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Classical logic as limit completion

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We define a constructive model for Δ^0_2-maps, that is, maps recursively definable from a map deciding the halting problem. Our model refines an existing constructive interpretation for classical reasoning over one-quantifier formulas: it is compositional (Modus Ponens is interpreted as an application) and semantical (rather than translating classical proofs into intuitionistic ones, we define a mathematical structure intuitionistically validating excluded middle for one-quantifier formulas).

1. Introduction

1.1. *A topological model for classical reasoning*

Let N denote the set of non-negative integers. In our model, we define a completion of N that is quite similar to the topological completion defining the set R of Real numbers out of the set Q of Rational numbers.

As we all know very well, some converging (that is, Cauchy) succession over Q has no limit. In order to have a limit for all converging successions over Q, we consider the set of converging successions over Q themselves as a new class of individual, the set R of Real Numbers. Rational numbers are identified with constant successions over Q.

In much the same way, since some converging succession over N has no computable result, we consider the set of converging successions to N as a new class of individual, and we denote it by N^*. The original elements of N are identified with constant successions. All convergent successions over N now have some computable result in N^*.

From a classical viewpoint, N^* is nothing but N, because each limit may be identified with its integer result. From a constructive viewpoint, several maps that were not computable over N are computable over N^* (but with a result in N, no longer in N). Fix any two element subset $\mathscr{B} = \{\mathtt{tt}, \mathtt{ff}\}$ of N, and any total recursive $f : N^2 \to \mathscr{B}$. Typical maps computable over N^*, but *not computable* over N, are the 'some maps' '$\mathtt{s}_f$', defined by $\mathtt{s}_f(x) = \mathtt{tt}$ if $f(x, y) = \mathtt{tt}$ for some y, and $\mathtt{s}_f(x) = \mathtt{ff}$ if $f(x, y) = \mathtt{ff}$ for all y. $\mathtt{s}_f$ decides solvability in y of integer equations $f(x, y) = \mathtt{tt}$ (one of the many formulations of the so-called Halting Problem, which is a problem without recursive solution).

Starting from this consideration, we build a purely intuitionistic (that is, not using excluded middle at all) model of recursive maps + '$\mathtt{s}_f$' maps: more precisely, of all Δ^0_2-maps. We only need some minor shifts in definition: the convergence of successions

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and equality between successions have to be reformulated. New definitions are classically equivalent to the more traditional ones, but from a constructive viewpoint they are more significant.

Any classical reasoning using only the existence of the '$\mathtt{s}_f$' map (essentially, any classical reasoning over one-quantifier formulas) may be reinterpreted as constructive reasoning in our model.

The main result we have for our topological model is that whenever an equation $g(x) = 0$ (for g recursive map) has some solution $x = l \in N^*$ (has some solution in N), it has a solution $x = n \in N$ (has some solution in the ordinary sense). Also, n is not found by blind search through all integers, but may be computed using ideas from the proof of $g(l) = 0$. In other words, by working in N^*, we may use non-recursive maps and one-quantifier classical reasoning in order to find effectively given solutions to recursive problems.

1.2. *A realisation semantics for classical equational reasoning*

By building on top of our model of Δ_2^0-maps, we may define a Heyting-like notion of 'realiser' r for equations $l = m$ between limits. A 'realiser' is a map that makes explicit the construction hidden in constructive reasoning. Unlike equational reasoning over integers, equational reasoning over limits hides some non-trivial construction, hence some non-trivial realiser. Eventually, we sketch an implementation of realisers, that is, an explicit constructive content for a classical proof using the '$\mathtt{s}_f$' maps.

All definition and proofs in this paper are purely intuitionistic. We use intuitionism as a tool to make the constructions usually hidden in classical reasoning explicit.

1.3. *Some features of classical logic not considered in this paper*

Often, a proof in classical logic hides a non-deterministic, parallel algorithm. However, we have not attempted to represent such features of classical logic in N^*.

There is another relevant topic not included in this paper. In both classical and intuitionistic arithmetic we may define, by a fixed point operator, maps that are convergent with respect to some evaluation strategy (say, call-by-name), yet divergent with respect to some other evaluation strategy (say, pure call-by value). In N^* we will only represent convergent evaluation strategies for any expression.

In a forthcoming paper (Berardi submitted) we plan to show how non-determinism, parallelism, and divergent evaluation strategies may be represented in N^*. The cost is having to choose a more general notion of convergence to define N^*.

1.4. *Structure of the paper*

We begin by introducing completion N^* of N, then a computational interpretation for equational reasoning on N^*.

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In Section 2 we introduce all the intuitionistic concepts we will need in the paper (well-founded relations, direct sets and so forth). In Section 3 we define our model N^* of limit reasoning over N. In Section 4 we characterise continuous maps on N^* as Δ^0_2-maps.

Finally, in Section 5 we define a realisability interpretation for equational reasoning, and in Section 6 we use realisers to constructivise a simple classical theorem requiring excluded middle over one-quantifier formulas (over formulas without nested quantifiers).

2. Some intuitionistic concepts

In this section, we will introduce all the ingredients we will need for our completion of N:

— direct sets,
— covering of N,
— sequences,
— successions,
— stationarity,
— limits,
— well-founded relations,
— change ordering,
— stability.

The key notion of the paper will be the notion of *stability* (a kind of convergence notion for limits).

Total recursive maps over subsets of N. Let $I, J \subseteq N$. A total recursive map $f : I \to J$ is any partial recursive map $f : N \to N$ that is total over $I \subseteq N$ and such that $f(I) \subseteq J$. This notion of total recursive may be generalised to maps of any arity.

Recursive direct sets. D is a direct set if D is an ordered set that is inhabited (there is some $d \in D$) and for which any two $d, e \in D$ have some upper bound $e' \geqslant d, e$ in D. We will only consider direct sets D that are *recursive subsets* of N, with a *recursive ordering* relation. We call these recursive direct sets.

Upper bound maps. An upper bound map is any total recursive map $j : D, D \to D$ picking, for any $d, e \in D$, some common upper bound $j(d, e) \geqslant d, e$. All recursive direct sets have at least one upper bound map j, defined by $j(d, e) =$ the first $x \in N$ such that $x \in D$ and $x \geqslant d, e$. This map j is total on D because all $d, e \in D$ have some common upper bound in D, by definition of direct set. This map j is recursive because the predicates $x \in D$ and $x \geqslant d, e$ are, by assumption, both recursive.

We suppose that for each recursive directed set there is fixed some upper bound map j. Usually, $j(d, e)$ will be the l.u.b. of d, e. For any $d_1, d_2, ..., d_n \in D$, we will write $j(d_1, d_2, ..., d_n)$ for $j(d_1, j(d_2, j(..., d_n)))$. Then $j(d_1, d_2, ..., d_n)$ is some common upper bound for $d_1, d_2, ..., d_n$ in D.

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The interpretation of D. D represents a tree of possible computation states. When $d \leqslant e$, we say that the state e is 'a possible future' for the state d. We consider any increasing succession of computation states

$$d_0 < d_1 < d_2 < d_3 < \dots$$

as the history of some computation. For the purpose of defining a model of Δ^0_2-maps, the example $D = N$ would suffice. In this simplified case, a canonical example of increasing succession of computation states is

$$0 < 1 < 2 < \dots.$$

We choose the map $j(n,m) = \max(n,m)$ as upper bound map j for $D = N$. Another relevant example is $D = \mathscr{P}_{\text{fin}}(N) = \{\text{finite subsets of } N\}$. We choose the map $j(J,H) = J \cup H$ as upper bound map j for $D = \mathscr{P}_{\text{fin}}(N)$.

Only at the end of this paper will we consider a more subtle choice for D. A particular choice for D may improve the description of the computations over N^*, and therefore the efficiency of the computational interpretation of classical proofs. However, different choices for D generate isomorphic models of Δ^0_2-maps, as we are going to prove. In the rest of the paper the reader may assume that $D = \mathscr{P}_{\text{fin}}(N)$.

Coverings of N. Let D be a directed set. Then a covering structure of N by D is any family $\{\mathtt{Cov}(d)\}_{d\in D}$ of finite subsets of N:

(i) Indexed by elements of D in a recursive and increasing way,
(ii) Such that we may effectively find, for any $x \in N$, some element $\mathtt{Cov}(d)$ such that $x \in \mathtt{Cov}(d)$.

Formally, a covering structure is a pair of recursive maps $\mathtt{Cov} : D \to \mathscr{P}_{\text{fin}}(N)$ and $\xi : N \to D$ satisfying the conditions:

(i) $\mathtt{Cov}$ is weakly increasing: if $d \leqslant e$, then $\mathtt{Cov}(d) \subseteq \mathtt{Cov}(e)$;
(ii) $x \in \mathtt{Cov}(\xi(x))$ for all $x \in N$.

When a covering structure by D is given, we say 'D covers N'. When $x \in \mathtt{Cov}(d)$, we say that d covers x. Intuitively, if we have a covering relation, elements of D may be identified with finite subsets of N covering N. Clauses (i), (ii) now read:

(i) The larger $d \in D$ is, the more it covers elements of N.
(ii) Given any $x \in N$, we may effectively find some $d = \xi(x) \in D$ covering x.

We will abbreviate $x \in \mathtt{Cov}(d)$ or 'd covers x' by $x \lhd d$.

Usually, we take as $\xi(x)$ the smallest $d \in D$ covering x. If $D = N$, we choose $\mathtt{Cov}(d) = [0,d]$, and $\xi(x) = x$: in this case, $x \lhd d$ if and only if $x \leqslant d$. If $D = \mathscr{P}_{\text{fin}}(N)$, we choose $\mathtt{Cov}(d) = d$, and $\xi(x) = \{x\}$: in this case, $x \lhd d$ iff $x \in d$.

In applications, we sometimes assume an *infinite effective family* $\{(\mathtt{Cov}_n, \xi_n) | n \in N\}$ of covering structures to cover N by a fixed D (see Section 6 for an example). We read '$x \lhd_l d$' as 'the l^{th} component of d covers x'. If we have a family of covering relations, any $d \in D$ may be identified with a record whose components are finite (and possibly overlapping) subsets $\{x \in N | x \lhd_l d\}$ of N, each of them labelled by some $l \in N$. Several ways of covering

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N are introduced, but only to improve the efficiency of the interpretation. Theoretically, the reader may always think that there is just one covering, that all coverings are equal to this one, and the index l in $\triangleleft_l, \xi_l$ is dummy. In fact, the reader may even assume that $D = \mathscr{P}_{\text{fin}}(N)$, and that each $d \in D$ is a finite subset of N.

Cofinality and stationarity on D. A subset $X \subseteq D$ is cofinal in D if and only if for all $d \in D$ there is some $e \in X$ such that $d \leqslant e$. Any cofinal subset is inhabited (there is at least one inhabitant of D, and therefore one inhabitant of X above it). A subset $X \subseteq D$ is stationary in D if and only if for some $e_0 \in X$ and all $e \geqslant e_0$, we have $e \in X$.

Every stationary subset of D is cofinal (but the converse is not true).

Proof. Assume that for some $e_0 \in X$ and all $e \geqslant e_0$ we have $e \in X$. Assume that $d \in D$ in order to prove $e \in X$ for some $e \geqslant d$. We choose $e = j(d, e_0) \geqslant e_0, d$: we deduce $e \in X$ from $e \geqslant e_0$. □

Sequences over D – successions. We call any total recursive map $f : D \to N$ a sequence: we think of f as a tree of streams of integers,

$$f(d_0), f(d_1), f(d_2), f(d_3), \ldots,$$

one for each increasing succession of computation states:

$$d_0 < d_1 < d_2 < d_3 < \ldots.$$

For instance, when $D = N$, the succession

$$0, 12, 3, \ldots$$

of computation states in D produces the stream of integers

$$f(0), f(1), f(2), f(3), \ldots.$$

We call any total recursive map $g : N \to I$ (with the elements of I coded in N) a succession (over I).

Stationarity and limits. Fix any sequence $f : D \to N$. We say that f is stationary with limit $n \in N$ if and only if the set $\{d \in D | f(d) = n\}$ is stationary (iff for some $d \in D$ and all $e \geqslant d$ we have $f(d) = n$). We say that f is stationary if and only if f is stationary with some limit $n \in N$. By directness of D, the limit of f, when it exists, is unique: if we have both $f(e) = n$ for all $e \geqslant d$, and $f(e) = m$ for all $e \geqslant d'$, then there is some $d'' \geqslant d, d'$, and we conclude $m = f(d") = n$.

There is a key example in the case $D = N$. Fix any total recursive $f : N \to \mathscr{B}$. There is a sequence $L = \{L_n\}_n \in N$, whose limit is `tt` when $f(x) = \texttt{tt}$ for some x, and `ff` if $f(x) = \texttt{ff}$ for all x. Just choose $L_n = f(0) \vee f(1) \vee \ldots \vee f(n) \in \mathscr{B}$. $\{L_n\}_n \in N$ is a stream of `ff`'s, turning to a stream of `tt`'s from the first x such that $f(x) = \texttt{tt}$ (if any exists).

The key point to note for an understanding of the rest of the paper is that the limit of L cannot be computed out of f. The limit of L is indeed `tt` or `ff`, according to whether $f(x) = \texttt{tt}$ for some $x \in N$ or $f(x) = \texttt{ff}$ for all $x \in N$, but in order to decide which is

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the case, we need either excluded middle or the map in the parameter f that decides the statement $\exists x.(f(x) = \mathtt{tt})$. This map is not recursive in f. In both cases, *the existence of a limit of L is not proved intuitionistically*.

Thus, if we want an intuitionistic theory of convergence encompassing L, we have to weaken (from an intuitionistic viewpoint) the notion of convergence. The new notion we will use is: L converges if and only if L changes its value only 'finitely many times'. This idea will be expressed by Well-foundation of a suitable 'change ordering' on the stream of values of L. From a classical viewpoint, this new notion, which we call 'stability', is quite close to the traditional one, and equivalent to it when $D = N$ (we skip the proof because it is classical and of no interest here). From a constructive viewpoint, however, 'stability' will make the difference. By defining convergence as 'stability', we will be able to prove convergence of limits with a non-recursive result. We will now introduce intuitionistic well-foundation, then stability.

Well-foundation: intuitionistic definition. Let (I, R) be any pair of a set I and some binary relation $R \subseteq I \times I$. We now define the notion '(I, R) is well-founded'.

If $P \subseteq I$, we consider P as a property defined over I, and we write $P(z)$ for $z \in P$. We say that a property P is (I, R)-inductive if and only if for all $x \in I$, if $P(y)$ for all $y \in I$ such that $R(y, x)$, then $P(x)$. We say that (I, R) is well-founded on $x \in I$ if and only if for all $P \subseteq I$ that are (I, R)-inductive, we have $P(x)$. We say (I, R) is well-founded iff (I, R) is well-founded on all $x \in I$; equivalently, if and only if for all $P \subseteq I$ that are (I, R)-inductive, we have $P = I$.

Here are two examples: $(N, <)$ is well-founded; but $(Z, <)$, with $Z = \{\text{relative integers}\}$, is not well-founded, because the empty set is inductive on Z, but $\varnothing \neq Z$.

Being well-founded is a proof principle: if (I, R) is well-founded and $P \subseteq I$ is (I, R)-inductive, then in order to prove $P = I$ it is enough to prove that P is (I, R)-inductive.

'I is R-well-founded on x' is itself an inductive property in the variable x. When no ambiguity will arise, we will drop any reference to I, or to R, or to both. In this case we will say: 'P is inductive', '$x \in I$ is well-founded', 'I is well-founded', 'R is well-founded'.

If $x \in I$ is well-founded, and R is recursive, then for all infinite successions

$$x_0, x_1, x_2, \ldots, x_n, \ldots \in I,$$

with $x_0 = x$, we have $R(x_n + 1, x_n)$ false for some $n \in N$ (the proof is by induction over x). Classically, we may prove something more for any R, namely that 'x is well-founded' is equivalent to the fact that all R-decreasing chains from x are finite. Intuitionistically, we may prove only a weak form of the converse by induction over $n \in N$: if all R-decreasing chain from x have length $\leqslant n$, then x is well-founded. There are well-founded x with no common bound $n \in N$ to the length of R-decreasing sequences from x. For instance, add an element ω to N, with the axioms $n < \omega$ for all $n \in N$. Then ω is well-founded in $(N \cup \{\omega\}, <)$, but no $n \in N$ is a bound to the length of decreasing sequences from ω. Indeed, for all n there are decreasing sequences of length $> n$ from ω:

$$\omega > n > n - 1 > n - 2 > \ldots > 1 > 0.$$

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Lexicographic ordering. Let $L \subseteq N$ be any set of integer codes of lists, ordered by list extension. Let $a =< x_0, \ldots, x_{n-1} > \in L$ and $b =< y_0, \ldots, y_{m-1} > \in L$. We define the lexicographic ordering over L by: $a > b$ iff either a is an extension of b, or for some $i < n$, m we have $x_0 = y_0, \ldots, x_{i-1} = y_{i-1}$, and $x_i > y_i$ (in the integer ordering). We now quote two properties of well-founded sets that will be useful later:

1 If I, J are orderings and $f : I \to J$ is any strictly increasing map, then J well-founded implies I well-founded.
2 If a set L of lists is well-founded by extension ordering, then L is also well-founded by lexicographic ordering.

2.1. *Stability*

Ershov's change ordering. Stability will express the convergence for a sequence f. Intuitively, f is a tree of streams of integers, and stability will say that, in all such streams, the value of f cannot keep on changing forever (eventually, it 'stabilises'). We say that f 'changes its mind' between $d \leqslant e \in D$ if and only if there are $d \leqslant d' \leqslant e' \leqslant e$ such that $f(d') \neq f(e')$. In this case we write $d <_f e$. The relation $<_f$ is a sub-ordering of the (strict) ordering on D, and $d \leqslant d' <_f e' \leqslant e$ implies $d <_f e$. We call $<_f$ the *change ordering*. It was first considered, but for a quite different purpose, by Ershov (Ash and Knight 2000).

Definition 2.1 (Stability). We say that f is *stable* if and only if $>_f$ (not $<_f$, but its dual, $>_f$) is well-founded.

Classically, f is *stable* if and only if on all *increasing* successions $\{a_n\}_{n \in N}$ on D, we have f changes its value only finitely many times. We interpreted increasing successions on D as a history of some computation: thus, intuitively speaking, f is stable if and only if f is stationary when restricted to the history of any computation. We skip the proof of this equivalence because it is classical, and of no interest here.

Intuitively, this means that we consider f a convergent sequence if and only if the succession $\{f(d_i)\}_{i \in N}$ is convergent for *all* $d_0 < d_1 < d_2 < d_3 < \ldots$ (for all computations described by some increasing succession over D). As we anticipated in the introduction, there is no room in this simplified picture for limits that are convergent with respect to some computation and divergent with respect to some other computation.

'f stable' is an intuitionistic proof principle. Let $P \subseteq D$, and assume P is $(D, >_f)$-inductive. That is, assume, for all $d \in D$: if $P(e)$ for all $e >_f d$, then $P(d)$. Then we may conclude that $P = D$.

A crucial example of stability. The sequence $L = \{L_n\}_{n \in N}$ defined above is not intuitionistically stationary, yet it is intuitionistically stable. L may change its value at most once, from some n such that $L_n = 1$, to some $m > n$ such that $L_m = 0$. Thus, all $>_L$-decreasing chains have length $\leqslant 1$. This implies that $(D, >_L)$ is well-founded. Thus, stability expresses convergence of L, even if we cannot decide whether L has limit 0 or 1.

More generally, if there is a bound n to the length of $>_L$-decreasing chains, then $>_L$ is well-founded and L is stable. This is a particular case of stability: usually, there is no

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bound $n \in N$ to the length of $>_L$-decreasing chains, or no recursive way to compute this bound from L.

Another example of stability. In the case $D = N$, a sequence $f : D \to N$ is a single stream of values

$$a, a, a, a, b, b, b, c, c, c, c, d, d, d, e, e, e, e, e, \ldots\ldots$$

If we split the stream into adjacent segments each time f changes its value, we get a succession of segments

$$a, a, a, a, \quad b, b, b, \quad c, c, c, c, \quad d, d, d, \quad e, e, e, e, e, \ldots.$$

We have $i >_f j$ if and only if when walking from i to j we change segment. f stable means $>_f$ well-founded, that is, that the segments above are well-founded by $>$. This is an indirect way of saying that there are only finitely many such segments (finitely many values of f), and, in particular, that there is a last segment (corresponding to the last and limit value of f). By expressing this fact in terms of well-foundation, we do not have to be precise about how many values f has, nor which is the last one (the limit of f). This weakening of the notion of convergence (which is only weakening intuitionistically) will allow us to prove convergence also (in an intuitionistic way) for limits with a non-recursive result.

3. A limit completion N^* of N

In this section we define the set N^* of converging sequences over N and the notions of effective maps over N^* and of properties over N^*. Then we check that our definitions are well-given.

Definition 3.1 (one-step completion N^* of N). We define one-step completion of N with respect to D, or just 'completion' for short, as the set

$$N^{*,D} = \{\text{stable (total, recursive) sequences } f : D \to N\}.$$

For all $I \subseteq N$, we use $I^{*,D}$ to mean the set of all stable (total, recursive) sequences $f : D \to I$.

When there is no risk of confusion we will drop the superscript D, and write I^* for $I^{*,D}$. We code the elements of N^* by the integer codes of the stable maps f. Each integer $n \in N$ is identified with the sequence $n^{\circ,D} \in N^*$, constantly equal to $n : n^{\circ,D}(d) = n$ for all $d \in D$. When no confusion arises, we will drop the superscript D, and write n°, or simply n.

Classically, all stable sequences have a (unique) limit, which *cannot* be computed uniformly on N^*: the existence of such a limit is not an intuitionistic theorem (see the example of the family of stable sequences L in the previous section).

We say that any bijection compatible with zero and successor is an isomorphism between N and N^*. Classically, N and N^* have the same elements, but there is no recursive isomorphism between them. Otherwise, the image of any $L \in N^*$ under this bijection would be the limit value of L, and this limit would be computable from L,

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contradicting the fact that limit is not computable. From an effective viewpoint, N^* and N have quite different structures: as we have already said, computable maps over N^* correspond, via identification of a limit with its own result, to a class of non-recursive maps (actually, to Δ^0_2-maps).

Fix any two element subset $\mathscr{B} = \{\mathtt{tt}, \mathtt{ff}\}$ of N. We say that any $b \in \mathscr{B}^*$ is a *stable Boolean*. We continue our model construction by defining (by a suitable equation) effective continuous maps over N^*. Then we define (by topological adherence) the predicate 'b is equal to true' for any stable boolean b. Eventually, we define equality over N^*. If $\underline{x} = x_1, \ldots, x_k$ is any vector of elements of N^*, we abbreviate $x_1(d) \ldots, x_k(d)$ by $\underline{x}(d)$.

Lifting. For each (total, recursive) map $g : N^k \to N^*$ (k-ary) we can define its *lifting*, or extension, to $(N^*)^k$. Lifting is a (total, recursive) map $g^{*,D} : (N^*)^k \to N^*$, defined by:

$$g^{*,D}(\underline{x})(d) = g(\underline{x}(d))(d) \in N, \quad \text{for all } \underline{x} \in (N^*)^k \text{ and } d \in D.$$

We say $g^{*,D}$ is the lifting of g. Our definition makes sense: we have $\underline{x}(d) \in N^k$, therefore $g(\underline{x}(d)) \in N^*$, thus $g(\underline{x}(d))(d) \in N$. We will have to prove that $g^{*,D} : (N^*)^k \to N^*$.

When there is no risk of confusion, we will drop the index D, and write g^* for $g^{*,D}$.

Continuous maps. We say that a (total, recursive) map $\phi : (N^*)^k \to N^*$ is *continuous* if and only if ϕ is extensionally equal to some lifting (to f^*, for some $f : N^k \to N^*$). Note that ϕ is determined by its restriction f to N^k. From a topological viewpoint, this means that N is dense within its completion N^*.

Extending recursive maps to continuous maps. Every total recursive $h : N^k \to N$ may be extended to some continuous map $h^{\circ,D}$. We define $h^{\circ,D}(\underline{x})(d) = h(\underline{x}(d))$, for all $\underline{x} \in (N^*)^k$ and $d \in D$. We will drop the index D if there is no risk of confusion. The two main properties of h° are:

— h° extends h to N^*: for all $\underline{x} \in (N^*)^k$ we have $h^\circ(\underline{x}^\circ) = h(\underline{x})^\circ$.

Proof. By unfolding definitions, we get $h^\circ(\underline{x}^\circ)(d) = h(\underline{x}^\circ(d)) = h(\underline{x}) = h(\underline{x})^\circ(d)$ for all $\underline{x} \in (N^*)^k$ and $d \in D$. □

— h is continuous.

Proof. Define $h' : N^k \to N^*$ by $h'(\underline{x}) = h(\underline{x})^\circ$ for all $\underline{x} \in N^k$. Then $h^\circ = (h')^*$, therefore h is continuous. Indeed, by unfolding the definition, we get, for all $\underline{x} \in (N^*)^k$ and $d \in D$: $h^\circ(\underline{x})(d) = h(\underline{x}(d)) = h(\underline{x}(d))^\circ(d) = h'(\underline{x}(d))(d) = (h')^*(\underline{x})(d)$. □

If we think of N as a subset of N^*, we may think of "$\circ$" as a special case of "*".

We can give now some examples of the extension of a total recursive map to some continuous map. First, if $x, y \in N^*$, then $(x +^\circ y) \in N^*$ is defined pointwise, by

$$(x +^\circ y)(d) = (x(d) + y(d)) \quad \text{for all } d \in D.$$

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For another example, let $\mathtt{if} : \mathscr{B}, N, N \to N$ be a map such that $\mathtt{if}(a,b,c) = b$ when $a = \mathtt{tt}, \mathtt{if}(a,b,c) = c$ when $a = \mathtt{ff}$, for all $b, c \in N$. Take $D = N$, and assume $l \in \mathscr{B}^*$ is any (stable) flow of $\mathtt{tt}$'s and $\mathtt{ff}$'s:

$$\mathtt{tt}, \mathtt{ff}, \mathtt{ff}, \mathtt{tt}, \mathtt{tt}, \mathtt{tt}, \mathtt{ff}, \mathtt{tt}, \ldots.$$

Then, $\mathtt{if}^\circ(l, b^\circ, c^\circ)$ is (for all $b, c \in N$) a (stable) flow of b's and c's:

$$b, c, c, b, b, b, c, b, \ldots.$$

Note that $\mathtt{if}^\circ$ does not decide whether l has limit 0 or has limit 1 (in fact, there is no way to make such a decision, in general). In order to introduce properties over N^*, we first have to define truth over $\mathscr{B}$.

Truth over $\mathscr{B}^*$. Let $b \in \mathscr{B}^*$ be a stable boolean. We say that 'b is equal to $\mathtt{tt}$', or that 'b is $\mathtt{tt}$' for short, if and only if the set $\{d \in D | b(d) = \mathtt{tt}\}$ is cofinal in D. From a topological viewpoint, when $D = N$, 'b is $\mathtt{tt}$' means that the integer $\mathtt{tt}$ is adherent to the range $\{b(0), b(1), b(2), \ldots\}$ of the sequence $b : N \to \mathscr{B}$. Classically, if $b \in \mathscr{B}^*$, then b is true if and only if b has limit $\mathtt{tt}$.

If $b \in \mathscr{B}^*$, then a sufficient condition for b true is that b has limit $\mathtt{tt}$: that is, that $b(e) = \mathtt{tt}$ for some $e_0 \in D$ and all $e \geqslant e_0$.

Proof. If $\{d \in D | b(d) = \mathtt{tt}\}$ is stationary in D, then it is cofinal in D. □

All constants and operations over $\mathscr{B}$,

$$\mathtt{tt}, \mathtt{ff}, \neg, \wedge, \vee, \Rightarrow,$$

may be extended to constants and operations over $\mathscr{B}^*$,

$$\mathtt{tt}^\circ, \mathtt{ff}^\circ, \neg^\circ, \wedge^\circ, \vee^\circ, \Rightarrow^\circ.$$

For instance, if $b, c \in \mathscr{B}^*$, then $(b \Rightarrow^\circ c) \in \mathscr{B}^*$ is defined by

$$(b \Rightarrow^\circ c)(d) = (b(d) \Rightarrow c(d)) \in \mathscr{B} \quad \text{for all } d \in D.$$

Properties over N^*. Fix any continuous map $P^* : (N^*)^k \to \mathscr{B}^*$. We may consider P^* a property over $(N^*)^k$ by defining, for all $\underline{x} \in (N^*)^k$,

$$\underline{x} \text{ satisfies } P^* \text{ if and only if } P^*(\underline{x}) \text{ is true (as element of } \mathscr{B}^*).$$

We say any P^* as above is a 'property over N^*'. Any decidable property $Q : N^k \to \mathscr{B}$ may be raised:

1 to a continuous map $Q^\circ : (N^*)^k \to \mathscr{B}^*$;
2 to a property over $(N^*)^k$, which we still denote by Q°.

Q° is an extension of Q.

Proof. By unfolding definitions, for all $\underline{x} \in N^k$, $Q^\circ(\underline{x}^\circ)$ is equivalent to $Q^\circ(\underline{x}^\circ)(d) = \mathtt{tt}$ cofinally in D. We have $Q^\circ(\underline{x}^\circ)(d) = Q(\underline{x}^\circ(d)) = Q(x)$ for all $d \in D$. We conclude that $Q^\circ(\underline{x}^\circ)(d) = \mathtt{tt}$ cofinally in D if and only if $Q(x) = \mathtt{tt}$. □

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Equality over N^*. Let $=^{\circ,D}$ be the lifting of the characteristic map for the equality. We say that $l, m \in N^*$ are equal if and only if $(l =^{\circ,D} m)$ is true, as element of $\mathscr{B}^*$. Again, we drop the index D when there is no risk of confusion.

For any $l, m \in N^*$, the element $(l =^{\circ} m) \in \mathscr{B}^*$ is a sequence, defined as follows: for all $d \in D$, we have $(l =^{\circ} m)(d) = (l(d) = m(d)) = \mathtt{tt}$ if $l(d), m(d)$ are equal integers, and $(l =^{\circ} m)(d) = \mathtt{ff}$ otherwise.

Thus, $l, m \in N^*$ are equal if and only if $\{d \in D | (l(d) = m(d)) \in N\}$ is cofinal in D (if and only if equality of $l(d), m(d)$ holds cofinally). Classically, l, m are equal if and only if l, m have the same limit. Hence the equality over N^* is not decidable.

This completes our model construction.

3.1. *Correctness of the limit construction*

We have to check that equality, projections, compositions of continuous maps and lifting of recursive maps are well-defined. Then we will check that continuous maps are exactly Δ^0_2-maps, and, in particular, that they include all decision maps for one-quantifier formulas. We need one more concept: pointwise truth and pointwise implication.

Pointwise truth and implication. Let $a, b \in \mathscr{B}^*$. We say that a is pointwise true if and only if $a(d) = \mathtt{tt}$ for all $d \in D$. If a is true pointwise, then, with some more reasoning, a is true cofinally, that is, true as element of $\mathscr{B}^*$. We say that a implies b pointwise if and only if $a(d) = \mathtt{tt}$ implies $b(d) = \mathtt{tt}$, for all $d \in D$ (if and only if $a \Rightarrow^{\circ} b$ is true pointwise). We start by proving a technical result, that the set of true stable booleans is a (proper) filter with respect to the pointwise implication.

Lemma 3.2 (Filter Lemma). Let $F = \{b \in \mathscr{B}^* | b \text{ is true}\}$. Then F is a proper filter with respect to the pointwise implication. By this we mean that for all $a, b \in \mathscr{B}^*$:

1 If a and b are true, then $(a \wedge^{\circ} b)$ is true.
2 If a is true, and a implies b *pointwise*, then b is true.
3 $\mathtt{tt}^{\circ}$ is true (pointwise); $\mathtt{ff}^{\circ}$ is *not* true.

Proof.

1 Fix any $d \in D$. We have to prove that there is some $e \geqslant d$ such that $a(e) = b(e) = \mathtt{tt}$. We argue by induction over $d \in D$, with respect to the order $>_a$, which is well-founded because $a \in \mathscr{B}^*$. By the assumption 'a is true', there is some $e \geqslant d$ such that $a(e) = \mathtt{tt}$. By the assumption 'b is true', there is some $e' \geqslant e \geqslant d$ such that $b(e') = \mathtt{tt}$. If $a(e') = a(e)$, then $a(e') = b(e') = \mathtt{tt}$. Now assume $a(e') \neq a(e)$. Since $e' \geqslant e \geqslant d$, we have $e' >_a d$, and by the induction hypothesis over e', we conclude that there is some $e'' \geqslant e' \geqslant d$ such that $a(e'') = b(e'') = \mathtt{tt}$.
2 For all d there is some $e \geqslant d$ such that $a(e) = \mathtt{tt}$. Then $b(e) = \mathtt{tt}$ by pointwise implication. Thus, $b(e) = \mathtt{tt}$ holds cofinally in D.
3 $\mathtt{tt}^{\circ}(d)$ is always equal to $\mathtt{tt}$, therefore it is pointwise true. $\mathtt{ff}^{\circ}(d)$ is always equal to $\mathtt{ff}$, therefore it is *not* cofinally true. □

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Point 1 of Lemma 3.2 says that the intersection of two particular cofinal subsets of D (the subset of $d \in D$ such that $a(d) = \mathtt{tt}$, and of $d \in D$ such that $b(d) = \mathtt{tt}$) is still cofinal in D. This is far from being true for generic cofinal subsets of D (think of even and odd integers in $D = N$). In this particular case, however, the intersection is still cofinal, thanks to the stability conditions for $a, b \in \mathscr{B}^*$.

Tautological consequences. The Filter Lemma (Lemma 3.2) implies that the set of true propositions defined using $\neg^\circ, \wedge^\circ, \vee^\circ, \Rightarrow^\circ$ over elements of $\mathscr{B}^*$ is closed under tautological consequences.

Proof. Assume $\alpha_1, \ldots, \alpha_n$ are true propositions, and α is a tautological consequence of $\alpha_1, \ldots, \alpha_n$. Then $\alpha_1 \wedge^\circ \ldots \wedge^\circ \alpha_n$ is true by Filter Lemma 3.2, and $(\alpha_1, \ldots, \alpha_n \Rightarrow^\circ \alpha)$ is true pointwise because it is a tautology. Again by the Filter Lemma, we conclude that α is true. □

Using the Filter Lemma, we can begin checking some correctness conditions for our model. Let $< _, _ >: N, N \to N$ be any coding of pairs in N.

Lemma 3.3 (Correctness).

(i) Any n-tuple of stable sequences is a stable sequence.
(ii) Equality on stable sequences is an equivalence relation.
(iii) Continuous maps include constant maps and projections, and are closed under compositions.
(iv) Lifting produces maps compatible with equality on N^*.

Proof.

(i) Take $n = 2$ (this case implies the general case). Let $a, b : D \to N$ be stable, and $c =< a, b >^\circ$ denote the map $d \mapsto < a(d), b(d) >$. We have to prove that D is well-founded with respect to $>_c$, that is, that any $d \in D$ is well-founded with respect to $>_c$. We will prove a little more, namely that for all $d_1, d_2 \in D$, and all $d \geqslant d_1, d_2$, we have d well-founded by $>_c$. The thesis then follows by taking any $d_1 = d_2 = d \in D$.
We argue by principal induction over $d_1, >_a$ and secondary induction over $d_2, >_b$. Well-foundation with respect to $>_c$ is inductive: thus, it is enough to check that for all $e >_c d$, we have e well-founded. By the definition of $>_c$, for some d', e' such that $d \leqslant d' < e' \leqslant e$, we have $< a(d'), b(d') > \neq < a(e'), b(e') >$, that is, either $a(d') \neq a(e')$ or $b(d') \neq b(e')$. This means either $(d_1 \leqslant d <_a e)$ or $(d_2 \leqslant d <_b e)$. In the first case we apply the principal induction hypothesis over the pair e, e. In the second case we apply the secondary induction hypothesis over the pair d_1, e.
(ii) Fix any three stable sequences $a, b, c : D \to N$.
— *Reflexivity*: $a =^\circ a$ is pointwise true.
— *Symmetry*: Assume $a =^\circ b$ is true. The implication $(a =^\circ b \Rightarrow^\circ b =^\circ a)$ is true pointwise: for all $d \in D$, we have $a(d) = b(d)$ implies $b(d) = a(d)$. By point 2 of Lemma 3.2, $b =^\circ a$ is true also.
— *Transitivity*: Assume $a =^\circ b$ and $b =^\circ c$ are true. Then by point 1 of Lemma 3.2, $(a =^\circ b \wedge b =^\circ c)$ is true. The implication $(a =^\circ b \wedge b =^\circ c \Rightarrow^\circ a =^\circ c)$ is pointwise

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true: for all $d \in D, a(d') = b(d') \wedge b(d') = c(d')$ implies $a(d') = c(d')$. By point 2 of Lemma 3.2, we conclude that $a =^\circ c$ is true.

(iii) — *Constant maps*: If $f(\underline{x})(d)$ is constantly n, then f^* is constantly $n^\circ : f^*(\underline{x})(d) = f(\underline{x}(d))(d) = n = n^\circ(d)$. Thus, constant maps are continuous.

— *Projections*: Let q_i be the i-th projection over N^k, and p_i the i-th projection over $(N^*)^k$. We check that p_i, q_i° are extensionally equal. Indeed, $q_i^\circ(\underline{x})(d) = q_i(\underline{x}(d)) = x_i(d) = p_i(\underline{x})(d)$ for all $\underline{x} \in (N^*)^k, d \in D$. Thus, p_i is continuous.

— *Compositions*: Let g be a map of arity n, and $\underline{h}$ be any vector of n maps of arity m. Then $g \circ \underline{h}$ is defined $:N^m \to N$. We check that $(g^* \circ \underline{h}^*)(\underline{x}) = (g^* \circ \underline{h})^*(\underline{x})$ for all $\underline{x} \in (N^*)^k$. By definition unfolding, for all $d \in D$ we have: $g^*(\underline{h}^*(\underline{x}))(d) = g(\underline{h}^*(\underline{x})(d))(d) = g(\underline{h}(\underline{x}(d))(d))(d) = g^*(\underline{h}(\underline{x}(d)))(d) = (g^* \circ \underline{h})(\underline{x}(d))(d) = (g^* \circ \underline{h})^*(\underline{x})(d)$. We conclude that $(g^* \circ \underline{h}^*)$ is the lifting of $(g^* \circ \underline{h})$. Thus, $(g^* \circ \underline{h}^*)$ is continuous.

(iv) Assume $f : N^k \to N^*$ in order to show $f^* : (N^*)^k \to N^*$. Let $\underline{x} \in (N^*)^k$: we have to check that the sequence $f^*(\underline{x})(.) : D \to N^k$ is stable, or that $>_f (\underline{x})$ is well-founded. We will prove the following Claim, which is slightly more general than our thesis.

Claim. For any three $d \geqslant d_1, d_2 \in D, d$ is well-founded under $>_{f(\underline{x})}$.

Our thesis 'all $d \in D$ are well-founded under $>_{f(\underline{x})}$' will then follow from this Claim by taking $d_1 = d_2 = d \in D$.

By point (i) of this lemma, $\underline{x}(.) : D \to N^k$ is stable, and $>_{\underline{x}}$ is well-founded. Thus, we may prove the Claim by principal induction over $d_1, >_{\underline{x}}$, and secondary induction over $d_2, >_{f(\underline{i})}$, with $\underline{i} = \underline{x}(d_1)$. Well-foundation of $d \in D$ is an inductive property, so it is enough to prove that all $e \in D$ with $e >_{f(\underline{x})} d$ are well-founded under $>_{f(\underline{x})}$.

By the definition of $e >_{f(\underline{x})} d$, and of f^*, for some d', e' such that $d_1, d_2 \leqslant d \leqslant d' < e' \leqslant e$, we have

$$f(\underline{x})(d') = f(\underline{x}(d'))(d') \neq f(\underline{x}(e'))(e') = f(\underline{x})(e').$$

There are two subcases:

— Assume either $\underline{x}(d_1) \neq \underline{x}(d')$, or $\underline{x}(d') \neq \underline{x}(e')$. Then $(d_1 <_{\underline{x}} e)$. By *the induction hypothesis over the pair* e, e, and $e \geqslant e$, we deduce that e is well-founded under $>_f (\underline{x})$.

— Assume $\underline{x}(d_1) = \underline{x}(d') = \underline{x}(e')$. Call $\underline{i}$ the common value of these three expressions. By the hypothesis on d', e', we deduce $f(\underline{i})(d') = f(\underline{x}(d'))(d') \neq f(\underline{x}(e'))(e') = f(\underline{i})(e')$. Thus, $(d_2 <_{f(\underline{i})} e)$. By $i = \underline{x}(d_1)$, and *the induction hypothesis over the pair* d_1, e, and $e \geqslant d_1, e$, we conclude that e is well-founded by $>_{f(\underline{x})}$. □

The last lemma to be proved says that any lifting f^* only depends on the extensional behaviour (modulo $=^\circ$) of the map f.

Lemma 3.4 (Compatibility of * and extensional equality). If $f, g : N^k \to N^*$ are extensionally equal, then f^*, g^* are extensionally equal over $(N^*)^k$.

Proof. Let $\underline{x} \in (N^*)^k$. We have to prove that $f^*(\underline{x}) =^\circ g^*(\underline{x})$, that is, that for all $d \in D$ there is some $e \geqslant d$ such that $f(\underline{x}(e))(e) = g(\underline{x}(e))(e)$. We argue by induction over $>_{\underline{x}}$ (which is well-founded by point (i) of Lemma 3.3). By assumption, f, g are extensionally

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equal over N^k : for all $\underline{i} \in N$ we have $f(\underline{i}) =^\circ g(\underline{i})$ in N^*, that is, $\{e \in D | f(\underline{i})(e) = g(\underline{i})(e)\}$ is cofinal in D. Taking $\underline{i} = \underline{x}(d)$, we deduce that for all d, there is some $e \geqslant d$ such that $f(\underline{x}(d))(e) = g(\underline{x}(d))(e)$. If $\underline{x}(d) = \underline{x}(e)$ we get the thesis. If $\underline{x}(d) \neq \underline{x}(e)$, we have $e >_{\underline{x}} d$. By the induction hypothesis on e, there is some p such that $p \geqslant e \geqslant d$ and $f(\underline{x}(p))(p) = g(\underline{x}(p))(p)$. Thus we also get our thesis in this case. □

Fix any map $f : N \to N^*$ that is constantly equal in N^* to n°: that is, $f(x) =^\circ n^\circ$ for all $x \in N$. Then f^* is constantly equal, in N^*, to n°.

Proof. Define a map $g : N \to N^*$ by $g(x) = n^\circ$ for all $x \in N$. Then f, g are extensionally equal. By Lemma 3.4, f^*, g^* are extensionally equal. By Lemma 3.3, g^* is constantly n°. Thus, f^* is constantly n°. □

The density argument. As a particular case, if $P : N \to \mathscr{B}^*$ is an always true property on N, then its lifting $P^* : N^* \to \mathscr{B}^*$ is an always true property on N^*. Thus, in order to prove $P^*(l)$ for all $l \in N^*$, it is enough to prove $P(x)$ for all $x \in N$. We call this argument 'the density argument', because, topologically, and if we think of P^* as an open subset of N^*, it means that N is dense in N^*.

By combining the previous Lemmas, we conclude correctness as follows.

Theorem 3.5 (Correctness of the topological model). All relations and maps used in our model construction are well-defined:

1 *Equality*: Equality on N^* is an equivalence relation.
2 *Continuous maps*: Continuous maps include projections and are closed under composition.
3 *Lifting*: The result of lifting is compatible with equality on N^*; lifting sends extensionally equal maps into extensionally equal maps.

The following is a simple consequence of the Correctness Theorem.

Lemma 3.6 (Lifting and completion). Let $X : N \to \mathscr{B}$ be any recursive subset of N. Then we may form both the property $X^\circ : N^* \to \mathscr{B}^*$ over N^*, and the subset $X^* \subseteq N^*$. Then X° and X^* are related as follows: for all $x \in N^*$,

$$X^\circ(x) \text{ iff } x =^\circ y \quad \text{for some} \quad y \in X^*.$$

Proof. Assume $X^\circ(x)(d)$ is cofinally true. Then $X(x(d_0))$ is true for some $d_0 \in D$, hence $x(d_0) \in X$. Define $y = \mathtt{if}^\circ(X^\circ(x), x, x(d_0)^\circ) \in N^*$. We have $y(d) \in X$ for all $d \in D$ because if $X(x(d)) = \mathtt{tt}$, then $y(d) = x(d) \in X$, while if $X(x(d)) = \mathtt{ff}$, then $y(d) = x(d_0) \in X$. Thus, $y \in X^*$. We have $x =^\circ y$ because $X^\circ(x) = \mathtt{tt} \Rightarrow^\circ \mathtt{if}^\circ(X^\circ(x), x, x(d_0)^\circ) =^\circ x$ follows by definition of `if` and Density.

Conversely, assume $x =^\circ y$ for some $y \in X^*$. Then $X^\circ(x) =^\circ X^\circ(y) =^\circ \mathtt{tt}^\circ$. □

3.2. *The limit construction is independent of D*

In the rest of this section we will prove, as we anticipated in the introduction, that the construction N^{*D} does not depend on the direct set D. For all direct sets D, E covering

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N, we will prove that the structures $N^{*,D}$ and $N^{*,E}$ are recursively isomorphic. These are recursive bijections between elements of $N^{*,D}$ and of $N^{*,E}$, and between continuous maps over $N^{*,D}$ and over $N^{*,E}$, and these bijections are compatible with equality, with extensionally equality, and with function application.

We need some definitions first. We say that a map $\alpha : D \to D$ is above the identity if and only if $\alpha(d) \geqslant d$ for all $d \in D$. We say that a pair of total recursive *weakly increasing* maps $\beta : D \to E, \gamma : E \to D$ connects two direct sets D, E if and only if both $\gamma\beta : D \to D$ and $\beta\gamma : E \to E$ are above the identity. We say that D, E are connected if and only if there are β, γ connecting D, E. We will prove that any two countable direct sets D, E covering N are connected, and that if D, E are connected, then $N^{*,D}, N^{*,E}$ are recursively isomorphic. The first step is to check:

If $\beta : D \to E$ is any (total recursive) map, we define a map $g \mapsto g^\beta$ from $(E \to N)$ to $(D \to N)$ by composition: $g^\beta = g\beta$.

Lemma 3.7 (Weakly increasing maps). Let D, E be recursive direct sets covering N. If $\beta : D \to E$ is weakly increasing, then $g \mapsto g^\beta$ is a map from $N^{*,D}$ to $N^{*,E}$.

Proof. Assume $>_g$ is well-founded over D in order to prove that $>_{g^\beta}$ is well-founded over E. If $e >_{g^\beta} e'$, then $g\beta(e_1) \neq g\beta(e_2)$ for some $e \geqslant e_1 > e_2 \geqslant e'$. Since β is weakly increasing, this implies $g(\beta(e_1)) \neq g(\beta(e_2))$ for some $\beta(e) \geqslant \beta(e_1) > \beta(e_2) \geqslant \beta(e')$. This latter, in turn, implies $\beta(e) >_g \beta(e')$. Thus, β is an increasing mapping from $(E, >_{g^\beta})$ to $(D, >_g)$. From $(D, >_g)$ well-founded, we then conclude $(E, >_{g^\beta})$ well-founded. □

Lemma 3.8 (Maps above the identity). Assume D is a direct set, $\alpha : D \to D$ is weakly increasing and above the identity. If $f, g \in N^{*,D}$ and $f =^{\circ,D} g$, then $f\alpha, g\alpha \in N^{*,D}, f\alpha =^{\circ,D} g\alpha$ and $f\alpha =^{\circ,D} f$.

Proof. We have $f\alpha, g\alpha \in N^{*,D}$ by Lemma 3.7 because α is weakly increasing. Now it is enough to check that $f\alpha =^{\circ,D} f$ for all $f \in N^{*,D}$. Then $f\alpha =^{\circ,D} g\alpha$ will follow from

$$f\alpha =^{\circ,D} f, f =^{\circ,D} g, g =^{\circ,D} g\alpha,$$

and transitivity of $=^{\circ,D}$. We have to check that for all $d \in D$ there is some $d' \geqslant d$ such that $f\alpha(d') = f(d')$. We argue by induction over d and $>_f$. If $f\alpha(d) = f(d)$, we are done. If $f\alpha(d) \neq f(d)$, then from $\alpha(d) \geqslant d$ we deduce $\alpha(d) >_f d$. By the induction hypothesis on $\alpha(d)$, we conclude that there is some $d' \geqslant \alpha(d) \geqslant d$ such that $f\alpha(d') = f(d')$. □

Lemma 3.9 (Connection Lemma). Let D, E be recursive direct sets covering N. If β, γ connects D, E, then $f \mapsto f^\gamma, g \mapsto g^\beta$ are maps compatible with $=^{\circ,D}, =^{\circ,E}$.

Proof.

— Assume $f =^{\circ,D} f'$ in order to prove $f\gamma =^{\circ,E} f'\gamma$.
By Lemma 3.8, and $\gamma\beta$ above the identity, we deduce $f\gamma\beta =^{\circ,D} f'\gamma\beta$. This means that the set of $d \in D$ such that $f\gamma\beta(d) = f'\gamma\beta(d)$ is cofinal in D. Now fix any $e \in E$. Our

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thesis is that there is some $e' \geqslant e$ such that $f\gamma(e') = f'\gamma(e')$. By what we said, for some $d' \geqslant \gamma(e)$ we have $f\gamma\beta(d') = f'\gamma\beta(d')$. By $d' \geqslant \gamma(e)$ and β weakly increasing, we deduce $\beta(d') \geqslant \beta\gamma(e) \geqslant$ (by $\beta\gamma$ above the identity) e. Thus, $f\gamma(\beta(d')) = f'\gamma(\beta(d'))$ and $\beta(d') \geqslant e$. The thesis now follows by choosing $e' = \beta(d') \in E$.

— Assume $g =^{\circ,E} g'$, in order to prove $g\beta =^{\circ,E} g'\beta$.
If D, E are connected, then E, D are connected. By what we just said, from $g =^{\circ,E} g'$ we conclude $g\beta =^{\circ,D} g'\beta$. □

Lemma 3.10 (Covering Lemma). Assume D is a recursive direct set covering N. Then N, D are connected by some $c : N \to D, h : D \to E$.

Proof. By assumption, there is some recursive enumeration $d_0, d_1, d_2, \ldots$ of the elements of D, and some covering relation $\mathtt{Cov}, \xi$ between D and N. We will now define c, h connecting N, D.

— *The map* $c : N \to D$. There is some total recursive $c : N \to D$ strictly increasing, with $\{c(x) | x \in N\}$ cofinal in D. Define $c(0) =$ any element of $D, c(n+1) = j(c(n), d_n, \xi(i))$, for $i =$ the first element of N not covered by $c(n) \in D$. c is well-defined because D is direct. c is recursive because the maps $j, d(.), \xi$ are. $\{c(x) | x \in N\}$ is cofinal in D, because each element of D is equal to some d_n, and $d_n \leqslant c(n)$ by construction.

— *The map* $h : D \to N$. For any $d \in D$, define $h(d) =$ the first $x \in N$ such that $d \leqslant c(x)$. There is some $x \in N$ such that $d \leqslant c(x)$ by cofinality of $\{c(n) | n \in N\}$. h is recursive because c is.

— c *is strictly increasing*. We have $c(n+1) > c(n)$ because $c(n+1) \geqslant c(n)$ by definition of c, and again by definition, $c(n+1)$ covers i, while $c(n)$ does not.

— $c \circ h$ *is above the identity*. If $h(d) = x$, then $c(x) \geqslant d$ by the definition of h. Thus, $c(h(d)) \geqslant d$.

— $h \circ c$ *is above the identity*. We have $h(c(n)) =$ the first $x \in N$ such that $(c(n) \leqslant c(x)) = n$, because c is strictly increasing, and if it were $n < x$ we would deduce $c(n) < c(x)$. □

We extend isomorphism to maps in the traditional way. Let $\phi : N^{*,D} \to N^{*,D}$. We define a map $\phi^{\beta,\gamma} : N^{*,E} \to N^{*,E}$ by $\phi^{\beta,\gamma}(g) = \phi(g^\beta)^\gamma$ for all $g \in N^{*,E}$.

Lemma 3.11 (Isomorphism Lemma). Let D, E be any two direct sets connected by β, g. Then $f \mapsto f\gamma, g \mapsto g\beta$ is an isomorphism pair between $N^{*,D}$ and $N^{*,E}$ commuting with application of maps.

Proof. By Lemma 3.9, $f \mapsto f\gamma, g \mapsto g\beta$ are maps from $N^{*,D}$ to $N^{*,E}$, and, conversely, compatible with equality.

We have still to prove that $f \mapsto f\gamma, g \mapsto g\beta$ is an isomorphism pair. By $\gamma\beta, \beta\gamma$ above the identity, and Lemma 3.8, we conclude $f\gamma\beta =^{\circ,D} f$ and $g\beta\gamma =^{\circ,E} g$.

To show isomorphism commutes with application, we have to check that for all $\phi : N^{*,D} \to N^{*,D}$ and $f \in N^{*,D}$, we have $\phi(f)^\gamma =^{\circ,E} \phi^{\beta,\gamma}(f^\gamma)$. By unfolding definitions, $\phi^{\beta,\gamma}(f^\gamma) = \phi(f\gamma\beta)\gamma =^{\circ,E}$ (by $f\gamma\beta =^{\circ,D} f$) $\phi(f)^\gamma$. In the same way we may check that for all $\psi : I^{*,E} \to J^{*,E}$ and $g \in N^{*,E}$, we have $\psi(g)^\beta =^{\circ,D} \psi^{\gamma,\beta}(g^\beta)$. □

Theorem 3.12 (Isomorphism Theorem). Let D, E be any two direct sets covering N. Then $N^{*,D}$ and $N^{*,E}$ are recursively isomorphic.

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Proof. By Lemma 3.11, we deduce that the completions taken with respect to D and N are effectively isomorphic, for any D covering N. By transitivity of isomorphism, we conclude that the completions taken with respect to any direct sets D, E covering N are effectively isomorphic. □

4. Characterisation of continuous maps on N^*

In this section we check that the set continuous maps over N^* coincide with the set of maps recursive with respect to the Halting Problem; classically, this means that they are exactly all Δ_2^0-maps. Then we prove, constructively, a conservativity result: equations $g(x) = 0$ (with g recursive) solvable in N^* are solvable in N, too.

Let $\mathscr{B} = \{\mathtt{tt}, \mathtt{ff}\}$ be any two element subset of N. Let $f : N^2 \to \mathscr{B}$ be any binary recursive property, and $\mathtt{s}_f(x), \mathtt{fst}_f(x)$ be (classically) defined by:

$$\begin{aligned}
\mathtt{s}_f(x) &= \mathtt{tt} \text{ if } f(x,y) = \mathtt{tt} \text{ for some } y,\\
&= \mathtt{ff} \text{ if } f(x,y) = \mathtt{ff} \text{ for all } y.\\
\mathtt{fst}_f(x) &= \text{the first } y \text{ such that } f(x,y) = \mathtt{tt}, \text{ if any},\\
&= 0 \text{ if } f(x,y) = \mathtt{ff} \text{ for all } y.
\end{aligned}$$

The maps are non-recursive in general (they solve the Halting Problem). They may be generalised to $(k+1)$-ary recursive properties $f(\underline{x}, y)$.

The maps $\mathtt{s}_f, first_f$ may be (intuitionistically) defined as continuous maps: $(N^*)^k \to \mathscr{B}^*$, provided D covers N. Take any D equipped with a covering relation $\mathtt{Cov}, x$. If we have a family $(\mathtt{Cov}_n, \xi_n)$ of covering for N, we choose (for a reason we will explain later) the covering with index $m =< f, \underline{x} >$ (some name for the map $f(\underline{x}, .)$ that we are considering). We abbreviate $y \in \mathtt{Cov}_m(d)$ by $y \lhd_m d$.

Then we may define, for all $\underline{x} \in N^k$ and all $d \in D$:

$$\begin{aligned}
\mathtt{s}_f(\underline{x})(d) &= \mathtt{tt} && \text{if } f(\underline{x}, y) = \mathtt{tt} \text{ for some } y \lhd_m d,\\
&= \mathtt{ff} && \text{if } f(\underline{x}, y) = \mathtt{ff} \text{ for all } y \lhd_m d.\\
\mathtt{fst}_f(\underline{x})(d) &= \text{the first } y \in \mathtt{Cov}_m(d) \text{ s.t. } f(\underline{x}, y) = 0 && \text{if } f(\underline{x}, y) = \mathtt{tt} \text{ for some } y \lhd_m d,\\
&= 0 && \text{if } f(\underline{x}, y) = \mathtt{ff} \text{ for all } y \lhd_m d.
\end{aligned}$$

We call $\mathtt{s}_f$ a 'some map', and we identify it with the existential quantification $\exists y \in N. f(\underline{x}, y) = \mathtt{tt}$. We define a universal quantification $\mathtt{all}_f(\underline{x})$ by $\neg \mathtt{s}_{\neg f}(\underline{x})$.

These maps are recursive because we assumed $\mathtt{Cov}_m(d)$ to be finite and recursive in m, d. Fix any $\underline{x} \in N^k$, and any increasing succession $d_1 < d_2 < d_2 < \ldots$ of computation states. This succession represents the history of some computation. The succession

$$\mathtt{s}_f(\underline{x})(d_1), \mathtt{s}_f(\underline{x})(d_2), \mathtt{s}_f(\underline{x})(d_2), \ldots$$

represents, instead, for each step d_i of the computation, the truth value $\mathtt{s}_f(\underline{x})(d_i)$ that we assign to $\exists y^* \in N. f(\underline{x}, y) = \mathtt{tt}$ in the step d_i. We start by assigning $\mathtt{ff}$ to $\exists y \in N. f(\underline{x}, y) = \mathtt{tt}$.

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We switch to $\mathtt{tt}$ if and when we reach some $d_i \in D$ that m-covers some solution of $f(\underline{x}, y) = \mathtt{tt}$.

Lemma 4.1 ('some' and 'first' maps).

(i) $\mathtt{s}_f(\underline{x}), \mathtt{fst}_{cf}(\underline{x}) \in \mathscr{B}^*$.

(ii) For all $x, y \in (N^*)^k + 1$, the implications:

(a) $\mathtt{s}_f(x) = \mathtt{tt} \Rightarrow^\circ f(x, \mathtt{fst}(x)) =^\circ \mathtt{tt}$

(b) $\mathtt{s}_f(x) = \mathtt{ff} \Rightarrow^\circ f(x, y) =^\circ \mathtt{ff}$

are true in $\mathscr{B}^*$ (are cofinally true).

Proof.

(i) We start by noting that, by monotonicity of covering, if $f(\underline{x}, y) = \mathtt{tt}$ for some $y \lhd_m d$, and $e \geqslant d$, then $f(\underline{x}, y) = \mathtt{tt}$ for some $y \lhd_m e$. That is, if $\mathtt{s}_f(\underline{x})(d) = \mathtt{tt}$, then $\mathtt{s}_f(\underline{x})(e) = \mathtt{tt}$ for all $e \geqslant d$, hence $\mathtt{s}_f(\underline{x})(d) = \mathtt{s}_f(\underline{x})(e)$. By contraposition, if $\mathtt{s}_f(\underline{x})(d) \neq \mathtt{s}_f(\underline{x})(e)$ and $e \geqslant d$, we deduce $\mathtt{s}_f(\underline{x})(d) = \mathtt{ff}$. We also deduce $\mathtt{s}_f(\underline{x})(e) = \mathtt{tt}$, because $\mathtt{s}_f(\underline{x})(e) \neq \mathtt{s}_f(\underline{x})(d) = \mathtt{ff}$.

Now we e claim that $e >_{\mathtt{s}_f(\underline{x})} d$ implies $\mathtt{s}_f(\underline{x})(d) = \mathtt{ff}$ and $\mathtt{s}_f(\underline{x})(e) = \mathtt{tt}$. Indeed, $e > \mathtt{s}_f(\underline{x})d$ unfolds to $\mathtt{s}_f(\underline{x})(d') \neq \mathtt{s}_f(\underline{x})(e')$ for some $d \leqslant d' \leqslant e' \leqslant e$. We deduce $\mathtt{s}_f(\underline{x})(d') = \mathtt{ff}, \mathtt{s}_f(\underline{x})(e') = \mathtt{tt}$. By $d \leqslant d'$ and contraposition, we deduce $\mathtt{s}_f(\underline{x})(d) = \mathtt{ff}$. By $e' \leqslant e$, we obtain $\mathtt{s}_f(\underline{x})(e) = \mathtt{tt}$.

Thus, $\mathtt{s}_f(\underline{x})$ is an increasing map from $D, > \mathtt{s}_f(\underline{x})$ to $\mathscr{B}$, ordered by $\mathtt{tt} > \mathtt{ff}$. The latter is a well-founded set, so we conclude that $>_{\mathtt{s}_f(\underline{x})}$ is well-founded.

(ii.a) Assume $\mathtt{s}_f(x)(d) = \mathtt{tt}$. Then, by the definition of $\mathtt{s}_f$, we have $f(x, \mathtt{fst}_f(x)(d)) = \mathtt{tt}$ for the same d. Thus, implication (a) is pointwise true, hence it is cofinally true.

(ii.b) Assume $\mathtt{s}_f(x)(d) = \mathtt{ff}$, and fix any $y \in N$. Then, by the definition of $\mathtt{s}_f$, whenever $y \lhd_m d$, we have $f(x, y)(d) = \mathtt{ff}$. By the definition of Covering, $y \lhd_m \xi_m(y)$. By the monotonicity of covering, for all $e \geqslant \xi_m(y)$, we have $y \lhd_m e$. Therefore, $\mathtt{s}_f(x)(e) = \mathtt{ff} \Rightarrow f(x, y)(e) = \mathtt{ff}$ for all $e \geqslant \xi_m(y)$. In particular, (b) is true for $e = j(d, \xi_m(y)) \geqslant d, \xi_m(y)$, for all $d \in D$. Therefore (b) is cofinally true, or true in $\mathscr{B}^*$. By a Density argument, (b) is true for all $y \in N^*$. □

Interpreting Quantifiers. By the definition of $\mathtt{s}_f(\underline{x})$, the implication

$$\mathtt{s}_f(\underline{x}) =^\circ \mathtt{tt}^\circ \Rightarrow^\circ f^*(\underline{x}^\circ, \mathtt{fst}_f(\underline{x})) =^\circ \mathtt{tt} \tag{1}$$

is true pointwise. The implication

$$(f^*(\underline{x}^\circ, n^\circ) =^\circ \mathtt{tt} \Rightarrow^\circ \mathtt{s}_f(\underline{x}) = \mathtt{tt})(d) \tag{2}$$

is true for all $d \in D$ covering n, and hence for all $d \geqslant \xi(n)$, so it is cofinally true. By density, it is true if we replace n° by any $y \in N^*$. We deduce that $\mathtt{s}_f(\underline{x})$ is a faithful interpretation of the existential over N^*:

$$\mathtt{s}_f(\underline{x}) =^\circ \mathtt{tt}^\circ \text{ iff } f^*(\underline{x}^\circ, y) =^\circ \mathtt{tt}^\circ \text{ for some } y \in N^*. \tag{3}$$

By taking the contraposition of the implications (1) and (2), we obtain for all $y \in N^*$

$$f^*(\underline{x}^\circ, \mathtt{fst}_f(\underline{x})) =^\circ \mathtt{ff}^\circ \Rightarrow \mathtt{s}_f(\underline{x}) =^\circ \mathtt{ff}^\circ \tag{4}$$

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$$(\mathtt{s}_f(\underline{x}) =^\circ \mathtt{ff}^\circ \Rightarrow^\circ f^*(\underline{x}^\circ, y) =^\circ \mathtt{ff}). \tag{5}$$

From (4), (5), if we replace f by $\neg f$ we deduce:

$$\mathtt{s}_{\neg f}(\underline{x}) =^\circ \mathtt{ff}^\circ \text{ iff } \neg f^*(\underline{x}^\circ, y^\circ) =^\circ \mathtt{ff}^\circ \text{ for all } y \in N^*. \tag{6}$$

From $b =^\circ \mathtt{ff}$ iff $\neg^\circ b =^\circ \mathtt{tt}$ for all $b \in \mathscr{B}$, we conclude

$$\neg\mathtt{s}_{\neg f}(\underline{x}) =^\circ \mathtt{tt}^\circ \text{ iff } f^*(\underline{x}^\circ, y^\circ) =^\circ \mathtt{tt}^\circ \text{ for all } y \in N^*. \tag{7}$$

Thus, $\mathtt{all}_f(\underline{x}) = \neg\mathtt{s}_{\neg f}(\underline{x})$ is a faithful interpretation of the universal quantifier over N^*. $\mathtt{s}_f$, $\mathtt{fst}_f$ may be lifted to maps:

$$\mathtt{s}_f^* : (N^*)^k \to \mathscr{B}^*$$

$$\mathtt{fst}_f^* : (N^*)^k \to \mathscr{B}^*$$

having the same properties. Indeed, (1), (2), (4), (5) are true for all $\underline{x} \in (N^*)^k$ by Density, and, for all $\underline{x} \in (N^*)^k$, (3), (6), (7) follow from them.

Lifting recursive maps to N^*. Maps on N^* include a corresponding f° for each total recursive map f over N. They are closed under composition. In order to prove that they include all Δ_2^0-maps, we only have to prove they are closed under minimalisation. We first express what we mean by minimalisation in N^*.

Fix any continuous property $f^* : (N^*)^{k+1} \to \mathscr{B}^*$, any $\underline{x} \in (N^*)^k$. We say that f^* is minimalisable on $\underline{x}$ if and only if $f^*(\underline{x}, y)$ is solvable in N^*:

$$\text{for some } y \in N^*, \quad f^*(\underline{x}, y) \text{ is true.}$$

We say that f^* is minimalisable if and only if f^* is minimalisable on all $\underline{x} \in (N^*)^k$.

Let $g^* : (N^*)^k \to N^*$ be any continuous map. We say that g^* is a minimalisation of f^* if and only if for all $\underline{x}, y \in (N^*)^{k+1}$, $y = g^*(\underline{x})$ is the minimum solution to $f^*(\underline{x}, y)$. In other words, we ask the following statements to be true in $\mathscr{B}^*$:

$$(f^*(\underline{x}, g^*(\underline{x})) =^\circ \mathtt{tt}) \quad \text{and} \quad {}^\circ(y < g^*(\underline{x}) \Rightarrow^\circ f^*(\underline{x}, y) =^\circ \mathtt{ff}).$$

We say that maps over N^{*D} are closed under minimalisation if and only if whenever a map f^* is minimalisable, there is some minimalisation g^* of f^*. It is immediate to show that minimalisation, when it exists, is unique up to extensional equality.

We prove that, at least in the case $D = N$, maps over N^{*D} are closed under minimalisation. Since all N^{*D} are recursively isomorphic, provided D covers N, the thesis will follow for all D. We introduce a map $(\mu x.f) : N^k \to N^*$, which is our candidate for g. We define, for all $\underline{x} \in (N^*)^k, d \in N$:

$$\begin{aligned}(\mu x.f)(\underline{x})(d) &= \text{the minimum } x \leqslant d \text{ such that } f(\underline{x}, x)(d) = \mathtt{tt}, \text{ if any,}\\ &= d \text{ otherwise.}\end{aligned}$$

We will prove that if f^* is minimalisable, then $(\mu x.f)^*$ is a minimalisation of f^*. The first step is to check that $(\mu x.f)$ is, indeed, a map: $N^k \to N^*$. Before this, we show that, in the case $D = N$, we may weaken the requirement of being a stable map.

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Lemma 4.2 (Cofinality and well-foundation). Choose $D = N$. Let X be a recursive cofinal subset of N. Fix any sequence $L : D \to N$. Assume $>_L$ is well-founded when restricted to X. Then L is stable.

Proof. We have to prove that all $d \in N$ are well-founded for $>_L$. Since X is recursive cofinal, we may define a total recursive map $\phi : N \to N$ by: $\phi(x) =$ the first $y \in X$ such that $y \geqslant x$ in N. We will prove d well-founded for $>_L$ by principal induction over $\phi(d)$ in $(X, >_L)$ and secondary induction over $\phi(d) - d$ (an integer $\geqslant 0$ by definition of ϕ). Since well-foundation is inductive, we have to check that all $e >_L d$ are well-founded with respect to $>_L$. We consider two cases, according to whether $e \leqslant \phi(d)$ or $e > \phi(d)$.

— *Assume* $e \leqslant \phi(d)$. Then $\phi(e) = \phi(d)$, but $(\phi(d) - d) > (\phi(e) - e) \geqslant 0$ because $e > d$. Well-foundation of e then follows by the secondary induction hypothesis.

— *Assume* $e > \phi(d)$. We will prove in this case that all $e' >_L e$ are well-founded for $>_L$. Since well-foundation is inductive, it will follow that e is well-founded. We have $\phi(e') \geqslant e' >_L e > \phi(d)$, and hence $\phi(e') >_L \phi(d)$ by the properties of $>_L$. By the principal induction hypothesis, we conclude that all $e' >_L e$ are well-founded for $>_L$, as required. □

We may now check that $(\mu x.f)$ is, indeed, a map: $N^k \to N^*$. We start with a general remark for all $y \in D^*$.

All $y \in D^*$ are cofinally $\leqslant$ their argument $d \in D$. Putting this another way gives the following lemma.

Lemma 4.3 (Bar Lemma). Assume $P : D^* \to (D \to \mathscr{B})$ is defined by

$$P(y)(d) = (d \geqslant y(d)), \text{ for all } y \in D^*, d \in D.$$

Then $P^* : D^* \to \mathscr{B}^*$, and $P^*(y)$ is true for all $y \in D^*$.

Proof. By Density, we only have to prove that $P^*(y^\circ) \in \mathscr{B}$, and that $P^*(y^\circ)$ is true for all $y \in D$. For all $d \in N$, we have $P^*(y^\circ)(d)$ unfolds to $(d \geqslant y^\circ(d))$, that is, to $(d \geqslant y)$. $P^*(y^\circ)(d)$ may change value at most once, from `ff` to `tt`, hence it is stable. To prove that $P^*(y^\circ)$ is true, we just note that it is stationary true (true for all $d \geqslant y$). □

For instance, if $D = N$, then for all $y \in N^* = D^*$ we have $y(n) \leqslant n$ cofinally. From this remark we can prove the following lemma.

Lemma 4.4 (Convergence of $\mu x.f$). Choose $D = N$. Let $f^* : (N^*)^{k+1} \to \mathscr{B}^*$ be any continuous property. If f^* is minimalisable for all $\underline{x} \in N^k$, then $(\mu x.f) : N^k \to N^*$.

Proof. We use the abbreviation $g = (\mu x.f)$. Fix any $\underline{x} \in N^k$. By hypothesis, f^* is minimalisable in $\underline{x} \in N^k$. Hence, for some $y \in N^*$, we have $f^*(\underline{x}, y)$ is true. Let $X = \{d \in D = N | f^*(\underline{x}, y)(d) = \mathtt{tt} \wedge y(d) \leqslant d\}$. X is recursive, and cofinal in D because both $f^*(\underline{x}, y)(.) = \mathtt{tt}$ and $(y(.) \leqslant .)$ are stable booleans, and are true: the first by assumption, the second by Lemma 4.3. By Lemma 4.2, it is enough to prove, for any $\underline{x} \in N^k$, that $>_{g(\underline{x})}$ is well-founded if restricted to X. As usual, we will in fact prove something more: for all $d, d_1, d_2 \in X$, if $d_1 \leqslant d_2 \leqslant d$ and $y(d_1) = y(d_2)$, then d is well-founded for $(X, >_{g(\underline{x})})$. The thesis will follow by selecting $d = d_1 = d_2$.

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We argue by:

— principal induction over d_1 and $>_y$,
— secondary induction over d_2 and $>_{<f(\underline{x},0),\ldots,f(\underline{x},y(d_1))>}$.

We have to prove that for all $e \in X$, if $e >_{g(\underline{x})} d$, then e is well-founded in $X, >_{g(\underline{x})}$. Since $e >_{g(\underline{x})} d$, there are d', e' such that $d_1, d_2 \leqslant d \leqslant d' \leqslant e' \leqslant e$ and $g(\underline{x})(d') \neq g(\underline{x})(e')$. In order to apply the induction hypothesis, we have first to check the following claim.

Claim. We cannot have:

$$f(\underline{x}, i)(d_2) = f(\underline{x}, i)(d') = f(\underline{x}, i)(e')$$

for all $i \leqslant y(d_2)$.

Proof of the Claim. We have $f(\underline{x}, y(d_2))(d_2) = f^*(\underline{x}, y)(d_2) = \mathtt{tt}$, and $y(d_2) \leqslant d_2 \leqslant d', e'$, because $d_2 \in X$. Hence for some $i_0 \leqslant y(d_2) \leqslant d', e'$ we have $f(\underline{x}, i_0)(d_2) = \mathtt{tt}$, while $f(\underline{x}, i)(d_2) = \mathtt{ff}$ for all $i < i_0$. Now assume $f(\underline{x}, i)(d_2) = f(\underline{x}, i)(d') = f(\underline{x}, i)(e')$ for all $i \leqslant i_0 \leqslant y(d_2)$. Then $f(\underline{x}, i_0)(d') = f(\underline{x}, i_0)(e') = \mathtt{tt}$, while $f(\underline{x}, i)(d') = f(\underline{x}, i)(e') = \mathtt{ff}$ for all $i < i_0 \leqslant d', e'$. We conclude $g(\underline{x})(d') = g(\underline{x})(e') = i_0$, contradicting $g(\underline{x})(d') \neq g(\underline{x})(e')$. □

We now continue the proof of the Lemma by applying the induction hypothesis, case reasoning and the Claim.

— Assume $y(d_1) \neq y(d_2)$, or $y(d_2) \neq y(d')$, or $y(d') \neq y(e')$, or $y(e') \neq y(e)$.
Since $d_1 \leqslant d_2 \leqslant d' \leqslant e' \leqslant e$, in all cases we deduce $e >_y d_1$. Well-foundation of e in $(X, >_{g(\underline{x})})$ follows by the principal induction hypothesis over the pair e, e.
— Assume $y(d_1) = y(d_2) = y(d') = y(e') = y(e)$. By the Claim, $f(\underline{x}, i)(d_2) \neq f(\underline{x}, i)(d')$, or $f(\underline{x}, i)(d') \neq f(\underline{x}, i)(e')$ for some $i \leqslant y(d_1)$.
Since $d_2 \leqslant d' \leqslant e' \leqslant e$, in all cases we deduce $e >_{\langle f(\underline{x},0),\ldots,f(\underline{x},y(d_1))\rangle} d_2$. Well-foundation of e in $(X, >_{g(\underline{x})})$ follows by the secondary induction hypothesis over the pair d_1, e, and $y(d_1) = y(e)$. □

Lemma 4.5 (Minimalisation Lemma). Choose $D = N$. Then maps over $N^{*,D}$ are closed under minimalisation.

Proof. Let $f^* : (N^*)^{k+1} \to \mathscr{B}^*$ be any continuous property, and $g = (\mu x.f)$. We have to show that, if f^* is minimalisable for all $\underline{x} \in N^k$, then g^* is a minimalisation of f^*. We proved in Lemma 4.4 that g^* is a continuous map. We have to prove that, for all $\underline{x}, y \in (N^*)^{k+1}$, the following statement is true in $\mathscr{B}^*$:

$$(f^*(\underline{x}, g^*(\underline{x})) =^\circ \mathtt{tt}) \wedge^\circ (y < g^*(\underline{x}) \Rightarrow^\circ f^*(\underline{x}, y) =^\circ \mathtt{ff}).$$

By Density, it is enough to prove this for all $\underline{x}, y \in N^{k+1}$. We prove first the truth of the right-hand side, then the truth of the left-hand side.

— *Right-hand side.* This is true pointwise. Indeed, for all $\underline{x}, y \in N^{k+1}$ we have $(y < g(\underline{x})(d) \Rightarrow f(\underline{x}, y)(d) = \mathtt{ff})$, because $g(\underline{x})(d)$ is the smallest solution $z \leqslant d$ of $f(\underline{x}, z)(d) = \mathtt{tt}$, if any.
— *Left-hand side.* Since f^* is minimalisable for all $\underline{x} \in N^k$, there is some $y \in N^*$ such that $(f^*(\underline{x}, y) = \mathtt{tt})$. The thesis unfolds to: $(f(\underline{x}, g(\underline{x})(d))(d) = \mathtt{tt})$ holds cofinally on $d \in D$.

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For all $d \in D = N$, the equation $(f(\underline{x}, g(\underline{x})(d))(d) = \mathtt{tt})$ is a consequence of $(d \geqslant y(d))$ and of $(f(\underline{x}, y(d))(d) = \mathtt{tt})$. Indeed, if there is some solution $z \leqslant d$ of $(f(\underline{x}, z)(d) = \mathtt{tt})$, then $(f(\underline{x}, g(\underline{x})(d))(d) = \mathtt{tt})$. Both $(d \geqslant y(.))$ and $(f^*(\underline{x}, y)(.) = \mathtt{tt})$ are stable booleans and cofinally true, the first by 4.3, the second by hypothesis. By the Filter Lemma 3.2, we conclude $(f^*(\underline{x}, g^*(\underline{x})) =^\circ \mathtt{tt}^\circ)$. □

From the previous Lemmas, we deduce that continuous maps include Δ_2^0-maps. Next step is to check that the reverse implication holds, and therefore continuous maps are exactly all maps recursive in the Halting problem (classically, all Δ_2^0-maps). We also show that Δ_2^0-maps may be obtained as the smallest class of maps definable using lifting of recursive maps, some suitable 'some' maps, and minimalisation.

Let $L \in N^*$. Then L is also a recursive map $L : D \to N$. Therefore it makes sense to consider its lifting $L^\circ : D^* \to N^*$. We say that $\delta \in D^*$ is a stationary point of L° on D^* if and only if:

$$\delta \leqslant^\circ \eta \Rightarrow^\circ L^\circ(\delta) =^\circ L^\circ(\eta) \text{ is true, for all } \eta \in D^*.$$

By Density, $\delta \in D^*$ is a stationary point of L° if and only if the implication above holds for $\eta = d^\circ$, and for all $d \in D$.

Lemma 4.6 (Stationarity Lemma). Let $L \in N^*$, and $L^\circ : D^* \to N^*$ be the lifting of L (as a map: $D \to N$) to D^*. Assume $\delta \in D^*$ is a stationary point of L° on D^*. Then L° reaches its limit value in

$$\delta \in D^* : L =^\circ L^\circ(\delta) \text{(equality between elements of } N^*).$$

Proof. We have to prove that $L(d) = L^\circ(\delta)(d))$ holds cofinally on $d \in D$: for every $d \in D$ there is some $e \geqslant d$ such that $L(e) = L^\circ(\delta)(d)) = L(\delta(e))$. We argue by induction over d and $>_L$. Define $\eta = j^\circ(d^\circ, \delta) \in D^*$. Then $\eta \geqslant^\circ \delta$ and $\eta \geqslant^\circ d^\circ$ are pointwise true. Since we assumed that d is a stationary point, we deduce that $L^\circ(\delta) =^\circ L^\circ(\eta)$ is true. By definition of truth, $L(\delta(e)) = L(\eta(e))$ for some $e \geqslant d$. By definition of η, we also have $\eta(e) \geqslant d$. We reason by cases:

— *Assume both* $L(\eta(e)) = L(d)$ *and* $L(d) = L(e)$. We deduce $L(\delta(e)) = L(\eta(e)) = L(d) = L(e)$. We conclude there is some $e \geqslant d$ such that $L(e) = L(\delta(e))$.

— *Assume either* $L(\eta(e)) \neq L(d)$, *or* $L(d) \neq L(e)$. We deduce either $\eta(e) >_L d$, or $e >_L d$. If $\eta(e) >_L d$, we apply the induction hypothesis over $\eta(e)$, if $e >_L d$, over e. We conclude there is some $e' \in D$ such that $e' \geqslant \eta(e) \geqslant d$, respectively, such that $e' \geqslant e \geqslant d$, and such that, in both cases, $L(e') = L(\delta(e'))$. □

Fix any recursive enumeration $\{d_k\}_k \in N$ of elements of D. Let $D(x)$ be the recursive predicate $(x \in D)$. Being a stationary point of L is a property $P \in N \to \mathscr{B}^*$, defined by $\mathtt{stat}_L(x) = \mathtt{all}_f(x)$, with

$$f(x)(e) = D(x) \wedge (D(y), x \leqslant y \Rightarrow L(x) = L(y)).$$

Then $\mathtt{stat}_L$ is a Δ_2^0-map $: D \to \mathscr{B}^*$, because it may be defined by composition of a negation and of an existential over some decidable property. We have already checked

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that universal quantifiers are faithfully interpreted in N^*. Thus, for all $x \in N^*$,

$$\mathtt{stat}_L(x) \text{ iff for all } y \in N^*, D^\circ(x) \wedge (D^\circ(y), x \leqslant^\circ y \Rightarrow^\circ L^\circ(x) =^\circ L^\circ(y)).$$

Using $D^\circ(z)$ if and only if $z =^\circ d$ for some $d \in D^*, D^\circ(y)$, we obtain

$$\mathtt{stat}_L(x) \text{ iff } x =^\circ \delta \text{ for some } \delta \in D^* \quad \text{and} \quad \text{for all } \eta \in D^*, (\delta \leqslant^\circ \eta \Rightarrow^\circ L^\circ(\delta) =^\circ L^\circ(\eta)).$$

That is, $\mathtt{stat}_L(x)$ if and only if for some $\delta \in D^*, x =^\circ \delta$ and δ is stationary for L° on D^*.

Theorem 4.7 (Characterisation of continuous maps). Continuous maps are exactly Δ_2^0-maps.

Proof.

$\Rightarrow$This follows from Lemma 4.5.

$\Leftarrow$Take any $f : N \to N^*$. Then $f(x)$ is a total recursive map : $D \to N$, and $f(x)^\circ : D^* \to N^*$ is a Δ_1^0-map. We will define a Δ_2^0-map

$$s(x) = \mu d.\mathtt{stat}_{f(x)^\circ}(d).$$

By Lemma 4.5, $s(x)$ is defined provided $f(x)^\circ$ has some stationarity point, and by Lemma 4.6, if $s(x)$ is always defined, then $f(x) =^\circ f(x)^\circ(s(x))$ for all x. The right-hand side, being a composition of a Δ_1^0-map with a Δ_2^0-map, is a Δ_2^0-map.

All we have to prove is that there is some stationary point $d \in D^*$ for $f(x)^\circ$ on D^*. We begin with some preliminaries. Fix any covering $(\mathtt{Cov}, \xi)$, and any effective enumeration $\{d_k\}_k$ of elements of D. Let $x@l$ denote the (integer code of) the concatenation of any integer x with some list l of integers, and $\mathtt{last}(l)$ denote the last element of a list l ($\mathtt{last}(l) = 0$ if the list is empty)). We abbreviate $x \in \mathtt{Cov}(d)$ to $x \lhd d$.

We will now define a map $g(x)(d)$, returning some finite succession of 'possible positions' in which f(x)(d) might assume its limit value:

$$d_{k_0}, d_{k_1}, \ldots, d_{k_n}.$$

We ask that each next value in the list be a 'better guess' for the point in which f assumes its limit value. That is, we want

$$d_{k_0} <_{f(x)} d_{k_1} <_{f(x)} \ldots <_{f(x)} d_{k_n}.$$

We ask, also, that $\{k_1, \ldots, k_n\} \subseteq Cov(d)$, and that the last guess has no better guess indexed by some $i \lhd d$ (we ask that $d_i >_{f(x)} d_{k_n}$ for no $i \lhd d$). We may recursively define some $g(x,k)(d)$ with the required property as follows:

$$\begin{array}{lcll} g(x,k)(d) & = & d_k & \text{if there is no } h \lhd d \text{ such that } d_h >_{f(x)} d_k, \\ g(x,k)(d) & = & d_k@g(x,h)(d) & \text{for the first } h \lhd d \text{ such that } d_h >_{f(x)} d_k, \text{ otherwise.} \end{array}$$

Then we set:

$$\begin{array}{rcll} g(x)(d) = g(x,0)(d) & = & \text{some list } d_{k_0}, d_{k_1}, \ldots, d_{k_n} \text{ such that} \\ & & \{k_1, \ldots, k_n\} \subseteq Cov(d), \\ & & \text{and} \\ & & d_i >_{f(x)} d_{k_n} \text{ for no } i \lhd d. \\ \mathtt{best}(x)(d) = \mathtt{last}(g(x)(d)) & = & \text{some } d_k \text{ such that } d_i >_{f(x)} d_k \text{ for no } i \lhd d. \end{array}$$

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— $g(x,k)(d)$ *and* $\mathtt{best}(x)(d)$ *are well-defined.* We may prove that $g(x,k)(d)$ is well-defined for all $d \in D$ by induction with respect to the stable ordering $>_{f(x)}$. Indeed, we use the recursive call $g(x,k)(d) = g(x,h)(d)$ only when there is some $d_h >_{f(x)} d_k$, for some $h \lhd d$.

— $\mathtt{best}(x) \in N^*$. It is enough to prove that $g(x) \in N^*$, that is, that $>_{g(x)}$ is well-founded over D. Let us order the set of lists increasing under $<_{f(x)}$ lexicographically. Recall that we have $l > l'$ lexicographically if and only if either l is an extension of l', or for some list l'' and some integers $a < b$ (in the integer ordering) we have $l \geqslant l''^* a, l' \geqslant l''^* b$ (by prefix ordering). This set of lists is well-founded because if l is an extension of l', then $\mathtt{last}(l) >_{f(x)} \mathtt{last}(l')$ because l is increasing with respect to $>_{f(x)}$. Thus, $\mathtt{last}(.)$ is an increasing map from our set of lists and $D, >_{f(x)}$. We conclude that lexicographic ordering is well-founded.

Claim. The map $g(x)$ is weakly increasing with respect to the lexicographic ordering.

Proof of the Claim. Assume $e \geqslant d$ (in D), in order to prove $g(x)(e) \geqslant g(x)(d)$ (lexicographically). Indeed, assume $e \geqslant d$. Let r, s be the length of $g(x)(e), g(x)(d)$. Assume the two lists are equal for all indices $< r, s$. Then, by induction on s, and the fact that $h \lhd d$ implies $h \lhd e$ for all $h \in N$, we may prove that the each element of $g(x)(d)$ is in $g(x)(e)$. Hence $g(x)(e)$ is an extension of $g(x)(d)$.
Assume $g(x)(d)$ and $g(x)(e)$ are different in a first index $i < r, s$. Then the first $h \lhd e$ such that $d_h >_{f(x)} d_i$ is smaller than the first $h \lhd d$ such that $d_h >_{f(x)} d_i$, again because $h \lhd d$ implies $h \lhd e$.

Proof of $\mathtt{best}(x) \in N^*$. If $e >_{g(x)} d$, then for some $e \geqslant e' \geqslant d' \geqslant d$ in D we have $g(x)(e') \neq g(x)(d')$. Therefore, $g(x)(e) > g(x)(e') \geqslant g(x)(d') > g(x)(e)$ lexicographically, hence $g(x)(e) > g(x)(d)$ (again, lexicographically). Thus, $g(x)(.)$ is an increasing map from $>_g (x)$ and D to some well-founded lexicographic ordering. We conclude that $>_g (x)$ is well-founded over D. Hence $g(x) \in N^*$.

— $\mathtt{best}(x)$ *is a stationary point of* $f(x)^\circ$. By definition, for all $i \lhd e$, if $d_i \geqslant \mathtt{best}(x)(e)$ in D, then $d_i >_{f(x)} \mathtt{best}(x)(e)$ is false, that is, $f(x)(d_i) = f(x)(\mathtt{best}(x)(e))$. By unfolding definitions, we obtain

$$(\mathtt{best}(x) \leqslant^\circ d_i \Rightarrow^\circ f(x)^\circ(\mathtt{best}(x)) = f(x)^\circ(d_i))(e) \tag{8}$$

is true for all $i \lhd e$. In particular, for all $d \in D$, (8) is true for $e = j(d, \xi(i)) \geqslant d$, because $i \lhd \xi(i) \leqslant e$ and covering is monotone. Thus, (8) is cofinally true, and hence true in $\mathscr{B}^*$, for all $i \in N$. □

There are definition schemata for Δ^0_2-maps, equivalent to minimalisation, but easier to use on N^*: for instance pointwise linear recursion. Fix any recursive well-founded relation R on N. Assume $a : N \to \mathscr{B}^*, b : N \to N^*, c : N \to N^*$, and suppose that

$$a(x) \Rightarrow^\circ R^\circ(c(x), x^\circ)$$

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is pointwise true for all $x \in N$. Then there is some (in fact, a unique) map f solving

$$f^*(x) = \mathtt{if}^\circ(a^*(x),\ b^*(x),\ f^*(c^*(x))) \qquad (LinRec)$$

for all $x \in N^*$. A continuous solution f^* may be obtained by solving the corresponding fixed point equation for recursive maps:

$$f(x)(d) = \mathtt{if}(a(x)(d),\ b(x)(d),\ f(c(x)(d))(d)).$$

We will skip the proof of this result.

The final result of this section has an immediate proof, yet it is quite relevant. Let g be any recursive map: the theorem below says that if there is a limit solution $l \in N^*$ for $g^\circ(x) = 0$, then l may be effectively turned into a solution $x \in N$ of the same $g^\circ(x) = 0$.

From a logical viewpoint, this is a conservativity result: extending N to N^* adds no new solution to any equation. This justifies the interest of our model construction: we may use limit reasoning just to find integer solutions faster.

From a topological viewpoint, the result is a density property (of N in N^*): the set $O = \{x | g(x) = 0\}$ is a basic open in effective discrete topology, and the theorem says that if O intersects N^*, then O intersects N.

Theorem 4.8 (Conservativity Theorem or Density Theorem). Let g be any recursive map. If $g^\circ(l) = 0$ for some $l \in N^*$, then $g(x) = 0$ for some $x \in N$.

Proof. We have $g(l(d)) = 0$ cofinally in D, hence at least for some $d \in D$. Take $x = l(d) \in N$. □

As a final remark, note that, in general, x cannot be the limit of l, because limits are not computable.

The situation is similar when we get a real number by completion from rational numbers. Suppose we want to find the square root of 2 with approximation $\epsilon = 10^{-9}$, that is, we want to find some $x \in Q$ such that $|x^2 - 2| < \epsilon$. This means that we have to find some inhabitant in the open set $O = \{x | x^2 \in (2 - \epsilon, 2 + \epsilon)\}$. We know that O is inhabited by $\sqrt{2}$: since Q is dense in R, this means that O is inhabited by some $x \in Q$ (usually, we find some trunking x of $\sqrt{2}$). The rational x is always different from the irrational $\sqrt{2}$. The notation $\sqrt{2}$ is only used as a step in the computation whose output is x: we are planning to use limit integers in the same way.

In the rest of the paper, we introduce a notion of realiser for equational statements in N^*. Let f be total recursive : $N \to N$. The constructive proof of

$$\forall l \in N^* : f^\circ(l) =^\circ 0 \text{ is true } \text{ implies } \exists x \in N. f(x) = 0$$

includes implicitly a (non-trivial) method for turning any proof of $f^\circ(l) =^\circ 0$, for some limit integer l, into the computation of an x such that $f(x) = 0$.

In Section 5, we describe such a method explicitly. The tool we use is the notion of a realiser. A realiser represents, in form of an effective map, the construction implicit in an intuitionistic reasoning. In Section 6, we will use realisers to constructivise a simple classical theorem requiring excluded middle on Δ_2^0-formulas.

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5. A realisation interpretation for equations between limits

In this section we introduce a notion of realiser for equations $t =^\circ u$, for $t, u \in N^*$. Then we will show that classical equational reasoning about non-recursive maps on N may be turned, first, into constructive reasoning about continuous maps on N^*, then, into operations building such realisers.

'Adherence maps', or 'realisers', of equations. Any equation $t =^\circ u$ is shorthand for the statement 'the sequences $t, u \in N^*$ are topologically adherent', or

$$\forall d \in D. \exists e \geqslant d. t(e) = u(e)$$

From a constructive point of view, this means that there is some effective way, given any $d \in D$, to find some $e \geqslant d$ such that $t(e) = u(e)$. We may represent such an 'effective way of computing e from d' by some recursive map $\Phi : D \to D$ computing e.

In this paper, we only use realisers for adherence statements with respect to discrete topology on N. For this reason, we will also call our realisers 'adherence maps'.

Definition 5.1 (Realisation of equality). Let $\Phi : D \to D$ be any (total) recursive map. We call Φ an 'adherence map' for $t =^\circ u$, and we write $\Phi : t =^\circ u$, if for all $d \in D, \Phi$ chooses some $\Phi(d) \geqslant d$ such that $t(\Phi(d)) = u(\Phi(d))$.

Φ may be considered as the *constructive content of topological adherence between limits*: Φ is a 'realiser' of $t =^\circ u$. If we do not have Φ, in order to find some $e \in D$ such that $t(e) = u(e)$ we may just do blind search (we assumed D effectively enumerated); if we do have Φ, we have some more relevant information about computing $e \in D$.

We propose a realisation semantics for *closed* equational reasoning over continuous functions f^*, g^* on N^* (we use no variables). For all $l \in N^*$, each logical rule

$$f_1^*(l) =^\circ g_1^*(l), f_2^*(l) =^\circ g_2^*(l), \ldots \vdash f^*(l) =^\circ g^*(l)$$

will be interpreted as an operation turning adherence maps $\Phi_1 : f_1^*(l) =^\circ g_1^*(l), \Phi_2 : f_2^*(l) =^\circ g_2^*(l), \ldots$ for each hypothesis into an adherence map $\Phi : f^*(l) =^\circ g^*(l)$ for the conclusion.

Thus, in principle, an adherence map would be automatically produced from any classical reasoning on limits. They would be a kind of 'microscope', singling out the construction hidden in the classical proof.

Realisation semantics may be used as follows. Again let f be total recursive $: N \to N$. From a classical proof of $f^\circ(l) =^\circ 0$, for some $l \in N^*$, we may *automatically generate* some Φ such that $\Phi : f(l) = 0$. Then (by the Conservativity Theorem), we conclude $f(l(e)) = 0$ for all $d \in D$ and all $e = \Phi(d)$. Now $x = l(e) = l(\Phi(d))$, for any $d \in D$, is some integer solution for $f(x) = 0$, computed using Φ, therefore using ideas from the original classical proof of $f(l) = 0$.

The possibility of automatic generation, however, should not be taken too seriously. In reality, it can only be used for short proofs to help get an intuitive understanding of the constructions hidden in classical reasonings. For long proofs, it takes less time to reformulate the goal constructively, in term of adherence maps, and then try to prove it directly, rather than by re-using the classical proof. In this case, however, the fact that *some* constructivisation exists, both for the theorem and for all of its lemmas, is of some help.

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Definition 5.2 (Realisation statements). Let $p : N^k \to N$ be any recursive map, $\Phi : D \to D$, and $\underline{t} \in (N^*)^k$. A realisation statement is any statement of the form $\Phi : p^\circ(\underline{t}) =^\circ \mathtt{tt}^\circ$, or just $\Phi : p^*(\underline{t})$ for short. We will sometimes drop the superscripts $(.)^\circ, (.)^*$ for the sake of readability.

Realisation statements include previous realisation statements $\Phi : (t =^\circ u)$, which may be expressed as $(t =^\circ u) =^\circ \mathtt{tt}$. We have already remarked that statements of the form $p^\circ(\underline{t})$ are also closed under lifting to $\mathscr{B}^*$ of boolean operation (negation, conjunction, disjunction).

We now introduce semi-formal rules to derive statements of the shape $\Phi : p^\circ(\underline{t})$, then we will interpret them as operations on realisers f.

Definition 5.3. (Equational Rules for Δ_2^0-maps).

(1) *Atomic Rules.* Let $p_1, \ldots, p_k, p$ be any total recursive properties of arity h. Then Atomic rules are all rules

$$p_1^\circ(\underline{l}), \ldots, pk^\circ(\underline{l}) \vdash p^\circ(\underline{l}) \text{ (for } \underline{l} \in (N^*)^h)$$

such that

$$p_1(\underline{x}), \ldots, pk(\underline{x}) \vdash p(\underline{x})$$

is true in N for all $\underline{x} \in N^h$.

(2) *Density of N.* Let $q : N^{h+j} \to \mathscr{B}$. Given any $\underline{l} \in (N^*)^h$, and any recursive succession of proofs of $q^\circ(\underline{x}^\circ, \underline{m})$, recursively indexed on $\underline{x} \in N^h$, we conclude $q^\circ(\underline{l}, \underline{m})$, for any $\underline{l} \in (N^*)^h$. This rule has infinitely many (recursively given) premises:

$$\ldots q^\circ(\underline{x}^\circ, \underline{m}) \ldots \text{(for all } \underline{x} \in N^k) \vdash q^\circ(\underline{l}, \underline{m}).$$

(3) *Some and First Axioms.* Let f be any total recursive property $: N^h \to \mathscr{B}$, and

$$s(\underline{x}) = \mathtt{s}_f(\underline{x}) \in \{\mathtt{tt}, \mathtt{ff}\}, \quad c(\underline{x}) = \mathtt{fst}_f(\underline{x}) \in N.$$

The 'some' rule (an axiom) says that if $s(\underline{x}) = \mathtt{ff}$, then $f(\underline{x}, y) = \mathtt{ff}$ for all $y \in N$:

$$\vdash (\neg^\circ s(\underline{x}) \Rightarrow^\circ \neg f(\underline{x}, y)^\circ)(\text{all y } \in N).$$

While the 'first' rule (another axiom) says that if $s(\underline{x}) = \mathtt{tt}$, then $y = c(\underline{x})$ picks some solution of $f(\underline{x}, y) = \mathtt{tt}$:

$$\vdash (s(\underline{x}) \Rightarrow^\circ f^\circ(\underline{x}^\circ, c(\underline{x}))).$$

Atomic rules in point (1) include, say

$$a \vee b, \neg a \vee b \vdash b.$$

That is, they include case reasoning and the Cut rule. By combining (2) and (3), we may generalise (3) to any $\underline{l} \in (N^*)^k$ and $m \in N^*$:

$$\vdash \neg^\circ s^*(\underline{l}) \Rightarrow^\circ f^\circ(\underline{l}, m) \neq 0 \qquad \vdash s^*(\underline{l}) \Rightarrow^\circ f^\circ(\underline{l}, c^*(\underline{l})) = 0.$$

From (2) we deduce, instead, induction over N^*.

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Lemma 5.4 (Induction Lemma). Let $\underline{m} \in (N^*)^k$, and p be any total recursive property: $N^{k+1} \to \mathscr{B}$. Then p^* satisfies the induction schema.

Proof. Assume $p^\circ(0^\circ, \underline{m})$ and $p^\circ(l, \underline{m}) \Rightarrow^\circ p^\circ(l+1, \underline{m})$ for all $l \in N^*$. By induction over N we deduce $p^\circ(x^\circ, \underline{m})$ for all $x \in N$. By Density, we conclude $p^\circ(l, \underline{m})$, for all $l \in N^*$. □

Total recursive properties : $N \to \mathscr{B}^*$ also satisfy an equivalent formulation of induction, like total induction.

Merging of realisation maps. Merging is an operation taking any sequence

$$\Phi_1 : p_1(\underline{l}), \ldots, \Phi_n : p_n(\underline{l})$$

of adherence maps, and returning an adherence map

$$\Phi : p_1(\underline{l}) \wedge \ldots \wedge p_n(\underline{l}).$$

Our realisation interpretation is parametric under such an operation (which was already implicitly defined in the Filter Lemma (Lemma 3.2)). We include here a very particular case of merging, left merging, and postpone the most general definition to the next paper. Merging is inspired by Coquand's game interpretation of Cut (Coquand 1995). Left merging is functional, deterministic and sequential. General merging will be an imperative, non-deterministic protocol solving conflicts among concurrent processes.

Left merging – two arguments. Assume $p_1, p_2 : N^k \to \mathscr{B}$ to be total recursive properties, $\underline{l} \in (N^*)^k$ be a vector of k stable maps, and $\Phi_1 : p_1(\underline{l}), \Phi_2 : p_2(\underline{l})$. That is,

$$\Phi_1(d) \geqslant d, \quad \Phi_2(d) \geqslant d, \quad p_1(\underline{l}(\Phi_1(d))) = \mathtt{tt}, \quad p_2(\underline{l}(\Phi_2(d))) = \mathtt{tt}$$

for all $d \in D$. We will define their left merging, which is some adherence map f such that

$$\Phi : (p_1(\underline{l}) \wedge p_2(\underline{l})),$$

that is,

$$\Phi(d) \geqslant d, \quad p_1(\underline{l}(f(d))) = \mathtt{tt}, \quad p_2(\underline{l}(f(d))) = \mathtt{tt}$$

for all $d \in D$. The left merging will be independent of p_1, p_2. The idea is to find some $\Phi(d) \geqslant d$ such that

$$\underline{l}(\Phi(d)) = \underline{l}(\Phi_1(d)) = \underline{l}(\Phi_2(d)).$$

Φ applies Φ_1, Φ_2 alternately to $d_0 = d$,

$$d_0 \leqslant d_1 = \Phi_1(d_0) \leqslant d_2 = \Phi_2(d_1) \leqslant d_3 = \Phi_1(d_2) \leqslant d_4 = \Phi_2(d_3) \leqslant \ldots,$$

until the value $\underline{l}(d)$ stops changing, that is, as long as we have

$$\underline{l}(d_1) \neq \underline{l}(d_2) \neq \underline{l}(d_3) \neq \underline{l}(d_4) \neq \ldots,$$

or, equivalently, as long as we have

$$d_1 <_{\underline{l}} d_2 <_{\underline{l}} \underline{l}d_3 <_{\underline{l}} d_4 <_{\underline{l}} \ldots.$$

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The result of $\Phi(d)$ is the first d_i, with $i \geqslant 2$, such that $\underline{l}(d_i) = \underline{l}(d_{i-1})$. Φ is well-defined because any recursive call is decreasing with respect to $>_{\underline{l}}$. Besides, for some d', d'', we have

$$\underline{l}(f(d)) = \underline{l}(d_i) = \underline{l}(d_{i-1}) = \underline{l}(\Phi_1(d')) = \underline{l}(\Phi_2(d''))$$

(because either i is odd and $d_i = \Phi_1(d_{i-1}), d_{i-1} = \Phi_2(d_{i-2})$, or i is even and $d_i = \Phi_2(d_{i-1}), d_{i-1} = \Phi_1(d_{i-2})$). We deduce:

$$\begin{array}{rcl} p_1(\underline{l})(f(d)) & = & p_1(\underline{l}(\Phi_1(d'))) = 0 \\ p_2(\underline{l})(f(d)) & = & p_2(\underline{l}(\Phi_2(d''))) = 0. \end{array}$$

We conclude that Φ is an adherence map.

Left merging – n arguments. We repeatedly apply binary left merging, using left association of arguments (we merge the first two maps, the result with the third one, and so on). A more general choice for merging will be considered in a forthcoming paper.

Definition 5.5 (Realisation semantics for Δ_2^0-maps). For each logical rule, we will now define some adherence maps $\Phi: p(\underline{l})$ for conclusion, from adherence maps $\Phi_1 : p_1(\underline{l}), \ldots, \Phi_n : p_n(\underline{l})$ for the hypothesis. Let $j(d,e) \geqslant d, e$ be any map picking some upper bound of d, e.

Case (1) (*Atomic rules*) We take $\Phi =$ left merging of $\Phi_1, \ldots, \Phi_n$. The map Φ realises $p_1(\underline{l}) \wedge \ldots \wedge p_n(\underline{l})$, hence it also realises $p(\underline{l})$, because, for all $d \in D$, $p_1(\underline{l}(d)) \wedge \ldots \wedge pn(\underline{l}(d))$ implies $p(\underline{l}(d))$.

Case (2) (*Density of N*) Assume $\Phi_{\underline{x}} : p^\circ(\underline{x}, \underline{m})$, recursively on $\underline{x}$. This means

$$p^\circ(\underline{x}, \underline{m})(f\,\underline{x}(d)) = 0$$

for all d. We define a map Φ such that $p^\circ(\underline{l}, \underline{m})(\Phi(d)) = 0$ for all $\underline{l} \in N^k$. Such a Φ repeatedly applies $d \mapsto \Phi_{\underline{l}(d)}(d)$, starting from $d_0 = d$,

$$d_0 \leqslant d_1 = \Phi_{\underline{l}(d)}(d_0) \leqslant d_2 = \Phi_{\underline{l}(d_1)}(d_1) \leqslant d_3 = \Phi_{\underline{l}(d_2)}(d_2) \leqslant d_4 = \Phi_{\underline{l}(d_3)}(d_3)) \ldots,$$

until the value $\underline{l}(d)$ stops changing, that is, as long as we have

$$\underline{l}(d_0) \neq \underline{l}(d_1) \neq \underline{l}(d_2) \neq \underline{l}(d_3) \neq \underline{l}(d_4) \neq \ldots,$$

or, equivalently, as long as we have

$$d_1 <_{\underline{l}} d_2 <_{\underline{l}} d_3 <_{\underline{l}} d_4 <_{\underline{l}} \ldots.$$

Formally, Φ is recursively defined by:

$$\begin{array}{rcll} \Phi(d) & = & \Phi_{\underline{l}(d)}(d) & \text{if } \underline{l}(\Phi_{\underline{l}(d)}(d)) = \underline{l}(d), \\ \Phi(d) & = & \Phi(\Phi_{\underline{l}(d)}(d)) & \text{if } \underline{l}(\Phi_{\underline{l}(d)}(d)) \neq \underline{l}(d). \end{array}$$

The result of $\Phi(d)$ is the first d_i (with $i \geqslant 1$) such that $\underline{l}(d_i) = \underline{l}(d_{i-1})$. $f(d)$ is well-defined by induction over $>_{\underline{l}}$. For all $i \in N$, just by definition unfolding, we

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get $p(\underline{l}(d_{i-1}), \underline{m}(d_i)) = \mathtt{tt}$. Indeed, if we set $\underline{x} = \underline{l}(d_{i-1})$, from the definition of d_i and $\Phi_{\underline{x}}(d)$, we deduce:

$$\begin{aligned} p(\underline{l}(d_{i-1}), \underline{m}(d_i)) &= p(\underline{x}, \underline{m})(d_i) \\ &= p(\underline{x}, \underline{m})(\Phi_{\underline{l}}(d(i-1))(d_{i-1})) \\ &= p(\underline{x}, \underline{m})(\Phi_{\underline{x}}(d_{i-1})) \\ &= \mathtt{tt}. \end{aligned}$$

We may now check that $p^\circ(\underline{l}, \underline{m})(\Phi(d)) = 0$, for all $d \in D$:

$$\begin{aligned} p^\circ(\underline{l}, \underline{m})(\Phi(d)) &= p^\circ(\underline{l}, \underline{m})(d_i) && \text{(by definition of } \Phi) \\ &= p(\underline{l}(d_i), \underline{m}(d_i)) && \text{(by definition of lifting)} \\ &= p(\underline{l}(d_{i-1}), \underline{m}(d_i)) && \text{(by } \underline{l}(d_i) = \underline{l}(d_{i-1})) \\ &= \mathtt{tt} && \text{(by the remark above).} \end{aligned}$$

Case (3) (*Some, First*) Let $m =< f, \underline{x} > \in N$ be the name of the map $f(\underline{x}, .)$. Then $\mathtt{s}_f(\underline{x})$ has been defined in terms of the covering $\lhd_m, \xi_m$, the covering number m in our list.

— *Some*. Fix any 'some' axiom

$$\neg \mathtt{s}_f(\underline{x}) \Rightarrow f(\underline{x}, y) \neq 0 (\underline{x} \in N^k, y \in N).$$

Define $\Phi(d) = j(d, e) \in D$, with $e = \xi_m(y)$. Then $y \lhd_m \Phi(d) (\Phi(d)$ covers $y)$. By the definition of $\mathtt{s}_f(\underline{x})$, if $\neg s_f(\underline{x})(d)$, then no point z covered by $\Phi(d)$ solves $f(\underline{x}, z) = 0$. We conclude

$$\neg \mathtt{s}_f(\underline{x})(d) \Rightarrow^\circ f(\underline{x}, y) = \mathtt{ff}.$$

— *First*. We just take $\Phi = \mathtt{id}$. We have to prove that $\mathtt{id}$ is an adherence map for any 'first' axiom, that is, that the statement

$$(s_f(\underline{x}) \Rightarrow^\circ f(\underline{x}, \mathtt{fst}_f(\underline{x})) = \mathtt{tt})(d)$$

is true for all $d \in D$. But we have already checked that it is pointwise true.

In this section we have defined a translation from classical proofs to programs. In the next section, inspired by this translation, we will define a constructivisation for some classical reasoning requiring excluded middle on Δ^0_2-formulas (formulas without nested quantifiers).

6. An example of constructivisation of Δ^0_2-classical reasoning

In this section we will consider a simple combinatorial result over N, the minimum principle, requiring excluded middle over Δ^0_2—formulas (one-quantifier formulas). Our goal is to prove them constructively over N^*. The constructive proof in this section was originally obtained mechanically, by applying our realisability interpretation on classical reasoning. For the sake of simplicity, however, we will not include details here, but just a few hints of how the constructive proof was obtained.

In fact, in further examples (not included here) realisation semantics will *only* be used to ensure that *some* constructive version of a classical result proved by Δ^0_2—excluded middle

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exists (provided we shift from N to N^*). It usually takes too long to use realisation as an effective method for automatically producing a constructive proof from some classical proof.

Fix any D equipped with some covering $(\lhd, \xi)$ of N. Let $\mathtt{Cov}(d) =$ the finite set of points covered by $d = \{y \in N | y \lhd d\}$. For the sake of simplicity, in this section we assume: that there is some $d \in D$ covering no points, that is, with $\mathtt{Cov}(d) = \varnothing$; that the set covered by $j(d, e) \in D$ is the union of the sets covered by $d, e \in D$, that is, $\mathtt{Cov}(j(d, e)) = \mathtt{Cov}(d) \cup \mathtt{Cov}(e)$; and that $\xi(x) \in D$ exactly covers $\{x\} : \mathtt{Cov}(\xi(x)) = \{x\}$. The reader might think of $D = \mathscr{P}_{\text{fin}}(N)$ as example.

Lemma 6.1 (Minimum principle for integer functions). Classically, for each (total recursive) $f : N \to N$ there is some minimum point $m \in N$. m is a minimum point for f if and only if $f(m) \leqslant f(y)$ for all $y \in N$.

Proof (*Classical*). Start with $x_0 = 0 \in N$ (some arbitrary choice). If for all $y \in N$ we have $f(x_0) \leqslant f(y)$, we may choose $m = x_0$. Suppose, instead, $f(x_0) > f(y)$ for some y. Let $x_1 =$ the first such y. If for all $y \in N$ we have $f(x_1) \leqslant f(y)$, we may choose $m = x_1$. Suppose, instead, $f(x_1) > f(y)$ for some y. Let $x_2 =$ the first such y. If we continue in this way, we produce a sequence

$$f(x_0) > f(x_1) > f(x_2) > f(x_3) > f(x_4) > \ldots$$

of length at most $f(x_0)$. After at most $f(x_0)$ steps, we find some $m = x_i$ such that for all $y \in N$ we have $f(x_i) \leqslant f(y)$. □

The minimum m as a stable map $m \in N^*$. The classical proof of the minimum principle defines m using a suitable 'some' map $s(x) = \exists y.(f(x) > f(y))$. Indeed, we may formally define $m = \mathtt{out}(0)$, with:

$$\begin{array}{ll} \mathtt{out}(x) = x & \text{if } f(x) \leqslant f(y) \text{ for all } y \in N, \text{ otherwise} \\ \mathtt{out}(x) = \mathtt{out}(y) & \text{for the first } y \in N \text{ such that } f(y) < f(x). \end{array}$$

(The choice of the first point is arbitrary). Thus, m is a Δ^0_2-map on the recursive code of f. Using our model of Δ^0_2-maps, we may turn the classical definition of $m \in N$ into the definition of a stable map $m \in N^*$. We only have to relativise the '$\mathtt{out}$' map to the points covered by some $d \in D$. We set $m(d) = \mathtt{out}(0)(d)$, with:

$$\begin{array}{ll} \mathtt{out}(x)(d) = x & \text{if } f(x) \leqslant f(y) \text{ for all } y \lhd d, \text{ otherwise} \\ \mathtt{out}(x)(d) = \mathtt{out}(y)(d) & \text{for the first point } y \lhd d \text{ such that } f(y) < f(x). \end{array}$$

m is defined using pointwise linear recursion over the value of $f(x)$, because in the case $\mathtt{out}(x)(d) = \mathtt{out}(y)(d)$ we must have $f(y) < f(x)$.

A simple characterisation of $m \in N^*$. $\mathtt{out}(0)(d)$ produces a sequence

$$f(0) = f(x_0) > f(x_1) > f(x_2) > f(x_3) > f(x_4) > \ldots$$

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with $x_1, x_2, \ldots, x_n \lhd d$. Then $m(d)$ in fact produces the (first) minimum point of f on $\mathtt{Cov}(d) \cup \{0\}$.

$$m(d) = \text{ the first minimum point of } f \text{ on } \mathtt{Cov}(d) \cup \{0\}.$$

A constructive realisation of the minimum principle. The constructive interpretation of '$f(m) \leqslant f(y)$ for all $y \in N$' in our realisation semantics is

$$\Phi : (f(m) \leqslant f(y)).$$

A realiser Φ of the statement $(f(m) \leqslant f(y))$ can be defined by $f(d) = j(d, \xi(y)) =$ some $e \geqslant d$ covering y. In order to check $\phi : f(m) \leqslant f(y)$, we have to prove $f(m(e)) \leqslant f(y)$ for $e = \phi(d)$. This follows from the fact that $m(e) =$ minimum of f on $\mathtt{Cov}(e)$, and $y \in \mathtt{Cov}(e)$.

The constructive minimum point principle – discussion. We will briefly compare the classical minimum principle with its constructive version. Instead of finding a minimum point $m \in N$ for f on N, we find a minimum $m(d) \in N$ of f on $\mathtt{Cov}(d) \cup \{0\}$, some finite set. All we know is $f(m(d)) \leqslant f(y)$, for $y \in \mathtt{Cov}(d)$ or $y = 0$. For a generic $y \in N$, we cannot guarantee $f(m(d)) \leqslant f(y)$. However, we may force $f(m(d)) \leqslant f(y)$ to be true by enlarging d to some $e = \phi(d) = j(d, \xi(y))$, also covering y. By the assumptions of this section, $\mathtt{Cov}(e) = \mathtt{Cov}(d) \cup \{y\}$. Adding the point y may change the value of $m(d)$. Indeed, if $f(m(d)) <_f (y)$, or $f(m(d)) = f(y)$ but $m(d) \leqslant y$, then $m(d)$ is the first minimum of f also in $\mathtt{Cov}(e) \cup \{y\}$ and $m(e) = m(d)$; if $f(m(d)) >_f (y)$, or $f(m(d)) = f(y)$ and $m(d) > y$, the first minimum is y, and $m(e) = y$.

Thus, *the 'minimum' we get is a flow* $\underline{m}(d)$ *of values indexed over* $d \in D$, *rather than a single value.* The key point to note is that $m(d)$ is stable on d: the successions of all possible changes of $m(d)$ form a well-founded tree. Indeed, whenever $m(d)$ changes, either $f(m(d)) \leqslant f(0)$ decreases, and it may decrease at most $f(0)$ times, or $f(m(d))$ stays the same, but $m(d)$ decreases: thus, the tree of all possible changes is well-founded.

By the Conservativity Theorem, whenever we use the classical minimum principle in order to prove the existence of some integer with some decidable property, we may use the constructive version of it in order to define some effective way to get such an integer. The following example is due to T. Coquand:

$$\textit{there is some } x \textit{ such that } f(x) \leqslant f(g(x)).$$

The *classical* proof is immediate: take

$$x = \text{ the minimum point } m \text{ of } f.$$

Since $f(m) \leqslant f(y)$ for all $y \in N$, we have, in particular, $f(m) \leqslant f(g(m))$. We may define an effective way of finding such an m in our realisation semantics.

Let $l = g(m)$. Then we have to define a realiser $\Phi : (f(m) \leqslant f(l))$ from a family of realisers

$$\Phi_y : (f(m) \leqslant f(y)).$$

In the previous paragraph we defined $\Phi_y(d) = j(d, \xi(y))$: Φ_y adds the point y to the points covered by d. We use the method outlined in the proof of the Density rule. We start from

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any d, and we keep on applying $d \mapsto \Phi_{l(d)}(d)$, starting from $d_0 = d$,

$$d_0 \leqslant d_1 = \Phi_{l(d_0)}(d_0) \leqslant d_2 = \Phi_{l(d_1)}(d_1) \leqslant d_3 = \Phi_{l(d_2)}(d_2) \leqslant \ldots,$$

until the value $l(d) = g(m(d))$ stops changing, that is, as long as we have

$$g(m(d_0)) \neq g(m(d_1)) \neq g(m(d_2)) \neq g(m(d_3)) \neq g(m(d_4)) \neq \ldots.$$

A solution to $f(x) \leqslant f(g(x))$. No matter what d_i is, when the succession $d_0, d_1, d_2, \ldots$ stops in d_i, we find a solution to $f(x) \leqslant f(g(x))$. Let us analyse which solution we get, assuming we start from some d_0 covering no points: $\mathtt{Cov}(d_0) = \varnothing$. Then,

$$m(d_0) = \text{ the minimum of } f \text{ on } \{0\} = 0.$$

By definition of $d_i = \Phi_{l(d(i-1))}(d_i), d_i$ covers the points covered by d_{i-1}, plus $g(m(d_{i-1}))$. By induction on i, we may prove

$$\mathtt{Cov}(d_i) = \{g(m(d_0)), \ldots, g(m(d_{i-1}))\}.$$

Besides, if $g(m(d_i))$ changes, $m(d_i)$ changes, and by the definition of m, we get $m(d_i)$ becomes $g(m(d_i - 1))$, which is the only new point covered by d_i. Thus, $m(d_i) = g(m(d_{i-1})) = g(g(m(d_{i-2}))) = g^i(m(d_0)) = g^i(0)$, and

$$\mathtt{Cov}(d_i) = \{g^0(0), \ldots, g^{i-1}(0)\}.$$

The succession $g^0(0), g^1(0), g^2(0), \ldots$ continues as long as we have

$$f(g^i(0)) > f(g^{i+1}(0))$$

or

$$f(g^i(0)) = f(g^{i+1}(0)) \quad \text{but} \quad g^i(0) > g^{i+1}(0).$$

(This last clause is a bit arbitrary, it is just there because we want $m(d)$ to be the *first* minimum of f on $\{g^0(0), g^1(0), g^2(0), ..\}$). When we stop, we have $f(g^i(0)) \leqslant f(g^{i+1}(0))$, and we have found some $x = g^i(0)$ such that $f(x) \leqslant f(g(x))$.

Conclusion of the section. We can compare this method of finding some x such that $f(x) \leqslant f(g(x))$ with the only method we usually associate to a classical proof of $\exists x. f(x) \leqslant f(g(x))$: blind search through $x = 0, 1, 2, \ldots$. In the first case, we only look through the values $x = g^0(0), g^1(0), g^2(0), \ldots$. In the second, we look through all possible values of x.

It is curious to note, however, that the search through $x = g^0(0), g^1(0), g^2(0), \ldots$ was in fact implicit in the classical proof, but invisible until we used realisation semantics to make it explicit. Realisation semantics is a kind of *microscope*, singling out constructive features hidden deeply in classical reasoning.

We end this section by outlining a canonical choice for the direct set D. We propose the choice $D = \mathscr{P}_{\text{fin}}(N) < \omega$ (finite sequences over $\mathscr{P}_{\text{fin}}(N)$) for efficiency reasons, which we will now explain.

The choice of the direct set D. We said earlier that D is intended to represent the computation state. More specifically, we want $d \in D$ to be a data structure, storing all

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numerical instances $y \in N$ of the 'some' axiom

$$\neg s(\underline{x}) \Rightarrow^{\circ} \neg f(\underline{x}, y)^{\circ} \quad (\underline{x} \in N^k, y \in N),$$

which are really used by a classical proof of some $p^{\circ}(\underline{l})$. Let $m = <_f, \underline{x}>$ be the name of this particular instance of the 'some' axiom. We interpret $y \lhd_m d$ as

'at computation stage d, we used the instance $y \in N$ of the 'some' axiom named m'.

For this reason, in the definition of both $s_d(\underline{x}), c_d(\underline{x})$, and associated realisers, we only considered the set of integers covered by the m-th component of d. Having all coverings $(\lhd_n, \xi_n)$ for $n \in N$ equal to the same is conceptually correct, but quite rough: in this way, we pile together integers used to instantiate different 'some' axioms. If all covering are the same, whenever we compute $s_d(\underline{x})$, that is, we decide if $f(\underline{x}, y) = \mathtt{tt}$ for some $y \lhd_m d$, we are in fact searching through all $y \lhd_n d$ for some $n \in N$. Computation of $s_d(\underline{x})$ then slows down considerably.

For this reason, we choose $D = \mathscr{P}_{\text{fin}}(N) < \omega$, the set of all sequences $J = \{J_n\}_{n \in \omega}$ of finite subsets of N such that $J_n = \varnothing$ from some n on. We order D by pointwise inclusion: $J \subseteq H$ if and only if $J_n \subseteq H_n$ for all n. We will use $\varnothing$ to denote the bottom of D (the sequence always empty). We define $j =$l.u.b., that is, we set

$$j(J, H)_n = J_n \cup H_n (\text{for all } n \in N).$$

We define $y \lhd_m J$ (y is covered by the m-th component of J) by $(y \in J_m)$, and $\xi_m(x)$ (the canonical J whose m-th component covers x) by the smallest J that m-covers x. Such a J is defined by:

$$\begin{aligned} J_n &= \{x\} \quad \text{if } n = m, \\ J_n &= \varnothing \quad \text{if } n \neq m. \end{aligned}$$

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