_1)}\\
&=\frac{(2q)!}{q!2^q}\sum_{r_1\leq q}(-1)^{r_1+1}\binom{q}{r_1}\frac{\gamma(k/2+q-r_1,u/2)}{\Gamma(k/2+q-r_1)}.
\end{aligned}$$

□

Proof of (15).We can write

$$\begin{aligned}
\alpha_{2,0,\ldots,0}(u)&=\mathbb{E}\big[\mathbb{E}\big[\mathbf{1}_{\sum_{j=1}^k N_j^2\geq u}H_2(N_1)\,|\,N_2,\ldots,N_k\big]\big]\\
&=\mathbb{E}\Big[\mathbb{E}\big[\mathbf{1}_{N_1^2+\sum_{j=2}^k x_j^2\geq u}H_2(N_1)\big]_{x_2=N_2,\ldots,x_k=N_k}\Big].
\end{aligned}$$

Now observe that, if $u-\sum_{j=2}^k x_j^2\leq 0$, then $\mathbb{E}\big[\mathbf{1}_{N_1^2+\sum_{j=2}^k x_j^2\geq u}H_2(N_1)\big]=\mathbb{E}[H_2(N_1)]=0$. On the other hand, if $u-\sum_{j=2}^k x_j^2>0$,

$$\mathbb{E}\Big[\mathbf{1}_{N_1^2+\sum_{j=2}^k x_j^2\geq u}H_2(N_1)\Big]=2\phi\left(\sqrt{u-\sum_{j=2}^k x_j^2}\right)H_1\left(\sqrt{u-\sum_{j=2}^k x_j^2}\right)>0.$$

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Hence,

$$\alpha_{2,0,\dots,0}(u) = \mathbb{E}\left[\mathbf{1}_{u-\sum_{j=2}^{k} N_j^2>0}\, 2\phi\left(\sqrt{u-\sum_{j=2}^{k} N_j^2}\right) H_1\left(\sqrt{u-\sum_{j=2}^{k} N_j^2}\right)\right] > 0,$$

thus concluding the proof. □

Remark 4. More generally, we can deduce the following formula: for $n_1 \in \mathbb{N}$,

$$\alpha_{2n_1,0,\dots,0}(u) = \mathbb{E}\left[\mathbf{1}_{u-\sum_{j=2}^{k} N_j^2>0}\, 2\phi\left(\sqrt{u-\sum_{j=2}^{k} N_j^2}\right) H_{2n_1-1}\left(\sqrt{u-\sum_{j=2}^{k} N_j^2}\right)\right].$$

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