109.43 "Look-see" proofs that $\pi > e$

Introduction

We can surely all agree on the significance and beauty of Euler's equation $e^{i\pi} + 1 = 0$, which was once memorably described by Michael Atiyah in [1] as, "the mathematical equivalent of Hamlet's phrase 'To be, or not to be' — very short, very succinct, but at the same time very deep." But I am also constantly intrigued by the proximity of the key players $0,\ 1,\ e,\ \pi,\ i$ in the Argand diagram, Figure 1, where the smallest circle that contains all of them has radius $\frac{1}{2}\sqrt{\pi^2 + 1}$ (which is surprisingly close to $\sqrt{e}$).

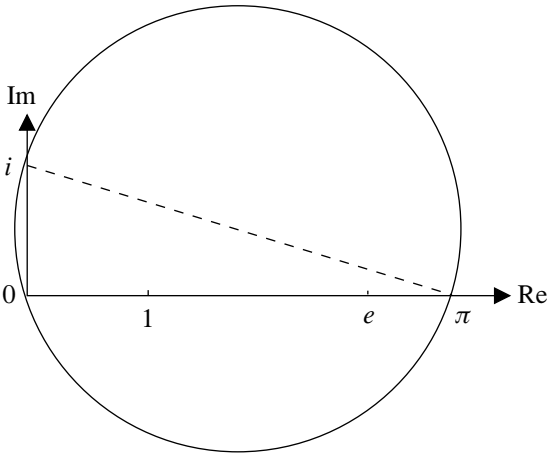
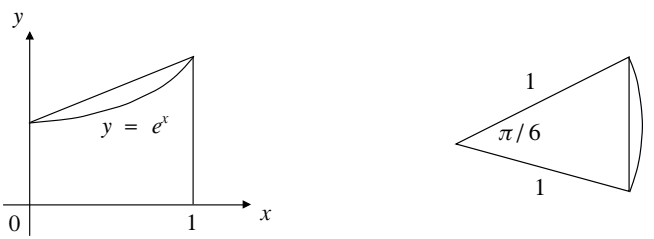


FIGURE 1: The smallest circle that contains $0,\ 1,\ e, \pi,\ i$

Particularly intriguing is the closeness of those two stalwarts of analysis e and π and, for many years, I have been on a perhaps rather idiosyncratic quest to find a "look-see" proof that $\pi > e$, where an expression involving π and e makes it immediately clear without any calculation that $\pi > e$. Thus I am ruling out even light-touch calculations that separately establish (say) $e < 3$ and $3 < \pi$ such as those in Figure 2, where comparison of areas gives $\frac{1}{2}(e + 1) > e - 1$ in (a) (from the trapezium and the area under the graph) and $\frac{1}{2}\sin\frac{\pi}{6} < \frac{1}{2}\cdot\frac{\pi}{6}$ (from the triangle and sector).



(a) (b)

FIGURE 2: (a) $e < 3$, (b) $3 < \pi$

https://doi.org/10.1017/mag.2025.10144 Published online by Cambridge University Press

At long last, I have found the proof I have been looking for: I warmly invite readers to find others!

Consider the integral $\int_0^{\pi/2} \ln\left(\frac{x}{\sin x}\right) dx$.

It is manifestly positive because $x > \sin x$ for $0 < x < \frac{1}{2}\pi$. (A proof of this in the spirit of this Note would be $x - \sin x = \int_0^x 1 - \cos t \, dt = 2\int_0^x \sin^2 \frac{1}{2}t \, dt < 0$.)

Thus

$$0 < \int_0^{\frac{1}{2}\pi} (\ln x - \ln \sin x)\, dx = \Big[x \ln x - x\Big]_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} (\ln x - \ln \sin x)\, dx$$

$$= \frac{\pi}{2} \ln \frac{\pi}{2} - \frac{\pi}{2} + \frac{\pi}{2} \ln 2,$$

using the integral $\int_0^{\frac{1}{2}\pi} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2$ (see below), so that

$$0 < \int_0^{\frac{1}{2}\pi} \ln\left(\frac{x}{\sin x}\right) dx = \frac{\pi}{2}(\ln \pi - 1) = \frac{\pi}{2} \ln\left(\frac{\pi}{e}\right), \tag{1}$$

from which $\pi > e$ follows.

(To evaluate $I = \int_0^{\frac{1}{2}\pi} \ln \sin x \, dx$, compare the ends of the following calculation:

$$I = \int_0^{\frac{1}{2}\pi} \ln \sin x \, dx = \frac{1}{2} \int_0^{\pi} \ln \sin x \, dx = \int_0^{\pi/2} \ln \sin 2t \, dt$$

$$= \int_0^{\pi} \left(\ln 2 + \ln \sin t + \ln \sin\left(\tfrac{1}{2}\pi - t\right)\right) dt = \frac{\pi}{2} \ln 2 + I + I.)$$

The rest of this Note describes my circuitous quest to reach this satisfying endpoint.

The quest

A casual glance through the pages of tables of integrals such as [2] reveals several definite integrals with values that are expressions involving both π and e. Typical examples are $\int_0^{\infty} \frac{2 \cos x}{1 + x^2} dx = \frac{\pi}{e}$ and $\int_0^{\infty} e^{-\frac{1}{4}\left(x^2 + \frac{1}{x^2}\right)} dx = \sqrt{\frac{\pi}{e}}$. But quickly establishing that the integrals are greater than 1 defeated me.

So my quest sharpened into a search for other definite integrals with values involving $\frac{\pi}{e}$. My "eureka moment" came when I serendipitously stumbled across Problem 82.2 in the excellent Problem section curated by Ian Short in the ever-interesting open-access *Irish Mathematical Society*

https://doi.org/10.1017/mag.2025.10144 Published online by Cambridge University Press

Bulletin, [3]. This challenged readers to show that $\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x} dx = \ln 2$, with a rider to show further that $\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x} e^{-x} dx = \ln \pi - 1 = \ln\left(\frac{\pi}{e}\right)$. A solution by contour integration to the initial problem was duly published, but a solution to the rider was not. And it was the rider that piqued my interest, for it implies that $\ln\left(\frac{\pi}{e}\right) > 0$ since the integrand is positive because $\sinh x - x > 0$. (Quick proof: $e^x - 2x - e^{-x} = \int_0^x \left(e^{t/2} - e^{-t/2}\right)^2 dt > 0$, for $x > 0$.)

So my quest distilled into trying to show that $\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x} e^{-x} dx = \ln\left(\frac{\pi}{e}\right)$: as it happens, the proof below can be modified to give an alternative proof that $\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x} dx = \ln 2$ as well.

The Maclaurin expansion for $\sinh x$ gives $\frac{\sinh x - x}{x^2} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n+1)!}$, so that

$$\int_0^\infty \frac{\sinh x - x}{x^2} \cdot \frac{e^{-x}}{\sinh x} = \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} \int_0^\infty x^{2n-1} \cdot \frac{2e^{-2x}}{1 - e^{-2x}} dx$$

$$= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{2}{(2n+1)!} \int_0^\infty x^{2n-1} e^{-(2k+2)x} dx,$$

on expanding $(1 - e^{-2x})^{-1}$ binomially.

But, substituting $t = (2k+2)x$, we obtain

$$\int_0^\infty x^{2n-1} e^{-(2k+2)x} dx = \frac{1}{(2k+2)^{2n}} \int_0^\infty t^{2n-1} e^{-t} dt = \frac{(2n-1)!}{(2k+1)^{2n}},$$

so that

$$\int_0^\infty \frac{\sinh x - x}{x^2 \sinh x} e^{-x} dx = 2 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{2n(2n+1)} \cdot \frac{1}{(2k+2)^{2n}} = 2 \sum_{n=1}^{\infty} \frac{4^{-n} \zeta(2n)}{2n(2n+1)}, \quad (2)$$

where, as usual, $\zeta(2n) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{2n}}$.

We next invoke from [4] (for example), Euler's generating function for $\zeta(2n)$, $2 \sum_{n=1}^{\infty} \zeta(2n) t^{2n} = 1 - \pi t \cot \pi t$.

From this, $2 \sum_{n=1}^{\infty} \zeta(2n) t^{2n-1} = \frac{1}{t} - \pi t \cot \pi t$ so, integrating from 0 to x and carefully inserting limits, $2 \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n} x^{2n} = \ln\left(\frac{\pi x}{\sin \pi x}\right)$.

https://doi.org/10.1017/mag.2025.10144 Published online by Cambridge University Press

Integrating again from 0 to $\frac{1}{2}$ then gives

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\zeta(2n)\,4^{-n}}{2n(2n+1)} &= \int_0^{\frac{1}{2}} \ln\left(\frac{\pi x}{\sin \pi x}\right) dx \\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \ln\left(\frac{u}{\sin u}\right) dx, \text{ on setting } u = \pi x \\ &= \frac{1}{\pi}\cdot\frac{\pi}{2}\ln\left(\frac{\pi}{e}\right), \text{ by (1)} \\ &= \frac{1}{2}\ln\left(\frac{\pi}{e}\right). \end{aligned} \tag{3}$$

Tracking back to (2), we finally see that $\int_0^{\infty} \frac{\sinh x - x}{x^2 \sinh x} e^{-x} dx = \ln\left(\frac{\pi}{e}\right)$. Establishing this, proved the catalyst for finding the simpler integral, (1), which emerged from the concluding steps of the proof. Even better, (3) provides another (and perhaps the ultimate) "look-see" proof that $\pi > e$, which does not rely on any auxiliary results such as $x > \sin x$ or $\sinh x - x > 0$: $\ln\left(\frac{\pi}{e}\right) = 2\sum_{n=1}^{\infty} \frac{\zeta(2n)\cdot 4^{-n}}{2n(n+1)}$. a series of positive terms!

Acknowledgement

I am grateful to the referee for carefully reading the first draft of this note and highlighting several areas for improvement.

References

1. Michael Atiyah's imaginative state of mind, *Quanta magazine*: available on-line at
   https://www.quantamagazine.org/michael-atiyahs-mathematical-dreams-20160303/
2. I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series and products*, Academic Press (1980).
3. Problem 82.2, *Irish Math. Soc. Bulletin* **82** (Winter 2018), p. 105 : available on-line at https://www.irishmathsoc.org/bulletin/
4. G. J. O. Jameson, Series involving $\zeta(n)$, *Math. Gaz.* **98** (March 2014), p. 64.

10.1017/mag.2025.10144 © The Authors, 2025
Published by Cambridge University Press
on behalf of The Mathematical Association

NICK LORD

https://doi.org/10.1017/mag.2025.10144 Published online by Cambridge University Press