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Circle actions and suspension operations on smooth manifolds

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Abstract

Let M be a smooth manifold with $\dim M \geq 3$ and a base point x_0. Surgeries along the oriented circle $S^1 \times \{x_0\}$ on the product $S^1 \times M$ yield two manifolds $\Sigma_0 M$ and $\Sigma_1 M$, called the suspensions of M.

These suspension operations play a fundamental role in the construction and classification of smooth manifolds admitting free circle actions. This paper presents corresponding results and supporting evidence.

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1. *Introduction and main results*

Let $\mathcal{T}_n$ be the set of diffeomorphism classes of closed, oriented and smooth manifolds of dimension n. A free circle action $S^1 \times M \to M$ on a manifold $M \in \mathcal{T}_n$ is called *regular* if the quotient space M/S^1, equipped with the induced topology and orientation, belongs to $\mathcal{T}_{n-1}$. For a manifold $M \in \mathcal{T}_n$ we would like to know whether there is a regular circle action on M. If the answer is affirmative, we would like to decide furthermore the diffeomorphism types of the quotients M/S^1. If M is a homotopy sphere, these problems were raised and studied by Montgemory–Yang, Edmonds, Schultz, Madsen–Milgram and Wall [**9**, **17**–**23**, **29**]. In the case M is an 1-connected 5-manifolds, the problems was solved by Duan and Liang in [**8**].

This paper introduces a general method to construct smooth manifolds M that admit regular circle actions, where the diffeomorphism types of the quotients M/S^1 can be made explicit. The central ingredient of this construction is the *suspension operations* $\Sigma_i : \mathcal{T}_n \to \mathcal{T}_{n+1}$ (with $i = 0, 1$) we are about to describe. In this paper D^n denotes the unit disc on the Euclidean n-space $\mathbb{R}^n$, with boundary the $(n-1)$-sphere $S^{n-1} = \partial D^n$. We shall assume that $n \geq 3$, unless otherwise stated.

The fundamental group $\pi_1(SO(n))$ of the special orthogonal group $SO(n)$ with order $n \geq 3$ is isomorphic to $\mathbb{Z}_2$ with the generator

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$$\alpha(t) = \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix} \oplus I_{n-2}, t \in [0, 1],$$

where I_r denotes the identity matrix of rank r. It gives rise to the diffeomorphism of the product $S^1 \times D^n$

$$\widetilde{\alpha} : S^1 \times D^n \to S^1 \times D^n \text{ by } \widetilde{\alpha}(t, x) = (t, \alpha(t)x), \tag{1·1}$$

whose restriction on the boundary $S^1 \times S^{n-1}$ is denoted by τ. Recall that any orientation preserving diffeomorphism of $S^1 \times S^{n-1}$ is isotopic either to identity id or to τ (e.g. [**29**, p.241]). For a manifold $N \in \mathcal{T}_n$ with a base point $x_0 \in N$, and an orientation preserving smooth embedding $(D^n, 0) \subset (N, x_0)$ centered at x_0, we introduce two adjoint manifolds $\Sigma_i N$, $i = 0, 1$, by

$$\begin{aligned} \Sigma_0 N &:= D^2 \times S^{n-1} \cup_{id} (S^1 \times (N - \overset{\circ}{D^n})), \\ \Sigma_1 N &:= D^2 \times S^{n-1} \cup_{\tau} (S^1 \times (N - \overset{\circ}{D^n})), \end{aligned} \tag{1·2}$$

respectively, where $\cup_f$ means gluing the boundaries together by the diffeomorphism f. Alternatively, the manifolds $\Sigma_0 N$ and $\Sigma_1 N$ are obtained by surgeries along the oriented circle $S^1 \times \{x_0\}$ on $S^1 \times N$ with the framings id and $\widetilde{\alpha}$, respectively. By the homogeneity lemma [**19**, section 4] the diffeomorphism types of $\Sigma_i N$ are irrelevant to the choice of a smooth embedding $(D^n, 0) \subset (N, x_0)$. This justifies the following notion.

Definition 1·1. The correspondences $\Sigma_i : \mathcal{T}_n \to \mathcal{T}_{n+1}$ by $N \to \Sigma_i N$ $(i = 0, 1)$ are called *the suspension operations on* $\mathcal{T}_n$.

The relationship between the two suspensions $\Sigma_0 N$ and $\Sigma_1 N$ is quite subtle. We can only derive partial information about it. Let $Diff(N, x_0)$ be the group of diffeomorphisms of $N \in \mathcal{T}_n$ that fix the base point x_0, and let $GL(n)$ be the general linear group of order n. Sending a diffeomorphism $h \in Diff(N, x_0)$ to the tangent map of h at x_0 yields a homomorphism

$$e_N : Diff(N, x_0) \to GL(n), h \to T_{x_0} h,$$

called *the linear representation* of the group $Diff(N, x_0)$. A manifold $N \in \mathcal{T}_n$ $(n \geq 3)$ is called Σ*-stable* if the induced action on the fundamental groups

$$e_{N*} : \pi_1(Diff(N, x_0), id) \to \pi_1(GL(n), I_n) = \mathbb{Z}_2$$

surjects. The following result will be shown in Section 3.

THEOREM A. *If $N \in \mathcal{T}_n$ is Σ-stable, then $\Sigma_0 N = \Sigma_1 N$.*

As supplements to Theorem A we shall see that the product $S^k \times M$ with $k \geq 3$ are all Σ-stable. On the contrary, $\Sigma_1 M \neq \Sigma_0 M$ for the quaternionic projective plane $M = \mathbb{H}P^2$.

For a regular circle action on a manifold $E \in \mathcal{T}_n$ we can regard the quotient map

$$p : E \to B := E/S^1$$

as an oriented circle bundle over B. For a manifold $N \in \mathcal{T}_{n-1}$ let $B\#N$ be the connected sum of B and N, let $f_N : B\#N \to B$ be the canonical projection. Denote the induced

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bundle f_N^*E by

$$p_N : E_N \to B\#N. \tag{1·3}$$

Our main result expresses the total space E_N in terms of E and N, where $w_2(M)$ stands for the second Stiefel–Whitney class of a manifold M.

THEOREM B. *Let $S^1 \times E \to E$ be a regular circle action on an 1-connected manifold $E \in \mathcal{T}_n$ with quotient map $p : E \to B$, where $n \geq 5$.*

Then for any $N \in \mathcal{T}_{n-1}$ we have:

$$E_N = \begin{cases} E\#\Sigma_0 N \text{ if } w_2(B) \neq 0; \\ E\#\Sigma_1 N \text{ if } w_2(B) = 0. \end{cases} \tag{1·4}$$

In addition, if $p^*(w_2(B)) = w_2(E) \neq 0$, then:

$$E_N = E\#\Sigma_0 N = E\#\Sigma_1 N. \tag{1·5}$$

In Theorem B the 1-connectedness restriction on E can not be excluded. For an integer $q \geq 2$ let $p : E \to \mathbb{C}P^2$ be the oriented circle bundle over the complex projective plane with Euler class $q \cdot x$, where $x \in H^2(\mathbb{C}P^2)$ is a generator. Then E is the 5-dimensional lens space $L^5(q)$ whose fundamental group is cyclic with order q. It can be shown that E_N is homeomorphic neither to $E\#\Sigma_0 N$ nor to $E\#\Sigma_1 N$, unless $N \in \mathcal{T}_4$ is the sphere S^4.

Theorem B is useful to produce manifolds that admit regular circle actions with explicit quotients. We present such examples.

COROLLARY 1·3. *Let $E \in \mathcal{T}_n$ be a 1-connected manifold that admits a regular circle action with quotient B, where $n \geq 5$.*

Then, for any $N \in \mathcal{T}_{n-1}$ one of the two connected sums $E\#\Sigma_i N$, $i = 0, 1$, admits a regular circle action with quotient $B\#N$.

Example 1·4. Let $p : S^{2n+1} \to \mathbb{C}P^n$ be the Hopf fibration over the complex projective n-space $\mathbb{C}P^n$, $n \geq 2$. By Corollary 1·3, for any $N \in \mathcal{T}_{2n}$ either $\Sigma_0 N$ or $\Sigma_1 N$ admits a regular circle action with quotient $\mathbb{C}P^n\#N$. In particular, by the homotopy exact sequence of circle fibrations one concludes that the homotopy groups of the suspension $\Sigma_i N$ are:

$$\pi_r(\Sigma_i N) = \begin{cases} \pi_r(N) \text{ if } r = 1 \text{ or } 2, \\ \pi_r(\mathbb{C}P^n\#N) \text{ if } r \geq 3, \end{cases} \tag{1·6}$$

where $n \equiv i \bmod 2$.

In general, by a *fake complex projective n-space* we mean a smooth manifold $F_{\mathbb{C}P^n}$ that is homotopy equivalent to the complex projective space $\mathbb{C}P^n$, $n \geq 2$. For the classifications of such manifolds, see [**17**, theorem 4·9] and [**29**, chapter 14C]. In particular, if we let $p : E \to F_{\mathbb{C}P^n}$ be the oriented circle bundle on $F_{\mathbb{C}P^n}$ whose Euler class generates the group $H^2(F_{\mathbb{C}P^n}) = \mathbb{Z}$, then E is a homotopy $(2n+1)$-sphere. In addition, since the second Stiefel–Whitney class is a homotopy invariant of smooth manifolds, we have $w_2(F_{\mathbb{C}P^n}) \neq 0$ if and only if n even. Thus, Theorem B implies that, for any fake complex projective space $F_{\mathbb{C}P^n}$ and any $N \in \mathcal{T}_{2n}$, the (topological) manifold $\Sigma_i N$ admits a smooth structure, together with a regular circle action, whose quotient is diffeomorphic to $F_{\mathbb{C}P^n}\#N$, where $n \equiv i \bmod 2$.

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Example 1·5. For an integer $n \geq 2$ let $S^2 \widetilde{\times} S^n$ be the only nontrivial smooth S^n-bundle over S^2. Take $B := S^2 \times S^{n-2}$ or $S^2 \widetilde{\times} S^{n-2}$ with $n \geq 4$, and let $p : E \to B$ be the circle bundle over B whose Euler class is the generator of $H^2(B)$ corresponding to the orientation class of the factor $S^2 \subset B$. Since the total space $E = S^3 \times S^{n-2}$ is 1-connected, $w_2(E) = 0$, and since $w_2(B) \neq 0$ if and only if $B = S^2 \widetilde{\times} S^{n-2}$, we conclude by Corollary 1·3 that for any $N \in \mathcal{T}_{n-1}$ with $n \geq 5$, both manifolds $S^3 \times S^{n-2} \# \Sigma_i N$ admit regular circle actions with quotients $S^2 \widetilde{\times} S^{n-2} \# N$ and $S^2 \times S^{n-2} \# N$, respectively.

Combined with general properties of the suspensions Σ_i to be developed in Section 4, Theorem B is applicable to study the classification problem of the manifolds that admit regular circle actions. In the 1960's Smale and Barden [**2**, **24**] classified the 1-connected 5-manifolds in terms of a collection of five basic ones $\{S^2 \times S^3, S^2 \widetilde{\times} S^3, W, M_k, X_{2^i}\}$, where $k \geq 2, i \geq 1$, see Section 5 for detailed constructions of these manifolds. On the other hand, in [**12–13**] R. Goldstein and L. Lininger initiated the problem of classifying the 1-connected 6-manifolds that admit regular circle actions. Our next result completes the project (compare with [**16**, theorem 4, theorem 6]).

THEOREM C. *If M is an 1-connected 6-manifold that admits regular circle actions, then*:

$$M = \begin{cases} S^3 \times S^3 \#_r \Sigma_0(S^2 \times S^3) \#_{1 \leq j \leq t} \Sigma_1 M_{k_j} \# \Sigma_1 H \text{ if } w_2(M) \neq 0, \\ S^3 \times S^3 \#_r \Sigma_0(S^2 \times S^3) \#_{1 \leq j \leq t} \Sigma_i M_{k_j} \text{ if } w_2(M) = 0, \end{cases} \tag{1·7}$$

where

$$H \in \{S^2 \widetilde{\times} S^3, W, X_k\},\ \Sigma_0(S^2 \times S^3) = S^3 \times S^3 \# S^2 \times S^4,$$

and where the notation $\#_r N$ means the connected sum of r-copies of N (and so forth).

Let $T^k = S^1 \times \cdots \times S^1$ (k-copies) be the k-dimensional torus group with classifying space BT^k. For a (topological) manifold M, and a set of cohomology classes $\{\alpha_1, \cdots, \alpha_k\} \subset H^2(M)$ of degree 2, let $f_i : M \to \mathbb{C}P^\infty$ be the classifying map of α_i, and let

$$\pi : M(\alpha_1, \ldots, \alpha_k) \to M \tag{1·8}$$

be the principal T^k-bundle on M whose classifying map is the product

$$f = (f_1, \ldots, f_k) : M \to BT^k = \mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty \ (k\text{-copies}).$$

Theorem B can be iterated to calculate the homeomorphism type of the total space $M(\alpha_1, \ldots, \alpha_k)$. We illustrate the calculation for the cases where M is an 1-connected 4-manifold, which is not necessarily smoothable.

For an integer $k \geq 1$ define the sequence of integers $(b_1, \ldots, b_{\left[\frac{k}{2}\right]})$ by

$$b_i := (k-1)\binom{k-1}{i} - \binom{k-1}{i+1} + (k-1)\binom{k-1}{i-1} - \binom{k-1}{i-2}, 1 \leq i \leq \left[\frac{k}{2}\right].$$

Accordingly, let Q_k be the connected sum of the products of spheres

$$Q_k := \#_{c_1}(S^3 \times S^{k+1}) \#_{c_2}(S^4 \times S^k) \#_{c_3} \cdots \#_{c_r}(S^{r+2} \times S^{k-r+2}), r = \left[\frac{k}{2}\right], \tag{1·9}$$

where $c_i = b_i$ with the only exception that $c_r = 1/2 b_r$ if $k \equiv 0 \bmod 2$.

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Orlik and Raymond [22, p.553] (resp. Davis and Januszkiewicz [7, example 1·20]) proved that any *toric* 4*-manifold* is homeomorphic to a connected sum of copies of $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$ and $S^2 \times S^2$. Buchstaber and Panov [4, theorem 4·6·12] have further shown that (e.g. [3, theorem 6·3], [18]), if M is a toric 4-manifold and if $\{\alpha_1, \ldots, \alpha_k\}$ is the canonical basis of $H^2(M)$, then the manifold $M(\alpha_1, \ldots, \alpha_k)$ is diffeomorphic to Q_k. Indeed, this phenomenon holds for all 1-connected topological 4-manifolds. Let $\beta_2(X)$ denote the second Betti number of a CW-complex X.

THEOREM D. *If M is a 1-connected 4-manifold and if $\{\alpha_1, \ldots, \alpha_k\}$ is a basis of $H^2(M)$, $k = \beta_2(X)$, then $M(\alpha_1, \ldots, \alpha_k)$ is diffeomorphic to Q_k.*

In particular, any 1-connected 4-manifold M with $k = \beta_2(M)$ is the quotient of a (topological) principal T^k-action on Q_k.

From the homotopy exact sequence of the bundle π one finds that:

$$\pi_r(M(\alpha_1, \ldots, \alpha_k)) = \begin{cases} 0 \text{ for } r = 0, 1, 2; \\ \pi_r(M) \text{ for } r \geq 3 \text{ (induced by } \pi_*). \end{cases}$$

From $M(\alpha_1, \ldots, \alpha_k) \cong Q_k$ by Theorem C we get:

COROLLARY 1·6. *The 2–connected cover of any 1-connected 4-manifold M is homeomorphic to Q_k, where $k = \beta_2(M)$.*

In particular, for $3 \leq r \leq k+3$ the homotopy group $\pi_r(M)$ can be expressed in terms of that of spheres by the formula:

$$\pi_r(M) = \pi_r\Big(\underset{c_1}{\vee}(S^3 \vee S^{k+1}) \underset{c_2}{\vee}(S^4 \vee S^k) \underset{c_3}{\vee} \cdots \underset{c_r}{\vee}(S^{r+2} \vee S^{k-r+2})\Big), \tag{1·10}$$

as well as the Hilton–Milnor formula [30, p.533].

The map π in (1·8) induces the fibration $\Omega\pi : \Omega M(\alpha_1, \cdots, \alpha_k) \to \Omega M$ in the loop spaces, where $\Omega M(\alpha_1, \cdots, \alpha_k) \cong \Omega Q_k$ by Theorem D. Since the space ΩQ_k is 1-connected, while the fiber ΩT^k is homotopy equivalent to the Eilenberg–McLane space $K(\oplus_k \mathbb{Z}, 0)$, we get:

COROLLARY 1·7. *For any 1-connected 4-manifold M the universal cover of the loop space ΩM is ΩQ_k, where $k = \beta_2(M)$.*

The remaining sections of the paper are organised as follows. Section 2 introduces isotopy invariants for the framed circles on a manifold, and establishes Theorem B. Section 3 is devoted to show Theorem A, and present relevant examples. Based on general properties of the suspension operators Σ_i developed in Section 4, the proofs of Theorem C and D are completed in Sections 5 and 6, respectively.

2. *Invariants of framed circles on a manifold.*

For an orientation preserving smooth embedding $F : S^1 \times D^{n-1} \to M$ into a manifold $M \in \mathcal{T}_n$ we set $S^1 = F(S^1 \times \{0\}) \subset M$, and call the pair (S^1, F) a *framed circle* on M. Two framed circles (S^1, F) and (S^1, F') are called *isotopic* if there is a smooth embedding $H : (S^1 \times D^{n-1}) \times I \to M \times I$ such that $H((S^1 \times D^{n-1}) \times \{t\}) \subset M \times \{t\}$ and

$$H(x, 0) = (F(x), 0), H(x, 1) = (F'(x), 1), x \in S^1 \times D^{n-1}.$$

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Let $[S^1, M]^{fr}$ be the set of isotopy classes $[S^1, F]$ of framed circles on M, and write

$$\gamma : [S^1, M]^{fr} \to \pi_1(M), \gamma([S^1, F]) = [S^1],$$

for the projection of isotopy classes to homotopy classes. Observe that if $\widetilde{\alpha}$ is the diffeomorphism of $S^1 \times D^{n-1}$ in (1·1), then a framed circle (S^1, F) on M determines a new one $(S^1, F\widetilde{\alpha})$.

LEMMA 2·1. *If $n \geq 5$ the map γ surjects. Moreover, for any framed circle (S^1, F) on M we have $\gamma^{-1}[S^1] = \{[S^1, F], [S^1, F\widetilde{\alpha}]\}$.*

Proof. With $n \geq 5$ any map $S^1 \to M$ is homotopic to a smooth embedding, and any two homotopic embeddings $S^1 \subset M$ are isotopic (by a result of Whitney). Furthermore, since both S^1 and M are oriented, the normal disc bundle $N(S^1)$ of an embedding $S^1 \subset M$ has the induced orientation, hence has a trivialisation $N(S^1) \cong S^1 \times D^{n-1}$, showing that γ surjects.

As the group of trivialisations of $N(S^1)$ is $\pi_1(SO(n-1)) = \mathbb{Z}_2$, which is generated by α, the pre-image $\gamma^{-1}[S^1]$ of any $S^1 \subset M$ contains at most two elements: if one of them is $[S^1, F]$, the other must be $[S^1, F\widetilde{\alpha}]$.

By the proof of Lemma 2·1 we may consider a framed circle (S^1, F) as a tubular neighbourhood of the embedding $S^1 \subset M$; project on the factor D^{n-1}, and shrink the boundary to a point ∞, giving a sphere S^{n-1}, and finally extend the map of the tubular neighbourhood to M by mapping the complement to ∞. We have defined a map $M \to S^{n-1}$ whose homotopy class is denoted by $\gamma_1(S^1, F) \in [M, S^{n-1}]$. We use $\varepsilon(S^1, F) = 0$ or 1 to denote whether $\gamma_1(S^1, F) = 0$ or not.

LEMMA 2·2. *Let $M \in \mathcal{T}_n$ be 1-connected with $n \geq 5$. Then the map $\varepsilon : [S^1, M]^{fr} \to \mathbb{Z}_2$ injects, and satisfies the following relations*:

(i) *if $\varepsilon(S^1, F) = 1$ then $\varepsilon(S^1, F\widetilde{\alpha}) = 0$;*
(ii) *if $w_2(M) = 0$ and $\varepsilon(S^1, F) = 0$ then $\varepsilon(S^1, F\widetilde{\alpha}) = 1$;*
(iii) *if $w_2(M) \neq 0$ and $\varepsilon(S^1, F) = 0$ then $\varepsilon(S^1, F\widetilde{\alpha}) = 0$.*

In addition, for any $[S^1, F] \in [S^1, M]^{fr}$ we can write $M = M\#S^n$ in which the sphere S^n has the decomposition:

$$S^n = \begin{cases} D^2 \times S^{n-2} \cup_{id} (S^1 \times D^{n-1}) \text{ if } \varepsilon(S^1, F) = 0; \\ D^2 \times S^{n-2} \cup_{\tau} (S^1 \times D^{n-1}) \text{ if } \varepsilon(S^1, F) = 1, \end{cases} \tag{2·1}$$

such that the second component $S^1 \times D^{n-1}$ is identical to F.

Proof. The first assertion, together with the properties (i)–(iii), have been shown by Goldstein and Linger in [**13**, theorem 1].

Let $S^1 \subset M$ be an oriented circle on M. Since M is 1-connected with dimension $n \geq 5$ there is an embedded disc $D^2 \subset M$ with $\partial D^2 = S^1 \subset M$. Taking a tubular neighbourhood D^n of the disc D^2 one gets a frame F of the circle S^1 such that $\operatorname{Im} F \subset \overset{\circ}{D}{}^n$. That is, the manifold M has a decomposition $M = M\#S^n$ so that (S^1, F) is a framed circle on the sphere S^n.

For $M = S^n$ we have $[M, S^{n-1}] = \mathbb{Z}_2$, while the map ε is one to one, implying that S^n possesses the two decompositions stated in (2·1).

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COROLLARY 2·3. *If $M \in \mathcal{T}_n$ is 1-connected with $n \geq 5$, then $[S^1, M]^{fr} = \mathbb{Z}_2$ or 0, where $[S^1, M]^{fr} = \mathbb{Z}_2$ happens if and only if $w_2(M) = 0$.*

For a $N \in \mathcal{T}_{n-1}$ we furnish the product $S^1 \times N$ with the induced orientation, and let $(D^{n-1}, 0) \subset (N, x_0)$ be an orientation preserving embedded disc centered at x_0. Then the natural inclusion $F_0 : S^1 \times D^{n-1} \subset S^1 \times N$ gives rise to the *canonical frame* of the oriented circle $S^1 \times \{x_0\}$ on $S^1 \times N$, for which $\gamma(S^1 \times \{x_0\}, F_0)$ is the generator of the first summand of $\pi_1(S^1 \times N) = \mathbb{Z} \oplus \pi_1(N)$ that corresponds to the orientation on $S^1 \times \{x_0\}$, while $\gamma_1(S^1 \times \{x_0\}, F_0)$ is the homotopy class of the composition

$$c : S^1 \times N \to N \to N/(N - \overset{\circ}{D}_n) = S^{n-1}.$$

For a frame F of the circle $S^1 \times \{x_0\}$ on $S^1 \times N$ we write $\delta(S^1 \times \{x_0\}, F) = 0$ or 1 to denote the statement that $\gamma_1(S^1 \times \{x_0\}, F) = [c]$ or not. From Lemma 2·1 we get:

LEMMA 2·4. *If $n \geq 4$ and if F is a frame of the oriented circle $S^1 \times \{x_0\}$ on $S^1 \times N$, then $\delta(S^1 \times \{x_0\}, F) = 1$ implies that $\delta(S^1 \times \{x_0\}, F\widetilde{\alpha}) = 0$.*

Now, let (S^1, F) be a framed circle on a 1-connected manifold $M \in \mathcal{T}_n$ with $n \geq 5$, and let F' be one of the two possible frames of the oriented circle $S^1 \times \{x_0\}$ on $S^1 \times N$, $N \in \mathcal{T}_{n-1}$. *The tunnel sum* $\circ$ of M and $S^1 \times N$ along the framed circles (S^1, F) and $(S^1 \times \{x_0\}, F')$ is the adjoint manifold

$$(M, F) \circ (S^1 \times N, F') := (M - \overset{\circ}{\mathrm{Im}}\, F) \cup_{f \circ f'^{-1}} (S^1 \times N - \overset{\circ}{\mathrm{Im}}\, F'), \tag{2·2}$$

where

$$f := F \mid S^1 \times S^{n-2},\ f' := F' \mid S^1 \times S^{n-2}.$$

Useful properties of the operation $\circ$ are collected in the following lemma.

LEMMA 2·5. *The tunnel sum operation $\circ$ satisfies the following relations*:

$$(M, F) \circ (S^1 \times N, F') = (M, F\widetilde{\alpha}) \circ (S^1 \times N, F'\widetilde{\alpha}); \tag{2·3}$$

$$(M, F) \circ (S^1 \times N, F') = \begin{cases} M \# \Sigma_0 N \text{ if } \varepsilon(S^1, F) = \delta(S^1, F'); \\ M \# \Sigma_1 N \text{ if } \varepsilon(S^1, F) \neq \delta(S^1, F'). \end{cases} \tag{2·4}$$

In addition,

$$M \# \Sigma_0 N = M \# \Sigma_1 N \text{ if } w_2(M) \neq 0. \tag{2·5}$$

Proof. Since the generator $\alpha \in \pi_1(SO(n-1))$ is of order 2, the relation (2·3) follows from the definition of the tunnel sum in (2·2).

Since M is 1-connected, we can assume by Lemma 2·2 that $M = S^n$, and take a decomposition of S^n relative to the index $\varepsilon(S^1, F)$ as in (2·1). If $\varepsilon(S^1, F) = \delta(S^1, F') = 0$ (resp. if $\varepsilon(S^1, F) = 1$ but $\delta(S^1, F') = 0$) we get (2·4) directly from the definition of Σ_i:

$$(S^n, F) \circ (S^1 \times N, F') = D^2 \times S^{n-1} \cup_{id} (S^1 \times (N - \overset{\circ}{D^{n-1}})) = \Sigma_0 N$$

$$(\text{resp.} = D^2 \times S^{n-1} \cup_{\tau} (S^1 \times (N - \overset{\circ}{D^{n-1}}) = \Sigma_1 N).$$

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For the remaining case $\varepsilon(S^1, F) = \delta(S^1, F') = 1$ (resp. $\varepsilon(S^1, F) = 0$ but $\delta(S^1, F') = 1$), we can compose both F and F' with the diffeomorphism $\widetilde{\alpha}$ of $S^1 \times D^{n-1}$ by (2·3), then apply (i) of Lemma 2·2 and Lemma 2·4 to reduce the current case to the previous one.

Finally, the relation (2·5) has been shown by (iii) of Lemma 2·2.

We are ready to show Theorem B stated in Section 1.

Proof of Theorem B. Let $p : E \to B$ an oriented circle bundle over a manifold $B \in \mathcal{T}_{n-1}$, where the total space E is 1-connected with $\dim E = n \geq 5$. Take an orientation preserving embedding $i : D^{n-1} \subset B$ and set $i(0) = b \in B$. Then the inclusion $F_b : E \mid D^{n-1} \subset E$ defines a frame $(p^{-1}(b), F_b)$ of the oriented circle $p^{-1}(b) \subset E$, to be called a *canonical frame of* $p^{-1}(b)$. Recall from [**13**, theorem 8] that the function $\varepsilon : B \to \{0, 1\}$ by $\varepsilon(b) := \varepsilon(p^{-1}(b), F_b)$ is constant, and satisfies that

$$\varepsilon(p^{-1}(b), F_b) = 0 \text{ or } 1 \text{ in accordance to } w_2(B) \neq 0 \text{ or } w_2(B) = 0. \tag{2·6}$$

On the other hand, let $p' : S^1 \times N \to N$ be the trivial S^1-bundle on $N \in \mathcal{T}_{n-1}$. Take an orientation preserving embedding $i : D^{n-1} \subset N$ and set $x_0 := i(0) \in N$. Since the inclusion $F_{x_0} : p' \mid D^{n-1} \subset S^1 \times N$ is precisely the canonical frame $(p'^{-1}(x_0), F_{x_0})$ of the oriented circle $p'^{-1}(x_0) = S^1 \times x_0$ on $S^1 \times N$ we get by the definition of the index δ that

$$\delta(p'^{-1}(x_0), F_{x_0}) = 0. \tag{2·7}$$

Consider now the projection $f_N : B\#N \to B$ ($f_N(N - \overset{\circ}{D}{}^{n-1}) = b$), and let $p_N : E_N \to B\#N$ be the induced bundle $f_N^* E$. Then the total space E_N admits the decomposition into the tunnel sum

$$E_N = (E, F_b) \circ (S^1 \times N, F_{x_0}),$$

where $b \in B$, $x_0 \in N$, and where both $(p^{-1}(b), F_b)$ and $(S^1 \times \{x_0\}, F_{x_0})$ are canonical. In view of (2·6) and (2·7), the desired formulae (1·4) and (1·5) of Theorem B have been shown by the relations (2·4) and (2·5) in Lemma 2·5, respectively.

3. *Relationship between* $\Sigma_1 N$ *and* $\Sigma_0 N$

This section proves Theorem A, and illustrates related applications and examples.

Proof of Theorem A. Let $N \in \mathcal{T}_n$ be a manifold with base point $x_0 \in N$, and let F_0 be the canonical frame of the oriented circle $S^1 \times \{x_0\}$ on $S^1 \times N$. A map $\beta : S^1 \to \mathit{Diff}(N, x_0)$ gives rise to a diffeomorphism of $S^1 \times N$

$$\widetilde{\beta} : S^1 \times N \to S^1 \times N \text{ by } \widetilde{\beta}(z, x) = (z, \beta(z) \cdot x)$$

that fixes the circle $S^1 \times x_0$ pointwisely. In particular, in addition to F_0 the composition $\widetilde{\beta} \circ F_0$ is a second frame of the circle $S^1 \times \{x_0\}$.

Assume that the manifold N is Σ-stable. Then there exists a β such that $e_N \circ \beta = \alpha$, implying $\widetilde{\beta} \circ F_0 = F_0 \widetilde{\alpha}$. Thus, surgeries along $S^1 \times x_0$ with respect to the two frames F_0 and $\widetilde{\beta} \circ F_0$, we obtain from $\widetilde{\beta}$ a desired diffeomorphism $\Sigma_0 N \to \Sigma_1 N$.

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For two manifolds $N \in \mathcal{T}_n$ and $M \in \mathcal{T}_m$ with base points $x_0 \in N$ and $y_0 \in M$ the linear representation $e_{N\times M}$ fits into the commutative diagram

$$\begin{array}{ccc} \mathit{Diff}(N, \{x_0\}) & \overset{e_N}{\to} & GL(n) \\ i \downarrow & & j \downarrow \\ \mathit{Diff}(N \times M, \{x_0, y_0\}) & \overset{e_{N\times M}}{\to} & GL(n+m) \end{array},$$

where $i(h) = h \times id_M$ and $j(g) = g \oplus I_m$. By the fact that j is an $(n-1)$-homotopy equivalence we get

COROLLARY 3·1. *If $N \in \mathcal{T}_n$ with $n \geq 3$ is Σ-stable, then the product $N \times M$ is also Σ-stable.*

In particular, for any $M \in \mathcal{T}_m$, $\Sigma_0(N \times M) = \Sigma_1(N \times M)$.

For a manifold $N \in T_n$ with an orientation preserving embedding $(D^n, 0) \subset (N, x_0)$ set $W_N := D^2 \times (N - \overset{\circ}{D^n})$. In addition to (1·2) we have (e.g. [**25**])

$$\Sigma_0 N = \partial W_N. \tag{3·1}$$

This relation, together with Theorem A, is useful to calculate the diffeomorphism types of some $\Sigma_i N$. We present such examples.

PROPOSITION 3·2. *The sphere S^n is Σ-stable with $\Sigma_0 S^n = \Sigma_1 S^n = S^{n+1}$.*

In particular, for any $p \leq q$ with $q \geq 3$ we have

$$\Sigma_0(S^p \times S^q) = \Sigma_1(S^p \times S^q) = S^p \times S^{q+1} \# S^{p+1} \times S^q. \tag{3·2}$$

Proof. Let $\{e_1, \ldots, e_{n+1}\}$ be the standard orthonormal basis of $\mathbb{R}^{n+1}$ and take $e_{n+1} \in S^n$ as the base point. The generator $\alpha \in \pi_1(SO(n))$ gives rise to the loop on $\mathit{Diff}(S^n, x_0)$

$$\beta : S^1 \to \mathit{Diff}(S^n, x_0), \beta(z)(x) = \alpha(z)x, z \in S^1, x \in S^n,$$

that satisfies $e_{S^n}(\beta) = \alpha$, implying that S^n is Σ-stable. We get $\Sigma_0 S^n = \Sigma_1 S^n$ by Theorem A. Moreover, from $W_{S^n} = D^{n+2}$ we get by (3·1) that

$$\Sigma_0 S^n = \partial W_{S^n} = S^{n+1},$$

showing the first statement of the proposition.

For (3·2) we get from (3·1) that (e.g. [**25**, lemma 1·3], [**11**, lemma 2])

$$\Sigma_0(S^p \times S^q) = \partial W_{S^p \times S^q} = S^p \times S^{q+1} \# S^{p+1} \times S^q.$$

Since $S^p \times S^q$ is Σ-stable by Corollary 3·1, we get (3·2) by Theorem A.

Let $\mathbb{F}$ denote either the field $\mathbb{R}$ of reals, the field $\mathbb{C}$ of complexes, or the algebra $\mathbb{H}$ of quaternions. Let $\mathbb{F}P^n$ be the n-dimensional projective space of the 1-dimensional $\mathbb{F}$-subspaces of the vector space $\mathbb{F}^{n+1}$. The canonical Hopf line bundle over $\mathbb{F}P^n$ is

$$\lambda^n_{\mathbb{F}} := \{(x, v) \in \mathbb{F}P^n \times \mathbb{F}^{n+1} \mid v \in x\}.$$

In accordance to $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ the real reduction of $\lambda^n_{\mathbb{F}}$ is a real Euclidean vector bundle over $\mathbb{F}P^n$ with dimension 1, 2 or 4, respectively. For an Euclidean vector bundle ξ over a

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CW-complex B write $D(\xi)$ and $S(\xi)$ to denote the total spaces of the associated disc-bundle and sphere-bundle, respectively.

PROPOSITION 3·3. *$\Sigma_0\mathbb{F}P^n = S(\lambda_{\mathbb{F}}^{n-1} \oplus \varepsilon^2)$, where ε^2 denotes the 2-dimensional trivial bundle over $\mathbb{F}P^{n-1}$.*

In particular, the homotopy groups of the suspension $\Sigma_0\mathbb{F}P^n$ are

$$\pi_r(\Sigma_0\mathbb{F}P^n) = \pi_r(\mathbb{F}P^{n-1}) \oplus \pi_r(S^d), \tag{3·3}$$

where $d = 2$, 3 or 5 in accordance to $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Proof. Set $r := \dim_{\mathbb{R}} \mathbb{F}$. In view of $\partial D(\lambda_{\mathbb{F}}^{n-1}) = S^{rn-1}$ the manifold $\mathbb{F}P^n$ has the decomposition $\mathbb{F}P^n = D(\lambda_{\mathbb{F}}^{n-1}) \cup_{id} D^{rn}$, implying

$$W_{\mathbb{F}P^n} = D^2 \times (\mathbb{F}P^n - \overset{\circ}{D^{rn}}) = D^2 \times D(\lambda_{\mathbb{F}}^{n-1}) = D(\lambda_{\mathbb{F}}^{n-1} \oplus \varepsilon^2).$$

One obtains by (3·1) that

$$\Sigma_0\mathbb{F}P^n = \partial W_{\mathbb{F}P^n} = S(\lambda_{\mathbb{F}}^{n-1} \oplus \varepsilon^2).$$

Since $\Sigma_0\mathbb{F}P^n$ is the total space of a d-dimensional sphere bundle over $\mathbb{F}P^{n-1}$ which has a section, we obtain (3·3) by the homotopy exact sequence of the fiber bundle $\Sigma_0\mathbb{F}P^n \to \mathbb{F}P^{n-1}$.

Example 3·4. Let Ω_n^{Spin} be the n-dimensional Spin bordism group. Then $\Sigma_0\mathbb{H}P^2 = 0$ in Ω_9^{Spin} by $\Sigma_0\mathbb{H}P^2 = \partial W_{\mathbb{H}P^2}$. On the other hand, there exists a surgery along the circle $S^1 \times \{x_0\} \subset S^1 \times \mathbb{H}P^2$ such that the resulting manifold M satisfies $M \neq 0$ in Ω_9^{Spin} (e.g. [**1**, p.258], [**6**]). That is

$$M \neq \Sigma_0\mathbb{H}P^2 \text{ but } M = \Sigma_1\mathbb{H}P^2.$$

In particular, the manifold $\mathbb{H}P^2$ fails to be Σ-stable by Theorem A.

Example 3·5. Formula (1·10) in Section 1 suggests that Theorem B may be applicable to derive the homotopy groups of certain manifolds. Combining (1·6) with (3·3) we get further evidence:

$$\pi_r(\mathbb{C}P^{2n}\#\mathbb{R}P^{4n}) = \begin{cases} \mathbb{Z}_2 \text{ if } r = 1, \\ \pi_r(S^{4n-1}) \oplus \pi_r(S^2) \text{ if } r \neq 1 \end{cases};$$

$$\pi_r(\mathbb{C}P^{2n}\#\mathbb{C}P^{2n}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \text{ if } r = 2 \\ \pi_r(S^{4n-1}) \oplus \pi_r(S^3) \text{ if } r \neq 2 \end{cases};$$

$$\pi_r(\mathbb{C}P^{2n}\#\mathbb{H}P^n) = \begin{cases} \mathbb{Z} \text{ if } r = 2, 4 \\ \pi_r(S^{4n-1}) \oplus \pi_r(S^5) \text{ if } r \neq 2, 4 \end{cases}.$$

Example 3·6. For $n = 2$ we have $\pi_1(SO(2)) = \mathbb{Z}$ with generator α. Therefore, the operation Σ_i applies also to the orientable surfaces to yield the correspondences $\Sigma_i : \mathcal{T}_2 \to \mathcal{T}_3$, $i = 0, 1$.

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Let $M_g \in \mathcal{T}_2$ be the orientable surface with genus $g \geq 1$, and let $D^2 \subset M_g$ be an embedded disc. Then the fundamental group $\pi_1(M_g - \overset{\circ}{D^2})$ is the free group generated by the standard elements

$$a_1, b_1, \ldots a_g, b_g \in \pi_1(M_g - \overset{\circ}{D^2}).$$

Applying the Van Kampen Theorem to (1·2) one gets the presentations of the fundamental groups $\pi_1(\Sigma_i M_g)$ by the generators and relations [**10**]

$$\pi_1(\Sigma_0 M_g) = \langle a_1, b_1, \cdots a_g, b_g \rangle;$$

$$\pi_1(\Sigma_1 M_g) = \Big\langle a_1, b_1, \cdots a_g, b_g, z \mid a_i z = z a_i, b_i z = z b_i,$$

$$z^{-1} = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \Big\rangle.$$

Precisely, $\Sigma_0 M_g = S^1 \times S^2 \# \cdots \# S^1 \times S^2$ ($2g$-copies), while $\Sigma_1 M_g$ is the total space of the oriented circle bundle over M_g whose Euler class is the orientation class on M_g (e.g. Freedman [**10**]). This shows that $\Sigma_0 M_g \neq \Sigma_1 M_g$.

4. *Properties of the suspensions* Σ_i

This section develops general properties of the suspensions Σ_i, required to show Theorems C and D.

PROPOSITION 4·1. *For a set of* 1*-connected manifolds* $N_1, \ldots, N_k \in \mathcal{T}_{n-1}$ *with* $n \geq 5$ *we have*

$$\Sigma_i(N_1 \# \cdots \# N_k) = \Sigma_i N_1 \# \cdots \# \Sigma_i N_k, \ i = 0, 1. \tag{4·1}$$

Proof. It suffices to show (4·1) for the case of $i = 1$. The remaining case $i = 0$ follows from the same idea.

By the definition of the space E_N in (1·3) we have $E_{N\#M} = (E_N)_M$. In addition, the Van–Kampen Theorem [**14**, p.43] implies that, if $N \in \mathcal{T}_{n-1}$ is 1-connected, then

$$\pi_1(\Sigma_1 N) = \pi_1(N) = 0, i = 0, 1.$$

Since $\Sigma_1 N_1, \cdots, \Sigma_1 N_k \in \mathcal{T}_{n-1}$ are all 1-connected, Theorem B can be applied repeatedly to show that, if $w_2(B) = 0$, then

$$E_{N_1 \# \cdots \# N_k} = E \# \Sigma_1 N_1 \# \cdots \# \Sigma_1 N_k.$$

In particular, taking p to be the circle bundle $S^3 \times S^{n-2} \to S^2 \times S^{n-2}$ in Example 1·5, we get

$$S^3 \times S^{n-2} \# \Sigma_1(N_1 \# \cdots \# N_k) = S^3 \times S^{n-2} \# \Sigma_1 N_1 \# \cdots \# \Sigma_1 N_k.$$

Since the surgery on the 3-sphere $S^3 \times \{*\} \subset S^3 \times S^{n-3}$ with trivial framing yields S^n, applying surgery on both sides to kill $S^3 \times \{*\}$ shows (4·1).

A manifold N is a *homology n-sphere* if its integral homology is isomorphic to that of the n-sphere S^n. The suspension Σ_i enables one to produce higher dimensional homology spheres from the lower dimensional ones. For a homology n-sphere N consider the circle

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bundle given by Example 1·5:

$$p_N : S^3 \times S^{n-2} \# \Sigma_i N \to S^2 \tilde{\times} S^{n-2} \# N \text{ (resp. } \to S^2 \times S^{n-2} \# N).$$

By the Gysin sequence of p_N [**20**, p.143] with integer coefficients we get

$$H^*(S^3 \times S^{n-2} \# \Sigma_i N) = H^*(S^3 \times S^{n-2}).$$

This shows that

PROPOSITION 4·2. *If N is a homology n-sphere with $n \geq 4$, then both $\Sigma_i N$, $i = 0, 1$, are homology $(n+1)$-sphere.*

We conclude this section by expressing the topological invariants (i.e. the integral homology, cohomology, and characteristic classes) of the suspensions $\Sigma_i N$ in term of that of N. Recall that *the reduced suspension* SX of a CW-complex X is the quotient space $S^1 \times X/(S^1 \vee X)$ [**14**, p.223]. Let $\widetilde{H}_*$ (resp. $\widetilde{H}^*$) denotes the reduced homology (resp. cohomology) of topological spaces.

PROPOSITION 4·3. *The homology and cohomology of the suspensions $\Sigma_i N$ have the following presentations*:

$$H_*(\Sigma_i N) = \widetilde{H}_*(SN) \oplus H_*(N - \{x_0\}); \tag{4·2}$$

$$H^*(\Sigma_i N) = \widetilde{H}^*(SN) \oplus H^*(N - \{x_0\}). \tag{4·3}$$

In addition, with respect to the decomposition of $H^*(\Sigma_i N)$ in (4·3), the Stiefel–Whitney classes w_r and Pontryagin classes p_r of $\Sigma_i N$ are given respectively by the formulae:

$$w_r(\Sigma_i N) \equiv 0 \oplus w_r(N - \{x_0\}) \bmod \mathbb{Z}_2;$$

$$p_r(\Sigma_i N) = 0 \oplus p_r(N - \{x_0\}). \tag{4·4}$$

Proof. It suffices to show formulae (4·3) and (4·4). In view of (1·2) we set $A := S^1 \times (N - \overset{\circ}{D^n})$ and $B := D^2 \times S^{n-1}$ to get

$$\Sigma_i N = A \cup B, A \cap B = S^1 \times S^{n-1},$$

as well as the Mayer–Vietoris sequence

$$\cdots \to H^{r-1}(A \cap B) \xrightarrow{\delta} H^r(\Sigma_i N) \xrightarrow{i_1^* \oplus i_2^*} H^r(A) \oplus H^r(B) \xrightarrow{j_1^* - j_2^*} H^r(A \cap B) \to \cdots$$

For a $M \in \mathcal{T}_n$ let $\omega_M \in H^n(M)$ be the orientation class. Then the graded group $H^*(A \cap B)$ is torsion free with basis

$$\{1, \omega_{S^1} \times 1, 1 \times \omega_{S^{n-1}}, \omega_{S^1 \times S^{n-1}}\},$$

where $\omega_{S^1} \times 1 \in \operatorname{Im} j_1^*$, $1 \times \omega_{S^{n-1}} \in \operatorname{Im} j_2^*$. Moreover, the Mayer–Vietoris sequence can be summarised as the short exact sequence

$$0 \to \mathbb{Z}\{\omega_{S^1 \times S^{n-1}}\} \xrightarrow{\delta} H^*(\Sigma_i N) \xrightarrow{i_1^*} \frac{H^*(S^1 \times (N - \{x_0\}))}{\mathbb{Z}\{\omega_{S^1} \times 1\}} \to 0, \tag{4·5}$$

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where

(a) $\mathbb{Z}\{\omega\}$ denotes the cyclic group with generator ω;

(b) $N - \overset{\circ}{D}{}^n$ has been replaced by $N - \{x_0\}$, and

(c) the map δ is an isomorphism onto $H^{n+1}(\Sigma_i N) = \mathbb{Z}$.

In particular, the sequence (4·5) is splittable to yields (4·3), under the convention that the orientation class $\omega_{\Sigma_i N} = \delta(\omega_{S^1 \times S^{n-1}})$ agrees with the top degree generator $s \wedge \omega_N$ of $\widetilde{H^{n+1}}(SN) = \mathbb{Z}$.

Finally, let TM be the tangent bundle of a smooth manifold M. Then the inclusion $i_1 : S^1 \times (N - \overset{\circ}{D}{}^n) \to \Sigma_i N$ satisfies that

$$i_1^* T(\Sigma_i N) = \varepsilon^1 \oplus T(N - \{x_0\}),$$

where ε^1 is the 1-dimensional trivial bundle on $S^1 \times (N - \overset{\circ}{D}{}^n)$ consisting of the tangent directions of the first factor S^1. The formulae (4·4) follows now from the naturality of the characteristic classes, together with the decomposition (4·3).

5. *Circle actions on the 1-connected 6-manifolds*

In [12–13] Goldstein and Lininger initiated the project to classify those 1-connected 6-manifolds that admit regular circle actions. In this section we complete the project by showing Theorem C, which makes the main results of [12, 13] precise.

Let A be an abelian group. An element $\omega \in A$ is called *primitive* if the cyclic subgroup generated by ω is a direct summand of A. By the homotopy exact sequence of fiber bundles one gets (e.g. [8]).

LEMMA 5·1. *Let M be an 1-connected 6-manifolds that has a regular circle action with quotient map $\pi : M \to M/S^1$. Then*:

(i) *M/S^1 is an 1-connected 5-manifolds*;

(ii) *The Euler class $\omega \in H^2(M/S^1)$ of π is primitive.*

Let γ be the Hopf complex line bundle over S^2, and consider the two handlebodies of dimension 5

$$A := S^2 \times D^3, B := D(\gamma \oplus \varepsilon),$$

where ε is the 1-dimensional trivial bundle over S^2, and $D(\eta)$ denotes the disc bundle of an Euclidean vector bundle η. For a connected manifold V with non-empty boundary ∂V write $V \cdot V$ to denote *the boundary connected sum* of two copies of V. As examples, one gets from $\partial A = S^2 \times S^2$ and $\partial B = \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ that

$$\partial(A \cdot A) = S^2 \times S^2 \# S^2 \times S^2, \partial(B \cdot B) = \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2.$$

Denote by $\{a, b\}$, $\{a_1, b_1, a_2, b_2\}$ and $\{x_1, y_1, x_2, y_2\}$ the canonical bases of the second cohomologies of the boundaries ∂B, $\partial(A \cdot A)$ and $\partial(B \cdot B)$, respectively. With these conventions we introduce in the table below a collection of five 1-connected 5-manifolds

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$S^2 \times S^3$	$S^2 \widetilde{\times} S^3$	W	M_k	X_{2^i}
$A \cup_{id} A$	$B \cup_{id} B$	$B \cup_{\tau} B$	$A \cdot A \cup_{f_k} A \cdot A$	$B \cdot B \cup_{g_k} B \cdot B$

(5·1)

where $\tau \in Diff(\partial B)$, $f_k \in Diff(\partial(A \cdot A))$, and $g_k \in Diff(\partial(B \cdot B))$ are the diffeomorphisms whose actions on the second homologies are (e.g. [28])

(i) $(\tau_*(a), \tau_*(b)) = (a, -b)$;

(ii) $(f_{k*}(a_1), f_{k*}(b_1), f_{k*}(a_2), f_{k*}(b_2)) = (a_1, b_1, a_2, b_2) \cdot C_k$,

(iii) $(g_{k*}(x_1), g_{k*}(y_1), g_{k*}(x_2), g_{k*}(y_2)) = (x_1, y_1, x_2, y_2) \cdot L_k$,

and where C_k and L_k denote, respectively, the 4×4 matrices

$$C_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ -k & 0 & 0 & 1 \end{pmatrix} \text{ and } L_k = \begin{pmatrix} 1 & 2^k & 2^k & 0 \\ 2^k & 1 & 0 & -2^k \\ -2^k & 0 & 1 & 2^k \\ 0 & 2^k & 2^k & 1 \end{pmatrix}.$$

From the Mayer–Vietoris sequence one gets (e.g. [5]):

LEMMA 5·2. *Let M be one of the five manifolds in* (5·1).

(i) *The second cohomology $H_2(M)$ is*:

$$H_2(S^2 \times S^3) = H_2(S^2 \widetilde{\times} S^3) = \mathbb{Z}; H_2(W) = \mathbb{Z}_2;$$

$$H_2(M_k) = \mathbb{Z}_k \oplus \mathbb{Z}_k; H_2(X_k) = \mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k};$$

(ii) $w_2(M) \neq 0$ *implies that* $M \in \{S^2 \widetilde{\times} S^3, W, X_k\}$.

In terms of the five manifolds constructed in (5·1), the classification of Smale and Barden on the closed 1-connected 5-manifolds [2, 24] may be rephrased as follows.

LEMMA 5·3 *For any 1-connected 5-manifold $N \neq S^5$ one has*

$$N = \begin{cases} \#_r S^2 \times S^3 \#_{1 \leq i \leq t} M_{k_i} \text{ if } w_2(N) \equiv 0; \\ \#_r S^2 \times S^3 \#_{1 \leq j \leq t} M_{k_j} \# H \text{ if } w_2(N) \neq 0, \end{cases} \tag{5·2}$$

where $H \in \{S^2 \widetilde{\times} S^3, W, X_k\}$, and where the notation $\#_r N$ means the connected sum of r-copies of N (and so forth).

In addition, for any primitive element $\omega \in H^2(N)$ one can modify the decomposition (5·2) *as*

$$N = \begin{cases} S^2 \times S^3 \# N', \text{ if } \omega \neq w_2(N) \bmod 2; \\ S^2 \widetilde{\times} S^3 \# N' \text{ with } w_2(N') \equiv 0, \text{ if } \omega \equiv w_2(N) \bmod 2, \end{cases} \tag{5·3}$$

so that ω is a generator of the first summand of $H^2(N) = H^2(S^2) \oplus H^2(N')$.

Proof of Theorem C. For a 1-connected 6-manifold M with a regular circle action let $\pi : M \to N := M/S^1$ be the quotient map. Since the quotient N is then an 1-connected

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and 5-dimensional manifold which is not diffeomorphic to S^5, formula (5·2) is applicable to decompose it as a connected sum of the basic ones. Moreover, since the Euler class $\omega \in H^2(N)$ of π is primitive by Lemma 5·1, we can take a decomposition of N relative to ω as that stated in (5·3), to get by the formula (1·4) in Theorem B the decomposition

$$M = S^3 \times S^3 \# \Sigma_i N', N' \in \mathcal{T}_5,$$

where $i = 0$ or 1 in accordance to $\omega \equiv w_2(N) \bmod 2$, or $\omega \neq w_2(N) \bmod 2$. Decomposing N' in the form of (5·2) and using (4·1) to expand $\Sigma_i N'$, we obtain formula (1·7) in Theorem C, in which

$$\Sigma_0(S^2 \times S^3) = \Sigma_1(S^2 \times S^3) = S^3 \times S^3 \# S^2 \times S^4 \text{ (by (3·2))}.$$

This completes the proof.

6. *Torus bundles over 4-manifolds*

The connected-sum operation furnishes the set $\mathcal{T}_n$ with the structure of a semi-abelian group with the sphere S^n as the null element. Write $\mathcal{S}_n \subset \mathcal{T}_n$ to denote the semi-subgroup generated by the products $S^r \times S^{n-r}$ of spheres. Since every $M \in \mathcal{S}_n$ has the unique decomposition

$$M = \#_{a_1}\left(S^1 \times S^{n-1}\right) \#_{a_2} \cdots \#_{a_{[\frac{n}{2}]}} \left(S^{[\frac{n}{2}]} \times S^{n-[\frac{n}{2}]}\right),$$

where the sequence of integers $\{a_1, \ldots, a_{[\frac{n}{2}]}\}$ is determined by the Poincarè polynomial of M as

$$P_t(M) = 1 + a_1\left(t^1 + t^{n-1}\right) + \cdots + a_{[\frac{n}{2}]}\left(t^{[\frac{n}{2}]} + t^{n-[\frac{n}{2}]}\right) + t^n,$$

we obtain:

LEMMA 6·1. *The map $\mathcal{S}_n \to \mathbb{Z}[t]$ by $M \to P_t(M)$ injects.*

Example 6·2. For $k \geq 1$ the manifold Q_k constructed in (1·9) satisfies that

$$P_t(Q_k) = 1 + c_1\left(t^3 + t^{k+1}\right) + c_2\left(t^4 + t^k\right) \cdots + c_r\left(t^{r+2} + t^{k-r-2}\right), r = \left[\frac{k}{2}\right], \tag{6·1}$$

where the sequence of integers $(c_1, \ldots, c_r)$ is determined by the integer k by

$$c_i := (k-1)\binom{k-1}{i} - \binom{k-1}{i+1} + (k-1)\binom{k-1}{i-1} - \binom{k-1}{i-2}, 1 \leq i < r$$

$$c_r := \frac{1}{2}\left((k-1)\binom{k-1}{r} - \binom{k-1}{r+1} + (k-1)\binom{k-1}{r-1} - \binom{k-1}{r-2}\right).$$

The proof of Theorem C (see Section 1) relies on the following two lemmas, whose proofs will be postponed.

LEMMA 6·3. *Let M be a manifold whose second cohomology $H^2(M)$ is torsion free with basis $\{\alpha_1, \ldots, \alpha_k\}$. Then, in the notation of* (1·8),

(i) *if $\{\alpha'_1, \cdots, \alpha'_k\}$ is another basis of $H^2(M)$, $M(\alpha_1, \ldots, \alpha_k) \cong M(\alpha'_1, \ldots, \alpha'_k)$;*
(ii) *if $n \geq 5$ and $M \in \mathcal{S}_n$, $M(\alpha_1, \ldots, \alpha_k) \in \mathcal{S}_{n+k}$.*

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LEMMA 6·4. *If* $N := \#_{1\le i\le k-1}(S^2\times S^3)$ *and if* $\{\omega_1,\dots,\omega_{k-1}\}$ *is the basis of* $H^2(N)$ *Kronecker dual to the canonical embeddings* $S^2\subset N$, *then* $N(\omega_1,\dots,\omega_{k-1}) = Q_k$.

Proof of Theorem D. Let M be an 1-connected 4-manifold with $\beta_2(M) = k$. According to [**8**, corollary 3] there is a primitive element $e\in H^2(M)$ so that the total space of the circle bundle $\pi : M(e)\to M$ with Euler class e is

$$M(e) = \#_{1\le i\le k-1}(S^2\times S^3)(= N \text{ in the notation of Lemma 6·4}).$$

Moreover, in term of the Gysin sequence of π [**20**, p.143], one finds a basis $\{\alpha_1,\dots,\alpha_k\}$ of $H^2(M)$ such that

$$\alpha_1 = e, \pi^*(\alpha_i) = \omega_{i-1}, 2\le i\le k,$$

and that $\{\omega_1,\dots,\omega_{k-1}\}$ is the basis of $H^2(M(e))$ specified in Lemma 6·4. That is, in the notation of (1·8),

$$M(\alpha_1,\dots,\alpha_k) = M(e)(\omega_1,\dots,\omega_{k-1}) = N(\omega_1,\dots,\omega_{k-1}).$$

We get $M(\alpha_1,\dots,\alpha_k) = Q_k$ by Lemma 6·4

Finally, if $\{\alpha'_1,\cdots,\alpha'_k\}$ is an arbitrary basis of $H^2(M)$, we have by i) of Lemma 6·3 a diffeomorphism

$$M(\alpha'_1,\dots,\alpha'_k)\cong M(\alpha_1,\dots,\alpha_k).$$

This completes the proof of Theorem D.

It remains to show Lemmas 6·3 and 6·4.

Proof of Lemma 6·3. Let ξ_k be the universal principal T^k-bundle on BT^k, and let M be a manifold whose second cohomology $H^2(M)$ is torsion free. For two bases $\{\alpha_1,\dots,\alpha_k\}$ and $\{\alpha'_1,\dots,\alpha'_k\}$ of $H^2(M)$ with classifying maps f and $f' : M\to BT^k$, respectively, we have by (1·8) that

$$M(\alpha_1,\dots,\alpha_k) = f^*\xi_k, M(\alpha'_1,\dots,\alpha'_k) = f'^*\xi_k. \tag{6·2}$$

On the other hand, expressing the basis elements α_i's in the α'_i's we get a $k\times k$ integral invertible matrix $C = (c_{ij})_{k\times k}$ such that

$$\{\alpha_1,\dots,\alpha_k\} = \{\alpha'_1,\dots,\alpha'_k\}\cdot C.$$

Regard S^1 as the group of the unit complexes $z\in\mathbb{C}$, define in term of C the group isomorphism

$$g : T^k\to T^k, g(z_1,\dots,z_k) = \left(z_1^{c_{11}}\cdots z_k^{c_{1k}},\dots,z_1^{c_{k1}}\cdots z_k^{c_{kk}}\right),$$

and let $B_g : BT^k\to BT^k$ be the induced map of g. From $f^* = f'^*B_g^*$ we get $f\simeq B_g\circ f'$. By the homotopy invariance of the induced bundles, together with the fact that the map B_g has a homotopy inverse, we obtain by (6·2) a diffeomorphism $M(\alpha_1,\dots,\alpha_k)\cong M(\alpha'_1,\dots,\alpha'_k)$, showing part (i) of Lemma 6·3.

For ii) we can assume by $M\in\mathcal{S}_n$ and $n\ge 5$ that

$$M = \#_k\left(S_i^2\times S_i^{n-2}\right)\#_{1\le j\le r}\left(S_j^{t_j}\times S_j^{n-t_j}\right), t_j\le n-t_j, t_j\ne 2, \tag{6·3}$$

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where $S^p_t \times S^{n-p}_t = S^p \times S^{n-p}$. For each $1 \le i \le k$ let $q_i : M \to S^2_i \times S^{n-2}_i$ be the quotient of M by the summands other than the component $S^2_i \times S^{n-2}_i$, $p_1 : S^2_i \times S^{n-2}_i \to S^2$ be the projection onto the first factor, and set

$$g_i =: p_1 \circ q_i : M \to S^2_i \times S^{n-2}_i \to S^2, \omega_i := g_i^*(\omega) \in H^2(M),$$

where $\omega \in H^2(S^2) = \mathbb{Z}$ is the orientation class of S^2. In terms of (i) it suffices to show $M(\omega_1, \dots, \omega_k) \in \mathcal{S}_{n+k}$. This can be done by induction on k.

Let π_1 be the circle bundle $M(\omega_1) \to M$ over M with Euler class $\omega_1 \in H^2(M)$. Then

$$M(\omega_1, \dots, \omega_k) = M(\omega_1)(\pi_1^*(\omega_2), \dots, \pi_1^*(\omega_{k-1})),$$

where, by the Gysin sequence [**20**, p.143] of spherical fibrations, the set $\{\pi_1^*(\omega_2), \dots, \pi_1^*(\omega_{k-1})\}$ is a basis of $H^2(M(\omega_1))$. However, it follows from Example 1·5, together with formulae (6·3) and (3·2), that

$$M(\omega_1) = \left(S^3 \times S^{n-2}\right) \#_{k-1} \Sigma_1 \left(S^2_i \times S^{n-2}_i\right) \#_r \Sigma_1 \left(S^{k_j}_j \times S^{n-k_j}_j\right) \in \mathcal{S}_{n+1}.$$

This completes the inductive procedure of showing part (ii).

For a finitely generated graded abelian group $A = A^0 \oplus A^1 \oplus A^2 \oplus \cdots$ define the Poincarè polynomial $P_t(A)$ of A by

$$P_t(A) := a_0 + a_1 t + a_2 t^2 + \cdots, a_i = \dim A^i \otimes \mathbb{R}.$$

Proof of Lemma 6·4. Suppose that $N = \#_{1 \le i \le k-1}(S^2_i \times S^3_i)$, and that $\{\omega_1, \dots, \omega_{k-1}\}$ is the basis of $H^2(N)$ specified as that in Lemma 6·4. Then the cohomology $H^*(N)$ has the basis

$$\{1, \omega_1, \dots, \omega_{k-1}, y_1, \dots, y_{k-1}, z\}, \deg \omega_i = 2, \deg y_i = 3, \deg z = 5, \tag{6·4}$$

that is subject to the multiplicative relations

$$\omega_i \cdot \omega_j = 0; \quad y_i \cdot y_j = 0; \quad \omega_i \cdot y_j = \delta_{ij} \cdot z, 1 \le i, j \le k-1, \tag{6·5}$$

where δ_{ij} is the Kronecker symbol. By (ii) of Lemma 6·3, for the principal T^{k-1}-bundle

$$\pi : N(\omega_1, \dots, \omega_{k-1}) \to N$$

over N we have $N(\omega_1, \cdots, \omega_{k-1}) \in S_{4+k}$. According to Lemma 6·1 it suffices to derive that

$$P_t(N(\omega_1, \dots, \omega_{k-1})) = P_t(Q_k) \text{ (see (6·1))}. \tag{6·6}$$

To this end we compute with the Leray–Serre spectral sequence $\{E_r^{*,*}, d_r\}$ of the bundle π [**30**, p.645].

Since both cohomologies $H^*(N)$ and $H^*(T^{k-1})$ are torsion free, while the base manifold N is 1-connected, the Leray–Serre theorem tells that

$$E_2^{*,*} = H^*(N) \otimes H^*(T^{k-1}).$$

In addition, there are unique elements $t_1, \dots, t_{k-1} \in H^1(T^{k-1})$ such that

$$H^*(T^{k-1}) = \Lambda(t_1, \dots, t_{k-1}); d_2(1 \otimes t_i) = \omega_i \otimes 1,$$

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implying

$$d_2(x \otimes t_i) = \omega_i \cdot x \otimes 1, d_2(x \otimes 1) = 0, x \in H^*(N). \tag{6·7}$$

For a multi-index $I \subseteq \{1, \ldots, k-1\}$ we set $t_I := \Pi_{i \in I} t_i$, and let $B_i \subset E_2^{*,*}$, $1 \leq i \leq 4$, be the subgroups with the basis:

$$B_1 : \{1 \otimes t_I \mid I \subseteq \{1, \ldots, k-1\}\};$$

$$B_2 : \{1, \omega_i \otimes 1, \omega_i \otimes t_I \mid \quad i \in \{1, \ldots, k-1\}, I \subseteq \{1, \ldots, k-1\}\};$$

$$B_3 : \{y_i \otimes 1, y_i \otimes t_I \mid i \in \{1, \ldots, k-1\}, I \subseteq \{1, \ldots, k-1\}\};$$

$$B_4 : \{z \otimes 1, z \otimes t_I \mid I \subseteq \{1, \ldots, k-1\}\},$$

respectively. Then $E_2^{*,*} = B_1 \oplus B_2 \oplus B_3 \oplus B_4$ by (6·4) and (as is clear):

$$P_t(B_1) = \sum_{1 \leq i \leq k-1} \binom{k-1}{i} t^i;$$

$$P_t(B_2) = 1 + (k-1) \sum_{0 \leq i \leq k-1} \binom{k-1}{i} t^{i+2};$$

$$P_t(B_3) = (k-1) \sum_{0 \leq i \leq k-1} \binom{k-1}{i} t^{i+3};$$

$$P_t(B_4) = \sum_{0 \leq i \leq k-1} \binom{k-1}{i} t^{i+5}.$$

Moreover, the d_2-action on the basis elements $x \in B_i$ has been decided by (6·5) and (6·7) as

$$d_2(x) = \begin{cases} 0 \text{ if } x = \omega_i \otimes 1, y_i \otimes 1, z \otimes t_I, \omega_i \otimes t_I, y_j \otimes t_J \text{ if } j \notin J; \\ \sum\limits_{1 \leq s \leq t} (-1)^{s-1} \omega_{i_s} \otimes t_{I_s} \text{ if } x = 1 \otimes t_I \text{ with } I = \{i_1, \cdots, i_t\}; \\ (-1)^{s-1} z \otimes t_{I_s} \text{ if } x = y_i \otimes t_I \text{ with } i = i_s \in I = \{i_1, \cdots, i_t\}, \end{cases}$$

where I_s denotes the multi-index obtained from I by deleting the sth coordinate i_s. It follows that:

(a) $d_2(B_2) = 0$, d_2 maps B_1 monomorphically into B_2;
(b) $d_2(B_4) = 0$, $d_2(B_3) \subset B_4$ with $B_4/d_2(B_3) = \mathbb{Z}$ generated by $z \otimes t$.

As results,

$$E_3^{*,*} = B_2/d_2(B_1) \oplus B_4/d_2(B_3) \oplus \overline{B}_3 \text{ with } \overline{B}_3 := \ker [d_2 : B_3 \to B_4]; \tag{6·8}$$

$$E_3^{*,*} = E_\infty^{*,*} = H^*(N(\omega_1, \ldots, \omega_{k-1})). \tag{6·9}$$

Since

$$P_t(B_2/d_2(B_1)) = P_t(B_2) - t \cdot P_t(B_1) \text{by a);}$$

$$P_t(B_4/d_2(B_3)) = t^{k+4} \text{by b),}$$

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and since

$$P_t(\overline{B}_3) = P_t(B_3) - t^{-1}(P_t(B_4) - P_t(B_4/d_2(B_3)),$$

by the exactness of the sequence of the free $\mathbb{Z}$-modules

$$0 \to \overline{B}_3 \to B_3 \xrightarrow{d_2} B_4 \to B_4/d_2(B_3)(=\mathbb{Z}) \to 0,$$

we get, in the order of (6·9), (6·8) and (6·1), the equalities

$$\begin{aligned} &P_t(N(\omega_1, \ldots, \omega_{k-1})) = P_t(E_3^{*,*}) \\ &= P_t(B_2) - t \cdot P_t(B_1) + P_t(B_3) - t^{-1} \cdot (P_t(B_4) - t^{k+4}) + t^{k+4} \\ &= P_t(Q_k). \end{aligned}$$

This verifies the relation (6·6), completing the proof of Lemma 6·4.

7. *Remarks*

The constructions, results and calculations of the present paper are likely to be generalised. We discuss such examples.

7·1. By Duan–Liang [**8**, corollary 2] and Theorem C, the classification of the 1-connected 5- and 6-dimensional manifolds that admit regular circle actions has been completed. In addition, Montgemory–Yang [**21**] showed that among all the 28 homotopy 7-spheres there are precisely 10 admit regular circle actions, which has been extended by Jiang [**15**] to the classification of regular circle actions on the 2-connected 7-manifolds, both in the topological and smooth categories. As suggested by the proofs of Theorem C and Theorem D, the suspension operations, combined with Theorem B, could provide a powerful tool to deal with the classification problems in the higher dimensional cases.

For instance, let $\mathcal{V}_{2n} \subset \mathcal{T}_{2n}$ (resp. $\mathcal{V}_{2n+1} \subset \mathcal{T}_{2n+1}$) be the subset of $(n-1)$-connected $2n$-manifolds (resp. the subset of $(n-1)$-connected $(2n+1)$-manifolds). The operators Σ_i clearly satisfy the relation $\Sigma_i(\mathcal{V}_{2n}) \subset \mathcal{V}_{2n+1}$. On the other hand, the manifolds of $\mathcal{V}_m$ ($m \geq 8$) have been classified by C.T.C. Wall [**26**, **27**] in term of a set of explicit invariants. It is possible to describe the map $\Sigma_i : \mathcal{V}_{2n} \to \mathcal{V}_{2n+1}$ in terms of the Wall's invariants, so that a classification for the manifolds of $\mathcal{V}_{2n+1}$ that admit regular circle actions is attainable.

7·2. The operations $\Sigma_i : \mathcal{T}_n \to \mathcal{T}_{n+1}$ admit a natural generalisation. Namely, for each pair $(\alpha, M) \in \pi_k(SO(n)) \times \mathcal{T}_n$ take an orientation persevering embedding $(D^n, 0) \subset (M, x_0)$, surgery along the oriented sphere $S^k \times \{x_0\} \subset S^k \times M$ relative to the framing $\alpha \in \pi_k(SO(n))$, and denote the resulting manifold by $\Sigma_\alpha M$ to get the pairing

$$\Sigma : \pi_k(SO(n)) \times \mathcal{T}_n \to \mathcal{T}_{n+k}, (\alpha, M) \to \Sigma_\alpha M.$$

As indicated by the construction (1·3), this operation may be use to describe the total space E_N the S^k-bundles $p_N : E_N \to B\#N$ induced from a S^k-bundles $E \to B$, where $\dim B = \dim N$. In particular, when $k = 3$ it may be applied to construct or classify the $S^3 = SU(2)$-actions on smooth manifolds.

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