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Regular Partitions of Hypergraphs: Regularity Lemmas

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Szemerédi's regularity lemma for graphs has proved to be a powerful tool with many subsequent applications. The objective of this paper is to extend the techniques developed by Nagle, Skokan, and the authors and obtain a stronger and more 'user-friendly' regularity lemma for hypergraphs.

1. Introduction

In the course of proving his celebrated *density theorem* concerning arithmetic progressions [30], Szemerédi established the *regularity lemma* for graphs [31]. This lemma turned out to be a very useful tool in extremal graph theory and theoretical computer science (see, *e.g.*, [18, 19] for a survey). Following the work of Frankl and Rödl on 3-uniform hypergraphs [5], Gowers [10, 12] and Nagle, Skokan, and the authors [20, 26] developed generalizations of the *regularity method* to k-uniform hypergraphs. Subsequently, Tao [32] also obtained such a generalization. Those extensions yield the following theorem (see [10, 20, 27]), which settles a conjecture of Erdős, Frankl and Rödl [4].

Theorem 1.1 (removal lemma). *Let $\ell \geqslant k \geqslant 2$ be integers and let $\varepsilon > 0$. There exist $\delta = \delta(\ell, k, \varepsilon) > 0$ and $n_0 = n_0(\ell, k, \varepsilon)$ such that the following holds.*

Suppose $\mathcal{F}^{(k)}$ is a fixed k-uniform hypergraph on ℓ vertices and $\mathcal{H}^{(k)}$ is a k-uniform hypergraph on $n \geqslant n_0$ vertices. If $\mathcal{H}^{(k)}$ contains at most δn^{ℓ} copies of $\mathcal{F}^{(k)}$, then one can delete εn^k edges of $\mathcal{H}^{(k)}$ to make it $\mathcal{F}^{(k)}$-free.

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This theorem can be viewed as an extension of the theorem of Ruzsa and Szemerédi [28], which addressed the case $k = 2$ and $\mathcal{F}^{(2)}$ a triangle, the complete graph on three vertices. In [4], Theorem 1.1 was verified for all graphs $\mathcal{F}^{(2)}$. It was shown by Frankl and Rödl [5], Solymosi [29], and Tengan, Tokushige, and the authors [25], that Theorem 1.1 implies Szemerédi's density theorem [30], as well as some of its multidimensional extensions due to Furstenberg and Katznelson [7, 8]. (It is not known, however, whether Theorem 1.1 also yields an alternative proof of the density version of the theorem of Hales and Jewett [15], which was established by Furstenberg and Katznelson [9] using *ergodic theory*.)

In this paper we continue the line of research from [5, 20, 26] and obtain a stronger and hopefully easier-to-use regularity lemma for hypergraphs: Theorem 2.3. The proof of a corresponding counting lemma will appear in a subsequent paper [23]. A standard application of those theorems, following the lines of [4, 5, 10, 20, 27], yields a proof of Theorem 1.1.

As a by-product we obtain a result for hypergraphs, Theorem 2.2, which might be of independent interest. Roughly speaking, in the context of graphs Theorem 2.2 says that, for every fixed $\nu > 0$, any graph on n vertices can be approximated, by adding and deleting at most νn^2 edges, by an ε-regular graph on a vertex partition into t parts, where $\varepsilon = \varepsilon(t)$ is an arbitrary function of t, and thus we may have $\varepsilon(t) \ll 1/t$. This may perhaps be somewhat surprising, since it follows from the work of Gowers [11] that there are graphs which, if not changed, admit only an ε-regular partition with t classes, where $t \gg 1/\varepsilon$. In fact Gowers constructed graphs with number of partition classes in any ε-regular partition being bigger than a tower of height polynomial in $1/\varepsilon$.

2. Main results

In what follows, we give a precise account of Szemerédi's regularity lemma. For a graph $G = (V, E)$ and two disjoint sets $A, B \subset V$, let $E(A, B)$ denote the set of edges $\{a, b\} \in E$ with $a \in A$ and $b \in B$ and set $e(A, B) = |E(A, B)|$. We also set $d(A, B) = e(A, B)/(|A||B|)$ for the *density* of the pair A, B.

The concept central to Szemerédi's lemma is that of an *ε-regular pair*. Let $\varepsilon > 0$ be given. We say that the pair A, B is ε-regular if $|d(A, B) - d(A', B')| < \varepsilon$ holds whenever $A' \subset A$, $B' \subset B$, and $|A'||B'| > \varepsilon|A||B|$.

We call a partition $\mathscr{P}^{(1)} = \{V_i : 0 \leqslant i \leqslant t\}$ of V *t-equitable* if it satisfies $|V_0| \leqslant t$ and $|V_i| = \lfloor |V|/t \rfloor$ for $i \in [t]$. We say the graph $G = (V, E)$ is *ε-regular w.r.t. $\mathscr{P}^{(1)}$* if all but εt^2 pairs V_i, V_j are ε-regular. Szemerédi's lemma [31] may then be stated as follows.

Theorem 2.1 (Szemerédi's regularity lemma). *For any positive real ε and any integer t_0, there are positive integers $t_{\mathrm{Sz}} = t_{\mathrm{Sz}}(\varepsilon, t_0)$ and $n_{\mathrm{Sz}} = n_{\mathrm{Sz}}(\varepsilon, t_0)$ such that, for every graph $G = (V, E)$ with $|V| = n \geqslant n_{\mathrm{Sz}}$ vertices, there exists a partition $\mathscr{P}^{(1)}$ of V such that*

(i) *$\mathscr{P}^{(1)} = \{V_i : 0 \leqslant i \leqslant t\}$ is t-equitable, where $t_0 \leqslant t \leqslant t_{\mathrm{Sz}}$, and*

(ii) *G is ε-regular w.r.t. $\mathscr{P}^{(1)}$.*

Moreover, if $(t_{\mathrm{Sz}})!$ divides n then V_0 can be chosen to be empty.

We note that our definition of an ε-regular pair differs slightly from the usual one of [31]. However, it is easy to see that both are equivalent. Also we point out that in an early version of

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the regularity lemma, which appeared in [30], the partition structure was a bit different from – and more complicated than – the one stated above, which appeared in [31].

In this paper we consider two extensions of Theorem 2.1 to hypergraphs (see Theorems 2.2 and 2.3). To simplify the notation we will restrict to the case where V_0 is empty. Since our result is of asymptotic nature, dealing with hypergraphs on n vertices where n is very large, and since every hypergraph can be altered to satisfy $(t_{\mathrm{Sz}})!|n$ by adding or deleting a constant number (independent of n) of vertices, this additional divisibility assumption has no essential bearing.

2.1. Basic notation

For real constants α, β, and a non-negative constants ξ, we sometimes write

$$\alpha = \beta \pm \xi, \quad \text{if } \beta - \xi \leqslant \alpha \leqslant \beta + \xi.$$

For a positive integer ℓ, we denote by $[\ell]$ the set $\{1, \dots, \ell\}$. For a set V and an integer $k \geqslant 1$, let $[V]^k$ be the set of all k-element subsets of V. We may drop one pair of brackets and write $[\ell]^k$ instead of $\big[[\ell]\big]^k$. A subset $\mathcal{H}^{(k)} \subseteq [V]^k$ is a *k-uniform hypergraph* on the vertex set V. We identify hypergraphs with their edge sets. For a given k-uniform hypergraph $\mathcal{H}^{(k)}$, we denote by $V(\mathcal{H}^{(k)})$ and $E(\mathcal{H}^{(k)})$ its vertex and edge set, respectively. For $U \subseteq V(\mathcal{H}^{(k)})$, we denote by $\mathcal{H}^{(k)}[U]$ the sub-hypergraph of $\mathcal{H}^{(k)}$ induced on U (*i.e.*, $\mathcal{H}^{(k)}[U] = \mathcal{H}^{(k)} \cap [U]^k$). A k-uniform *clique* of *order* j, denoted by $K_j^{(k)}$, is a k-uniform hypergraph on $j \geqslant k$ vertices consisting of all $\binom{j}{k}$ different k-tuples.

In this paper ℓ-partite, j-uniform hypergraphs play a special rôle, where $j \leqslant \ell$. Given vertex sets $V_1, \dots, V_\ell$, we denote by $K_\ell^{(j)}(V_1, \dots, V_\ell)$ the *complete* ℓ-partite, j-uniform hypergraph (*i.e.*, the family of all j-element subsets $J \subseteq \bigcup_{i\in[\ell]} V_i$ satisfying $|V_i \cap J| \leqslant 1$ for every $i \in [\ell]$). If $|V_i| = m$ for every $i \in [\ell]$, then an *(m, ℓ, j)-hypergraph* $\mathcal{H}^{(j)}$ on $V_1 \cup \cdots \cup V_\ell$ is any subset of $K_\ell^{(j)}(V_1, \dots, V_\ell)$. Note that the vertex partition $V_1 \cup \cdots \cup V_\ell$ is an $(m, \ell, 1)$-hypergraph $\mathcal{H}^{(1)}$. (This definition may seem artificial right now, but it will simplify later notation.) For $j \leqslant i \leqslant \ell$ and set $\Lambda_i \in [\ell]^i$, we denote by $\mathcal{H}^{(j)}[\Lambda_i] = \mathcal{H}^{(j)}\big[\bigcup_{\lambda\in\Lambda_i} V_\lambda\big]$ the sub-hypergraph of the (m, ℓ, j)-hypergraph $\mathcal{H}^{(j)}$ induced on $\bigcup_{\lambda\in\Lambda_i} V_\lambda$.

For an (m, ℓ, j)-hypergraph $\mathcal{H}^{(j)}$ and an integer $j \leqslant i \leqslant \ell$, we denote by $\mathcal{K}_i(\mathcal{H}^{(j)})$ the family of all i-element subsets of $V(\mathcal{H}^{(j)})$ which span complete sub-hypergraphs in $\mathcal{H}^{(j)}$ of order i. Note that $|\mathcal{K}_i(\mathcal{H}^{(j)})|$ is the number of all copies of $K_i^{(j)}$ in $\mathcal{H}^{(j)}$.

Given an $(m, \ell, j-1)$-hypergraph $\mathcal{H}^{(j-1)}$ and an (m, ℓ, j)-hypergraph $\mathcal{H}^{(j)}$ such that $V(\mathcal{H}^{(j)}) \subseteq V(\mathcal{H}^{(j-1)})$, we say an edge J of $\mathcal{H}^{(j)}$ *belongs to* $\mathcal{H}^{(j-1)}$ if $J \in \mathcal{K}_j(\mathcal{H}^{(j-1)})$, *i.e.*, J corresponds to a clique of order j in $\mathcal{H}^{(j-1)}$. Moreover, $\mathcal{H}^{(j-1)}$ *underlies* $\mathcal{H}^{(j)}$ if $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$, *i.e.*, every edge of $\mathcal{H}^{(j)}$ belongs to $\mathcal{H}^{(j-1)}$. This brings us to one of the main concepts of this paper, the notion of a *complex*.

Definition 1 ((m, ℓ, h)-complex). Let $m \geqslant 1$ and $\ell \geqslant h \geqslant 1$ be integers. An *(m, ℓ, h)-complex* $\boldsymbol{\mathcal{H}}$ is a collection of (m, ℓ, j)-hypergraphs $\{\mathcal{H}^{(j)}\}_{j=1}^h$ such that

(a) $\mathcal{H}^{(1)}$ is an $(m, \ell, 1)$-hypergraph, *i.e.*, $\mathcal{H}^{(1)} = V_1 \cup \cdots \cup V_\ell$ with $|V_i| = m$ for $i \in [\ell]$,
(b) $\mathcal{H}^{(j-1)}$ underlies $\mathcal{H}^{(j)}$ for $2 \leqslant j \leqslant h$, *i.e.*, $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$.

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Remark 1. We may also define hypergraphs and complexes in the same way for underlying vertex sets $V_1, \dots, V_\ell$ with different cardinalities. In such a case we will drop the m and say $\mathcal{H}^{(j)}$ is an (ℓ, j)-hypergraph or $\boldsymbol{\mathcal{H}}$ is an (ℓ, h)-complex.

2.2. Regular complexes

We begin with a notion of relative density of a j-uniform hypergraph w.r.t. a $(j-1)$-uniform hypergraph on the same vertex set.

Definition 2 (relative density). Let $\mathcal{H}^{(j)}$ be a j-uniform hypergraph and let $\mathcal{H}^{(j-1)}$ be a $(j-1)$-uniform hypergraph on the same vertex set. We define the *density of* $\mathcal{H}^{(j)}$ *w.r.t.* $\mathcal{H}^{(j-1)}$ as

$$d\big(\mathcal{H}^{(j)} \mid \mathcal{H}^{(j-1)}\big) = \begin{cases} \frac{|\mathcal{H}^{(j)} \cap \mathcal{K}_j(\mathcal{H}^{(j-1)})|}{|\mathcal{K}_j(\mathcal{H}^{(j-1)})|} & \text{if } |\mathcal{K}_j(\mathcal{H}^{(j-1)})| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We now define a notion of regularity of an (m, j, j)-hypergraph with respect to an $(m, j, j-1)$-hypergraph.

Definition 3. Let reals $\varepsilon > 0$ and $d_j \geqslant 0$ be given along with an (m, j, j)-hypergraph $\mathcal{H}^{(j)}$ and an underlying $(m, j, j-1)$-hypergraph $\mathcal{H}^{(j-1)}$. We say $\mathcal{H}^{(j)}$ is (ε, d_j)-*regular* w.r.t. $\mathcal{H}^{(j-1)}$ if, whenever $\mathcal{Q}^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$ satisfies

$$|\mathcal{K}_j(\mathcal{Q}^{(j-1)})| \geqslant \varepsilon |\mathcal{K}_j(\mathcal{H}^{(j-1)})|, \quad \text{then } d\big(\mathcal{H}^{(j)} \mid \mathcal{Q}^{(j-1)}\big) = d_j \pm \varepsilon.$$

Next we extend the notion of (ε, d_j)-regularity from (m, j, j)-hypergraphs to (m, ℓ, j)-hypergraphs $\mathcal{H}^{(j)}$.

Definition 4 ((ε, d_j)-regular hypergraph). We say an (m, ℓ, j)-hypergraph $\mathcal{H}^{(j)}$ is (ε, d_j)-*regular* w.r.t. an $(m, \ell, j-1)$-hypergraph $\mathcal{H}^{(j-1)}$ if for every $\Lambda_j \in [\ell]^j$ the restriction $\mathcal{H}^{(j)}[\Lambda_j] = \mathcal{H}^{(j)}\big[\bigcup_{\lambda \in \Lambda_j} V_\lambda\big]$ is (ε, d_j)-regular w.r.t. to the restriction $\mathcal{H}^{(j-1)}[\Lambda_j] = \mathcal{H}^{(j-1)}\big[\bigcup_{\lambda \in \Lambda_j} V_\lambda\big]$.

We sometimes write ε-*regular* to mean $\big(\varepsilon, d\big(\mathcal{H}^{(j)} \mid \mathcal{H}^{(j-1)}\big)\big)$-regular.

Finally, we close this section with the notion of a regular complex.

Definition 5 ($(\varepsilon, \boldsymbol{d})$-regular complex). Let $\varepsilon > 0$ and let $\boldsymbol{d} = (d_2, \dots, d_h)$ be a vector of non-negative reals. We say an (m, ℓ, h)-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ is $(\varepsilon, \boldsymbol{d})$-*regular* if $\mathcal{H}^{(j)}$ is (ε, d_j)-regular w.r.t. $\mathcal{H}^{(j-1)}$ for every $j = 2, \dots, h$.

2.3. Partitions

The regularity lemmas for k-uniform hypergraphs which we prove in this paper provide a well-structured family of partitions $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ of vertices, pairs, …, and $(k-1)$-tuples of some vertex set. We now discuss the structure of these partitions following the approach of [26]. First we define the *refinement* of a partition.

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Definition 6 (refinement). Suppose $A \supseteq B$ are sets, $\mathscr{A}$ is a partition of A, and $\mathscr{B}$ is a partition of B. We say $\mathscr{A}$ *refines* $\mathscr{B}$ and write $\mathscr{A} \prec \mathscr{B}$ if, for every $\mathcal{A} \in \mathscr{A}$, there either exists a $\mathcal{B} \in \mathscr{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$ or $\mathcal{A} \subseteq A \setminus B$.

Let k be a fixed integer and V be a set of vertices. Throughout this paper we require a family of partitions $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ on V to satisfy properties which we are going to describe below (see Definition 7).

Let $\mathscr{P}^{(1)} = \{V_1, \dots, V_{|\mathscr{P}^{(1)}|}\}$ be a partition of V. For every $1 \leqslant j \leqslant k$ let $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ be the family of all crossing j-tuples J, *i.e.*, the set of j-tuples which satisfy $|J \cap V_i| \leqslant 1$ for every $V_i \in \mathscr{P}^{(1)}$.

Suppose that partitions $\mathscr{P}^{(i)}$ of $\mathrm{Cross}_i(\mathscr{P}^{(1)})$ into (i,i)-hypergraphs have been defined for $1 \leqslant i \leqslant j-1$. Then, for every $(j-1)$-tuple I in $\mathrm{Cross}_{j-1}(\mathscr{P}^{(1)})$ there exist a unique $\mathcal{P}^{(j-1)} = \mathcal{P}^{(j-1)}(I) \in \mathscr{P}^{(j-1)}$ so that $I \in \mathcal{P}^{(j-1)}$. Moreover, for every j-tuple J in $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ we define the *polyad of* J:

$$\hat{\mathcal{P}}^{(j-1)}(J) = \bigcup \{\mathcal{P}^{(j-1)}(I) \colon I \in [J]^{j-1}\}.$$

In other words, $\hat{\mathcal{P}}^{(j-1)}(J)$ is the unique collection of j partition classes of $\mathscr{P}^{(j-1)}$ in which J spans a copy of $K_j^{(j-1)}$. Observe that $\hat{\mathcal{P}}^{(j-1)}(J)$ can be viewed as a $(j, j-1)$-hypergraph, *i.e.*, a j-partite, $(j-1)$-uniform hypergraph. More generally, for $1 \leqslant i < j$, we set

$$\hat{\mathcal{P}}^{(i)}(J) = \bigcup \{\mathcal{P}^{(i)}(I) \colon I \in [J]^i\} \quad \text{and} \quad \boldsymbol{\mathcal{P}}(J) = \{\hat{\mathcal{P}}^{(i)}(J)\}_{i=1}^{j-1}. \tag{2.1}$$

Next, we define $\hat{\mathscr{P}}^{(j-1)}$, the family of all polyads:

$$\hat{\mathscr{P}}^{(j-1)} = \{\hat{\mathcal{P}}^{(j-1)}(J) \colon J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})\}.$$

Note that $\hat{\mathcal{P}}^{(j-1)}(J)$ and $\hat{\mathcal{P}}^{(j-1)}(J')$ are not necessarily distinct for different j-tuples J and J'. However, we view $\hat{\mathscr{P}}^{(j-1)}$ as a set and, consequently, $\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$. The structural requirement on the partition $\mathscr{P}^{(j)}$ of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ we have in this paper is

$$\mathscr{P}^{(j)} \prec \{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}. \tag{2.2}$$

In other words, we require that the set of cliques spanned by a polyad in $\hat{\mathscr{P}}^{(j-1)}$ is sub-partitioned in $\mathscr{P}^{(j)}$ and every partition class in $\mathscr{P}^{(j)}$ belongs to precisely one polyad in $\hat{\mathscr{P}}^{(j-1)}$. Note, that due to (2.2) we inductively infer that $\boldsymbol{\mathcal{P}}(J)$ defined in (2.1) is a $(j, j-1)$-complex.

Throughout this paper we also want to have control over the number of partition classes in $\mathscr{P}^{(j)}$, and more specifically, over the number of classes contained in $\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})$ for a fixed polyad $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$. We render this precisely in the following definition.

Definition 7 (family of partitions). Suppose V is a set of vertices, $k \geqslant 2$ is an integer and $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ is a vector of positive integers. We say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}) = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ is a *family of partitions on* V, if it satisfies the following:

(i) $\mathscr{P}^{(1)}$ is a partition of V into a_1 classes,

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(ii) $\mathscr{P}^{(j)}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ satisfying

$$\mathscr{P}^{(j)} \quad \text{refines} \quad \{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$$

$$\text{and} \quad |\{\mathcal{P}^{(j)} \in \mathscr{P}^{(j)} \colon \mathcal{P}^{(j)} \subseteq \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\}| = a_j \quad \text{for every } \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}.$$

Moreover, we say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ is t-bounded, if $\max\{a_1, \dots, a_{k-1}\} \leqslant t$.

We now combine Definitions 6 and 7 and define the *refinement of a family of partitions*.

Definition 8 (refinement of families). Suppose $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ and $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}^{\mathscr{R}})$ are families of partitions on the same vertex set V. We say $\mathscr{P}$ *refines* $\mathscr{R}$ and write $\mathscr{P} \prec \mathscr{R}$, if $\mathscr{P}^{(j)} \prec \mathscr{R}^{(j)}$ (*cf.* Definition 6) for every $j \in [k-1]$.

2.4. Main results

In this paper we prove two hypergraph regularity lemmas, which may be viewed as strengthened versions of the hypergraph regularity lemma from [26]. Those new lemmas were already applied in [2, 3, 21, 22, 24]. As in Szemerédi's regularity lemma, such hypergraph regularity lemmas should ensure the existence of partitions of the edge set of a k-uniform hypergraph which satisfy certain properties. Besides the structural conditions discussed in the last section, the partitions ensured by the main theorems in this paper will satisfy two more properties which we define below. More specifically, the family of partitions $\mathscr{P}$ has to satisfy properties analogous to (i) and (ii) of Theorem 2.1. We first extend the notion of equitability.

Definition 9 ($(\eta, \varepsilon, \boldsymbol{a})$-equitable). Suppose V is a set of n vertices, η and ε are positive reals, $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ is a vector of positive integers, and a_1 divides n.

We say a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on V (as defined in Definition 7) is $(\eta, \varepsilon, \boldsymbol{a})$-*equitable* if it satisfies the following:

(a) $|[V]^k \setminus \mathrm{Cross}_k(\mathscr{P}^{(1)})| \leqslant \eta \binom{n}{k}$;

(b) $\mathscr{P}^{(1)} = \{V_i \colon i \in [a_1]\}$ is an equitable vertex partition, *i.e.*, $|V_i| = |V|/a_1$ for $i \in [a_1]$;

(c) for every k-tuple $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$ we have that $\boldsymbol{\mathcal{P}}(K) = \{\hat{\mathcal{P}}^{(j)}\}_{j=1}^{k-1}$ (*cf.* (2.1)) is an $(\varepsilon, \boldsymbol{d})$-regular $(n/a_1, k, k-1)$-complex, where $\boldsymbol{d} = (1/a_2, \dots, 1/a_{k-1})$.

Next, we extend (ii) of Theorem 2.1. In this paper we consider two possible extensions, which give rise to the two different regularity lemmas below.

Definition 10 (perfectly ε-regular). Suppose ε is some positive real. Let $\mathcal{G}^{(k)}$ be a k-uniform hypergraph with vertex set V and let $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be a family of partitions on V. We say $\mathcal{G}^{(k)}$ *is perfectly ε-regular w.r.t.* $\mathscr{P}$ if, for every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$, we have that $\mathcal{G}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ is ε-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$.

The following theorem is one of the two main results in this paper.

Theorem 2.2 (regular approximation lemma). *Let $k \geqslant 2$ be a fixed integer. For all positive constants η and ν, and every function $\varepsilon \colon \mathbb{N}^{k-1} \to (0, 1]$ there are integers $t_{\text{Thm 2.2}}$ and $n_{\text{Thm 2.2}}$ so that the following holds.*

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For every k-uniform hypergraph $\mathcal{H}^{(k)}$ with $|V(\mathcal{H}^{(k)})| = n \geqslant n_{\text{Thm 2.2}}$ and $(t_{\text{Thm 2.2}})!$ dividing n, there exist a k-uniform hypergraph $\mathcal{G}^{(k)}$ on the same vertex set and a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that

(i) *$\mathscr{P}$ is $(\eta, \varepsilon(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and $t_{\text{Thm 2.2}}$-bounded,*
(ii) *$\mathcal{G}^{(k)}$ is perfectly $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular w.r.t. $\mathscr{P}$, and*
(iii) $|\mathcal{G}^{(k)} \triangle \mathcal{H}^{(k)}| \leqslant \nu n^k$.

Let us briefly compare Theorem 2.2 for $k = 2$ with Theorem 2.1. Note that, as discussed in [19, Section 1.8], there are graphs with irregular pairs in any partition. Therefore, due to the 'perfectness' in (ii) of Theorem 2.2, one has to alter $H = \mathcal{H}^{(2)}$ to obtain $G = \mathcal{G}^{(2)}$.

The main difference between Theorem 2.2 for $k = 2$ and Theorem 2.1, however, is in the choice of ε being a function of $a_1^{\mathscr{P}}$. It follows from the work of Gowers in [11] that it is not possible to regularize a graph H with an ε in such a way that, *e.g.*, $\varepsilon < 1/a_1^{\mathscr{P}}$ can be ensured, where $a_1^{\mathscr{P}} = |\mathscr{P}^{(1)}|$ is the number of vertex classes. Properties (i) and (iii) of Theorem 2.2 assert, however, that by adding or deleting at most νn^2 edges from H one can obtain a graph G which admits an $\varepsilon(a_1^{\mathscr{P}})$-regular partition, with $\varepsilon(a_1^{\mathscr{P}}) < 1/a_1^{\mathscr{P}}$. Such a lemma for graphs can also be deduced from the iterated regularity lemma in [1].

The other result of this paper, Theorem 2.3, concerns the case in which we do not change the given hypergraph $\mathcal{H}^{(k)}$. Due to the discussion above such a lemma needs to allow exceptional pairs (or polyads for $k \geqslant 3$) in the partition $\mathscr{P}$. Moreover, the measure of regularity of $\mathcal{H}^{(k)}$ w.r.t. $\mathscr{P}$ (called δ_k here) cannot depend on $a_1^{\mathscr{P}}$. In fact, in our proof of Theorem 2.3, δ_k is a constant independent of each $a_1^{\mathscr{P}}, \dots, a_{k-1}^{\mathscr{P}}$. On the other hand, as in [5, 26] we will infer that $\mathcal{H}^{(k)}$ is $(\delta_k, *, r)$-regular (defined below), where r may depend on $a_1^{\mathscr{P}}, \dots, a_{k-1}^{\mathscr{P}}$. We first extend Definition 4.

Definition 11 ((δ_k, d_k, r)-regular hypergraph). Let δ_k and d_k be positive reals and r be a positive integer. Suppose $\mathcal{H}^{(k-1)}$ is an $(m, k, k-1)$-hypergraph spanning at least one $K_k^{(k-1)}$. We say an (m, k, k)-hypergraph $\mathcal{H}^{(k)}$ is (δ_k, d_k, r)-*regular* w.r.t. $\mathcal{H}^{(k-1)}$ if, for every collection $\boldsymbol{\mathcal{Q}}^{(k-1)} = \{\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\}$ of not necessarily disjoint sub-hypergraphs of $\mathcal{H}^{(k-1)}$ which satisfy

$$\left|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right| \geqslant \delta_k |\mathcal{K}_k(\mathcal{H}^{(k-1)})| > 0, \quad \text{we have} \quad \frac{|\mathcal{H}^{(k)} \cap \bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})|}{|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})|} = d_k \pm \delta_k.$$

We write $(\delta_k, *, r)$-*regular* to mean $\big(\delta_k, d\big(\mathcal{H}^{(k)} \mid \mathcal{H}^{(k-1)}\big), r\big)$-regular. Moreover, if $r = 1$, then a $(\delta_k, d_k, 1)$-regular hypergraph is (ε, d_k)-regular with $\varepsilon = \delta_k$ (*cf.* Definition 4) and *vice versa*.

Finally, we give the second extension of (ii) of Theorem 2.1, which will be ensured by Theorem 2.3.

Definition 12 ($(\delta_k, *, r)$-regular w.r.t. $\mathscr{P}$). Suppose δ_k is a positive real and r is a positive integer. Let $\mathcal{H}^{(k)}$ be a k-uniform hypergraph with vertex set V and $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be a family

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of partitions on V. We say $\mathcal{H}^{(k)}$ *is* $(\delta_k, *, r)$*-regular w.r.t.* $\mathscr{P}$, if

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}):\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \text{ and } \mathcal{H}^{(k)} \text{ is not } (\delta_k, *, r)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}\Big\}\Big| \leqslant \delta_k |V|^k.$$

The following theorem is a strengthening of the main result of [26].

Theorem 2.3 (regularity lemma). *Let $k \geqslant 2$ be a fixed integer. For all positive constants η and δ_k, and all functions $r : \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta : \mathbb{N}^{k-1} \to (0, 1]$, there are integers $t_{\text{Thm 2.3}}$ and $n_{\text{Thm 2.3}}$ so that the following holds.*

For every k-uniform hypergraph $\mathcal{H}^{(k)}$ with $|V(\mathcal{H}^{(k)})| = n \geqslant n_{\text{Thm 2.3}}$ and $(t_{\text{Thm 2.3}})!$ dividing n, there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that

(i) *$\mathscr{P}$ is $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and $t_{\text{Thm 2.3}}$-bounded and*

(ii) *$\mathcal{H}^{(k)}$ is $(\delta_k, *, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}$.*

3. Auxiliary results

In this section we review a few results that are essential for our proofs of Theorem 2.2 and Theorem 2.3.

The following theorem can be used to estimate the number of copies of $K_\ell^{(h)}$ in an appropriate collection of dense and regular blocks within a regular partition provided by the regular approximation lemma, Theorem 2.2. Moreover, it can be applied to count the number of $K_k^{(k-1)}$s in the polyads of the partitions obtained by Theorem 2.2 and Theorem 2.3.

Theorem 3.1 (dense counting lemma). *For all integers $2 \leqslant h \leqslant \ell$ and all positive constants γ and d_0, there exist $\varepsilon_{\text{DCL}} = \varepsilon_{\text{DCL}}(h, \ell, \gamma, d_0) > 0$ and an integer $m_{\text{DCL}} = m_{\text{DCL}}(h, \ell, \gamma, d_0)$ so that, if $\boldsymbol{d} = (d_2, \dots, d_h) \in \mathbb{R}^{h-1}$ satisfying $d_j \geqslant d_0$ for $2 \leqslant j \leqslant h$ and $m \geqslant m_{\text{DCL}}$, and if $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ is an $(\varepsilon_{\text{DCL}}, \boldsymbol{d})$-regular (m, ℓ, h)-complex, then*

$$|\mathcal{K}_\ell(\mathcal{H}^{(h)})| = (1 \pm \gamma) \prod_{j=2}^{h} d_j^{\binom{\ell}{j}} \times m^\ell.$$

This theorem was proved by Kohayakawa, Rödl and Skokan in [17, Theorem 6.5]. For completeness we give a short proof of a generalization of Theorem 3.1 in the subsequent paper [23].

The following two facts regard regularity properties of the union of regular hypergraphs. The first of those two propositions states that the union of regular (m, j, j)-hypergraphs which share the same underlying $(m, j, j-1)$-hypergraph is regular. The proof is straightforward and we refrain from presenting it here.

Proposition 3.2. *Let $j \geqslant 2$, m, t, $r \geqslant 1$ be fixed integers and let δ and $d(1), \dots, d(t)$ be positive reals. Suppose $\mathcal{P}_1^{(j)}, \dots, \mathcal{P}_t^{(j)}$ is a family of pairwise edge disjoint (m, j, j)-hypergraphs with the same underlying $(m, j, j-1)$-hypergraph $\hat{\mathcal{P}}^{(j-1)}$.*

If $\mathcal{P}_\tau^{(j)}$ is $(\delta, d(\tau), r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$ for every $\tau \in [t]$, then $\mathcal{P}^{(j)}$ is $(t\delta, d, r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$, where $\mathcal{P}^{(j)} = \bigcup_{\tau\in[t]} \mathcal{P}_\tau^{(j)}$ and $d = \sum_{\tau\in[t]} d(\tau)$. □

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The next proposition gives us control when we union hypergraphs having different underlying polyads. Before we make this precise, we define the set-up for our proposition.

Set-up 3.3. *Let $j \geqslant 2$, m, $t \geqslant 1$ be fixed integers and let δ and d be positive reals. Let $\{\hat{\mathcal{P}}_\tau^{(j-1)}\}_{\tau\in[t]}$ be a family of $(m, j, j-1)$-hypergraphs such that*

$$\bigcup_{\tau\in[t]} \hat{\mathcal{P}}_\tau^{(j-1)} \text{ is a } j\text{-partite } (j-1)\text{-uniform hypergraph,}$$

$$\mathcal{K}_j\Big(\bigcup_{\tau\in[t]} \hat{\mathcal{P}}_\tau^{(j-1)}\Big) = \bigcup_{\tau\in[t]} \mathcal{K}_j\big(\hat{\mathcal{P}}_\tau^{(j-1)}\big), \tag{3.1}$$

$$\text{and} \quad \mathcal{K}_j\big(\hat{\mathcal{P}}_\tau^{(j-1)}\big) \cap \mathcal{K}_j\big(\hat{\mathcal{P}}_{\tau'}^{(j-1)}\big) = \varnothing \quad \text{for } 1 \leqslant \tau < \tau' \leqslant t.$$

Let $\{\mathcal{P}_\tau^{(j)}\}_{\tau\in[t]}$ be a family of (m, j, j)-hypergraphs such that $\hat{\mathcal{P}}_\tau^{(j-1)}$ underlies $\mathcal{P}_\tau^{(j)}$ for any $\tau \in [t]$. Set $\hat{\mathcal{P}}^{(j-1)} = \bigcup_{\tau\in[t]} \hat{\mathcal{P}}_\tau^{(j-1)}$ and $\mathcal{P}^{(j)} = \bigcup_{\tau\in[t]} \mathcal{P}_\tau^{(j)}$.

Proposition 3.4. *Let $r \geqslant 1$ be a fixed integer and let $\{\mathcal{P}_\tau^{(j)}\}_{\tau\in[t]}$ and $\{\hat{\mathcal{P}}_\tau^{(j-1)}\}_{\tau\in[t]}$ satisfy Set-up 3.3. If $\mathcal{P}_\tau^{(j)}$ is (δ, d, r)-regular w.r.t. $\hat{\mathcal{P}}_\tau^{(j-1)}$ for every $\tau \in [t]$, then $\mathcal{P}^{(j)}$ is $(2\sqrt{\delta}, d, r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$.* □

For $r = 1$ a proof of Proposition 3.4 appeared in [20] and the proof presented there works verbatim for general $r \geqslant 1$.

The proof of the following lemma is based on Chernoff's inequality and the fact that randomly chosen sub-hypergraphs of a regular hypergraph are regular. Similar statements were proved in [5, 26] and we will omit the technical details here.

Proposition 3.5 (slicing lemma). *Let $j \geqslant 2$, s_0, $r \geqslant 1$ be integers and let δ_0, ϱ_0, and p_0 be positive real numbers. There is an integer $m_{\mathrm{SL}} = m_{\mathrm{SL}}(j, s_0, r, \delta_0, \varrho_0, p_0)$ so that the following holds. If $m \geqslant m_{\mathrm{SL}}$,*

(i) *$\hat{\mathcal{P}}^{(j-1)}$ is a $(m, j, j-1)$-hypergraph satisfying $|\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})| \geqslant m^j/\ln m$ and*

(ii) *$\mathcal{P}^{(j)} \subseteq \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})$ is an (δ, ϱ, r)-regular (m, j, j)-hypergraph with $\varrho \geqslant \varrho_0 \geqslant 2\delta \geqslant 2\delta_0$.*

Then, for any positive integer $1 \leqslant s \leqslant s_0$ and all positive reals $p_1, \dots, p_s$ satisfying

(iii) *$\sum_{\sigma\in[s]} p_\sigma \leqslant 1$ and $p_\sigma \geqslant p_0$ for $\sigma \in [s]$,*

there exists a partition $\{\mathcal{T}_0^{(j)}, \mathcal{T}_1^{(j)}, \dots, \mathcal{T}_s^{(j)}\}$ of $\mathcal{P}^{(j)}$ such that $\mathcal{T}_\sigma^{(j)}$ is $(3\delta, p_\sigma\varrho, r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$ for every $\sigma = 1, \dots, s$. Moreover, $\mathcal{T}_0^{(j)}$ is $(3\delta, (1 - \sum_{\sigma\in[s]} p_\sigma)\varrho, r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$ and $\mathcal{T}_0^{(j)} = \varnothing$ if $\sum_{\sigma\in[s]} p_\sigma = 1$. □

4. Outline of the proofs

Roughly speaking, our proof of both theorems, Theorem 2.2 and Theorem 2.3, is based on the following induction scheme:

$$\text{Theorem 2.3 for } k \Longrightarrow \text{Theorem 2.2 for } k \Longrightarrow \text{Theorem 2.3 for } k+1.$$

To carry out the technical details for such an induction scheme, we need to strengthen the statements of Theorem 2.3 (**r**egularity **l**emma) and of Theorem 2.2 (**r**egularity **a**pproximation

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lemma) to the more general but, unfortunately, less aesthetically pleasing statement RL(k), Lemma 4.1, and RAL(k), Lemma 4.2.

Before we start to discuss these more general statements we will briefly outline why they are needed. While the proof of the implication that Theorem 2.2 for $k \Longrightarrow$ Theorem 2.3 for $k+1$ could follow the lines of [5, 26] (now using Theorem 2.2 for k to regularize the witnesses, which provides the cleaner partition $\mathscr{P}$), the need for generalizing the statements comes from the implication that Theorem 2.3 for $k \Longrightarrow$ Theorem 2.2 for k. In our proof of this implication we need to apply Theorem 2.3 for k twice. After the first application we obtain an $(\eta, \varepsilon(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable partition $\mathscr{P}$ which is bounded. However, the hypergraph $\mathcal{H}$ will only be δ_k-regular w.r.t. $\mathscr{P}$, where δ_k is a constant independent of $\boldsymbol{a}^{\mathscr{P}}$, and not $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular, as required by part (ii) of Theorem 2.2. To obtain such an $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular hypergraph $\mathcal{G}^{(k)}$, which will be 'ν close to $\mathcal{H}^{(k)}$' (*cf.* (iii) of Theorem 2.2) we need to apply Theorem 2.3 again. It will be essential for us that the partition obtained in the second application of Theorem 2.3 will refine $\mathscr{P}$, the partition obtained in the first application. This is the reason why we will strengthen the statement of Theorem 2.3 (see Lemma 4.1). This change is due to the induction scheme requiring a corresponding strengthening of Theorem 2.2 (see Lemma 4.2).

We now state the strengthened variant of Theorem 2.3. It allows us to enter the regularity lemma with an initial equitable family of partitions $\mathscr{O}$ and a family of k-uniform hypergraphs $\mathcal{H}_1^{(k)}, \dots, \mathcal{H}_s^{(k)}$. It then guarantees the existence of an equitable refinement $\mathscr{P}$ of $\mathscr{O}$ for which each $\mathcal{H}_i^{(k)}$ is regular. (Since it might not be completely obvious that Theorem 2.3 follows from Lemma 4.1 stated below, we give the formal reduction after Remark 2.)

Lemma 4.1 (RL(k))**.** *For all positive integers o and s, all positive reals η and δ_k, and all functions $r\colon \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta\colon \mathbb{N}^{k-1} \to (0,1]$ there is a positive real μ_{RL} and positive integers t_{RL} and n_{RL} such that the following holds. Suppose*

(a) *V is a set of cardinality $n \geqslant n_{\mathrm{RL}}$ and $(t_{\mathrm{RL}})!$ divides n,*
(b) *$\mathscr{O} = \mathscr{O}(k-1, \boldsymbol{a}^{\mathscr{O}})$ is an $(\eta^{\mathscr{O}}, \mu_{\mathrm{RL}}, \boldsymbol{a}^{\mathscr{O}})$-equitable (for some $\eta^{\mathscr{O}} > 0$) and o-bounded family of partitions on V, and*
(c) *$\mathscr{H}^{(k)} = \{\mathcal{H}_1^{(k)}, \dots, \mathcal{H}_s^{(k)}\}$ is a partition of $[V]^k$.*

Then there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that

(P1) *$\mathscr{P}$ is $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and t_{RL}-bounded,*
(P2) *$\mathscr{P} \prec \mathscr{O}$,*

and for every $i \in [s]$

(H) *$\mathcal{H}_i^{(k)}$ is $(\delta_k, *, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}$.*

Remark 2. In the inductive proof we will apply Lemma 4.1 twice. In the second application in Section 5.2 it will be convenient to use a variant of Lemma 4.1, where assumptions (a) and (b) are replaced by

(a′) $V = V_1 \cup \dots \cup V_k$, $|V_i| = m \geqslant n_{\mathrm{RL}}/k$ and $t_{\mathrm{RL}}!$ divides m,
(b′) $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\mu_{\mathrm{RL}}/3, \boldsymbol{d})$-regular $(m, k, k-1)$-complex, where the vertex set $\mathcal{R}^{(1)} = V_1 \cup \dots \cup V_k$ and $\boldsymbol{d} = (1/a_2, \dots, 1/a_{k-1})$, $a_i \in \mathbb{N}$ and $a_i \leqslant o$ for $2 \leqslant i < k$.

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Moreover, we weaken conclusion (P2) in this context, insisting only that $\mathscr{P}$ 'refines' the given complex $\mathcal{R}$, as follows.

(P2′) $\mathscr{P}^{(1)} \prec \mathcal{R}^{(1)} = V_1 \cup \cdots \cup V_k$, and for every $2 \leqslant j < k$ and every $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}$ we have either $\mathcal{P}^{(j)} \subseteq \mathcal{R}^{(j)}$ or $\mathcal{P}^{(j)} \cap \mathcal{R}^{(j)} = \varnothing$.

Note that this version of Lemma 4.1 is in fact a consequence of Lemma 4.1.

We now verify that Lemma 4.1 implies Theorem 2.3 for the same k.

Proof: RL(k) $\Longrightarrow$ Theorem 2.3 for k. Let k be a fixed integer and let constants η and δ_k and functions $r\colon \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta\colon \mathbb{N}^{k-1} \to (0,1]$ be given by Theorem 2.3. We want to apply Lemma 4.1. For that we will define an auxiliary family of partitions $\mathscr{O}$. In fact any sufficiently equitable partition would do. In order to avoid trivial cases we are going to split the vertex set into k parts of the same size and any part of the partition $\mathscr{O}^{(j)}$ will be isomorphic to the complete j-partite j-uniform hypergraph of the appropriate order for $2 \leqslant j \leqslant k-1$ (see (4.1) below). With this in mind we apply Lemma 4.1 with $o = k$, $s = 2$, and the given constants η and δ_k, and functions r and δ to obtain μ_{RL}, t_{RL} and n_{RL}. We then set $t_{\mathrm{Thm\,2.3}} = t_{\mathrm{RL}}$ and $n_{\mathrm{Thm\,2.3}} = n_{\mathrm{RL}}$.

Now let $n \geqslant n_{\mathrm{Thm\,2.3}}$ be a multiple of $t_{\mathrm{Thm\,2.3}} = (t_{\mathrm{RL}})!$ and let $\mathcal{H}^{(k)}$ be a hypergraph with vertex set V, where $|V| = n$. Set $a_1^{\mathscr{O}} = k$, $a_j^{\mathscr{O}} = 1$ for $j = 2, \ldots, k-1$, $\boldsymbol{a}^{\mathscr{O}} = (a_1^{\mathscr{O}}, \ldots, a_{k-1}^{\mathscr{O}})$ and let $V = V_1 \cup \cdots \cup V_{a_1^{\mathscr{O}}} = \mathscr{O}^{(1)}$ be some arbitrary equitable vertex partition. Moreover, set

$$\mathscr{O}^{(j)} = \{K_j^{(j)}(V_{i_1}, \ldots, V_{i_j})\colon 1 \leqslant i_1 < \cdots < i_j \leqslant a_1^{\mathscr{O}} = k\} \tag{4.1}$$

and $\mathscr{H}^{(k)} = \{\mathcal{H}^{(k)}, [V]^k \setminus \mathcal{H}^{(k)}\}$. Clearly, $\mathscr{O}$ constructed that way is $(\eta^{\mathscr{O}}, \mu, \boldsymbol{a}^{\mathscr{O}})$-equitable for some $\eta^{\mathscr{O}} > 0$ and every $\mu > 0$. Consequently, V, $\mathscr{O}$ and $\mathscr{H}^{(k)}$ satisfy the assumptions (a)–(c) of Lemma 4.1 for n_{RL}, t_{RL}, $o = a_1^{\mathscr{O}} = k$, $s = 2$ and any μ_{RL}. Then, (P1) and (H) yield conclusions (i) and (ii) of Theorem 2.3. □

Next we state a similarly strengthened version of Theorem 2.2.

Lemma 4.2 (RAL(k))**.** *For all positive integers o and s, all positive reals η and ν, and every function $\varepsilon\colon \mathbb{N}^{k-1} \to (0,1]$ there is a positive real μ_{RAL} and positive integers t_{RAL} and n_{RAL} such that the following holds. Suppose*

(a) *V is a set of cardinality $n \geqslant n_{\mathrm{RAL}}$ and $(t_{\mathrm{RAL}})!$ divides n,*

(b) *$\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ is an $(\eta^{\mathscr{O}}, \mu_{\mathrm{RAL}}, \boldsymbol{a}^{\mathscr{O}})$-equitable (for some $\eta^{\mathscr{O}} > 0$) and o-bounded family of partitions on V, and*

(c) *$\mathscr{H}^{(k)} = \{\mathcal{H}_1^{(k)}, \ldots, \mathcal{H}_s^{(k)}\}$ is a partition of $[V]^k$ so that $\mathscr{H}^{(k)} \prec \mathscr{O}^{(k)}$.*

Then there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that

(P1) *$\mathscr{P}$ is $(\eta, \varepsilon(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and t_{RAL}-bounded, and*

(P2) *$\mathscr{P} \prec \mathscr{O}(k-1) = \{\mathscr{O}^{(j)}\}_{j=1}^{k-1}$.*

Furthermore, there exists a partition $\mathscr{G}^{(k)} = \{\mathcal{G}_1^{(k)}, \ldots, \mathcal{G}_s^{(k)}\}$ of $[V]^k$ such that, for every $i \in [s]$, the following holds:

(G1) *$\mathcal{G}_i^{(k)}$ is perfectly $\varepsilon(\boldsymbol{a}^{\mathscr{P}})$-regular w.r.t. $\mathscr{P}$,*

(G2) *$|\mathcal{G}_i^{(k)} \triangle \mathcal{H}_i^{(k)}| \leqslant \nu n^k$, and*

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(G3) *if* $\mathcal{H}_i^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{O}^{(1)})$ *then* $\mathcal{G}_i^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{O}^{(1)})$ *and* $\mathscr{G}^{(k)} \prec \mathscr{O}^{(k)}$.

Lemma 4.2 yields Theorem 2.2 for the same k in a similar way as Lemma 4.1 implies Theorem 2.3 and we omit the details. Hence it suffices to show

$$\mathrm{RL}(2) \quad \text{and} \quad \mathrm{RL}(k) \Longrightarrow \mathrm{RAL}(k) \Longrightarrow \mathrm{RL}(k+1) \quad \text{for } k \geqslant 2,$$

in order to establish Theorem 2.3 and Theorem 2.2 inductively.

We outline the basis of the induction, the proof of RL(2), in Section 4.1. The proofs of each of the two implications establishing the induction step are the content of Section 5 and Section 6, respectively.

4.1. Sketch of the proof of RL(2)

Observe that in the statement of RL(2), Lemma 4.1 for $k = 2$, the constant μ and the function δ have no bearing. Consequently, RL(2) reduces to the following statement.

Lemma 4.3 (RL(2))**.** *For all positive integers o and s, all positive reals η and δ_2, and any function $r\colon \mathbb{N} \to \mathbb{N}$, there are positive integers t_{RL} and n_{RL} such that the following holds.*

Suppose

(a) *V is a set of cardinality $n \geqslant n_{\mathrm{RL}}$ and $(t_{\mathrm{RL}})!$ divides n,*

(b) *$\mathscr{O}^{(1)}$ is a vertex partition $V_1 \cup \cdots \cup V_{a_1^{\mathscr{O}}}$ of V, where $|V_1| = \cdots = |V_{a_1^{\mathscr{O}}}|$ and $a_1^{\mathscr{O}} \leqslant o$,*

(c) *$\mathscr{H} = \{H_1, \ldots, H_s\}$ is a partition of $[V]^2$ the complete graph on n vertices.*

Then there exists a partition $\mathscr{P}^{(1)} = \{W_1, \ldots, W_{a_1^{\mathscr{P}}}\}$ of V so that

(P1) *$|W_1| = \cdots = |W_{a_1^{\mathscr{P}}}|$, $\mathrm{Cross}_2(\mathscr{P}^{(1)}) \geqslant (1-\eta)\binom{n}{2}$, and $a_1^{\mathscr{P}} \leqslant t_{\mathrm{RL}}$,*

(P2) *for every $i \in [a_1^{\mathscr{P}}]$ we have $W_i \subseteq V_j$ for some $j \in [a_1^{\mathscr{O}}]$,*

and for every $i \in [s]$

(H) *H_i is $(\delta_2, *, r(a_1^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}^{(1)}$.*

The proof of RL(2) follows closely the lines of the proof of Szemerédi's regularity lemma [31] (Theorem 2.1). There are three differences, however. The first and the last of them are standard.

(1) Rather than one graph we have a fixed number of graphs $H_1, \ldots, H_s$ to regularize. Such a regularity lemma was used in a number of applications and is discussed for example in [19, Section 1.9].

(2) This difference, which regards the concept of regularity in (H), is perhaps most significant. Instead of a single pair $A' \subseteq A$, $B' \subseteq B$, $|A'||B'| \geqslant \varepsilon|A||B|$ that witnesses the irregularity of a bipartite graph with vertex classes A and B, we consider here a more complicated witness; namely an r-tuple of pairs (A_i, B_i) of sets where $A_1, \ldots, A_r \subseteq A$, $B_1, \ldots, B_r \subseteq B$ and $|\bigcup_{i\in[r]} A_i \times B_i| \leqslant \varepsilon|A||B|$ (*cf.* Definition 11 with $k = 2$ and $\mathcal{H}^{(1)} = (A, B)$).

We recall that the proof of Szemerédi's regularity lemma [31] is based on a procedure in which, having an initial partition $\mathscr{P}_0^{(1)}$, one constructs a sequence $\mathscr{P}_0^{(1)}, \mathscr{P}_1^{(1)}, \ldots$ of partitions. To each partition a quantity (the *index*) is associated, which is known to satisfy $\mathrm{ind}(\mathscr{P}^{(1)}) \leqslant 1$ for every vertex partition $\mathscr{P}^{(1)}$. On the other hand, one proves that if $\mathscr{P}_i^{(1)}$ is irregular, then $\mathrm{ind}(\mathscr{P}_{i+1}^{(1)}) \geqslant \mathrm{ind}(\mathscr{P}_i^{(1)}) + \delta_2^4/10$. Consequently, one infers that, after at most $10/\delta_2^4$ iterations, one arrives at a partition which is δ_2-regular.

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In [31], if $\mathscr{P}_i^{(1)}$ was partitioned into $a_1^{\mathscr{P}_i}$ parts then $\mathscr{P}_{i+1}^{(1)}$ was a partition into at most $4^{a_1^{\mathscr{P}_i}}$ parts; while, in our proof (due to the fact that the witness has $r(a_1^{\mathscr{P}_i})$ parts for each pair), we may have as many as $4^{r(a_1^{\mathscr{P}_i})\times a_1^{\mathscr{P}_i}}$ partition classes in $\mathscr{P}_{i+1}^{(1)}$. Consequently, t_{RL} (which is an upper bound for the number of classes in the final partition) depends not only on δ_2, but also on the function $r(\cdot)$. It is independent, however, of the cardinality of the vertex set V.

(3) In order to avoid the exceptional class V_0 we assume that the cardinality of V is divisible by $(t_{\mathrm{RL}})!$. This allows us to redistribute all the vertices in V_i which would remain from subdividing the witnesses. Such a lemma was considered, *e.g.*, in [26].

5. Proof of $\mathrm{RL}(k) \Longrightarrow \mathrm{RAL}(k)$

In order to simplify the presentation we split the proof into two parts. In the first part we deduce $\mathrm{RAL}(k)$ from $\mathrm{RL}(k)$ and the following lemma.

Lemma 5.1. *For every positive integer s, all positive reals ν and ε, and every vector $\boldsymbol{d} = (d_2, \dots, d_{k-1})$ satisfying $1/d_i \in \mathbb{N}$ for $2 \leqslant i \leqslant k-1$, there exist positive reals $\delta_{5.1}$ and $\xi_{5.1}$ and integers $t_{5.1}$ and $m_{5.1}$ such that the following holds. Suppose*

(a) *$m \geqslant m_{5.1}$ and $(t_{5.1})!$ divides m,*
(b) *$\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\delta_{5.1}, \boldsymbol{d})$-regular $(m, k, k-1)$-complex,*
(c) *$\mathcal{F}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ is $\xi_{5.1}$-regular w.r.t. $\mathcal{R}^{(k-1)}$, and*
(d) *$\{\mathcal{H}_1^{(k)}, \dots, \mathcal{H}_s^{(k)}\}$ is a partition of $\mathcal{F}^{(k)}$, where each $\mathcal{H}_i^{(k)}$ is $(\nu/12, *, t_{5.1}^{2k})$-regular w.r.t. $\mathcal{R}^{(k-1)}$ for every $i \in [s]$.*

Then there exists a partition $\{\mathcal{G}_1^{(k)}, \dots, \mathcal{G}_s^{(k)}\}$ of $\mathcal{F}^{(k)}$ so that for every $i \in [s]$ the following holds:

(i) *$\mathcal{G}_i^{(k)}$ is $(\varepsilon, d(\mathcal{H}_i^{(k)} \mid \mathcal{R}^{(k-1)}))$-regular w.r.t. $\mathcal{R}^{(k-1)}$ and*
(ii) *$|\mathcal{G}_i^{(k)} \triangle \mathcal{H}_i^{(k)}| \leqslant \nu |\mathcal{K}_k(\mathcal{R}^{(k-1)})|$.*

In Section 5.1 we derive $\mathrm{RAL}(k)$ from Lemma 5.1 and $\mathrm{RL}(k)$; then, in Section 5.2, we give the proof of Lemma 5.1, which is based on another application of $\mathrm{RL}(k)$.

5.1. Lemma 5.1 and $\mathrm{RL}(k)$ imply $\mathrm{RAL}(k)$

The idea of this reduction is as follows. Let $\mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ and $\mathscr{H}^{(k)}$ be given by $\mathrm{RAL}(k)$. We apply $\mathrm{RL}(k)$ to $\mathscr{O}(k-1) = \{\mathscr{O}^{(j)}\}_{j=1}^{k-1}$ and $\mathscr{H}^{(k)}$. The constants will be chosen in such a way that, after that application of $\mathrm{RL}(k)$, a 'typical' polyad $\hat{\mathcal{P}}^{(k-1)}$ with its underlying complex $\boldsymbol{\mathcal{P}} = \{\hat{\mathcal{P}}^{(j)}\}_{j=1}^{k-1}$ matches the assumptions of Lemma 5.1 for $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{P}}$, $\mathcal{F}^{(k)} = \mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{k-1})$ (where $\mathcal{O}^{(k)} \in \mathscr{O}^{(k)}$), and $\{\widetilde{\mathcal{H}}_h^{(k)} = \mathcal{H}_h^{(k)} \cap \mathcal{F}^{(k)} \colon \mathcal{H}_h^{(k)} \in \mathscr{H}^{(k)} \text{ and } \mathcal{H}_h^{(k)} \subseteq \mathcal{O}^{(k)}\}$. Lemma 5.1 then yields hypergraphs $\widetilde{\mathcal{G}}_h^{(k)}$ satisfying (i) and (ii) of Lemma 5.1. Repeating this for all 'typical' polyads $\hat{\mathcal{P}}^{(k-1)}$ and $\mathcal{O}^{(k)} \in \mathscr{O}^{(k)}$, and taking appropriate care of the 'untypical' case, then yields the promised hypergraphs $\mathcal{G}_1^{(k)} \cdots \mathcal{G}_s^{(k)}$ with properties (G1)–(G3) of $\mathrm{RAL}(k)$. We give the technical details of this outline below.

Proof: $\mathrm{RL}(k) \wedge \text{Lemma 5.1} \Longrightarrow \mathrm{RAL}(k)$**.** Let positive constants o_{RAL}, s_{RAL}, η_{RAL}, and ν_{RAL}, and a function $\varepsilon_{\mathrm{RAL}} \colon \mathbb{N}^{k-1} \to (0, 1]$ be given (w.l.o.g. we may assume that $\varepsilon_{\mathrm{RAL}}$ is monotone in every coordinate). We have to determine μ_{RAL}, t_{RAL}, and n_{RAL} (see (5.7)). Our proof relies

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on an application of RL(k) followed by an application of Lemma 5.1. In order to match the assumptions of Lemma 5.1 the parameters for the application of RL(k) have to match these assumptions. Consequently, 'constant-wise' we first apply Lemma 5.1 to foresee what is needed for its application, which will be provided by RL(k). With this in mind we set

$$s_{5.1} = s_{\mathrm{RAL}}, \qquad v_{5.1} = v_{\mathrm{RAL}}/2. \tag{5.1}$$

Note that for every choice of ε and $d_1, \dots, d_{k-1}$ (satisfying $1/d_i \in \mathbb{N}$), Lemma 5.1 yields constants $\delta_{5.1}$, $\xi_{5.1}$, $t_{5.1}$, and $m_{5.1}$. Accordingly, we define functions δ_{aux}, $\xi_{\mathrm{aux}} \colon \mathbb{N}^{k-1} \to (0,1]$ and t_{aux}, $m_{\mathrm{aux}} \colon \mathbb{N}^{k-1} \to \mathbb{N}$ mapping any given $\boldsymbol{a} = (a_1, \dots, a_{k-1}) \in \mathbb{N}^{k-1}$ to the corresponding constant from Lemma 5.1 with $\varepsilon = \varepsilon_{\mathrm{RAL}}(\boldsymbol{a})$ and $d_2 = 1/a_2, \dots, d_{k-1} = 1/a_{k-1}$. More precisely, we set for $x \in \{\delta, \xi, t, m\}$

$$x_{\mathrm{aux}}(\boldsymbol{a}) = x_{\mathrm{L.5.1}}\big(s = s_{5.1}, v = v_{5.1}, \varepsilon = \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}), d_2 = 1/a_2, \dots, d_{k-1} = 1/a_{k-1}\big), \tag{5.2}$$

where $x_{\mathrm{L.5.1}}(s, v, \varepsilon, d_2, \dots, d_{k-1})$ is given by Lemma 5.1 applied with constants s, v, ε, and $d_2, \dots, d_{k-1}$. Without loss of generality we assume that the functions defined in (5.2) are monotone in every coordinate.

We now choose the parameters for the application of RL(k). For that we set

$$o_{\mathrm{RL}} = o_{\mathrm{RAL}}, \quad s_{\mathrm{RL}} = s_{\mathrm{RAL}}, \quad \eta_{\mathrm{RL}} = \eta_{\mathrm{RAL}}, \quad \text{and} \quad \delta_{k,\mathrm{RL}} = \min\left\{\frac{v_{5.1}}{12}, \frac{v_{\mathrm{RAL}}}{2s_{\mathrm{RAL}}}\right\} \tag{5.3}$$

and consider functions $r_{\mathrm{RL}} \colon \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta_{\mathrm{RL}} \colon \mathbb{N}^{k-1} \to (0,1]$ defined for every integer vector $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ by

$$r_{\mathrm{RL}}(\boldsymbol{a}) = \big(t_{\mathrm{aux}}(\boldsymbol{a})\big)^{2^k} \quad \text{and} \tag{5.4}$$

$$\delta_{\mathrm{RL}}(\boldsymbol{a}) = \min\Big\{\varepsilon_{\mathrm{RAL}}(\boldsymbol{a}), \delta_{\mathrm{aux}}(\boldsymbol{a}), \varepsilon_{\mathrm{DCL}}\Big(h = k-1, \ell = k, \gamma = \tfrac{1}{2}, d_0 = \min_{2 \leqslant i < k} a_i^{-1}\Big)\Big\}, \tag{5.5}$$

where $\varepsilon_{\mathrm{DCL}}(h, \ell, \gamma, d_0)$ is given by Theorem 3.1.

Having defined all input parameters of RL(k), Lemma 4.1, in (5.3), (5.4) and (5.5), Lemma 4.1 now yields positive constants μ_{RL}, t_{RL}, and n_{RL}. We use t_{RL} to establish 'worst case' estimates on the functions ξ_{aux}, t_{aux}, and m_{aux} and set

$$\xi_{\mathrm{worst}} = \xi_{\mathrm{aux}}(t_{\mathrm{RL}}, \dots, t_{\mathrm{RL}}), \quad t_{\mathrm{worst}} = t_{\mathrm{aux}}(t_{\mathrm{RL}}, \dots, t_{\mathrm{RL}}), \quad \text{and} \quad m_{\mathrm{worst}} = m_{\mathrm{aux}}(t_{\mathrm{RL}}, \dots, t_{\mathrm{RL}}). \tag{5.6}$$

Finally, we define μ_{RAL}, t_{RAL}, and n_{RAL} promised by RAL(k). For that we set

$$\begin{aligned} &\mu_{\mathrm{RAL}} = \min\left\{\mu_{\mathrm{RL}}, \frac{\varepsilon_{\mathrm{RAL}}(t_{\mathrm{RL}}, \dots, t_{\mathrm{RL}})}{2t_{\mathrm{RL}}^{2^k}}, \frac{\xi_{\mathrm{worst}}}{2t_{\mathrm{RL}}^{2^k}}\right\}, \quad t_{\mathrm{RAL}} = t_{\mathrm{RL}} + t_{\mathrm{worst}}, \quad \text{and} \\ &n_{\mathrm{RAL}} = \max\{n_{\mathrm{RL}}, t_{\mathrm{RL}} m_{\mathrm{worst}}, t_{\mathrm{RL}} m_{\mathrm{DCL}}\big(h = k-1, \ell = k, \gamma = \tfrac{1}{2}, d_0 = t_{\mathrm{RL}}^{-1}\big)\}. \end{aligned} \tag{5.7}$$

Note that, for the given input parameters o_{RAL}, s_{RAL}, η_{RAL}, v_{RAL}, and $\varepsilon_{\mathrm{RAL}} \colon \mathbb{N}^{k-1} \to (0,1]$ of RAL(k) in (5.7), we defined the corresponding output parameters. Now we need to show that with this choice we will be able to verify RAL(k), Lemma 4.2.

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Let V, $\mathscr{O}_{\mathrm{RAL}}$, and $\mathscr{H}^{(k)}$ satisfying (a)–(c) of RAL(k), Lemma 4.2, be given, *i.e.*,

(RAL.a) $|V| = n \geqslant n_{\mathrm{RAL}}$ and $(t_{\mathrm{RAL}})!$ divides n,
(RAL.b) $\mathscr{O}_{\mathrm{RAL}} = \mathscr{O}_{\mathrm{RAL}}(k, \boldsymbol{a}^{\mathscr{O}_{\mathrm{RAL}}}) = \{\mathscr{O}_{\mathrm{RAL}}^{(j)}\}_{j=1}^{k}$ is $(\eta^{\mathscr{O}_{\mathrm{RAL}}}, \mu_{\mathrm{RAL}}, \boldsymbol{a}^{\mathscr{O}_{\mathrm{RAL}}})$-equitable (for some $\eta^{\mathscr{O}_{\mathrm{RAL}}} > 0$) and o_{RAL}-bounded, and
(RAL.c) $|\mathscr{H}^{(k)}| = s_{\mathrm{RAL}}$ and $\mathscr{H}^{(k)} \prec \mathscr{O}_{\mathrm{RAL}}^{(k)}$.

Our objective is to find a family of partitions $\mathscr{P}_{\mathrm{RAL}} = \mathscr{P}_{\mathrm{RAL}}(k-1, \boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}})$ on V and a partition $\mathscr{G}^{(k)} = \{\mathcal{G}_1^{(k)}, \dots, \mathcal{G}_{s_{\mathrm{RAL}}}^{(k)}\}$ of $[V]^k$ so that

(RAL.P1) $\mathscr{P}_{\mathrm{RAL}}$ is $(\eta_{\mathrm{RAL}}, \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}}), \boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}})$-equitable and t_{RAL}-bounded,
(RAL.P2) $\mathscr{P}_{\mathrm{RAL}} \prec \mathscr{O}_{\mathrm{RAL}}(k-1) = \{\mathscr{O}_{\mathrm{RAL}}^{(j)}\}_{j=1}^{k-1}$,
(RAL.G1) $\mathcal{G}_i^{(k)}$ is perfectly $\varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}})$-regular w.r.t. $\mathscr{P}_{\mathrm{RAL}}$ for every $i \in [s_{\mathrm{RAL}}]$,
(RAL.G2) $|\mathcal{G}_i^{(k)} \triangle \mathcal{H}_i^{(k)}| \leqslant \nu_{\mathrm{RAL}} n^k$ for every $i \in [s_{\mathrm{RAL}}]$, and
(RAL.G3) if $\mathcal{H}_i^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{O}_{\mathrm{RAL}}^{(1)})$ then $\mathcal{G}_i^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{O}_{\mathrm{RAL}}^{(1)})$ for every $i \in [s_{\mathrm{RAL}}]$ and $\mathscr{G}^{(k)} \prec \mathscr{O}_{\mathrm{RAL}}^{(k)}$.

Without loss of generality we may assume that

$$\mathcal{H}_i^{(k)} \neq \varnothing \quad \text{for every } \ i \in [s_{\mathrm{RAL}}]. \tag{5.8}$$

Otherwise we simply set $\mathcal{G}_i^{(k)} = \varnothing$ for every $i \in [s_{\mathrm{RAL}}]$ for which $\mathcal{H}_i^{(k)} = \varnothing$, and obviously (RAL.G1)–(RAL.G3) holds for those $\mathcal{G}_i^{(k)}$ for any family of partitions $\mathscr{P}$.

As we have already mentioned, we want to apply RL(k) to V, $\mathscr{O}_{\mathrm{RL}} = \mathscr{O}_{\mathrm{RAL}}(k-1) = \{\mathscr{O}_{\mathrm{RAL}}^{(j)}\}_{j=1}^{k-1}$, $\boldsymbol{a}^{\mathscr{O}_{\mathrm{RL}}} = (a_1^{\mathscr{O}_{\mathrm{RAL}}}, \dots, a_{k-1}^{\mathscr{O}_{\mathrm{RAL}}})$, and $\mathscr{H}^{(k)}$ with the constants o_{RL}, s_{RL}, η_{RL}, and $\delta_{k,\mathrm{RL}}$ defined in (5.3) and functions r_{RL} and δ_{RL} defined in (5.4) and (5.5). For this we have to verify that

(RL.a) $|V| = n \geqslant n_{\mathrm{RL}}$ and $(t_{\mathrm{RL}})!$ divides n,
(RL.b) $\mathscr{O}_{\mathrm{RL}} = \mathscr{O}_{\mathrm{RL}}(k-1, \boldsymbol{a}^{\mathscr{O}_{\mathrm{RL}}}) = \{\mathscr{O}_{\mathrm{RL}}^{(j)}\}_{j=1}^{k-1}$ is $(\eta^{\mathscr{O}_{\mathrm{RL}}}, \mu_{\mathrm{RL}}, \boldsymbol{a}^{\mathscr{O}_{\mathrm{RL}}})$-equitable (for some $\eta^{\mathscr{O}_{\mathrm{RL}}} > 0$) and o_{RL}-bounded, and
(RL.c) $|\mathscr{H}^{(k)}| = s_{\mathrm{RL}}$.

We note that (RL.a) is an easy consequence of the choice of $n_{\mathrm{RAL}} \geqslant n_{\mathrm{RL}}$ and $t_{\mathrm{RAL}} \geqslant t_{\mathrm{RL}}$ in (5.7) and (RAL.a). Similarly, (RL.b) follows from the choice of $\mu_{\mathrm{RAL}} \leqslant \mu_{\mathrm{RL}}$ in (5.7) and (RAL.b), while (RL.c) is a consequence of (RAL.c) and the choice of $s_{\mathrm{RL}} = s_{\mathrm{RAL}}$ in (5.3). Having verified that (RL.a)–(RL.c) hold, we reason that there is a family of partitions $\mathscr{P}_{\mathrm{RL}} = \mathscr{P}_{\mathrm{RL}}(k-1, \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ on V which satisfies properties (P1), (P2), and (H) of Lemma 4.1:

(RL.P1) $\mathscr{P}_{\mathrm{RL}}$ is $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$-equitable and t_{RL}-bounded,
(RL.P2) $\mathscr{P}_{\mathrm{RL}} \prec \mathscr{O}_{\mathrm{RL}} = \mathscr{O}_{\mathrm{RAL}}(k-1)$, and
(RL.H) $\mathcal{H}_i^{(k)}$ is $(\delta_{k,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))$-regular w.r.t. $\mathscr{P}_{\mathrm{RL}}$ for every $i \in [s_{\mathrm{RL}}]$.

We set

$$\mathscr{P}_{\mathrm{RAL}} = \mathscr{P}_{\mathrm{RL}} \quad \text{and} \quad \boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}} = \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}. \tag{5.9}$$

It then follows from (RL.P1) and (RL.P2) and the choices of $\eta_{\mathrm{RL}} = \eta_{\mathrm{RAL}}$ in (5.3), $\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}) \leqslant \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ in (5.5), and $t_{\mathrm{RAL}} \geqslant t_{\mathrm{RL}}$ in (5.7), that

$$\mathscr{P}_{\mathrm{RAL}} \ \text{ satisfies (RAL.P1)} \ \text{ and } \ \text{(RAL.P2)}. \tag{5.10}$$

It is left to ensure the existence of the partition $\mathscr{G}^{(k)}$ of $[V]^k$ which satisfies (RAL.G1)–(RAL.G3).

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Before we prove the existence of $\mathscr{G}^{(k)}$ we make some preparations, which simplify the presentation. We complete the partition $\mathscr{O}^{(k)}_{\mathrm{RAL}}$ (which partitions $\mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})$), to a partition of $[V]^k$. For that we set

$$\widetilde{\mathscr{O}}^{(k)} = \mathscr{O}^{(k)}_{\mathrm{RAL}} \cup \big([V]^k \setminus \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})\big). \tag{5.11}$$

We also define for every $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$

$$I(\mathcal{O}^{(k)}) = \{i \in [s_{\mathrm{RAL}}] \colon \mathcal{H}^{(k)}_i \subseteq \mathcal{O}^{(k)} \text{ and } \mathcal{H}^{(k)}_i \neq \varnothing\}. \tag{5.12}$$

Note that, due to (RAL.c), (5.8), and (5.11), the family $\{I(\mathcal{O}^{(k)}) \colon \mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}\}$ forms a partition of $[s_{\mathrm{RAL}}]$. Before we continue we make the following observation.

Claim 5.2. *For every $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ and $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$ the following holds. Set $\mathcal{F}^{(k)} = \mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$, then $\mathcal{F}^{(k)}$ is $(2t^{2^k}_{\mathrm{RL}}\mu_{\mathrm{RAL}})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$.*

The claim is trivial if $\mathcal{F}^{(k)} = \mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) = \varnothing$, and hence we assume that

$$\mathcal{F}^{(k)} = \mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \neq \varnothing. \tag{5.13}$$

We distinguish two cases. From $\mathscr{P}_{\mathrm{RL}} \prec \mathscr{O}_{\mathrm{RAL}}(k-1)$ (*cf.* (RL.P2)), we infer that either $\hat{\mathcal{P}}^{(k-1)}$ is contained in some polyad $\hat{\mathcal{O}}^{(k-1)} \in \hat{\mathscr{O}}^{(k-1)}_{\mathrm{RAL}}$ or $\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \cap \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}}) = \varnothing$. If $\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \cap \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}}) = \varnothing$, then we have $\mathcal{O}^{(k)} = [V]^k \setminus \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})$ (using (5.13)) and, consequently, $\mathcal{F}^{(k)} = \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$. Hence, $\mathcal{F}^{(k)}$ is ξ-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ for every $\xi > 0$ which yields the claim in that case.

On the other hand, if $\hat{\mathcal{P}}^{(k-1)} \subseteq \hat{\mathcal{O}}^{(k-1)}$ for some $\hat{\mathcal{O}}^{(k-1)} \in \hat{\mathscr{O}}^{(k-1)}_{\mathrm{RAL}}$, then, due to (5.13) and the fact that $\mathscr{O}_{\mathrm{RAL}}$ is a family of partitions (*cf.* Definition 7), we have that $\mathcal{O}^{(k)} \subseteq \mathcal{K}_k(\hat{\mathcal{O}}^{(k-1)})$. Therefore, if follows from (RAL.b) and the definition of regularity (Definition 4) that

$$\mathcal{F}^{(k)} \text{ is } \left(\mu_{\mathrm{RAL}} \frac{|\mathcal{K}_k(\hat{\mathcal{O}}^{(k-1)})|}{|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|}\right)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}. \tag{5.14}$$

Clearly, $|\mathcal{K}_k(\hat{\mathcal{O}}^{(k-1)})| \leqslant n^k$ and, due to the choice of $\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}) \leqslant \varepsilon_{\mathrm{DCL}}(h = k-1, \ell = k, \gamma = 1/2, d_0 = \min_{2\leqslant i<k} 1/a_i^{\mathscr{P}_{\mathrm{RL}}})$ in (5.5), the appropriate choice of $n_{\mathrm{RAL}} \geqslant t_{\mathrm{RL}} \times m_{\mathrm{DCL}}(h = k-1, \ell = k, \gamma = 1/2, d_0 = t_{\mathrm{RL}}^{-1})$ in (5.7), and (RL.P1), by Theorem 3.1, we infer

$$|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})| \geqslant \frac{1}{2}\prod_{j=2}^{k-1}\left(\frac{1}{a_j^{\mathscr{P}_{\mathrm{RL}}}}\right)^{\binom{k}{j}} \times \left(\frac{n}{a_1^{\mathscr{P}_{\mathrm{RL}}}}\right)^k \geqslant \frac{n^k}{2t^{2^k}_{\mathrm{RL}}},$$

and the claim follows. □

We now continue with the proof of the existence of the partition $\mathscr{G}^{(k)}$ of $[V]^k$ which satisfies (RAL.G1)–(RAL.G3). For that we will mainly use Lemma 5.1 applied to the polyads of $\mathscr{P}_{\mathrm{RL}}$. However, we distinguish between two types of polyads, and set

$$\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\cdot\mathrm{reg}} = \Big\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}} \colon \mathcal{H}^{(k)}_i \text{ is } (\delta_{k,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)} \text{ for every } i \in [s_{\mathrm{RL}}]\Big\}.$$

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Case 1 ($\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\cdot\mathrm{reg}}$)**.** In this case let $K \in \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ and set $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{P}}(K) = \{\hat{\mathcal{P}}^{(j)}(K)\}_{j=1}^{k-1}$ with $\hat{\mathcal{P}}^{(k-1)}(K) = \hat{\mathcal{P}}^{(k-1)}$ (see (2.1)). Let $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ be such that

$$\mathcal{F}^{(k)} = \mathcal{O}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \neq \varnothing, \tag{5.15}$$

and set

$$\widetilde{\mathcal{H}}_i^{(k)} = \mathcal{H}_i^{(k)} \cap \mathcal{F}^{(k)} = \mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \quad \text{for } i \in I(\mathcal{O}^{(k)}). \tag{5.16}$$

We want to apply Lemma 5.1 with parameters $s = s_{5.1}$, $\nu = \nu_{5.1}$, $\varepsilon = \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$, and $d_i = 1/a_i^{\mathscr{P}_{\mathrm{RL}}}$ for $2 \leqslant i < k$. Note that, due to the definition of the functions δ_{aux}, ξ_{aux}, t_{aux}, and m_{aux} in view of (5.2), Lemma 5.1 yields constants $\delta_{5.1}$, $\xi_{5.1}$, $t_{5.1}$ and $m_{5.1}$ which satisfy

$$\delta_{5.1} = \delta_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), \quad \xi_{5.1} = \xi_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), \quad t_{5.1} = t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), \quad \text{and} \quad m_{5.1} = m_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}).$$

In order to apply Lemma 5.1 with the chosen parameters to $m = n/a_1^{\mathscr{P}_{\mathrm{RL}}}$, $\boldsymbol{\mathcal{R}}$, $\mathcal{F}^{(k)}$, and $\{\widetilde{\mathcal{H}}_i^{(k)} : i \in I(\mathcal{O}^{(k)})\}$, we have to verify

(L.5.1.a) $n/a_1^{\mathscr{P}_{\mathrm{RL}}} = m \geqslant m_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ and $(t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))!$ divides m,

(L.5.1.b) $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\delta_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), \boldsymbol{d})$-regular $(m, k, k-1)$-complex, where $\boldsymbol{d} = (1/a_2^{\mathscr{P}_{\mathrm{RL}}}, \ldots, 1/a_{k-1}^{\mathscr{P}_{\mathrm{RL}}})$,

(L.5.1.c) $\mathcal{F}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ is $\xi_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$-regular w.r.t. $\mathcal{R}^{(k-1)}$, and

(L.5.1.d) the family $\{\widetilde{\mathcal{H}}_i^{(k)} : i \in I(\mathcal{O}^{(k)})\}$ partitions $\mathcal{F}^{(k)}$, $|I(\mathcal{O}^{(k)})| \leqslant s_{5.1}$, and each $\widetilde{\mathcal{H}}_i^{(k)}$ is $\big(\nu_{5.1}/12, *, (t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))^{2^k}\big)$-regular w.r.t. $\mathcal{R}^{(k-1)}$ for every $i \in I(\mathcal{O}^{(k)})$.

The verification of (L.5.1.a)–(L.5.1.d) is straightforward, but somewhat technical. We give the details below.

Due to (RAL.a), (5.7), (5.6), (RL.b) and the monotonicity of the function m_{aux}, we have

$$n \geqslant t_{\mathrm{RL}} \times m_{\mathrm{worst}} \geqslant a_1^{\mathscr{P}_{\mathrm{RL}}} \times m_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}).$$

In order to verify (L.5.1.a) it is left to show that $(t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))!$ divides $m = n/a_1^{\mathscr{P}_{\mathrm{RL}}}$. For this we note that, due to the definition of t_{RAL} in (5.7), we have $t_{\mathrm{RAL}} = t_{\mathrm{RL}} + t_{\mathrm{worst}}$, which due to (RAL.a) yields $(t_{\mathrm{RL}} + t_{\mathrm{worst}})!$ divides n. Consequently, $(t_{\mathrm{RL}})!(t_{\mathrm{worst}})!$ divides n (to see this consider $\binom{t_{\mathrm{RL}}+t_{\mathrm{worst}}}{t_{\mathrm{worst}}}$). Hence, from $a_1^{\mathscr{P}_{\mathrm{RL}}} \leqslant t_{\mathrm{RL}}$ (*cf.* (RL.b)) it follows that $n/a_1^{\mathscr{P}_{\mathrm{RL}}} = m$ is divisible by $(t_{\mathrm{worst}})!$. It now follows that $(t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))!$ divides m since $t_{\mathrm{worst}} \geqslant t_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ due to the monotonicity of the function t_{aux}.

Part (L.5.1.b) follows easily from (RL.b) and the choice of the function δ_{RL} in (5.5), ensuring that $\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}) \leqslant \delta_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$.

Next we verify (L.5.1.c). It follows from the definition of $\mathcal{F}^{(k)}$ that $\mathcal{R}^{(k-1)} = \hat{\mathcal{P}}^{(k-1)}$ underlies $\mathcal{F}^{(k)}$. The second assertion of (L.5.1.c) follows from $2t_{\mathrm{RL}}^{2^k}\mu_{\mathrm{RAL}} \leqslant \xi_{\mathrm{worst}} \leqslant \xi_{\mathrm{aux}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ (*cf.* (5.7) and (5.6)) and Claim 5.2.

Finally, it is left to verify (L.5.1.d). It follows from the definitions in (5.12) and (5.16) and the fact that $\mathscr{H}^{(k)}$ is a partition of $[V]^k$ (*cf.* (RAL.d)) that $\{\widetilde{\mathcal{H}}_i^{(k)} : i \in I(\mathcal{O}^{(k)})\}$ partitions $\mathcal{F}^{(k)}$. Clearly, $|I(\mathcal{O}^{(k)})| \leqslant s_{5.1}$. Moreover, from the assumption of this case ($\mathcal{R}^{(k-1)} = \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\cdot\mathrm{reg}}$) we know that $\widetilde{\mathcal{H}}_i^{(k)}$ is $(\delta_{k,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))$-regular w.r.t. $\mathcal{R}^{(k-1)} = \hat{\mathcal{P}}^{(k-1)}$ for every $i \in I(\mathcal{O}^{(k)})$. Therefore, (L.5.1.d) follows from the choice of $\delta_{k,\mathrm{RL}} \leqslant \nu_{5.1}/12$ in (5.3) and $r_{\mathrm{RL}}(a^{\mathscr{P}_{\mathrm{RL}}}) = (t_{\mathrm{aux}}(a^{\mathscr{P}_{\mathrm{RL}}}))^{2^k}$ in (5.4).

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Having verified (L.5.1.a)–(L.5.1.d), we are now able to apply Lemma 5.1 and infer the existence of a partition $\{\widetilde{\mathcal{G}}_i^{(k)}:\ i\in I(\mathcal{O}^{(k)})\}$ of $\mathcal{F}^{(k)}$ so that, for every $i\in I(\mathcal{O}^{(k)})$,

(L.5.1.i) $\widetilde{\mathcal{G}}_i^{(k)}$ is $\varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}=\mathcal{R}^{(k-1)}$, and

(L.5.1.ii) $|\widetilde{\mathcal{G}}_i^{(k)}\triangle\widetilde{\mathcal{H}}_i^{(k)}|\leqslant \nu_{5.1}|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|$.

For $i\in I(\mathcal{O}^{(k)})$ each $\widetilde{\mathcal{G}}_i^{(k)}$ given above defines $\mathcal{G}_i^{(k)}$ restricted to the polyad $\hat{\mathcal{P}}^{(k-1)}$. Formally we set

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})=\widetilde{\mathcal{G}}_i^{(k)}\quad\text{for } i\in I(\mathcal{O}^{(k)}),\tag{5.17}$$

and repeat the procedure for every $\mathcal{O}^{(k)}\in\widetilde{\mathscr{O}}^{(k)}$ satisfying (5.15). ◊

Case 2 ($\hat{\mathcal{P}}^{(k-1)}\notin\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}$)**.** Again, let $K\in\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ and set

$$\boldsymbol{\mathcal{P}}=\boldsymbol{\mathcal{P}}(K)=\{\hat{\mathcal{P}}^{(j)}(K)\}_{j=1}^{k-1}$$

with $\hat{\mathcal{P}}^{(k-1)}(K)=\hat{\mathcal{P}}^{(k-1)}$ (see (2.1)). Let $\mathcal{O}^{(k)}\in\widetilde{\mathscr{O}}^{(k)}$ be such that

$$\mathcal{F}^{(k)}=\mathcal{O}^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\neq\varnothing.\tag{5.18}$$

In this case fix some index $i_0\in I(\mathcal{O}^{(k)})$. We then define for $i\in I(\mathcal{O}^{(k)})$

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})=\begin{cases}\mathcal{F}^{(k)} & \text{for } i=i_0,\\ \varnothing & \text{for } i\neq i_0\in I(\mathcal{O}^{(k)}).\end{cases}\tag{5.19}$$

For later reference we note that for every $i\in I(\mathcal{O}^{(k)})$

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\subseteq\mathcal{O}^{(k)}\tag{5.20}$$

and

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\ \text{ is }\ \varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})\text{-regular w.r.t. }\ \hat{\mathcal{P}}^{(k-1)}.\tag{5.21}$$

Indeed, (5.20) is trivial for every $i\in I(\mathcal{O}^{(k)})$ and (5.21) is trivial for $i\neq i_0$. In the case $i=i_0$ we have $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})=\mathcal{F}^{(k)}=\mathcal{O}^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$, and (5.21) follows from Claim 5.2 and the choice of μ_{RAL} in (5.7), ensuring $2t_{\mathrm{RL}}^{2^k}\times\mu_{\mathrm{RAL}}\leqslant\varepsilon_{\mathrm{RAL}}(t_{\mathrm{RL}},\dots,t_{\mathrm{RL}})\leqslant\varepsilon_{\mathrm{RAL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$.

Again we repeat this procedure for every $\mathcal{O}^{(k)}\in\widetilde{\mathscr{O}}^{(k)}$ satisfying (5.18). ◊

We note that, due to both cases above, the following statement holds:

(*) For every $\hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$ and every $\mathcal{O}^{(k)}\in\widetilde{\mathscr{O}}^{(k)}$ satisfying $\mathcal{O}^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\neq\varnothing$, we have that $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}):\ i\in I(\mathcal{O}^{(k)})\}$ is a partition of $\mathcal{O}^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$.

Now we define the partition $\mathscr{G}^{(k)}$ and verify (RAL.G1)–(RAL.G3). For that we set for $i\in[s_{\mathrm{RAL}}]$

$$\mathcal{G}_i^{(k)}=\bigcup\Big\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}):\ \hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}\Big\}.\tag{5.22}$$

Since $\widetilde{\mathscr{O}}^{(k)}$ is a partition of $[V]^k$, we infer from (*) that $\mathscr{G}^{(k)}=\{\mathcal{G}_1^{(k)},\dots,\mathcal{G}_{s_{\mathrm{RAL}}}^{(k)}\}$ forms a partition of $[V]^k$.

Next we verify (RAL.G1). From (L.5.1.i) (combined with (5.17)) and (5.21) we conclude that, for all $i\in[s_{\mathrm{RAL}}]$ and all $\hat{\mathcal{P}}^{(k-1)}\in\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$, the defined $\mathcal{G}_i^{(k)}$ is $\varepsilon_{\mathrm{RAL}}(a^{\mathscr{P}_{\mathrm{RL}}})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$. Consequently, the definition of $\mathscr{P}_{\mathrm{RAL}}=\mathscr{P}_{\mathrm{RL}}$ and $\boldsymbol{a}^{\mathscr{P}_{\mathrm{RAL}}}=\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}$ in (5.9) yields (RAL.G1).

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In order to show (RAL.G2), let $i \in [s_{\mathrm{RAL}}]$ be fixed. It follows from (L.5.1.ii) that

$$\sum\Big\{|\big(\mathcal{G}_i^{(k)} \triangle \mathcal{H}_i^{(k)}\big) \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})| : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\cdot\mathrm{reg}}\Big\} \overset{(5.1)}{\leqslant} \frac{1}{2}\nu_{\mathrm{RAL}} n^k. \tag{5.23}$$

Moreover, from (RL.H) and Definition 11 we infer

$$\begin{aligned}\sum\Big\{|\big(\mathcal{G}_i^{(k)} \triangle \mathcal{H}_i^{(k)}\big) \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})| : \hat{\mathcal{P}}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\cdot\mathrm{reg}}\Big\}& \\ \leqslant \sum\Big\{|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})| : \hat{\mathcal{P}}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\cdot\mathrm{reg}}\Big\} \leqslant s_{\mathrm{RL}}\delta_{k,\mathrm{RL}} n^k \overset{(5.3)}{\leqslant} \frac{1}{2}\nu_{\mathrm{RAL}} n^k.&\end{aligned} \tag{5.24}$$

In view of (5.9) the inequalities (5.23) and (5.24) then yield (RAL.G2).

Finally, we consider (RAL.G3). For each $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$ we set

$$J(\mathcal{O}^{(k)}) = \{i \in [s_{\mathrm{RAL}}] : \mathcal{G}_i^{(k)} \cap \mathcal{O}^{(k)} \neq \varnothing\}.$$

Since (5.8) and $\mathscr{G}^{(k)}$ is a partition of $[V]^k$, the two assertions in (RAL.G3) are implied by the following two statements which we verify below:

$$J\big([V]^k \setminus \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})\big) \subseteq I\big([V]^k \setminus \mathrm{Cross}_k(\mathscr{O}^{(1)}_{\mathrm{RAL}})\big), \quad \text{and} \tag{5.25}$$

$$J(\mathcal{O}_1^{(k)}) \cap J(\mathcal{O}_2^{(k)}) = \varnothing \quad \text{for all } \mathcal{O}_1^{(k)} \neq \mathcal{O}_2^{(k)} \in \widetilde{\mathscr{O}}^{(k)}. \tag{5.26}$$

From (∗) we infer for every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$ that if $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \cap \mathcal{O}^{(k)} \neq \varnothing$ then $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \subseteq \mathcal{O}^{(k)}$. Consequently, (∗) yields

$$J(\mathcal{O}^{(k)}) \subseteq I(\mathcal{O}^{(k)}) \tag{5.27}$$

for every $\mathcal{O}^{(k)} \in \widetilde{\mathscr{O}}^{(k)}$, which gives (5.25).

Moreover, since $\mathscr{H}^{(k)} \prec \mathscr{O}^{(k)}$, and therefore $\mathscr{H}^{(k)} \prec \widetilde{\mathscr{O}}^{(k)}$ (see (5.11)), we have $I(\mathcal{O}_1^{(k)}) \cap I(\mathcal{O}_2^{(k)}) = \varnothing$ for all distinct $\mathcal{O}_1^{(k)}$ and $\mathcal{O}_2^{(k)}$ from $\widetilde{\mathscr{O}}^{(k)}$. Hence, (5.26) holds as well, and consequently (RAL.G3) follows.

From the discussion above and (5.10) we infer that $\mathscr{P}_{\mathrm{RAL}}$ defined in (5.9) and $\mathscr{G}^{(k)}$ defined in (5.22) satisfy the conclusions of RAL(k), Lemma 4.2, *i.e.*, (RAL.P1)–(RAL.G3). □

5.2. RL(k) implies Lemma 5.1

The proof of Lemma 5.1 is the heart of the implication RL(k) $\Longrightarrow$ RAL(k) and its idea resembles the main idea from the work of B. Nagle and the authors in [20]. Before we give the detailed proof below, we briefly discuss the main idea.

Recall that in Lemma 5.1 a $(\delta_{5.1}, \boldsymbol{d})$-regular $(m,k,k-1)$-complex $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ and a ξ-regular k-uniform hypergraph $\mathcal{F}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ are given. Moreover, we are given a partition $\mathscr{H}^{(k)} = \{\mathcal{H}_i^{(k)} : i \in [s_{5.1}]\}$ of $\mathcal{F}^{(k)}$, where every $\mathcal{H}_i^{(k)}$ is $(\nu, *, t_{5.1}^{2^k})$-regular w.r.t. $\mathcal{R}^{(k-1)}$. We will apply RL(k) to regularize every $\mathcal{H}_i^{(k)} \in \mathscr{H}^{(k)}$ with some appropriately chosen δ_k less than the given ε. For this regularization we apply the variant of RL(k) discussed in Remark 2, which allows us to find a t_{RL}-bounded family of partitions $\mathscr{P}_{\mathrm{RL}} = \mathscr{P}_{\mathrm{RL}}(k-1, \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}) = \{\mathscr{P}^{(j)}_{\mathrm{RL}}\}_{j=1}^{k-1}$ such that, for each $j = 1,\dots,k-1$ and each $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_{\mathrm{RL}}$, either $\mathcal{P}^{(j)} \subseteq \mathcal{R}^{(j)}$ or $\mathcal{P}^{(j)} \cap \mathcal{R}^{(j)} = \varnothing$. Since each $\mathcal{H}_i^{(k)} \subseteq \mathcal{F}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$, we will focus on the 'interesting' part of the partition $\mathscr{P}_{\mathrm{RL}}$ and consider only those polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$ which are subsets of $\mathcal{R}^{(k-1)}$. For that we

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set

$$\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}}) = \{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)} \colon \hat{\mathcal{P}}^{(k-1)} \subseteq \mathcal{R}^{(k-1)}\}.$$

From RL(k) we infer that, for every 'typical' $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}})$,

(i) $\mathcal{H}_i^{(k)}$ is $(\delta_k, d(\mathcal{H}_i^{(k)} \mid \hat{\mathcal{P}}^{(k-1)}), 1)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ for every $i \in [s_{5.1}]$.

Moreover, we will prove (*cf.* Claim 5.4) that for every $i \in [s_{5.1}]$ the typical density $d(\mathcal{H}_i^{(k)} \mid \hat{\mathcal{P}}^{(k-1)})$ will be 'near' to the density of $\mathcal{H}_i^{(k)}$ in $\mathcal{R}^{(k-1)}$, *i.e.*,

(ii) $|d(\mathcal{H}_i^{(k)} \mid \hat{\mathcal{P}}^{(k-1)}) - d(\mathcal{H}_i^{(k)} \mid \mathcal{R}^{(k-1)})| \leqslant \nu/6$ for 'most' $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}})$.

Property (ii) is the key observation in the proof of Lemma 5.1. Its proof is based on our choice of $t_{5.1} \geqslant t_{\mathrm{RL}}$ and $t_{\mathrm{RL}}^{2^k} \geqslant |\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}})|$. The proof of (ii) is then simple. Assuming that there is a constant fraction of polyads in $\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}})$ which violate (ii) gives rise to a witness that is $(\nu/12, *, t_{5.1}^{2^k})$-irregular w.r.t. $\mathcal{R}^{(k-1)}$. (The choice of $t_{5.1} \geqslant t_{\mathrm{RL}}$ allows us to 'look' into a constant fraction of polyads in $\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}})$.)

Combining (i) and (ii) with an appropriate use of the slicing lemma, Proposition 3.5, allows us to prove that, for a typical $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}})$, $\mathcal{H}_i^{(k)}$ needs to be altered only slightly (in less than $\nu/6$ proportion of the number of cliques in $\hat{\mathcal{P}}^{(k-1)}$) to become $(\varepsilon^2/4, d(\mathcal{H}_i^{(k)} \mid \mathcal{R}^{(k-1)}))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$. In other words, the resulting graph, which we will denote by $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$, maintains a large degree of regularity (we will choose $\delta_k \ll \varepsilon$), while its density will be $\sim d(\mathcal{H}_i^{(k)} \mid \mathcal{R}^{(k-1)})$.

On the other hand, in the rare case of an atypical polyad $\hat{\mathcal{P}}^{(k-1)}$ for which (i) or (ii) does not hold for $\mathcal{H}_i^{(k)}$ we use the slicing lemma to replace $\mathcal{H}_i^{(k)}$ by a randomly chosen (and therefore extremely regular) $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$, with $d(\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \mid \hat{\mathcal{P}}^{(k-1)}) \sim d(\mathcal{H}_i^{(k)} \mid \mathcal{R}^{(k-1)})$.

For each $i \in [s_{5.1}]$ we then set $\mathcal{G}_i^{(k)} = \bigcup \mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$, where the union is taken over all (typical and atypical) $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}})$. Since $\mathcal{G}_i^{(k)}$ obtained in that way is $(\varepsilon^2/4, d(\mathcal{H}_i^{(k)} \mid \mathcal{R}^{(k-1)}))$-regular for every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}})$, Proposition 3.2 then yields that $\mathcal{G}_i^{(k)}$ is $(\varepsilon, d(\mathcal{H}_i^{(k)} \mid \mathcal{R}^{(k-1)}))$-regular w.r.t. $\mathcal{R}^{(k-1)}$. Moreover, since in the typical case we changed $\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ only 'slightly' to become $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ and since the atypical case, in which more drastic changes are needed, happens rarely, we will be able to prove that $|\mathcal{G}_i^{(k)} \triangle \mathcal{H}_i^{(k)}| \leqslant \nu n^k$.

We now give the technical details of the proof of Lemma 5.1, sketched above.

Proof: RL(k) $\Longrightarrow$ Lemma 5.1**.** Let positive reals $s_{5.1}$, $\nu_{5.1}$, and $\varepsilon_{5.1}$ and a vector $\boldsymbol{d}_{5.1} = (d_2, \dots, d_{k-1})$ satisfying $1/d_i \in \mathbb{N}$ for $2 \leqslant i < k$ be given. Lemma 5.1 is trivial for $\nu_{5.1} > 1$. Moreover, without loss of generality we may assume that

$$\varepsilon_{5.1} < \nu_{5.1} \leqslant 1. \tag{5.28}$$

We will apply RL(k). For that we set[1]

$$o_{\mathrm{RL}} = \max_{2 \leqslant i < k} 1/d_i, \qquad s_{\mathrm{RL}} = s_{5.1} + 1, \qquad \eta_{\mathrm{RL}} = 10^{-2}, \tag{5.29}$$

$$\text{and} \quad \delta_{k,\mathrm{RL}} = \min\left\{ \frac{\nu_{5.1} \prod_{h=2}^{k-1} d_h^{(k)}}{6 \times s_{5.1} \times k^k}, \frac{\varepsilon_{5.1}^2}{384 s_{5.1}}, \frac{\nu_{5.1}}{18} \right\}, \tag{5.30}$$

[1] Since we are later only interested in partition classes $\mathcal{P}^{(j)}$ that are sub-hypergraphs of the given $\mathcal{R}^{(j)}$ (see, *e.g.*, (5.36)), the constant η_{RL} is not essential for our proof and any positive constant value would do.

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and consider functions $r_{\mathrm{RL}}\colon \mathbb{N}^{k-1}\to\mathbb{N}$, and $\delta_{\mathrm{RL}}\colon \mathbb{N}^{k-1}\to(0,1]$ defined for every $\boldsymbol{a}=(a_1,\dots,a_{k-1})\in\mathbb{N}^{k-1}$ by

$$r_{\mathrm{RL}}(\boldsymbol{a})=1 \quad\text{and}\quad \delta_{\mathrm{RL}}(\boldsymbol{a})=\varepsilon_{\mathrm{DCL}}\Big(h=k-1,\ell=k,\gamma=\tfrac{\nu_{5.1}}{48},d_0=\min_{2\leqslant i<k}a_i^{-1}\Big), \tag{5.31}$$

where $\varepsilon_{\mathrm{DCL}}(h,\ell,\gamma,d_0)$ is given by Theorem 3.1.

Having defined all input parameters of RL(k), Lemma 4.1, in (5.29), (5.30), and (5.31), Lemma 4.1 now yields positive constants μ_{RL}, t_{RL}, and n_{RL}. We then define $\delta_{5.1}$, $\xi_{5.1}$, $t_{5.1}$, and $m_{5.1}$ promised by Lemma 5.1. For that we set $t_{5.1}=t_{\mathrm{RL}}$,

$$\delta_{5.1}=\min\Big\{\tfrac{\mu_{\mathrm{RL}}}{3},\varepsilon_{\mathrm{DCL}}\Big(h=k-1,\ell=k,\gamma=\tfrac12,d_0=\min_{2\leqslant i<k}d_i\Big)\Big\},\quad \xi_{5.1}=\frac{\varepsilon_{5.1}^2}{192t_{\mathrm{RL}}^{2^k}}, \tag{5.32}$$

and let $m_{5.1}$ be sufficiently large.

Having defined all the parameters of Lemma 5.1, now let m, $\boldsymbol{\mathcal{R}}$, $\mathcal{F}^{(k)}$, and $\mathscr{H}^{(k)}$ satisfying (a)–(d) of Lemma 5.1 for these parameters be given, *i.e.*,

(L.5.1.a) $m\geqslant m_{5.1}$ and $(t_{5.1})!$ divides m,

(L.5.1.b) $\boldsymbol{\mathcal{R}}=\{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\delta_{5.1},\boldsymbol{d}_{5.1})$-regular $(m,k,k-1)$-complex with vertex set $V=V_1\cup\cdots\cup V_k$,

(L.5.1.c) $\mathcal{F}^{(k)}\subseteq\mathcal{K}_k(\mathcal{R}^{(k-1)})$ is $\xi_{5.1}$-regular w.r.t. $\mathcal{R}^{(k-1)}$, and

(L.5.1.d) $\mathscr{H}^{(k)}=\{\mathcal{H}_1^{(k)},\dots,\mathcal{H}_{s_{5.1}}^{(k)}\}$ is a partition of $\mathcal{F}^{(k)}$ and every $\mathcal{H}_i^{(k)}$ is $\big(\nu_{5.1}/12,*,t_{5.1}^{2^k}\big)$-regular w.r.t. $\mathcal{R}^{(k-1)}$ for $i\in[s_{5.1}]$.

We have to ensure the existence of a partition $\mathscr{G}^{(k)}=\{\mathcal{G}_1^{(k)},\dots,\mathcal{G}_{s_{5.1}}^{(k)}\}$ of $\mathcal{F}^{(k)}$ so that, for every $i\in[s_{5.1}]$,

(L.5.1.i) $\mathcal{G}_i^{(k)}$ is $\varepsilon_{5.1}$-regular w.r.t. $\mathcal{R}^{(k-1)}$, and

(L.5.1.ii) $|\mathcal{G}_i^{(k)}\triangle\mathcal{H}_i^{(k)}|\leqslant\nu_{5.1}|\mathcal{K}_k(\mathcal{R}^{(k-1)})|$.

Before we start, we note for later use that, due to (L.5.1.b) and the choice of $\delta_{5.1}\leqslant\varepsilon_{\mathrm{DCL}}(h=k-1,\ell=k,\gamma=1/2,d_0=\min_{2\leqslant i<k}d_i)$ in (5.32), we infer for sufficiently large m by DCL, Theorem 3.1, that

$$|\mathcal{K}_k(\mathcal{R}^{(k-1)})|=\left(1\pm\frac12\right)\prod_{h=2}^{k-1}d_h^{\binom{k}{h}}\times m^k. \tag{5.33}$$

Our proof is based on the variant of RL(k), Lemma 4.1, discussed in Remark 2. More precisely, we want to apply this variant of Lemma 4.1 with the constants and functions chosen in (5.29), (5.30), and (5.31) to V, $\boldsymbol{\mathcal{R}}$, and $\mathcal{H}_0^{(k)}\cup\{\mathcal{H}_1^{(k)},\dots,\mathcal{H}_{s_{5.1}}^{(k)}\}$, where

$$\mathcal{H}_0^{(k)}=[V]^k\setminus\mathcal{F}^{(k)}=[V]^k\setminus\bigcup_{i\in[s_{5.1}]}\mathcal{H}_i^{(k)}. \tag{5.34}$$

We artificially add $\mathcal{H}_0^{(k)}$ only to obtain a partition of $[V]^k$, to formally match the assumption (c) of RL(k) (see (RL.c) below). We have to verify the following assumptions of Lemma 4.1 (see also Remark 2).

(RL.a′) $|V|=km\geqslant n_{\mathrm{RL}}$, $V=V_1\cup\cdots\cup V_k$ with $V_i=m$ and $t_{\mathrm{RL}}!$ divides m,

(RL.b′) $\boldsymbol{\mathcal{R}}=\{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\mu_{\mathrm{RL}}/3,\boldsymbol{d}_{5.1})$-regular $(m,k,k-1)$-complex, where $\boldsymbol{d}_{5.1}=(d_2,\dots,d_{k-1})$, $1/d_i\in\mathbb{N}$ and $1/d_i\leqslant o_{\mathrm{RL}}$ for $2\leqslant i<k$, and $\mathcal{R}^{(1)}=V_1\cup\cdots\cup V_k$, and

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(RL.c) $\{\mathcal{H}_0^{(k)}, \mathcal{H}_1^{(k)}, \dots, \mathcal{H}_{s_{5.1}}^{(k)}\}$ is a partition of $[V]^k$ into s_{RL} parts.

Note that (RL.a′) follows from (L.5.1.a) and the choice of $t_{5.1}$ in (5.32) for sufficiently large m. Moreover, (RL.b′) is a consequence of the assumption on $\boldsymbol{d}_{5.1}$, and (L.5.1.b) combined with the choice of $\delta_{5.1}$ in (5.32) and o_{RL} in (5.29). Similarly, (RL.c) follows from (L.5.1.d) in conjunction with (5.34) and the choice of s_{RL} in (5.29).

Having verified that (RL.a′), (RL.b′), and (RL.c) hold, Lemma 4.1 then ensures the existence of a family of partitions $\mathscr{P}_{\mathrm{RL}} = \mathscr{P}_{\mathrm{RL}}(k-1, \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$ on V which satisfies the following properties:

(RL.P1) $\mathscr{P}_{\mathrm{RL}}$ is $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), \boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}})$-equitable and t_{RL}-bounded,
(RL.P2′) $\mathscr{P}^{(1)} \prec \mathcal{R}^{(1)} = V_1 \cup \cdots \cup V_k$, and for every $2 \leqslant j < k$ and every $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}$ we have either $\mathcal{P}^{(j)} \subseteq \mathcal{R}^{(j)}$ or $\mathcal{P}^{(j)} \cap \mathcal{R}^{(j)} = \varnothing$, and
(RL.H) $\mathcal{H}_i^{(k)}$ is $(\delta_{k,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))$-regular w.r.t. $\mathscr{P}_{\mathrm{RL}}$ for every $i \in [s_{\mathrm{RL}}]$.

Before we continue with the proof we make a few observations and develop some notation. To an arbitrary polyad $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}$ consider its corresponding $(m/a_1^{\mathscr{P}_{\mathrm{RL}}}, k, k-1)$-complex $\boldsymbol{\mathcal{P}} = \{\hat{\mathcal{P}}^{(j)}\}_{j=1}^{k-1}$. (More precisely, recalling (2.1), $\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}(K) = \{\hat{\mathcal{P}}^{(j)}(K)\}_{j=1}^{k-1}$ for any $K \in \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$). Due to (RL.P1) and Definition 9(c), $\boldsymbol{\mathcal{P}}$ is an $(\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}), (1/a_2^{\mathscr{P}_{\mathrm{RL}}}, \dots, 1/a_{k-1}^{\mathscr{P}_{\mathrm{RL}}})$-regular $(m/a_1^{\mathscr{P}_{\mathrm{RL}}}, k, k-1)$-complex. From the choice of the function δ_{RL} in (5.31) we infer, for sufficiently large m, by Theorem 3.1, that

$$|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})| = \left(1 \pm \frac{v_{5.1}}{48}\right) \prod_{h=2}^{k-1} \left(\frac{1}{a_h^{\mathscr{P}_{\mathrm{RL}}}}\right)^{\binom{k}{h}} \times \left(\frac{m}{a_1^{\mathscr{P}_{\mathrm{RL}}}}\right)^k \tag{5.35}$$

holds for every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}$.

Since each $\mathcal{H}_i^{(k)} \subseteq \mathcal{F}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$, for the rest of the proof we will focus on the 'interesting' part of the partition $\mathscr{P}_{\mathrm{RL}}$ and consider only those polyads which are sub-hypergraphs of $\mathcal{R}^{(k-1)}$. To this end we set

$$\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}}) = \left\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)} \colon \hat{\mathcal{P}}^{(k-1)} \subseteq \mathcal{R}^{(k-1)}\right\}. \tag{5.36}$$

Note that, due to (RL.P2′) and the properties of $\mathscr{P}_{\mathrm{RL}}$, we have that

$$\left\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \colon \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}})\right\} \text{ partitions } \mathcal{K}_k(\mathcal{R}^{(k-1)}). \tag{5.37}$$

To simplify the notation we set

$$d_{\mathscr{H},\mathcal{R}}(i) = d(\mathcal{H}_i^{(k)} \mid \mathcal{R}^{(k-1)}).$$

The following claim, Claim 5.3, ensures the existence of a partition $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \colon i \in [s_{5.1}]\}$ of $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ for every polyad $\hat{\mathcal{P}}^{(k-1)}$ in $\hat{\mathscr{P}}_{\mathrm{RL}}^{(k-1)}(\boldsymbol{\mathcal{R}})$ with the property that $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\varepsilon_{5.1}^2/4, d_{\mathscr{H},\mathcal{R}}(i))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ for each $i \in [s_{5.1}]$. This property will enable us to use Proposition 3.4 to infer property (L.5.1.i) for $\mathcal{G}_i^{(k)}$ defined in the obvious way.

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To verify (L.5.1.ii) we need some additional information concerning the $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}):\ i \in [s_{5.1}]\}$. Here our analysis splits into two cases and we define[2]

$$\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\cdot\mathrm{reg}}(\boldsymbol{\mathcal{R}}) = \Big\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}}):\ \mathcal{H}_i^{(k)} \text{ is } (\delta_{k,\mathrm{RL}},*,r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}_{\mathrm{RL}}}))\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)} \text{ for every } i \in [s_{5.1}]\Big\}. \quad (5.38)$$

Below we present two claims from which we will be able to ensure the existence of $\{\mathcal{G}_i^{(k)}:\ i \in [s_{5.1}]\}$ with the desired properties (L.5.1.i) and (L.5.1.ii). We then give the proofs of the claims.

Claim 5.3. *For every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$ there exist a partition $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}):\ i \in [s_{5.1}]\}$ of $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ such that, for every $i \in [s_{5.1}]$,*

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \text{ is } (\varepsilon_{5.1}^2/4, d_{\mathscr{H},\mathcal{R}}(i))\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}. \quad (5.39)$$

Moreover, if $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\cdot\mathrm{reg}}(\boldsymbol{\mathcal{R}})$, then the partition $\{\mathcal{G}_i^{(k)}:\ i \in [s_{5.1}]\}$ has the additional property that, for every $i \in [s_{5.1}]$,

$$|\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \triangle \big(\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)| \leqslant \big(|d_{\mathscr{H},\mathcal{R}}(i) - d(\mathcal{H}_i^{(k)} \mid \hat{\mathcal{P}}^{(k-1)})| + \tfrac{v_{5.1}}{6}\big)|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|. \quad (5.40)$$

In order to verify (L.5.1.ii) we need further control over the quantity considered in (5.40). The following claim ensures that 'typically' $|d_{\mathscr{H},\mathcal{R}}(i) - d(\mathcal{H}_i^{(k)} \mid \hat{\mathcal{P}}^{(k-1)})| \leqslant \frac{v_{5.1}}{6}$. For that we define, for every $i \in [s_{5.1}]$,

$$\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)}) = \{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}}):\ |d_{\mathscr{H},\mathcal{R}}(i) - d(\mathcal{H}_i^{(k)} \mid \hat{\mathcal{P}}^{(k-1)})| > \tfrac{v_{5.1}}{6}\}. \quad (5.41)$$

Claim 5.4. *For every $i \in [s_{5.1}]$,*

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\Big\}\Big| \leqslant \frac{v_{5.1}}{3}|\mathcal{K}_k(\mathcal{R}^{(k-1)})|. \quad (5.42)$$

We now finish the proof of Lemma 5.1 based on Claim 5.3 and Claim 5.4. We use Claim 5.3 and set $\mathcal{G}_i^{(k)}$ for every $i \in [s_{5.1}]$ equal to

$$\mathcal{G}_i^{(k)} = \bigcup\Big\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}):\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})\Big\}. \quad (5.43)$$

From $\mathcal{F}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ (*cf.* (L.5.1.c)) combined with (5.37) and Claim 5.3, we infer that $\mathscr{G}^{(k)} = \{\mathcal{G}_i^{(k)}:\ i \in [s_{5.1}]\}$ defined in (5.43) is a partition of $\mathcal{F}^{(k)}$.

Now, we have to verify (L.5.1.i) and (L.5.1.ii) for every fixed $i \in [s_{5.1}]$ and this choice of $\mathscr{G}^{(k)}$. So let $i \in [s_{5.1}]$ be fixed.

[2] Note that we exclude the artificially added hypergraph $\mathcal{H}_0^{(k)}$ in the definition of $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\cdot\mathrm{reg}}(\boldsymbol{\mathcal{R}})$.

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We start with property (L.5.1.i). Due to (5.37) the two families $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$ and

$$\{\mathcal{G}^{(k)}_i(\hat{\mathcal{P}}^{(k-1)})\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})\}$$

satisfy Set-up 3.3 for $j = k$, and $t = |\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})|$. Consequently, in view of (5.39) we can apply Proposition 3.4 with $r = 1$, $\delta = \varepsilon^2_{5.1}/4$, and $d = d_{\mathscr{H},\mathcal{R}}(i)$, to infer

$$\mathcal{G}^{(k)}_i \ \text{ is } \ \big(\varepsilon_{5.1}, d_{\mathscr{H},\mathcal{R}}(i)\big)\text{-regular w.r.t. } \bigcup\Big\{\hat{\mathcal{P}}^{(k-1)}\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})\Big\} = \mathcal{R}^{(k-1)},$$

and therefore (L.5.1.i) holds.

We now focus on (L.5.1.ii) for a fixed $i \in [s_{5.1}]$. We will estimate $|\mathcal{G}^{(k)}_i \triangle\, \mathcal{H}^{(k)}_i|$ as the sum of the symmetric difference taken over all polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$. In this sum we distinguish among polyads in which some $\mathcal{H}^{(k)} \in \mathscr{H}^{(k)}$ is 'irregular', in which $\mathcal{H}^{(k)}_i$ has 'bad' (atypical) density and the remaining 'typical' polyads in which $\mathcal{H}^{(k)}_i$ has the correct density and every $\mathcal{H}^{(k)} \in \mathscr{H}^{(k)}$ is regular. With this in mind we set

$$\begin{aligned}
\mathfrak{D}_{\mathrm{irreg}}(i) &= \sum\Big\{|\mathcal{G}^{(k)}_i(\hat{\mathcal{P}}^{(k-1)}) \triangle \big(\mathcal{H}^{(k)}_i \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)| : \\
&\qquad\qquad \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}}) \setminus \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}})\Big\},\\
\mathfrak{D}_{\mathrm{typ}}(i) &= \sum\Big\{|\mathcal{G}^{(k)}_i(\hat{\mathcal{P}}^{(k-1)}) \triangle \big(\mathcal{H}^{(k)}_i \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)| : \\
&\qquad\qquad \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}}) \setminus \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}}, \mathcal{H}^{(k)}_i)\Big\},\\
\mathfrak{D}_{\mathrm{bad}}(i) &= \sum\Big\{|\mathcal{G}^{(k)}_i(\hat{\mathcal{P}}^{(k-1)}) \triangle \big(\mathcal{H}^{(k)}_i \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)| : \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}}, \mathcal{H}^{(k)}_i)\Big\},
\end{aligned}$$

and note that

$$|\mathcal{G}^{(k)}_i \triangle\, \mathcal{H}^{(k)}_i| \leqslant \mathfrak{D}_{\mathrm{irreg}}(i) + \mathfrak{D}_{\mathrm{typ}}(i) + \mathfrak{D}_{\mathrm{bad}}(i). \tag{5.44}$$

In the following we bound each of the terms of (5.44) separately. We start with $\mathfrak{D}_{\mathrm{irreg}}(i)$. Due to (RL.H) and the definition of $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}})$ in (5.38) we have

$$\sum\Big\{|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}}) \setminus \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}})\Big\} \leqslant s_{5.1} \times \delta_{k,\mathrm{RL}} k^k m^k.$$

Clearly, the left-hand side of the last inequality is an upper bound on $\mathfrak{D}_{\mathrm{irreg}}(i)$ and we infer

$$\mathfrak{D}_{\mathrm{irreg}}(i) \leqslant s_{5.1}\delta_{k,\mathrm{RL}} k^k m^k \overset{(5.30)}{\leqslant} \frac{\nu_{5.1}}{6}\prod_{h=2}^{k-1} d^{\binom{k}{h}}_h \times m^k \overset{(5.33)}{\leqslant} \frac{\nu_{5.1}}{3}|\mathcal{K}_k(\mathcal{R}^{(k-1)})|. \tag{5.45}$$

Next we consider $\mathfrak{D}_{\mathrm{typ}}(i)$. Since, in view of (5.41), for each $\hat{\mathcal{P}}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}}, \mathcal{H}^{(k)}_i)$ we have $|d_{\mathscr{H},\mathcal{R}}(i) - d(\mathcal{H}^{(k)}_i \mid \hat{\mathcal{P}}^{(k-1)})| \leqslant \nu_{5.1}/6$, we infer from (5.40) that for every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}} \setminus \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}}, \mathcal{H}^{(k)}_i)$

$$|\mathcal{G}^{(k)}_i(\hat{\mathcal{P}}^{(k-1)}) \triangle \big(\mathcal{H}^{(k)}_i \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)| \leqslant \frac{\nu_{5.1}}{3}|\mathcal{K}_k(\mathcal{R}^{(k-1)})|.$$

Consequently, directly from the definition of $\mathfrak{D}_{\mathrm{typ}}(i)$, we infer

$$\begin{aligned}
\mathfrak{D}_{\mathrm{typ}}(i) &\leqslant \frac{\nu_{5.1}}{3}\sum\Big\{|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\text{-reg}}(\boldsymbol{\mathcal{R}}) \setminus \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}}, \mathcal{H}^{(k)}_i)\Big\}\\
&\leqslant \frac{\nu_{5.1}}{3}|\mathcal{K}_k(\mathcal{R}^{(k-1)})|.
\end{aligned} \tag{5.46}$$

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Finally, we derive the same bound for $\mathfrak{D}_{\mathrm{bad}}(i)$ directly from the definition of $\mathfrak{D}_{\mathrm{bad}}(i)$ and (5.42):

$$\mathfrak{D}_{\mathrm{bad}}(i) \leqslant \frac{v_{5.1}}{3}|\mathcal{K}_k(\mathcal{R}^{(k-1)})|. \tag{5.47}$$

We now conclude (L.5.1.ii) from (5.45), (5.46), and (5.47), applied to (5.44). In order to complete the proof of Lemma 5.1 we still have to verify Claim 5.3 and Claim 5.4, which we will do below. □

Proof of Claim 5.3. Let $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})$ be fixed. First we recall (5.35). Below we will apply the slicing lemma, Proposition 3.5, to sub-hypergraphs of $\hat{\mathcal{P}}^{(k-1)}$. For this, among others, we have to verify the assumption (i) of Proposition 3.5, *i.e.*,

$$|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})| \geqslant m^k/\ln m. \tag{5.48}$$

This, however, follows from (5.35) for sufficiently large m. Therefore, we do not have to verify this condition in future applications of the slicing lemma. We begin with the following consequence of the choice of $\xi_{5.1} \leqslant \varepsilon_{5.1}^2/(192t_{\mathrm{RL}}^{2^k})$ in (5.32):

$$\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \text{ is } \big(\varepsilon_{5.1}^2/96, d(\mathcal{F}^{(k)} \mid \mathcal{R}^{(k-1)})\big)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}. \tag{5.49}$$

The proof of Claim 5.3 splits into two main cases.

Case 1 ($d(\mathcal{F}^{(k)} \mid \mathcal{R}^{(k-1)}) > \varepsilon_{5.1}^2/16$). In this case we will treat 'thin' hypergraphs $\mathcal{H}_i^{(k)}$ w.r.t. $\mathcal{R}^{(k-1)}$ somewhat differently. To this end we set

$$R_{\mathrm{THIN}} = \left\{ i \in [s_{5.1}] : \, d_{\mathscr{H},\mathcal{R}}(i) < \frac{\varepsilon_{5.1}^2}{192 s_{5.1}} \right\}. \tag{5.50}$$

Due to the definition of R_{THIN} and the assumption of Case 1 we have

$$[s_{5.1}] \setminus R_{\mathrm{THIN}} \neq \varnothing. \tag{5.51}$$

We distinguish two sub-cases of Case 1 depending on whether $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\cdot\mathrm{reg}}(\boldsymbol{\mathcal{R}})$.

Case 1.1 ($\hat{\mathcal{P}}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\cdot\mathrm{reg}}(\boldsymbol{\mathcal{R}})$). In this particular case it suffices to prove the existence of a partition $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) : \, i \in [s_{5.1}]\}$ of $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ which satisfies (5.39) only. For this we will simply use the slicing lemma, Proposition 3.5, to decompose $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ into hypergraphs with the appropriate densities (as required for (5.39)). More precisely, we apply Proposition 3.5, with

$$j = k, \quad s_0 = s_{5.1}, \quad r = 1, \quad \delta_0 = \frac{\varepsilon_{5.1}^2}{96}, \quad \varrho_0 = \frac{\varepsilon_{5.1}^2}{16}, \quad \text{and} \quad p_0 = \frac{\varepsilon_{5.1}^2}{192 s_{5.1}},$$
$$\text{to } \hat{\mathcal{P}}^{(k-1)}, \quad \text{and} \quad \mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \quad \text{with } s = |[s_{5.1}] \setminus R_{\mathrm{THIN}}|,$$
$$\delta = \frac{\varepsilon_{5.1}^2}{96}, \quad \varrho = d(\mathcal{F}^{(k)} \mid \mathcal{R}^{(k-1)}), \quad \text{and} \quad \left\{ p_i = \frac{d_{\mathscr{H},\mathcal{R}}(i)}{d(\mathcal{F}^{(k)} \mid \mathcal{R}^{(k-1)})} : \, i \in [s_{5.1}] \setminus R_{\mathrm{THIN}} \right\}.$$

The conditions of Proposition 3.5 are immediate consequences of (5.48)–(5.50), and the assumption of Case 1. Proposition 3.5 yields a family $\mathcal{T}_0^{(k)} \cup \{\mathcal{T}_i^{(k)} : \, i \in [s_{5.1}] \setminus R_{\mathrm{THIN}}\}$ satisfying the

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following properties:

$$\mathcal{T}_0^{(k)} \cup \{\mathcal{T}_i^{(k)} : i \in [s_{5.1}] \setminus R_{\mathrm{THIN}}\} \text{ partitions } \mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}), \tag{5.52}$$

$$\mathcal{T}_i^{(k)} \text{ is } (\varepsilon_{5.1}^2/32, d_{\mathscr{H},\mathcal{R}}(i))\text{-regular w.r.t } \hat{\mathcal{P}}^{(k-1)} \text{ for } i \in [s_{5.1}] \setminus R_{\mathrm{THIN}}, \tag{5.53}$$

$$\mathcal{T}_0^{(k)} \text{ is } (\varepsilon_{5.1}^2/32, d_{\mathcal{T}_0^{(k)}})\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}, \quad \text{where} \tag{5.54}$$

$$d_{\mathcal{T}_0^{(k)}} = d(\mathcal{F}^{(k)} \mid \mathcal{R}^{(k-1)}) - \sum\{d_{\mathscr{H},\mathcal{R}}(i) : i \in [s_{5.1}] \setminus R_{\mathrm{THIN}}\} \overset{(5.50)}{\leqslant} \frac{\varepsilon_{5.1}^2}{192}. \tag{5.55}$$

Fix some $i_0 \in [s_{5.1}] \setminus R_{\mathrm{THIN}}$ (due to (5.51) such an i_0 exists). We then define the family $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ for $i \in [s_{5.1}]$ as follows:

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \begin{cases} \varnothing & \text{if } i \in R_{\mathrm{THIN}}, \\ \mathcal{T}_i^{(k)} \cup \mathcal{T}_0^{(k)} & \text{if } i = i_0, \\ \mathcal{T}_i^{(k)} & \text{otherwise.} \end{cases}$$

From (5.52) we infer that $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \in [s_{5.1}]\}$ defined in this way forms a partition of $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$, and it is left to verify (5.39) for every $i \in [s_{5.1}]$.

First, let $i \in R_{\mathrm{THIN}}$. From the definition of $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \varnothing$ we infer that $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\varepsilon', 0)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ for all $\varepsilon' > 0$. Since $i \in R_{\mathrm{THIN}}$, $d_{\mathscr{H},\mathcal{R}}(i) < \varepsilon_{5.1}^2/4$, and consequently the $(\varepsilon', 0)$-regularity for every $\varepsilon' > 0$ implies that $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\varepsilon_{5.1}^2/4, d_{\mathscr{H},\mathcal{R}}(i))$-regular (*i.e.*, (5.39) for $i \in R_{\mathrm{THIN}}$).

If $i \in [s_{5.1}] \setminus R_{\mathrm{THIN}}$ and $i \neq i_0$, then (5.53) and the definition $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \mathcal{T}_i^{(k)}$ immediately implies (5.39).

It is left to verify (5.39) for $i = i_0$. In that case Proposition 3.2 applied to $\mathcal{T}_{i_0}^{(k)}$ and $\mathcal{T}_0^{(k)}$ implies by (5.53) and (5.54) that $\mathcal{G}_{i_0}^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\varepsilon_{5.1}^2/16)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$, with density between $d_{\mathscr{H},\mathcal{R}}(i_0)$ and $d_{\mathscr{H},\mathcal{R}}(i_0) + \varepsilon_{5.1}^2/192$ (*cf.* (5.55)). Consequently, $\mathcal{G}_{i_0}^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $((\varepsilon_{5.1}^2/16 + \varepsilon_{5.1}^2/192), d_{\mathscr{H},\mathcal{R}}(i_0))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$, which yields (5.39).

Having verified (5.39) for every $i \in [s_{5.1}]$, we conclude Case 1.1. ◊

Case 1.2 ($\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL},\mathscr{H}\cdot\mathrm{reg}}(\mathcal{R})$). In this case we have to guarantee the existence of a partition of $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ which satisfies both (5.39) and (5.40) of Claim 5.3. Due to (5.40) we have to be more careful in defining the desired partition. On the other hand, the assumption in this case says that $\mathcal{H}_i^{(k)}$ is $\delta_{k,\mathrm{RL}}$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ for every $i \in [s_{5.1}]$. This allows us to apply the slicing lemma, to any $\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$.

Below we give a short outline of how we use this additional assumption. To simplify the notation, we set for every $i \in [s_{5.1}]$

$$d_{\mathscr{H},\hat{\mathcal{P}}}(i) = d(\mathcal{H}_i^{(k)} \mid \hat{\mathcal{P}}^{(k-1)}).$$

We first consider the hypergraphs $\mathcal{H}_i^{(k)}$ which are too 'fat' in $\hat{\mathcal{P}}^{(k-1)}$, *i.e.*, we consider

$$I_{\mathrm{FAT}}(\hat{\mathcal{P}}) = \left\{ i \in [s_{5.1}] \setminus R_{\mathrm{THIN}} : d_{\mathscr{H},\hat{\mathcal{P}}}(i) > d_{\mathscr{H},\mathcal{R}}(i) + \frac{\varepsilon_{5.1}^2}{192 s_{5.1}} \right\}. \tag{5.56}$$

We apply the slicing lemma to split each $\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ for $i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$ into a 'main' part $\mathcal{M}_i^{(k)}$ of density $d_{\mathscr{H},\mathcal{R}}(i)$ and a 'left-over' $\mathcal{L}_i^{(k)}$. The $\mathcal{M}_i^{(k)}$ will be used to define $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$.

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Furthermore, since each $\mathcal{L}_i^{(k)}$ is regular, and since each $\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ for $i \in R_{\mathrm{THIN}}$ is regular, as well (by the assumption of the case), we will infer that their union

$$\mathcal{U}^{(k)} = \bigcup_{i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}})} \mathcal{L}_i^{(k)} \cup \bigcup_{i \in R_{\mathrm{THIN}}} (\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$$

is regular with density 'very close' to

$$\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}}) = \sum \{ d_{\mathscr{H},\mathcal{R}}(i) - d_{\mathscr{H},\hat{\mathcal{P}}}(i) : \ i \in I_{\mathrm{SLIM}}(\hat{\mathcal{P}}) \}, \tag{5.57}$$

where

$$I_{\mathrm{SLIM}}(\hat{\mathcal{P}}) = \left\{ i \in [s_{5.1}] \setminus R_{\mathrm{THIN}} : \ d_{\mathscr{H},\hat{\mathcal{P}}}(i) < d_{\mathscr{H},\mathcal{R}}(i) - \frac{\varepsilon_{5.1}^2}{192 s_{5.1}} \right\}. \tag{5.58}$$

We then apply the slicing lemma again, this time to $\mathcal{U}^{(k)}$, to split it into regular pieces of densities $\{ d_{\mathscr{H},\mathcal{R}}(i) - d_{\mathscr{H},\hat{\mathcal{P}}}(i) : \ i \in I_{\mathrm{SLIM}}(\hat{\mathcal{P}}) \}$. For $i \in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$ uniting $\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ with the appropriate slice from $\mathcal{U}^{(k)}$ then gives rise to the desired partition. We now implement the technical details of this plan.

Let $I_{\mathrm{FAT}}(\hat{\mathcal{P}})$ and $I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$ be defined as in (5.56) and (5.58). We set

$$I_{\mathrm{OK}}(\hat{\mathcal{P}}) = \left\{ i \in [s_{5.1}] \setminus R_{\mathrm{THIN}} : \ d_{\mathscr{H},\hat{\mathcal{P}}}(i) = d_{\mathscr{H},\mathcal{R}}(i) \pm \frac{\varepsilon_{5.1}^2}{192 s_{5.1}} \right\} \tag{5.59}$$

and note that $[s_{5.1}]$ is the disjoint union of $I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, $I_{\mathrm{OK}}(\hat{\mathcal{P}})$, $I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$, and R_{THIN}. We will later need the following observation:

$$\begin{aligned}
&\left| \left(\sum_{i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}})} \big(d_{\mathscr{H},\hat{\mathcal{P}}}(i) - d_{\mathscr{H},\mathcal{R}}(i) \big) + \sum_{i \in R_{\mathrm{THIN}}} d_{\mathscr{H},\hat{\mathcal{P}}}(i) \right) - \sum_{i \in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})} \big(d_{\mathscr{H},\mathcal{R}}(i) - d_{\mathscr{H},\hat{\mathcal{P}}}(i) \big) \right| \\
&\quad = \Big| \sum \{ d_{\mathscr{H},\hat{\mathcal{P}}}(i) : \ i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}}) \cup R_{\mathrm{THIN}} \cup I_{\mathrm{SLIM}}(\hat{\mathcal{P}}) \} \\
&\qquad\qquad - \sum \{ d_{\mathscr{H},\mathcal{R}}(i) : \ i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}}) \cup I_{\mathrm{SLIM}}(\hat{\mathcal{P}}) \} \Big| \\
&\quad = \Big| \sum \{ d_{\mathscr{H},\hat{\mathcal{P}}}(i) : \ i \in [s_{5.1}] \setminus I_{\mathrm{OK}}(\hat{\mathcal{P}}) \} \\
&\qquad\qquad - \sum \{ d_{\mathscr{H},\mathcal{R}}(i) : \ i \in [s_{5.1}] \setminus \big(I_{\mathrm{OK}}(\hat{\mathcal{P}}) \cup R_{\mathrm{THIN}} \big) \} \Big| \\
&\quad \leqslant |d(\mathcal{F}^{(k)} \mid \hat{\mathcal{P}}^{(k-1)}) - d(\mathcal{F}^{(k)} \mid \mathcal{R}^{(k-1)})| \\
&\qquad\qquad + \sum_{i \in I_{\mathrm{OK}}(\hat{\mathcal{P}})} |d_{\mathscr{H},\mathcal{R}}(i) - d_{\mathscr{H},\hat{\mathcal{P}}}(i)| + \sum_{i \in R_{\mathrm{THIN}}} d_{\mathscr{H},\mathcal{R}}(i).
\end{aligned}$$

Thus, in view of (5.57) and (5.49), (5.50), and (5.59) we derive the following bound on the left-hand side from above:

$$\left| \left(\sum_{i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}})} \big(d_{\mathscr{H},\hat{\mathcal{P}}}(i) - d_{\mathscr{H},\mathcal{R}}(i) \big) + \sum_{i \in R_{\mathrm{THIN}}} d_{\mathscr{H},\hat{\mathcal{P}}}(i) \right) - \Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}}) \right| \leqslant \frac{\varepsilon_{5.1}^2}{48}. \tag{5.60}$$

Case 1.2 splits into two sub-cases depending on the size of $\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})$.

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Case 1.2.1 ($\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}}) > \varepsilon_{5.1}^2/12$)**.** For every $i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$ we have, due to the assumption of Case 1.2, that $\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ is $(\delta_{k,\mathrm{RL}}, d_{\mathscr{H},\hat{\mathcal{P}}}(i))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$. We apply the slicing lemma, Proposition 3.5, with

$$j = k, \quad s_0 = 2, \quad r = 1, \quad \delta_0 = \delta_{k,\mathrm{RL}}, \quad \varrho_0 = \frac{\varepsilon_{5.1}^2}{192 s_{5.1}}, \quad p_0 = \frac{\varepsilon_{5.1}^2}{192 s_{5.1}},$$
$$\text{to} \quad \hat{\mathcal{P}}^{(k-1)}, \quad \text{and} \quad \mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \quad \text{with}$$
$$s = 2, \quad \delta = \delta_{k,\mathrm{RL}}, \quad \varrho = d_{\mathscr{H},\hat{\mathcal{P}}}(i), \quad p_1 = \frac{d_{\mathscr{H},\mathcal{R}}(i)}{d_{\mathscr{H},\hat{\mathcal{P}}}(i)}, \quad \text{and} \quad p_2 = \frac{d_{\mathscr{H},\hat{\mathcal{P}}}(i) - d_{\mathscr{H},\mathcal{R}}(i)}{d_{\mathscr{H},\hat{\mathcal{P}}}(i)}.$$

For this choice of parameters the assumptions of Proposition 3.5 are satisfied. Indeed we have (5.48), $\varrho \geqslant \varrho_0$ since $i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$ and (5.56), $\varrho_0 \geqslant 2\delta = \delta_{k,\mathrm{RL}}$ (*cf.* (5.30)), $p_1 \geqslant p_0$ since $i \notin R_{\mathrm{THIN}}$ by definition of $I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, and $p_2 \geqslant p_0$ since $i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$.

Since $p_1 + p_2 = 1$, Proposition 3.5 yields a partition $\mathcal{M}_i^{(k)} \cup \mathcal{L}_i^{(k)}$ of $\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ for every $i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, where

$$\mathcal{M}_i^{(k)} \text{ is } (3\delta_{k,\mathrm{RL}}, d_{\mathscr{H},\mathcal{R}}(i))\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)} \text{ and} \tag{5.61}$$
$$\mathcal{L}_i^{(k)} \text{ is } (3\delta_{k,\mathrm{RL}}, (d_{\mathscr{H},\hat{\mathcal{P}}}(i) - d_{\mathscr{H},\mathcal{R}}(i)))\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}. \tag{5.62}$$

We now collect all 'left-overs' and distribute them among the hypergraphs $\mathcal{H}_i^{(k)}$ which are too 'slim' in $\hat{\mathcal{P}}^{(k-1)}$. For that we set

$$\mathcal{U}^{(k)} = \bigcup_{i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}})} \mathcal{L}_i^{(k)} \cup \bigcup_{i \in R_{\mathrm{THIN}}} \big(\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big).$$

From (5.62) and the assumption of Case 1.2, we infer from Proposition 3.2 that, w.r.t. $\hat{\mathcal{P}}^{(k-1)}, \mathcal{U}^{(k)}$ is $(3s_{5.1}\delta_{k,\mathrm{RL}})$-regular. Moreover, by the choice of $\delta_{k,\mathrm{RL}}$ in (5.30) we have $3s_{5.1}\delta_{k,\mathrm{RL}} \leqslant \varepsilon_{5.1}^2/48$ and by (5.60) it follows that

$$\mathcal{U}^{(k)} \text{ is } \big(\varepsilon_{5.1}^2/24, \Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})\big)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}. \tag{5.63}$$

We then apply the slicing lemma again, this time with

$$j = k, \quad s_0 = s_{5.1}, \quad r = 1, \quad \delta_0 = \frac{\varepsilon_{5.1}^2}{24}, \quad \varrho_0 = \frac{\varepsilon_{5.1}^2}{12}, \quad p_0 = \frac{\varepsilon_{5.1}^2}{192 s_{5.1}},$$
$$\text{to} \quad \hat{\mathcal{P}}^{(k-1)}, \quad \text{and} \quad \mathcal{U}^{(k)} \quad \text{with} \quad s = |I_{\mathrm{SLIM}}(\hat{\mathcal{P}})|,$$
$$\delta = \frac{\varepsilon_{5.1}^2}{24}, \quad \varrho = \Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}}), \quad \text{and} \quad \left\{ p_i = \frac{d_{\mathscr{H},\mathcal{R}}(i) - d_{\mathscr{H},\hat{\mathcal{P}}}(i)}{\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})} : \, i \in I_{\mathrm{SLIM}}(\hat{\mathcal{P}}) \right\}.$$

Here the assumptions of Proposition 3.5 are immediate consequences of (5.48) (showing (i) of Proposition 3.5), (5.63) (showing that $\mathcal{U}^{(k)}$ is sufficiently regular), the assumption of Case 1.2.1 (yielding $\varrho \geqslant \varrho_0$), and the definition of $I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$ in (5.58) combined with $\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}}) \leqslant 1$ (yielding $p_i \geqslant p_0$).

Also, note that $\sum_{i \in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})} p_i = 1$ and, consequently, Proposition 3.5 yields a partition $\{\mathcal{T}_i^{(k)} : \, i \in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})\}$ of $\mathcal{U}^{(k)}$, which by (5.63) has density 'close' to $\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})$, so that

$$\mathcal{T}_i^{(k)} \text{ is } \big(\varepsilon_{5.1}^2/8, (d_{\mathscr{H},\mathcal{R}}(i) - d_{\mathscr{H},\hat{\mathcal{P}}}(i))\big)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}. \tag{5.64}$$

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Finally, we are ready to define the family $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}):\ i\in[s_{5.1}]\}$. Set

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})=\begin{cases}\varnothing & \text{if } i\in R_{\mathrm{THIN}},\\ \mathcal{M}_i^{(k)} & \text{if } i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}}),\\ \mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) & \text{if } i\in I_{\mathrm{OK}}(\hat{\mathcal{P}}),\\ \big(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)\cup\mathcal{T}_i^{(k)} & \text{if } i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}}).\end{cases}$$

It is obvious that $\{\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}):\ i\in[s_{5.1}]\}$ defined this way is a partition of $\mathcal{F}^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$. We still have to verify (5.39) and (5.40).

We start by showing (5.39). First let $i\in R_{\mathrm{THIN}}$. By definition of $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$, it is $(\varepsilon',0)$-regular for every $\varepsilon'>0$, and hence it is $((\varepsilon'+d_{\mathscr{H},\mathcal{R}}(i)),d_{\mathscr{H},\mathcal{R}}(i))$-regular. Therefore, (5.39) follows from $d_{\mathscr{H},\mathcal{R}}(i)\leqslant\varepsilon_{5.1}^2/(192s_{5.1})<\varepsilon_{5.1}^2/4$ (*cf.* (5.50)).

If $i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, then (5.39) follows from (5.61) and $3\delta_{k,\mathrm{RL}}<\varepsilon_{5.1}^2/4$ (*cf.* (5.30)).

Now let $i\in I_{\mathrm{OK}}(\hat{\mathcal{P}})$. Then $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})=\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ is $(\delta_{k,\mathrm{RL}},d_{\mathscr{H},\hat{\mathcal{P}}}(i))$-regular due to the assumption of Case 1.2. Moreover, by (5.59), $|d_{\mathscr{H},\hat{\mathcal{P}}}(i)-d_{\mathscr{H},\mathcal{R}}(i)|\leqslant\varepsilon_{5.1}^2/(192s_{5.1})$, and therefore $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\delta_{k,\mathrm{RL}}+\varepsilon_{5.1}^2/(192s_{5.1}),d_{\mathscr{H},\mathcal{R}}(i))$-regular. Now (5.39) follows, since $\delta_{k,\mathrm{RL}}+\varepsilon_{5.1}^2/(192s_{5.1})\leqslant\varepsilon_{5.1}^2/4$ (*cf.* (5.30)).

Finally, let $i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$. Then Proposition 3.2 applied to $\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ and $\mathcal{T}_i$ yields that $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $((\delta_{k,\mathrm{RL}}+\varepsilon_{5.1}^2/8),d_{\mathscr{H},\mathcal{R}}(i))$-regular (*cf.* the assumption of Case 1.2 and (5.64)). Consequently, (5.39) follows since $\delta_{k,\mathrm{RL}}+\varepsilon_{5.1}^2/8\leqslant\varepsilon_{5.1}^2/4$ (*cf.* (5.30)).

It is left to verify (5.40) for $i\in[s_{5.1}]$ to conclude this case, Case 1.2.1. Again our argument is different for each partition classes R_{THIN}, $I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, $I_{\mathrm{OK}}(\hat{\mathcal{P}})$, and $I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$ of $[s_{5.1}]$.

For $i\in R_{\mathrm{THIN}}$, owing to notational reasons it will be easier to verify (5.40) in terms of the corresponding density

$$d\big(\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle\,\mathcal{H}_i^{(k)}\mid\hat{\mathcal{P}}^{(k-1)}\big)=\frac{|\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle\big(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big)|}{|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|}.$$

If $i\in R_{\mathrm{THIN}}$, then $d_{\mathscr{H},\mathcal{R}}(i)\leqslant\varepsilon_{5.1}^2/(192s_{5.1})$, and consequently

$$d\big(\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle\,\mathcal{H}_i^{(k)}\mid\hat{\mathcal{P}}^{(k-1)}\big)=d_{\mathscr{H},\hat{\mathcal{P}}}(i)\leqslant|d_{\mathscr{H},\mathcal{R}}(i)-d_{\mathscr{H},\hat{\mathcal{P}}}(i)|+\varepsilon_{5.1}^2/(192s_{5.1}).$$

Therefore, (5.40) follows for $i\in R_{\mathrm{THIN}}$ from $\varepsilon_{5.1}^2/(192s_{5.1})\leqslant v_{5.1}/6$ (*cf.* (5.28)).

If $i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, then by definition of $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})=\mathcal{M}_i^{(k)}$ we have

$$|\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}))|=|\mathcal{L}_i^{(k)}|.$$

Moreover, due to (5.62) we have $|\mathcal{L}_i^{(k)}|\leqslant(d_{\mathscr{H},\hat{\mathcal{P}}}(i)-d_{\mathscr{H},\mathcal{R}}(i)+3\delta_{k,\mathrm{RL}})|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|$, which combined with the choice of $\delta_{k,\mathrm{RL}}\leqslant v_{5.1}/18$ (*cf.* (5.30)) yields (5.40) for $i\in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$.

If $i\in I_{\mathrm{OK}}(\hat{\mathcal{P}})$, then (5.40) is a consequence of the definition $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})=\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$, which yields that the left-hand side in (5.40) is 0.

Finally, we consider the case $i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$. It follows from the definition of $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ and (5.64) that

$$|\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})\triangle(\mathcal{H}_i^{(k)}\cap\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}))|=|\mathcal{T}_i^{(k)}|\leqslant(d_{\mathscr{H},\mathcal{R}}(i)-d_{\mathscr{H},\hat{\mathcal{P}}}(i)+\varepsilon_{5.1}^2/8)|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|.$$

Consequently, (5.40) for $i\in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$ follows from (5.28).

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Having verified (5.39) and (5.40) for every $i \in [s_{5.1}]$, we conclude Claim 5.3 in Case 1.2.1. In order to finish Case 1.2, we have to consider the complementing and rather trivial sub-case when $\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})$ is small. ◊

Case 1.2.2 ($\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}}) \leqslant \varepsilon_{5.1}^2/12$)**.** In this case we set for every $i \in [s_{5.1}]$

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}).$$

Therefore, (5.40) of Claim 5.3 holds trivially, and we only have to show (5.39). For this we note that, due to the assumption of Case 1.2, we have $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\delta_{k,\mathrm{RL}}, d_{\mathscr{H},\hat{\mathcal{P}}}(i))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ and consequently, for every $i \in [s_{5.1}]$,

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \text{ is } \big(\delta_{k,\mathrm{RL}} + |d_{\mathscr{H},\hat{\mathcal{P}}}(i) - d_{\mathscr{H},\mathcal{R}}(i)|, d_{\mathscr{H},\mathcal{R}}(i)\big)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}. \tag{5.65}$$

In what follows we show that

$$|d_{\mathscr{H},\mathcal{R}}(i) - d_{\mathscr{H},\hat{\mathcal{P}}}(i)| \leqslant \frac{\varepsilon_{5.1}^2}{6} \quad \text{for every } i \in [s_{5.1}], \tag{5.66}$$

which, combined with (5.65) and $\delta_{k,\mathrm{RL}} + \varepsilon_{5.1}^2/6 \leqslant \varepsilon_{5.1}^2/4$ (*cf.* (5.30)), yields (5.39) for every $i \in [s_{5.1}]$.

First we consider $i \in R_{\mathrm{THIN}}$. Due to (5.60) and the assumption of Case 1.2.2, we have

$$\sum_{i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}})} \big(d_{\mathscr{H},\hat{\mathcal{P}}}(i) - d_{\mathscr{H},\mathcal{R}}(i)\big) + \sum_{i \in R_{\mathrm{THIN}}} d_{\mathscr{H},\hat{\mathcal{P}}}(i) \leqslant \left(\frac{1}{12} + \frac{1}{48}\right)\varepsilon_{5.1}^2 < \frac{\varepsilon_{5.1}^2}{6}, \tag{5.67}$$

where all terms on the left-hand side are positive (*cf.* (5.56)). Therefore, $d_{\mathscr{H},\hat{\mathcal{P}}}(i) \leqslant \varepsilon_{5.1}^2/6$ for every $i \in R_{\mathrm{THIN}}$. Moreover, since $d_{\mathscr{H},\mathcal{R}}(i) \leqslant \varepsilon_{5.1}^2/(192 s_{5.1})$ for every $i \in R_{\mathrm{THIN}}$, (5.66) holds for every $i \in R_{\mathrm{THIN}}$.

If $i \in I_{\mathrm{FAT}}(\hat{\mathcal{P}})$, then (5.67) yields $0 \leqslant d_{\mathscr{H},\hat{\mathcal{P}}}(i) - d_{\mathscr{H},\mathcal{R}}(i) \leqslant \varepsilon_{5.1}^2/6$ and consequently (5.66) holds for those i.

For $i \in I_{\mathrm{OK}}(\hat{\mathcal{P}})$, (5.66) follows from the definition of $I_{\mathrm{OK}}(\hat{\mathcal{P}})$ in (5.59).

Finally, we consider $i \in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$. From the assumption of this case, Case 1.2.2, and the definition of $\Delta_{\mathrm{SLIM}}(\hat{\mathcal{P}})$ in (5.57), we infer $0 \leqslant d_{\mathscr{H},\mathcal{R}}(i) - d_{\mathscr{H},\hat{\mathcal{P}}}(i) \leqslant \varepsilon_{5.1}^2/12$, which clearly implies (5.66) for $i \in I_{\mathrm{SLIM}}(\hat{\mathcal{P}})$.

This concludes Case 1.2.2 the last sub-case of Case 1. ◊

Case 2 ($d(\mathcal{F}^{(k)} \mid \mathcal{R}^{(k-1)}) \leqslant \varepsilon_{5.1}^2/16$)**.** In this case we set

$$\mathcal{G}_1^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \quad \text{and} \quad \mathcal{G}_2^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \cdots = \mathcal{G}_{s_{5.1}}^{(k)}(\hat{\mathcal{P}}^{(k-1)}) = \varnothing.$$

Again we must show (5.39) and (5.40) of Claim 5.3. We start with (5.39). Note that $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $(\varepsilon_{5.1}^2/96)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ for every $i \in [s_{5.1}]$. (This is trivial for $i \geqslant 2$ and follows from (5.49) for $i = 1$.) In order to show that $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is also $(\varepsilon_{5.1}^2/4, d_{\mathscr{H},\mathcal{R}}(i))$-regular recall the assumption $d(\mathcal{F}^{(k)} \mid \mathcal{R}^{(k-1)}) \leqslant \varepsilon_{5.1}^2/16$, which implies that $d_{\mathscr{H},\mathcal{R}}(i) \leqslant \varepsilon_{5.1}^2/16$ for every $i \in [s_{5.1}]$. Consequently, $\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)})$ is $\big((\varepsilon_{5.1}^2/96 + \varepsilon_{5.1}^2/16), d_{\mathscr{H},\mathcal{R}}(i)\big)$-regular for every $i \in [s_{5.1}]$ and (5.39) follows.

In order to infer (5.40) we observe that, for $i \in [s_{5.1}]$,

$$\mathcal{G}_i^{(k)}(\hat{\mathcal{P}}^{(k-1)}) \triangle \big((\mathcal{H}_i^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big) \subseteq \mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}).$$

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Moreover, due to the assumption $d(\mathcal{F}^{(k)} \mid \mathcal{R}^{(k-1)}) \leqslant \varepsilon_{5.1}^2/16$ and (5.49), we have

$$|\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})| \leqslant (\varepsilon_{5.1}^2/16 + \varepsilon_{5.1}^2/96)|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|.$$

Property (5.40) then follows from $\varepsilon_{5.1}^2/16 + \varepsilon_{5.1}^2/96 \leqslant \nu_{5.1}/6$ (see (5.28)). ◊

In all cases we ensured the existence of a partition of $\mathcal{F}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$, which satisfies the conclusions of Claim 5.3. This concludes the proof of Claim 5.3. □

Proof of Claim 5.4. We assume the contrary, *i.e.*,

$$\left|\bigcup\left\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\right\}\right| > \frac{\nu_{5.1}}{3}|\mathcal{K}_k(\mathcal{R}^{(k-1)})|.$$

Without loss of generality we may assume that $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)}) \subseteq \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,BAD}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})$ defined by

$$\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)}) = \left\{\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}(\boldsymbol{\mathcal{R}})\colon\ d(\mathcal{H}_i^{(k)} \mid \hat{\mathcal{P}}^{(k-1)}) > d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{5.1}}{6}\right\}$$

satisfies

$$\left|\bigcup\left\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\right\}\right| \geqslant \frac{\nu_{5.1}}{6}|\mathcal{K}_k(\mathcal{R}^{(k-1)})|. \tag{5.68}$$

(The complementing case concerning $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,SLIM}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})$ instead of $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})$ is very similar.) In what follows we will show that (5.68) contradicts the $(\nu_{5.1}/12, *, t_{5.1}^{2^k})$-regularity of $\mathcal{H}_i^{(k)}$ w.r.t. $\mathcal{R}^{(k-1)}$ (see (L.5.1.d)). Since

$$|\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})| \leqslant |\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}| = \prod_{h=1}^{k-1}(a_h^{\mathscr{P}_{\mathrm{RL}}})^{\binom{k}{h}} \leqslant t_{\mathrm{RL}}^{2^k} \overset{(5.32)}{=} t_{5.1}^{2^k},$$

this contradiction follows once we establish the following inequality:

$$\frac{|\mathcal{H}_i^{(k)} \cap \bigcup\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\}|}{|\bigcup\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\}|} \geqslant d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{5.1}}{12}. \tag{5.69}$$

By definition of $\hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})$ we have $d(\mathcal{H}_i^{(k)} \mid \hat{\mathcal{P}}^{(k-1)}) \geqslant d_{\mathscr{H},\mathcal{R}}(i) + \nu_{5.1}/6$ for every $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})$ and, consequently, since $\mathcal{K}_k(\hat{\mathcal{P}}_1^{(k-1)}) \cap \mathcal{K}_k(\hat{\mathcal{P}}_2^{(k-1)}) = \varnothing$ for all distinct $\hat{\mathcal{P}}_1^{(k-1)}, \hat{\mathcal{P}}_2^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)}) \subseteq \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL}}$ (*cf.* 5.37), it suffices to verify

$$\left(d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{5.1}}{6}\right)\frac{\min\{|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\}}{\max\{|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})|\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{RL,FAT}}(\boldsymbol{\mathcal{R}}, \mathcal{H}_i^{(k)})\}} \geqslant d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{5.1}}{12}$$

to infer (5.69). In view of (5.35) we derive the following upper bound on the right-hand side of the last inequality:

$$\left(d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{5.1}}{6}\right)\frac{1 - \nu_{5.1}/48}{1 + \nu_{5.1}/48} \geqslant \left(d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{5.1}}{6}\right)\left(1 - \frac{\nu_{5.1}}{24}\right) \geqslant d_{\mathscr{H},\mathcal{R}}(i) + \frac{\nu_{5.1}}{12},$$

which concludes the proof of Claim 5.4. □

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6. Proof of $\mathrm{RAL}(k) \Longrightarrow \mathrm{RL}(k+1)$

In what follows we give a proof of $\mathrm{RL}(k+1)$, Lemma 4.1, based on $\mathrm{RAL}(k)$, Lemma 4.2. The proof presented here resembles the main ideas from [5, 26, 31] combined with some techniques from [20]. In the next section we recall the concept of an *index* of a partition (*cf.* Definition 13) and derive some facts about it. We then give the proof of $\mathrm{RL}(k+1)$ in Section 6.2.

6.1. The index of a partition

The following propositions centre around the notion of an *index*. Throughout this section we will work under the following set-up.

Set-up 6.1. *Let $\mathscr{R}_0^{(1)}$ be a fixed partition of some vertex set V and $\mathscr{H}^{(k+1)}$ be a partition of $[V]^k$. Moreover, let $\mathscr{X}^{(k)}$ be a partition refining $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$, i.e., for every $\mathcal{X}^{(k)} \in \mathscr{X}^{(k)}$ we have $\mathcal{X}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ or $\mathcal{X}^{(k)} \cap \mathrm{Cross}_k(\mathscr{R}_0^{(1)}) = \varnothing$. Let $\mathcal{U}(\mathscr{X}^{(k)}) = \bigcup\{\mathcal{X}^{(k)} \colon \mathcal{X}^{(k)} \in \mathscr{X}^{(k)}\} \supseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ be the set of k-tuples partitioned by $\mathscr{X}^{(k)}$.*

For any $K \in \mathcal{U}(\mathscr{X}^{(k)})$ let $\mathcal{X}^{(k)}(K)$ be that partition class of $\mathscr{X}^{(k)}$ which contains K, i.e.,

$$\mathcal{X}^{(k)}(K) = \mathcal{X}^{(k)} \in \mathscr{X}^{(k)} \quad \text{so that} \quad K \in \mathcal{X}^{(k)}.$$

Moreover, for every $(k+1)$-tuple $K' \in [V]^{k+1}$ satisfying $[K']^k \subseteq \mathcal{U}(\mathscr{X}^{(k)})$ we set

$$\hat{\mathcal{X}}^{(k)}(K') = \bigcup\{\mathcal{X}^{(k)}(K) \colon K \in [K']^k\}$$
$$\text{and} \quad \hat{\mathscr{X}}^{(k)} = \{\hat{\mathcal{X}}^{(k)}(K') \colon K' \in [V]^{k+1} \text{ s.t. } [K']^k \subseteq \mathcal{U}(\mathscr{X}^{(k)})\}.$$

Note that every $K' \in \mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})$ satisfies $[K']^k \subseteq \mathcal{U}(\mathscr{X}^{(k)})$, because $\mathscr{X}^{(k)}$ refines $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$. □

We then define the index of a partition $\mathscr{X}^{(k)}$ (satisfying the above set-up) with respect to $\mathscr{R}_0^{(1)}$ and $\mathscr{H}^{(k+1)}$ as follows.

Definition 13 (index). Given V, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{X}^{(k)}$ as in Set-up 6.1. We set the index of $\mathscr{X}^{(k)}$ w.r.t. $\mathscr{R}_0^{(1)}$ and $\mathscr{H}^{(k+1)}$ equal to

$$\begin{aligned}\mathrm{ind}(\mathscr{X}^{(k)}) &= \frac{1}{|V|^{k+1}} \sum_{\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}} \sum_{K' \in \mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})} d^2\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{X}}^{(k)}(K')\big) \\ &= \frac{1}{|V|^{k+1}} \sum_{\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}} \sum_{\substack{\hat{\mathcal{X}}^{(k)} \in \hat{\mathscr{X}}^{(k)} \\ \hat{\mathcal{X}}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})}} d^2\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{X}}^{(k)}\big) |\mathcal{K}_{k+1}(\hat{\mathcal{X}}^{(k)})|.\end{aligned}$$

The next observation follows straight from the definition of the index.

Fact 6.2. *For all V, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{X}^{(k)}$ as in Set-up 6.1,* $\mathrm{ind}(\mathscr{X}^{(k)})$ *is bounded between* 0 *and* 1. □

We now derive a few more propositions related to the index, which allow a simpler presentation of the proof of $\mathrm{RL}(k+1)$.

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Proposition 6.3. *Let V, $\mathscr{R}_0^{(1)}$, and $\mathscr{H}^{(k+1)}$ be as in Set-up 6.1. Let $\mathscr{X}^{(k)} = \{\mathcal{X}_1^{(k)}, \dots, \mathcal{X}_s^{(k)}\}$ and $\mathscr{Y}^{(k)} = \{\mathcal{Y}_1^{(k)}, \dots, \mathcal{Y}_s^{(k)}\}$ be partitions which refine $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$. Moreover, let v be a given positive real. If for every $\ell \in [s]$ we have*

(i) $|\mathcal{X}_\ell^{(k)} \triangle \mathcal{Y}_\ell^{(k)}| \leqslant v|V|^k$, *and*
(ii) *if $\mathcal{X}_\ell^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ then $\mathcal{Y}_\ell^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$,*

then

$$\mathrm{ind}(\mathscr{Y}^{(k)}) \geqslant \mathrm{ind}(\mathscr{X}^{(k)}) - 3(k+1)s^{k+1}|\mathscr{H}^{(k+1)}|v. \tag{6.1}$$

Proof. For every $(k+1)$-tuple $I \in [s]^{k+1}$ we set

$$\hat{\mathcal{X}}_I^{(k)} = \bigcup_{i \in I} \mathcal{X}_i^{(k)} \quad \text{and} \quad \hat{\mathcal{Y}}_I^{(k)} = \bigcup_{i \in I} \mathcal{Y}_i^{(k)}.$$

From (i) we infer that for every $I \in [s]^{k+1}$ we have

$$\Big| |\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})| - |\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})| \Big| \leqslant |\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)}) \triangle \mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})| \leqslant v(k+1)|V|^{k+1}. \tag{6.2}$$

Furthermore, suppose the partition classes of $\mathscr{H}^{(k+1)}$ are labelled $\mathcal{H}_1^{(k+1)}, \dots, \mathcal{H}_h^{(k+1)}$. For a more concise notation we set for every $I \in [s]^{k+1}$ and $\zeta \in [h]$

$$d(\zeta \mid \hat{\mathcal{X}}_I^{(k)}) = d\big(\mathcal{H}_\zeta^{(k+1)} \mid \hat{\mathcal{X}}_I^{(k)}\big) \quad \text{and} \quad d(\zeta \mid \hat{\mathcal{Y}}_I^{(k)}) = d\big(\mathcal{H}_\zeta^{(k+1)} \mid \hat{\mathcal{Y}}_I^{(k)}\big).$$

The triangle inequality and (6.2) gives for every $I \in [s]^{k+1}$ and $\zeta \in [h]$

$$\begin{aligned}
&\Big| |\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})| d^2(\zeta \mid \hat{\mathcal{X}}_I^{(k)}) - |\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})| d^2(\zeta \mid \hat{\mathcal{Y}}_I^{(k)}) \Big| \\
&\qquad \leqslant \Big| |\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})| d(\zeta \mid \hat{\mathcal{X}}_I^{(k)}) - |\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})| d(\zeta \mid \hat{\mathcal{Y}}_I^{(k)}) \Big| d(\zeta \mid \hat{\mathcal{X}}_I^{(k)}) \\
&\qquad\qquad + \Big| |\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})| - |\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})| \Big| d(\zeta \mid \hat{\mathcal{Y}}_I^{(k)}) d(\zeta \mid \hat{\mathcal{X}}_I^{(k)}) \\
&\qquad\qquad + \Big| |\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})| d(\zeta \mid \hat{\mathcal{X}}_I^{(k)}) - |\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})| d(\zeta \mid \hat{\mathcal{Y}}_I^{(k)}) \Big| d(\zeta \mid \hat{\mathcal{Y}}_I^{(k)}) \\
&\qquad \leqslant 3 \Big| |\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})| - |\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})| \Big| \overset{(6.2)}{\leqslant} 3v(k+1)|V|^{k+1}.
\end{aligned} \tag{6.3}$$

Now let $\hat{\mathscr{X}}^{(k)}$ and $\hat{\mathscr{Y}}^{(k)}$ be defined as in Set-up 6.1. Clearly, for every $\hat{\mathcal{X}}^{(k)} \in \hat{\mathscr{X}}^{(k)}$ there exists a unique $I \in [s]^{k+1}$ so that $\hat{\mathcal{X}}^{(k)} = \hat{\mathcal{X}}_I^{(k)}$, while the converse fails to be true in general. We define

$$S(\mathscr{X}^{(k)}) = \{I \in [s]^{k+1} \colon \hat{\mathcal{X}}_I^{(k)} \in \hat{\mathscr{X}}^{(k)} \text{ and } \hat{\mathcal{X}}_I^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})\}.$$

Then we have

$$\mathrm{ind}(\mathscr{X}^{(k)}) = \frac{1}{|V|^{k+1}} \sum_{\zeta \in [h]} \sum_{I \in S(\mathscr{X}^{(k)})} |\mathcal{K}_{k+1}(\hat{\mathcal{X}}_I^{(k)})| d^2(\zeta \mid \hat{\mathcal{X}}_I^{(k)}),$$

and applying (6.3) for every $\zeta \in [h]$ and $I \in S(\mathscr{X}^{(k)})$ yields

$$\mathrm{ind}(\mathscr{X}^{(k)}) \leqslant \frac{1}{|V|^{k+1}} \sum_{\zeta \in [h]} \sum_{I \in S(\mathscr{X}^{(k)})} |\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})| d^2(\zeta \mid \hat{\mathcal{Y}}_I^{(k)}) + 3v(k+1)h|S(\mathscr{X}^{(k)})|. \tag{6.4}$$

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By assumption (ii) we have that $\hat{\mathcal{Y}}_I^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ for every $I \in S(\mathscr{X}^{(k)})$. Consequently, $\hat{\mathcal{Y}}_I^{(k)}$ is either in $\hat{\mathscr{Y}}^{(k)}$ or $\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)}) = \varnothing$ for every $I \in S(\mathscr{X}^{(k)})$, and hence

$$\mathrm{ind}(\mathscr{Y}^{(k)}) \geqslant \frac{1}{|V|^{k+1}} \sum_{\zeta \in [h]} \sum \{|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}_I^{(k)})| d^2(\zeta \mid \hat{\mathcal{Y}}_I^{(k)}) \colon I \in S(\mathscr{X}^{(k)})\}.$$

Therefore, the last inequality combined with (6.4) implies

$$\mathrm{ind}(\mathscr{X}^{(k)}) \leqslant \mathrm{ind}(\mathscr{Y}^{(k)}) + 3\nu(k+1)hs^{k+1},$$

which concludes the proof of Proposition 6.3. □

The following proposition is a simple consequence of Jensen's inequality.

Proposition 6.4. *Suppose $\hat{\mathcal{Y}}^{(k)}$ is an $(m, k+1, k)$-hypergraph and $\{\hat{\mathcal{Z}}_1^{(k)}, \dots, \hat{\mathcal{Z}}_z^{(k)}\}$ is a family of $(m, k+1, k)$-hypergraphs such that $\{\mathcal{K}_{k+1}(\hat{\mathcal{Z}}_i^{(k)}) \colon i \in [z]\}$ partitions $\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})$. Then*

$$d^2\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{Y}}^{(k)}\big) |\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})| \leqslant \sum_{i \in [z]} d^2\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{Z}}_i^{(k)}\big) |\mathcal{K}_{k+1}(\hat{\mathcal{Z}}_i^{(k)})| \tag{6.5}$$

for every hypergraph $\mathcal{H}^{(k+1)} \subseteq \mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})$.

Proof. For all $K' \in \mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})$, let $\hat{\mathcal{Z}}^{(k)}(K')$ be the unique member of $\{\hat{\mathcal{Z}}_1^{(k)}, \dots, \hat{\mathcal{Z}}_z^{(k)}\}$ so that $K' \in \mathcal{K}_{k+1}(\hat{\mathcal{Z}}^{(k)}(K'))$. Then we have

$$\begin{aligned} d\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{Y}}^{(k)}\big) &= \frac{\sum_{i \in [z]} d\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{Z}}_i^{(k)}\big) |\mathcal{K}_{k+1}(\hat{\mathcal{Z}}_i^{(k)})|}{|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})|} \\ &= \frac{\sum_{K' \in \mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})} d\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{Z}}^{(k)}(K')\big)}{|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})|}, \end{aligned}$$

and Jensen's inequality yields (6.5), since

$$\begin{aligned} d^2\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{Y}}^{(k)}\big) |\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})| &= \frac{\big(\sum_{K' \in \mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})} d\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{Z}}^{(k)}(K')\big)\big)^2}{|\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})|} \\ &\leqslant \sum_{K' \in \mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})} d^2\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{Z}}^{(k)}(K')\big) \\ &= \sum_{i \in [z]} d^2\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{Z}}_i^{(k)}\big) |\mathcal{K}_{k+1}(\hat{\mathcal{Z}}_i^{(k)})|. \end{aligned}$$

□

The following proposition is a corollary of Proposition 6.4 and asserts that the refinement of a family of partitions has the same or bigger index.

Proposition 6.5. *Let V, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{Y}^{(k)}$ be given as in Set-up 6.1 and let $\mathscr{Z}^{(k)}$ be a partition refining* $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$. *If $\mathscr{Z}^{(k)} \prec \mathscr{Y}^{(k)}$, then* $\mathrm{ind}(\mathscr{Y}^{(k)}) \leqslant \mathrm{ind}(\mathscr{Z}^{(k)})$.

Proof. We observe that for every $\hat{\mathcal{Y}}^{(k)} \in \hat{\mathscr{Y}}^{(k)}$ satisfying $\hat{\mathcal{Y}}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$, the family $\{\mathcal{K}_{k+1}(\hat{\mathcal{Z}}^{(k)}) \colon \hat{\mathcal{Z}}^{(k)} \in \hat{\mathscr{Z}}^{(k)} \text{ and } \hat{\mathcal{Z}}^{(k)} \subseteq \hat{\mathcal{Y}}^{(k)}\}$ partitions $\mathcal{K}_{k+1}(\hat{\mathcal{Y}}^{(k)})$. Consequently, we can

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apply Proposition 6.4 to every $\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}$ and $\hat{\mathcal{Y}}^{(k)} \in \hat{\mathscr{Y}}^{(k)}$, which yields the proposition. □

In the proof of RL$(k+1)$ we will also deal with partitions which 'almost' refine each other (see Definition 14 below) and we need approximations of their index (Proposition 6.6).

Definition 14. Given V, $\mathscr{R}_0^{(1)}$, and $\mathscr{Z}^{(k)}$ as in Set-up 6.1. Moreover, let $\beta \geqslant 0$ and let $\mathscr{T}^{(k)}$ be a partition refining $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$. We say the partition $\mathscr{T}^{(k)}$ is a *β-refinement of $\mathscr{Z}^{(k)}$* if

$$\sum \{|\mathcal{T}^{(k)}| \colon \mathcal{T}^{(k)} \in \mathscr{T}^{(k)},\ \mathcal{T}^{(k)} \not\subseteq \mathcal{Z}^{(k)} \text{ for every } \mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}\} \leqslant \beta |V|^k.$$

The following proposition extends Proposition 6.5 and its proof is very similar to that of [5, Lemma 3.6].

Proposition 6.6. *Let V, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{Z}^{(k)}$ be given as in Set-up 6.1, let $\mathscr{T}^{(k)}$ be a β-refinement of $\mathscr{Z}^{(k)}$ for some $\beta \geqslant 0$. Then* $\mathrm{ind}(\mathscr{T}^{(k)}) \geqslant \mathrm{ind}(\mathscr{Z}^{(k)}) - \beta$.

Proof. We first define an auxiliary partition $\mathscr{S}^{(k)}$ which is a refinement of $\mathscr{T}^{(k)}$ and $\mathscr{Z}^{(k)}$. For this, set

$$\mathscr{S}^{(k)} = \{\mathcal{T}^{(k)} \cap \mathcal{Z}^{(k)} \colon \mathcal{T}^{(k)} \in \mathscr{T}^{(k)} \text{ and } \mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}\}.$$

Due to Proposition 6.5 we have

$$\mathrm{ind}(\mathscr{Z}^{(k)}) \leqslant \mathrm{ind}(\mathscr{S}^{(k)}). \tag{6.6}$$

Let $\hat{\mathscr{T}}_0^{(k)}$ be the family of polyads $\hat{\mathcal{T}}^{(k)} \in \hat{\mathscr{T}}^{(k)}$ which are sub-hypergraphs of $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ and for which there exists a $\mathcal{T}^{(k)} \in \mathscr{T}^{(k)}$ such that

$$\mathcal{T}^{(k)} \subseteq \hat{\mathcal{T}}^{(k)} \quad \text{and} \quad \mathcal{T}^{(k)} \not\subseteq \mathcal{Z}^{(k)} \quad \text{for all } \mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}.$$

Since $\mathscr{H}^{(k+1)}$ is a partition of $[V]^{k+1}$ and $\mathscr{T}^{(k)}$ is a β-refinement of $\mathscr{Z}^{(k)}$ we have

$$\sum_{\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}} \sum_{\hat{\mathcal{T}}^{(k)} \in \hat{\mathscr{T}}_0^{(k)}} \sum_{\substack{\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}^{(k)} \\ \hat{\mathcal{S}}^{(k)} \subseteq \hat{\mathcal{T}}^{(k)}}} d\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{S}}^{(k)}\big) |\mathcal{K}_{k+1}(\hat{\mathcal{S}}^{(k)})| \leqslant \beta |V|^k \times |V|. \tag{6.7}$$

Note that for every $\hat{\mathcal{T}}^{(k)} \notin \hat{\mathscr{T}}_0^{(k)}$ there exist some $\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}^{(k)}$ such that $\hat{\mathcal{S}}^{(k)} = \hat{\mathcal{T}}^{(k)}$. Consequently,

$$\begin{aligned}
&\mathrm{ind}(\mathscr{S}^{(k)}) - \mathrm{ind}(\mathscr{T}^{(k)}) \\
&= \frac{1}{|V|^{k+1}} \sum_{\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}} \sum_{\hat{\mathcal{T}}^{(k)} \in \hat{\mathscr{T}}_0^{(k)}} \Bigg(\sum_{\substack{\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}^{(k)} \\ \hat{\mathcal{S}}^{(k)} \subseteq \hat{\mathcal{T}}^{(k)}}} d^2\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{S}}^{(k)}\big) |\mathcal{K}_{k+1}(\hat{\mathcal{S}}^{(k)})| \\
&\qquad\qquad - d^2\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{T}}^{(k)}\big) |\mathcal{K}_{k+1}(\hat{\mathcal{T}}^{(k)})| \Bigg) \\
&\leqslant \frac{1}{|V|^{k+1}} \sum_{\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}} \sum_{\hat{\mathcal{T}}^{(k)} \in \hat{\mathscr{T}}_0^{(k)}} \sum_{\substack{\hat{\mathcal{S}}^{(k)} \in \hat{\mathscr{S}}^{(k)} \\ \hat{\mathcal{S}}^{(k)} \subseteq \hat{\mathcal{T}}^{(k)}}} d\big(\mathcal{H}^{(k+1)} \mid \hat{\mathcal{S}}^{(k)}\big) |\mathcal{K}_{k+1}(\hat{\mathcal{S}}^{(k)})| \overset{(6.7)}{\leqslant} \beta,
\end{aligned}$$

and the proposition follows from (6.6). □

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The last proposition in this section concerns the index of a family of partitions $\mathscr{R}$ failing to satisfy (H) of RL($k+1$). It can be shown that a certain refinement of $\mathscr{R}$ has an index of at least the index of $\mathscr{R}$ plus some positive constant depending on δ_{k+1}. This observation is the crucial idea in the proof of RL($k+1$). Since it roughly shows (together with Fact 6.2) that there are only finitely many refinements which violate (H). The same idea was already used in [5, 26, 31].

Proposition 6.7. *Let V, $\mathscr{R}_0^{(1)}$, and $\mathscr{H}^{(k+1)}$ be given as in Set-up 6.1 and let $\mathscr{R}^{(k)}$ be a partition refining* $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$. *Moreover, let δ be a positive real and $r \geqslant 1$ be an integer. If*

$$|\mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})| \geqslant \left(1-\frac{\delta}{2}\right)\binom{|V|}{k+1} \tag{6.8}$$

and if there is some $\mathcal{H}_{\mathrm{irr}}^{(k+1)} \in \mathscr{H}^{(k+1)}$ which is $(\delta,*,r)$-irregular[3] *w.r.t. $\mathscr{R}^{(k)}$, then there exists a partition $\mathscr{X}^{(k)}$ of $[V]^{k+1}$ satisfying*

(i) $\mathscr{X}^{(k)} \prec \mathscr{R}^{(k)}$,
(ii) $|\mathscr{X}^{(k)}| \leqslant |\mathscr{R}^{(k)}| \times 2^{r\times|\hat{\mathscr{R}}^{(k)}|}$, *and*
(iii) $\mathrm{ind}(\mathscr{X}^{(k)}) \geqslant \mathrm{ind}(\mathscr{R}^{(k)}) + \delta^4/2$.

In the proof of Proposition 6.7 we will use the defect form of the Cauchy–Schwarz inequality, which we state first (see, *e.g.*, [31] for a similar statement).

Proposition 6.8 (defect Cauchy–Schwarz inequality). *Suppose $\varnothing \neq J \subsetneq I$ are some index sets and $d_i \geqslant 0$ is some non-negative real number for every $i \in I$. If*

$$\frac{1}{|J|}\sum_{j\in J} d_j = \frac{1}{|I|}\sum_{i\in I} d_i + \alpha \tag{6.9}$$

for some (not necessarily non-negative) real α and if $|\alpha| \geqslant \delta$ and $|J| \geqslant \delta|I|$ for some $\delta \geqslant 0$, then

$$\sum_{i\in I} d_i^2 \geqslant \frac{1}{|I|}\left(\sum_{i\in I} d_i\right)^2 + \delta^3|I|.$$

□

Proof of Proposition 6.7. Let $\hat{\mathscr{R}}_{\mathrm{irr},0}^{(k)}$ be the set of those polyads $\hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}$ such that

$$\mathcal{H}_{\mathrm{irr}}^{(k+1)} \text{ is } (\delta,*,r)\text{-irregular w.r.t. } \hat{\mathcal{R}}^{(k)} \text{ and} \tag{6.10}$$

$$\hat{\mathcal{R}}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)}). \tag{6.11}$$

From the definition of $(\delta,*,r)$-regularity w.r.t. $\mathscr{R}^{(k)}$ (see footnote 3) and (6.8) we infer that

$$\sum\left\{|\mathcal{K}_k(\hat{\mathcal{R}}^{(k)})| :\ \hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}_{\mathrm{irr},0}^{(k)}\right\} \geqslant \frac{\delta}{2}|V|^{k+1}. \tag{6.12}$$

[3] Strictly speaking, in Definition 12 we only defined the regularity with respect to a family of partitions, while here we only have a partition $\mathscr{R}^{(k)}$ of k-tuples. However, we can easily alter the definition based on $\hat{\mathscr{R}}^{(k)}$, meaning that $\mathcal{H}^{(k+1)}$ is $(\delta,*,r)$-regular w.r.t. $\mathscr{R}^{(k)}$ if $\left|\bigcup\{\mathcal{K}_k(\hat{\mathcal{R}}^{(k)}) :\ \hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)} \text{ and } \mathcal{H}^{(k+1)} \text{ is not } (\delta,*,r)\text{-regular w.r.t. } \hat{\mathcal{R}}^{(k)}\}\right| \leqslant \delta|V|^{k+1}$.

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For each $\hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$ there exists a witness of irregularity, *i.e.*, there exists $\hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)}) = \{\hat{\mathcal{Q}}^{(k)}_1, \dots, \hat{\mathcal{Q}}^{(k)}_r\}$ such that $\hat{\mathcal{Q}}^{(k)}_i \subseteq \hat{\mathcal{R}}^{(k)}$ for every $i \in [r]$ and

$$\left| \bigcup_{i\in[r]} \mathcal{K}_k\big(\hat{\mathcal{Q}}^{(k)}_i\big) \right| \geqslant \delta |\mathcal{K}_{k+1}\big(\hat{\mathcal{R}}^{(k)}\big)| > 0, \tag{6.13}$$

$$d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)})\big) = d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{R}}^{(k)}\big) + \alpha_{\hat{\mathcal{R}}^{(k)}} \text{ for } \alpha_{\hat{\mathcal{R}}^{(k)}} \text{ with } |\alpha_{\hat{\mathcal{R}}^{(k)}}| > \delta, \tag{6.14}$$

where $d\big(\mathcal{H}^{(k)} \mid \hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)})\big) = |\mathcal{H}^{(k+1)}_{\mathrm{irr}} \cap \bigcup_{i\in[r]} \mathcal{K}_{k+1}(\hat{\mathcal{Q}}^{(k)})| \,/\, |\bigcup_{i\in[r]} \mathcal{K}_{k+1}(\hat{\mathcal{Q}}^{(k)})|$. Moreover, we define for every $\mathcal{R}^{(k)} \in \mathscr{R}^{(k)}$ the family $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$ of those sub-hypergraphs of $\mathcal{R}^{(k)}$ which are contained in some witness $\hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)})$ with $\mathcal{R}^{(k)} \subseteq \hat{\mathcal{R}}^{(k)}$. More precisely, we set

$$\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)}) = \{\mathcal{R}^{(k)} \cap \hat{\mathcal{Q}}^{(k)} \colon \hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0} \text{ with } \mathcal{R}^{(k)} \subseteq \hat{\mathcal{R}}^{(k)} \text{ and } \hat{\mathcal{Q}}^{(k)} \in \hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)})\}.$$

We observe that $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$ might be empty (*e.g.*, if $\mathcal{R}^{(k)} \nsubseteq \mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$), that the hypergraphs in $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$ are not necessarily disjoint, and that for every $\mathcal{R}^{(k)} \in \mathscr{R}^{(k)}$ we have the following trivial upper bound $w_{\mathcal{R}^{(k)}}$ on the number of hypergraphs in $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$

$$w_{\mathcal{R}^{(k)}} = |\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})| \leqslant r \times |\hat{\mathscr{R}}^{(k)}|. \tag{6.15}$$

We now define the promised refinement $\mathscr{X}^{(k)}$ of $\mathscr{R}^{(k)}$. We construct $\mathscr{X}^{(k)}$ for each $\mathcal{R}^{(k)} \in \mathscr{R}^{(k)}$ separately. This partition of $\mathcal{R}^{(k)}$ will be called $\mathscr{X}^{(k)}(\mathcal{R}^{(k)})$ and is given by the atoms arising from the intersection of the hypergraphs in $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$ (*i.e.*, the regions of the Venn diagram of the family $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$). More precisely, if $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)}) \neq \varnothing$ let $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)}) = \{\mathcal{W}^{(k)}_i \colon i \in [w_{\mathcal{R}^{(k)}}]\}$ be some enumeration of the elements of $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)})$ and set

$$\mathscr{X}^{(k)}(\mathcal{R}^{(k)}) = \left\{ \bigcap_{i\in I} \mathcal{W}^{(k)}_i \cap \bigcap_{i\in I^c} \big(\mathcal{R}^{(k)} \setminus \mathcal{W}^{(k)}_i\big) \colon \{I, I^c\} \text{ partitions } [w_{\mathcal{R}^{(k)}}] \right\}.$$

If $\boldsymbol{\mathcal{W}}^{(k)}(\mathcal{R}^{(k)}) = \varnothing$, then we set $\mathscr{X}^{(k)}(\mathcal{R}^{(k)}) = \{\mathcal{R}^{(k)}\}$. Collecting 'contributions' for every $\mathcal{R}^{(k)} \in \mathscr{R}^{(k)}$ in that way defines $\mathscr{X}^{(k)}$

$$\mathscr{X}^{(k)} = \bigcup \{\mathscr{X}^{(k)}(\mathcal{R}^{(k)}) \colon \mathcal{R}^{(k)} \in \mathscr{R}^{(k)}\}.$$

Owing to the construction above, the partition $\mathscr{X}^{(k)}$ clearly refines $\mathscr{R}^{(k)}$, *i.e.*, it satisfies (i) of Proposition 6.7. Moreover, (6.15) and the construction yields (ii) of the proposition.

It is left to verify (iii). For that we first fix some $\hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$ and consider the witness of irregularity $\hat{\boldsymbol{\mathcal{Q}}}^{(k)}(\hat{\mathcal{R}}^{(k)}) = \{\hat{\mathcal{Q}}^{(k)}_1, \dots, \hat{\mathcal{Q}}^{(k)}_r\}$. Since $\mathscr{X}^{(k)}$ refines $\mathscr{R}^{(k)}$ it satisfies the assumptions of Set-up 6.1 with V, $\mathscr{R}^{(1)}_0$, and $\mathscr{H}^{(k+1)}$. In particular, the family of polyads $\hat{\mathscr{X}}^{(k)}$ is well defined and for every $K' \in \mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})$ there exist a $\hat{\mathcal{X}}^{(k)}(K') \in \hat{\mathscr{X}}^{(k)}$ so that $K' \in \mathcal{K}_{k+1}(\hat{\mathcal{X}}^{(k)})$. We are heading towards an application of Proposition 6.8 with

$$I = \mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)}), \quad J = \bigcup_{i=1}^{r} \mathcal{K}_{k+1}(\hat{\mathcal{Q}}^{(k)}_i), \quad \text{and} \quad d_{K'} = d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{X}}^{(k)}(K')\big) \tag{6.16}$$

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for every $K' \in I$, and verify (6.9) below for $\alpha_{\hat{\mathcal{R}}^{(k)}}$, and the choice above, that

$$\frac{1}{|J|}\sum_{K'\in J} d_{K'} \overset{(6.16)}{=} d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{Q}}^{(k)}(\hat{\mathcal{R}}^{(k)})\big) \overset{(6.14)}{=} d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{R}}^{(k)}\big) + \alpha_{\hat{\mathcal{R}}^{(k)}} \overset{(6.16)}{=} \frac{1}{|I|}\sum_{K'\in I} d_{K'} + \alpha_{\hat{\mathcal{R}}^{(k)}}.$$

Since $|\alpha_{\hat{\mathcal{R}}^{(k)}}| \geqslant \delta$ (*cf.* (6.14)) and

$$|J| \overset{(6.16)}{=} \Big|\bigcup_{i=1}^{r} \mathcal{K}_{k+1}(\hat{\mathcal{Q}}^{(k)}_i)\Big| \overset{(6.13)}{\geqslant} \delta|\mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})| \overset{(6.16)}{=} \delta|I|,$$

Proposition 6.8 yields

$$\sum_{K'\in I} d^2_{K'} \geqslant \frac{1}{|I|}\Big(\sum_{K'\in I} d_{K'}\Big)^2 + \delta^3|I|. \tag{6.17}$$

In view of (6.16) and since

$$\begin{aligned}\sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})} d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{X}}^{(k)}(K')\big) &= \sum_{\substack{\hat{\mathcal{X}}^{(k)}\in\hat{\mathscr{X}}^{(k)}\\ \hat{\mathcal{X}}^{(k)}\subseteq\hat{\mathcal{R}}^{(k)}}} d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{X}}^{(k)}\big)|\mathcal{K}_{k+1}(\hat{\mathcal{X}}^{(k)})|\\ &= d\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{R}}^{(k)}\big)|\mathcal{K}_{k+1}(\mathcal{R}^{(k)})|,\end{aligned}$$

we can reformulate inequality (6.17) to

$$\sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})} d^2\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{X}}^{(k)}(K')\big) \geqslant \sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})} \big(d^2\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{R}}^{(k)}\big) + \delta^3\big). \tag{6.18}$$

Note that (6.18) holds for every irregular polyad $\hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$. Summing over all such polyads, inequality (6.18) together with (6.12) yields

$$\sum_{\hat{\mathcal{R}}^{(k)}\in\hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}}\sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})} d^2\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{X}}^{(k)}(K')\big) \geqslant \sum_{\hat{\mathcal{R}}^{(k)}\in\hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}}\sum_{K'\in\mathcal{K}_{k+1}(\hat{\mathcal{R}}^{(k)})} d^2\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{R}}^{(k)}\big) + \frac{\delta^4}{2}|V|^{k+1}.$$

Since $\mathscr{X}^{(k)}$ refines $\mathscr{R}^{(k)}$, we can apply Proposition 6.4 to every $\hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)} \setminus \hat{\mathscr{R}}^{(k)}_{\mathrm{irr},0}$ which is contained in $\mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$ and we infer

$$\sum_{K'\in\mathrm{Cross}_{k+1}(\mathscr{R}^{(1)}_0)} d^2\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{X}}^{(k)}(K')\big) \geqslant \sum_{K'\in\mathrm{Cross}_{k+1}(\mathscr{R}^{(1)}_0)} d^2\big(\mathcal{H}^{(k+1)}_{\mathrm{irr}} \mid \hat{\mathcal{R}}(K')\big) + \frac{\delta^4}{2}|V|^{k+1}.$$

Finally, part (iii) of Proposition 6.7 follows from the last inequality and Proposition 6.4 applied to every $\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}$, $\mathcal{H}^{(k+1)} \neq \mathcal{H}^{(k+1)}_{\mathrm{irr}}$ and every $\hat{\mathcal{R}}^{(k)} \in \hat{\mathscr{R}}^{(k)}$ contained in $\mathrm{Cross}_k(\mathscr{R}^{(1)}_0)$. □

6.2. Proof of RL$(k+1)$

In what follows we give a proof of RL$(k+1)$ based on RAL(k), or more precisely, based on Lemma 4.2. In the next section, Section 6.2.1, we define all constants involved in the proof of this

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implication. In Section 6.2.2 we state the so called *index pumping lemma* and deduce $\mathrm{RL}(k+1)$ from it. We then prove the index pumping lemma in Section 6.3.

6.2.1. Constants. We first recall the quantification of $\mathrm{RL}(k+1)$, Lemma 4.1, for $k+1$:

$$\forall\, o_{\mathrm{RL}},\ s_{\mathrm{RL}},\ \eta_{\mathrm{RL}},\ \delta_{k+1,\mathrm{RL}},\ r_{\mathrm{RL}}\colon \mathbb{N}^k \to \mathbb{N},\ \delta_{\mathrm{RL}}\colon \mathbb{N}^k \to (0,1] \qquad \exists\, \mu_{\mathrm{RL}} > 0,\ t_{\mathrm{RL}},\ n_{\mathrm{RL}}.$$

So let positive integers o_{RL} and s_{RL}, positive reals η_{RL} and $\delta_{k+1,\mathrm{RL}}$, and positive functions r_{RL} and δ_{RL} be given. Without loss of generality we assume that

$$\eta_{\mathrm{RL}} \leqslant \delta_{k+1,\mathrm{RL}}/2 \quad \text{and } r_{\mathrm{RL}} \text{ and } \delta_{\mathrm{RL}} \text{ are monotone in every variable.} \tag{6.19}$$

For the definition of the promised constants μ_{RL}, t_{RL}, and n_{RL} we need auxiliary sequences of constants t_i, o_i, s_i, η_i, and v_i and a sequence of functions $\varepsilon_i\colon \mathbb{N}^{k-1} \to (0,1]$ for $i \geqslant 0$. First we define t_0:

$$t_0 = \min\left\{ t \geqslant \left\lceil \tfrac{(k+1)^{k+1}}{2\eta_{\mathrm{RL}}} \right\rceil :\ (o_{\mathrm{RL}})!\ \text{ divides }\ t \right\} > o_{\mathrm{RL}}. \tag{6.20}$$

Without loss of generality we may assume that the given function δ_{RL} is bounded for every $\boldsymbol{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$ by

$$\delta_{\mathrm{RL}}(\boldsymbol{a}) \leqslant \frac{\delta_{k+1,\mathrm{RL}}^4}{24t_0} < \frac{2}{t_0}, \quad \text{and} \quad \delta_{\mathrm{RL}}(\boldsymbol{a}) \leqslant \frac{3}{2a_k}. \tag{6.21}$$

For convenience we define the following integer-valued function $f\colon \mathbb{N} \to \mathbb{N}$:

$$f(s) = \min\left\{ x \in \mathbb{N}:\ x \geqslant \frac{24t_0 s}{\delta_{k+1,\mathrm{RL}}^4} \text{ and } (t_0)!\ \text{ divides } x \right\}. \tag{6.22}$$

We then define o_i, s_i, η_i, and v_i in terms of t_i, $\delta_{k+1,\mathrm{RL}}$, η_{RL}, and $r_{\mathrm{RL}}(t_i, \dots, t_i)$:

$$o_i = t_0, \quad s_i = t_i^{2^k} 2^{r_{\mathrm{RL}}(t_i,\dots,t_i)t_i^{2^{k+1}}}, \quad \eta_i = \eta_{\mathrm{RL}}, \quad \text{and} \quad v_i = \tfrac{\delta_{k+1,\mathrm{RL}}^4}{12(k+1)s_{\mathrm{RL}}s_i^{k+1}}. \tag{6.23}$$

Moreover, for $i \geqslant 0$, we define the function

$$\varepsilon_i\colon \mathbb{N}^{k-1} \to (0,1],$$

defined for every $\boldsymbol{a} = (a_1, \dots, a_{k-1}) \in \mathbb{N}^{k-1}$ by

$$\varepsilon_i(\boldsymbol{a}) = \min\left\{ \tfrac{\delta_{\mathrm{RL}}(a_1,\dots,a_{k-1},f(s_i))}{18s_i}, \varepsilon_{\mathrm{DCL}}\left(k-1, k, \tfrac{1}{2}, \min_{2\leqslant j\leqslant k-1} \tfrac{1}{a_j}\right), \tfrac{1}{2f(s_i)}, \tfrac{\delta_{k+1,\mathrm{RL}}^4}{72s_i t_0} \right\}, \tag{6.24}$$

where $\varepsilon_{\mathrm{DCL}}$ is given by Theorem 3.1. Moreover, without loss of generality we assume that ε_i is monotone in every variable.

We then define t_{i+1} using $t_{\mathrm{RAL}}(o, s, \eta, v, \varepsilon(\cdot, \dots, \cdot))$ given by Lemma 4.2, and set

$$t_{i+1} = \max\{t_i, t_{\mathrm{RAL}}\big(o_i, s_i, \eta_i, v_i, \varepsilon_i(\cdot,\dots,\cdot)\big), f(s_i)\} \overset{(6.22)}{\geqslant} s_i. \tag{6.25}$$

This concludes the definition of the sequences t_i, o_i, s_i, η_i, v_i and $\varepsilon_i\colon \mathbb{N}^{k-1} \to (0,1]$ for $i \geqslant 0$. We note that the sequence t_i is monotone by definition. In a similar way we define the monotone sequences μ_i for $i \geqslant 1$ by setting $\mu_1 = \delta_{\mathrm{RL}}(t_0, \dots, t_0)$ and

$$\mu_{i+1} = \min\left\{ \mu_i, \mu_{\mathrm{RAL}}\big(o_i, s_i, \eta_i, v_i, \varepsilon_i(\cdot,\dots,\cdot)\big), \frac{\delta_{\mathrm{RL}}(\overbrace{t_{i+1},\dots,t_{i+1}}^{(k-1)\text{-times}}, f(s_i))}{12t_{i+1}^{2^k}} \right\}, \tag{6.26}$$

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and we define n_i by setting $n_1 = 1$ and

$$\begin{aligned} n_{i+1} = \max\Big\{ & n_i, n_{\mathrm{RAL}}\big(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot,\dots,\cdot)\big), t_{i+1} m_{\mathrm{DCL}}\big(k-1, k, \tfrac{1}{2}, \tfrac{1}{t_{i+1}}\big), \\ & t_{i+1} m_{\mathrm{SL}}\big(k, f(s_i), 1, \varepsilon_i(t_{i+1},\dots,t_{i+1}), \tfrac{1}{f(s_i)}, \tfrac{1}{f(s_i)}\big), \\ & t_{i+1} m_{\mathrm{SL}}\big(k, f(s_i), 1, \tfrac{1}{3}\delta_{\mathrm{RL}}(t_{i+1},\dots,t_{i+1}, f(s_i)), \\ & \qquad \tfrac{1}{2}\delta_{\mathrm{RL}}(t_{i+1},\dots,t_{i+1}, f(s_i)), \tfrac{1}{f(s_i)}\big)\Big\}. \end{aligned} \tag{6.27}$$

We also define auxiliary constants

$$\begin{aligned} \mu^* &= \min\Big\{ \min_{2\leqslant j\leqslant k} \{\varepsilon_{\mathrm{DCL}}\big(j, j+1, \tfrac{1}{2}, \tfrac{1}{t_0}\big)\},\ \mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}\Big\}, \\ n^* &= \max_{2\leqslant j\leqslant k} \max\{t_0 m_{\mathrm{DCL}}\big(j, j+1, \tfrac{1}{2}, \tfrac{1}{t_0}\big),\ t_0 m_{\mathrm{SL}}\big(j+1, o_{\mathrm{RL}}, 1, \tfrac{\mu^*}{3}, 1, \tfrac{1}{o_{\mathrm{RL}}}\big)\}. \end{aligned} \tag{6.28}$$

Finally, we fix the constants μ_{RL}, t_{RL}, and n_{RL} promised by Lemma 4.1 in the following way:

$$\mu_{\mathrm{RL}} = \mu^*/(2t_0^{2^k}), \quad t_{\mathrm{RL}} = t_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}, \quad \text{and} \quad n_{\mathrm{RL}} = \max\{n^*, n_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}\}. \tag{6.29}$$

For the rest of this section let all constants and functions be fixed as stated in (6.20)–(6.29).

6.2.2. The index pumping lemma. Now let a set V, a family of partitions $\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ and a family of $(k+1)$-uniform hypergraphs $\mathscr{H}^{(k+1)}$ satisfying the assumptions (a)–(c) of $\mathrm{RL}(k+1)$ be given, *i.e.*,

(RL.a) $|V| = n \geqslant n_{\mathrm{RL}}$ and $(t_{\mathrm{RL}})!$ divided n,
(RL.b) $\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$ is an $(\eta^{\mathscr{O}}, \mu_{\mathrm{RL}}, \boldsymbol{a}^{\mathscr{O}})$-equitable (for some $\eta^{\mathscr{O}} > 0$) and o_{RL}-bounded family of partitions on V, and
(RL.c) $\mathscr{H}^{(k+1)} = \{\mathcal{H}_1^{(k+1)}, \dots, \mathcal{H}_{s_{\mathrm{RL}}}^{(k+1)}\}$ is a partition of $[V]^{k+1}$.

The main idea of the proof is to inductively define a sequence of families of partitions $\mathscr{R}_i = \mathscr{R}_i(k, \boldsymbol{a}^{\mathscr{R}_i})$ on V for $i \geqslant 0$, which will satisfy

(R_0.1) $\mathscr{R}_0 = \{\mathscr{R}_0^{(j)}\}_{j=1}^k$ is $(\eta_{\mathrm{RL}}, \mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil}, \boldsymbol{a}^{\mathscr{R}_0})$-equitable and t_0-bounded,
(R_0.2) $\mathscr{R}_0 \prec \mathscr{O}$,
(R_i.1) $\mathscr{R}_i = \{\mathscr{R}_i^{(j)}\}_{j=1}^k$ is an $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i}), \boldsymbol{a}^{\mathscr{R}_i})$-equitable and t_i-bounded, and
(R_i.2) $\mathscr{R}_i \prec \mathscr{R}_0$.

Note that due to the fact that $\mu_{\lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil} \leqslant \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_0})$ (*cf.* (6.26)), a family of partitions $\mathscr{R}_0$ which satisfies (R_0.1) and (R_0.2) also satisfies (R_i.1) and (R_i.2) for $i = 0$.

Moreover, we will show that if there is a hypergraph

$$\mathcal{H}^{(k+1)} \in \mathscr{H}^{(k+1)}$$

which is not $(\delta_{k+1,\mathrm{RL}}, *, r(\boldsymbol{a}^{\mathscr{R}_i}))$-regular w.r.t. $\mathscr{R}_i$, then $\mathscr{R}_{i+1}$ can be chosen in such a way that the index increases by $\delta^4_{k+1,\mathrm{RL}}/8$. More precisely, we will show the following so-called *index pumping lemma*, which proof merges some ideas from [26] and [20, Cleaning Phase I].

Lemma 6.9 (index pumping lemma). *Let $0 \leqslant i < \lceil 8/\delta^4_{k+1,\mathrm{RL}}\rceil$ be an integer and let $\mathscr{R}_0$ be a family of partitions satisfying* (R_0.1) *and* (R_0.2)*.*

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If $\mathscr{R}_i = \mathscr{R}_i(k, \boldsymbol{a}^{\mathscr{R}_i})$ *satisfies* (R_i.1) *and* (R_i.2) *and fails to satisfy* (H) *of* $\mathrm{RL}(k+1)$ *for* $r(\boldsymbol{a}^{\mathscr{R}_i})$*, then there exists a family of partitions* $\mathscr{R}_{i+1} = \mathscr{R}_{i+1}(k, \boldsymbol{a}^{\mathscr{R}_{i+1}})$ *satisfying* (R_{i+1}.1) *and* (R_{i+1}.2) *and*

$$\mathrm{ind}(\mathscr{R}_{i+1}^{(k)}) \geqslant \mathrm{ind}(\mathscr{R}_i^{(k)}) + \delta_{k+1,\mathrm{RL}}^4/8, \tag{6.30}$$

where the index is defined with respect to $\mathscr{R}_0^{(1)}$ *and* $\mathscr{H}^{(k+1)}$ (*cf. Definition 13*).

Next we deduce $\mathrm{RL}(k+1)$ (*i.e.*, Lemma 4.1) from Lemma 6.9. We then give the proof of Lemma 6.9 in Section 6.3.

Proof of Lemma 4.1. Suppose all constants are fixed as in Section 6.2.1 and let V, $\mathscr{O} = \mathscr{O}(k, \boldsymbol{a}^{\mathscr{O}})$, and $\mathscr{H}^{(k+1)}$ satisfying (RL.a)–(RL.c) be given. We have to ensure the existence of a family of partitions $\mathscr{P} = \mathscr{P}(k, \boldsymbol{a}^{\mathscr{P}})$ on V satisfying

(RL.P1) $\mathscr{P}$ is $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and t_{RL}-bounded,
(RL.P2) $\mathscr{P} \prec \mathscr{O}$, and
(RL.H) $\mathcal{H}_i^{(k+1)}$ is $(\delta_{k+1,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}$ for every $i \in [s_{\mathrm{RL}}]$.

Construction of a family $\mathscr{R}_0$. In view of Lemma 6.9 we first need an appropriate family of partitions $\mathscr{R}_0$. We distinguish two cases depending on the size of $\eta^{\mathscr{O}}$.

Case 1 ($\eta^{\mathscr{O}} \leqslant \eta_{\mathrm{RL}}$). In this case we simply set $\mathscr{R}_0 = \mathscr{O}$. It then follows from (RL.b) that $\mathscr{R}_0$ is $(\eta_{\mathrm{RL}}, \mu_{\lceil 8/\delta_{k+1,\mathrm{RL}}^4 \rceil}, \boldsymbol{a}^{\mathscr{R}_0})$-equitable, since $\mu_{\mathrm{RL}} \leqslant \mu^* \leqslant \mu_{\lceil 8/\delta_{k+1,\mathrm{RL}}^4 \rceil}$ by (6.28) and (6.29). Also $\mathscr{R}_0 = \mathscr{O}$ is o_{RL}-bounded by (RL.b) and, hence, it is t_0-bounded by (6.20). Therefore, $\mathscr{R}_0$ chosen this way satisfies (R_0.1). Moreover, (R_0.2) holds trivially. ◊

Case 2 ($\eta^{\mathscr{O}} > \eta_{\mathrm{RL}}$). We construct a refinement $\mathscr{R}_0$ of $\mathscr{O}$ so that $|\mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})| \geqslant (1-\eta_{\mathrm{RL}})\binom{n}{k+1}$. We construct $\mathscr{R}_0 = \{\mathscr{R}_0^{(1)}, \dots, \mathscr{R}_0^{(k)}\}$ inductively. More precisely, we show for every $j = 1, \dots, k$ that the following statement ($\mathfrak{S}_j$) holds:

($\mathfrak{S}_j$) there exist an $\big(\eta_{\mathrm{RL}}, \mu^*, (a_1^{\mathscr{R}_0}, \dots, a_j^{\mathscr{R}_0})\big)$-equitable and t_0-bounded family of partitions $\mathscr{R}_0(j) = \{\mathscr{R}_0^{(1)}, \dots, \mathscr{R}_0^{(j)}\}$ on V, which refines $\mathscr{O}(j) = \{\mathscr{O}^{(1)}, \dots, \mathscr{O}^{(j)}\}$.

Since $\mu^* \leqslant \mu_{\lceil 8/\delta_{k+1,\mathrm{RL}}^4 \rceil}$, it then follows that there is a family of partitions $\mathscr{R}_0$ so that (R_0.1) and (R_0.2) are satisfied.

Induction start $j = 1$. We split each vertex class $W \in \mathscr{O}^{(1)}$ into $t_0/a_1^{\mathscr{O}}$ classes of size $n/(a_1^{\mathscr{O}} t_0)$, where t_0 is given in (6.20). Note that $t_0/a_1^{\mathscr{O}}$ is an integer by definition of t_0 and $o_{\mathrm{RL}} \geqslant a_1^{\mathscr{O}}$. Moreover, $n/(a_1^{\mathscr{O}} t_0)$ is an integer due to the choice of $t_{\mathrm{RL}} \geqslant t_0 > o_{\mathrm{RL}} \geqslant a_1^{\mathscr{O}}$ (*cf.* (6.29) and (6.20)) and (RL.a). This defines the partition $\mathscr{R}_0^{(1)}$ with $a_1^{\mathscr{R}_0} = t_0$. Note that

$$|[V]^{k+1} \setminus \mathrm{Cross}_{k+1}(\mathscr{R}_0^{(1)})| \leqslant t_0 \binom{n/t_0}{2} n^{k-1} \leqslant \frac{n^{k+1}}{2t_0} \overset{(6.20)}{\leqslant} \eta_{\mathrm{RL}} \binom{n}{k+1}.$$

Consequently, $\mathscr{R}_0^{(1)}$ is an $\big(\eta_{\mathrm{RL}}, \mu^*, (a_1^{\mathscr{R}_0})\big)$-equitable, t_0-bounded refinement of $\mathscr{O}^{(1)}$, which establishes the induction start.

Induction step. Assume there exists an $\big(\eta_{\mathrm{RL}}, \mu^*, (a_1^{\mathscr{R}_0}, \dots, a_j^{\mathscr{R}_0})\big)$-equitable, t_0-bounded family of partitions $\mathscr{R}_0(j) = \{\mathscr{R}_0^{(1)}, \dots, \mathscr{R}_0^{(j)}\}$ refining $\mathscr{O}(j)$. We define $\mathscr{R}_0^{(j+1)}$ for each polyad $\hat{\mathcal{R}}^{(j)} \in \hat{\mathscr{R}}_0^{(j)}$. We set $a_{j+1}^{\mathscr{R}_0} = a_{j+1}^{\mathscr{O}}$ and in view of statement ($\mathfrak{S}_{j+1}$) we have to show that, for every $\hat{\mathcal{R}}^{(j)} \in$

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$\hat{\mathscr{R}}_0^{(j)}$, there exists a partition $\{\mathcal{R}_a^{(j+1)}:\ a\in[a_{j+1}^{\mathscr{R}_0}]\}$ of $\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})$ so that, for every $a\in[a_{j+1}^{\mathscr{R}_0}]$, the following two assertions hold:

(I) $\mathcal{R}_a^{(j+1)}$ is $(\mu^*, 1/a_{j+1}^{\mathscr{R}_0})$-regular w.r.t. $\hat{\mathcal{R}}^{(j)}$, and

(II) either $\mathcal{R}_a^{(j+1)}\subseteq \mathrm{Cross}_{j+1}(\mathscr{R}_0^{(1)})\setminus \mathrm{Cross}_{j+1}(\mathscr{O}^{(1)})$ or we have $\mathcal{R}_a^{(j+1)}\subseteq\mathcal{O}^{(j+1)}$ for some $\mathcal{O}^{(j+1)}\in\mathscr{O}^{(j+1)}$.

So let $\hat{\mathcal{R}}^{(j)}\in\hat{\mathscr{R}}_0^{(j)}$ and let $\boldsymbol{\mathcal{R}}$ be the corresponding $(n/a_1^{\mathscr{R}_0}, j+1, j)$-complex, *i.e.*, $\boldsymbol{\mathcal{R}}=\boldsymbol{\mathcal{R}}(J')=\{\hat{\mathcal{R}}^{(h)}(J')\}_{h=1}^{j}$ for any $J'\in\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})$ (see (2.1)). From the induction assumption we infer that $\boldsymbol{\mathcal{R}}$ is an $(\mu^*,(1/a_1^{\mathscr{R}_0},\dots,1/a_j^{\mathscr{R}_0}))$-regular complex. Therefore, by the choice of μ^* and $n_{\mathrm{RL}}\geqslant n^*$ in (6.28) and (6.29) we can apply Theorem 3.1 and infer that

$$|\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})|\geqslant\frac{1}{2}\prod_{h=2}^{j}\left(\frac{1}{a_h^{\mathscr{R}_0}}\right)^{\binom{j+1}{h}}\times\left(\frac{n}{a_1^{\mathscr{R}_0}}\right)^{j+1}\geqslant\frac{n^{j+1}}{2t_0^{2^{j+1}}}. \tag{6.31}$$

Case 2.1 ($\hat{\mathcal{R}}^{(j)}\not\subseteq\mathrm{Cross}_j(\mathscr{O}^{(1)})$)**.** In this case we simply apply the slicing lemma, Proposition 3.5, with

$$j_{\mathrm{SL}}=j+1,\quad s_{0,\mathrm{SL}}=o_{\mathrm{RL}},\quad r_{\mathrm{SL}}=1,\quad \delta_{0,\mathrm{SL}}=\mu^*/3,\quad \varrho_{0,\mathrm{SL}}=1,\quad\text{and}\quad p_{0,\mathrm{SL}}=1/o_{\mathrm{RL}},$$
$$\text{to}\quad \hat{\mathcal{P}}_{\mathrm{SL}}^{(j)}=\hat{\mathcal{R}}^{(j)},\quad\text{and}\quad \mathcal{P}_{\mathrm{SL}}^{(j+1)}=\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})\quad\text{with}$$
$$s_{\mathrm{SL}}=a_{j+1}^{\mathscr{O}},\quad \delta_{\mathrm{SL}}=\mu^*/3,\quad \varrho_{\mathrm{SL}}=1,\quad\text{and}\quad \left\{p_{\xi,\mathrm{SL}}=1/a_{j+1}^{\mathscr{O}}:\ \xi\in[a_{j+1}^{\mathscr{O}}]\right\}.$$

It follows from (6.31) and the choice of n^* in (6.28) that all assumptions of Proposition 3.5 are satisfied for this choice of parameters. Consequently, there exist a partition of $\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})$ into $a_{j+1}^{\mathscr{O}}$ distinct $(n/a_1^{\mathscr{R}_0}, j+1, j+1)$-hypergraphs which are $(\mu^*, 1/a_{j+1}^{\mathscr{R}_0})$-regular w.r.t. $\hat{\mathcal{R}}^{(j)}$, *i.e.*, (I) holds. Moreover, since we assume $\hat{\mathcal{R}}^{(j)}\not\subseteq\mathrm{Cross}_j(\mathscr{O}^{(1)})$ each of these $(n/a_1^{\mathscr{R}_0}, j+1, j+1)$-hypergraphs is contained in $\mathrm{Cross}_{j+1}(\mathscr{R}_0^{(1)})\setminus\mathrm{Cross}_{j+1}(\mathscr{O}^{(1)})$ and (II) holds. ◊

Case 2.2 ($\hat{\mathcal{R}}^{(j)}\subseteq\mathrm{Cross}_j(\mathscr{O}^{(1)})$)**.** Then there exists some $\hat{\mathcal{O}}^{(j)}\in\hat{\mathscr{O}}^{(j)}$ such that $\hat{\mathcal{R}}^{(j)}\subseteq\hat{\mathcal{O}}^{(j)}$, since $\mathscr{R}_0(j)\prec\mathscr{O}(j)$ by the induction hypothesis. Moreover, there exists a family

$$\left\{\mathcal{O}_1^{(j+1)},\dots,\mathcal{O}_{a_{j+1}^{\mathscr{O}}}^{(j+1)}\right\}\subseteq\mathscr{O}^{(j+1)}$$

of $(\mu_{\mathrm{RL}}, 1/a_{j+1}^{\mathscr{O}})$-regular (w.r.t. $\hat{\mathcal{O}}^{(j)}$) $(n/a_1^{\mathscr{O}}, j+1, j+1)$-hypergraphs partitioning $\mathcal{K}_{j+1}(\hat{\mathcal{O}}^{(j)})$. Hence (6.31) yields that $\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})\cap\mathcal{O}_a^{(j+1)}$ is $(2t_0^{2^{j+1}}\mu_{\mathrm{RL}}, 1/a_{j+1}^{\mathscr{O}})$-regular w.r.t. $\hat{\mathcal{R}}^{(j)}$ for every $a\in[a_{j+1}^{\mathscr{O}}]$. Therefore, from the choice of μ_{RL} in (6.29) we infer that

$$\left\{\mathcal{R}_a^{(j+1)}=\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})\cap\mathcal{O}_a^{(j+1)}:\ a\in[a_{j+1}^{\mathscr{O}}]\right\}$$

is a partition of $\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})$ which satisfies (I). Moreover, (II) holds trivially. ◊

In both cases, Case 2.1 and Case 2.2, we defined a partition of $\mathcal{K}_{j+1}(\hat{\mathcal{R}}^{(j)})$ which satisfies (I) and (II). Repeating the argument for every $\hat{\mathcal{R}}^{(j)}\in\hat{\mathscr{R}}_0^{(j)}$ gives rise to $\mathscr{R}_0^{(j+1)}$ and establishes the induction step. Consequently, there exists a partition $\mathscr{R}_0$ which satisfies ($\mathrm{R}_0.1$) and ($\mathrm{R}_0.2$) in this case, Case 2. ◊

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Having constructed an appropriate family of partitions $\mathscr{R}_0$, the rest of the proof of Lemma 4.1 is based on successive applications of Lemma 6.9. This idea was introduced by Szemerédi in [31] and also used in [5, 6, 10, 13, 14, 16, 26].

Since $\mathscr{R}_0$ was constructed in such a way that (R_0.1) and (R_0.2) hold, we note that due to

$$\mu_{\lceil 8/\delta_{k+1,\mathrm{RL}}^4\rceil} \overset{(6.26)}{\leqslant} \delta_{\mathrm{RL}}(t_0,\dots,t_0) \overset{(6.19)}{\leqslant} \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_0}),$$

and $t_0 \leqslant t_{\mathrm{RL}}$ (*cf.* (6.25) and (6.29)) the partition $\mathscr{P} = \mathscr{R}_0$ satisfies (RL.P1) and (RL.P2). If (RL.H) holds as well, then we are done.

Otherwise we iterate Lemma 6.9 and infer the existence of a sequence of partitions $\mathscr{R}_i$ for $i \geqslant 0$, which satisfy (R_i.1) and (R_i.2). It then follows from Fact 6.2 and (6.30) that there must be some $0 \leqslant i_0 \leqslant \lceil 8/\delta_{k+1,\mathrm{RL}}^4\rceil$ such that $\mathscr{R}_{i_0}$ also admits (RL.H) for $r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_{i_0}})$. Since $t_i \leqslant t_{\mathrm{RL}}$ (*cf.* (6.25) and (6.29)) and and $\mathscr{R}_i \prec \mathscr{R}_0 \prec \mathscr{O}$ (*cf.* (R_i.2) and (R_0.2)) for every $0 \leqslant i \leqslant \lceil 8/\delta_{k+1,\mathrm{RL}}^4\rceil$, $\mathscr{P} = \mathscr{R}_{i_0}$ satisfies (RL.P1), (RL.P2), and (RL.H). This concludes the proof of Lemma 4.1 based on Lemma 6.9. □

6.3. Proof of the index pumping lemma

We prove Lemma 6.9 in this section. The proof is mainly based on Lemma 4.2 and the propositions developed in Section 6.1.

Proof of Lemma 6.9. Recall the definition of the constants and functions in (6.20)–(6.29) in Section 6.2.1. Let $0 \leqslant i < \lceil 8/\delta_{k+1,\mathrm{RL}}^4\rceil$ be some integer and suppose $\mathscr{R}_i = \mathscr{R}_i(k, \boldsymbol{a}^{\mathscr{R}_i})$ satisfies (R_i.1) and (R_i.2) and fails to satisfy (H) of $\mathrm{RL}(k+1)$ for $r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i})$. In other words

($\neg H_i$) there exists $s_0 \in [s_{\mathrm{RL}}]$ such that $\mathcal{H}_{s_0}^{(k+1)}$ is $\big(\delta_{k+1,\mathrm{RL}}, *, r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i})\big)$-irregular w.r.t. $\mathscr{R}_i$.

Then V, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, and $\mathscr{R}_i^{(k)}$ satisfy the assumptions of Proposition 6.7 with $\delta = \delta_{k+1,\mathrm{RL}}$ and $r = r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i})$, due to ($\mathrm{R}_0$.1) combined with (6.19) and ($\neg H_i$). Consequently, there exists a partition $\mathscr{X}^{(k)}$ of $[V]^k$ satisfying the conclusions (i)–(iii) of Proposition 6.7, *i.e.*,

(P.6.7.i) $\mathscr{X}^{(k)} \prec \mathscr{R}_i^{(k)} \prec \mathscr{R}_0^{(k)}$ (*cf.* (R_i.2) for the second '$\prec$'),

(P.6.7.ii) $|\mathscr{X}^{(k)}| \leqslant |\mathscr{R}_i^{(k)}| \times 2^{r_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_i})\times|\hat{\mathscr{R}}_i^{(k)}|} \leqslant s_i$ (*cf.* t_i-boundedness of $\mathscr{R}_i$ in (R_i.1), the monotonicity of $r_{\mathrm{RL}}(\cdot,\dots,\cdot)$ in (6.19), and the definition of s_i in (6.23)), and

(P.6.7.iii) $\mathrm{ind}(\mathscr{X}^{(k)}) \geqslant \mathrm{ind}(\mathscr{R}_i^{(k)}) + \delta_{k+1,\mathrm{RL}}^4/2$.

The next step is to apply $\mathrm{RAL}(k)$, Lemma 4.2 to V, $\mathscr{O} = \mathscr{R}_0$, and $\mathscr{H}^{(k)} = \mathscr{X}^{(k)}$, with constants o_i, s_i, η_i, ν_i, and the function $\varepsilon_i \colon \mathbb{N}^{k-1} \to (0,1]$ defined in (6.20)–(6.24). For this we have to check the assumptions of Lemma 4.2:

(RAL.a) $|V| = n \geqslant n_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot,\dots,\cdot))$ and $(t_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot,\dots,\cdot)))!$ divides n,

(RAL.b) $\mathscr{R}_0 = \mathscr{R}_0(k, \boldsymbol{a}^{\mathscr{R}_0})$ is an $(\eta', \mu', \boldsymbol{a}^{\mathscr{R}_0})$-equitable family of partitions (for some $\eta' > 0$ and $\mu' \leqslant \mu_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot,\dots,\cdot))$) and o_i-bounded, and

(RAL.c) $s' = |\mathscr{X}^{(k)}| \leqslant s_i$ and $\mathscr{X}^{(k)} \prec \mathscr{R}_0^{(k)}$.

Property (RAL.a) is implied by assumption (RL.a) and the fact that, for $i < \lceil 8/\delta_{k+1,\mathrm{RL}}^4\rceil$,

$$n_{\mathrm{RL}} \overset{(6.29)}{\geqslant} n_{\lceil 8/\delta_{k+1,\mathrm{RL}}^4\rceil} \overset{(6.27)}{\geqslant} n_{i+1} \overset{(6.27)}{\geqslant} n_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot,\dots,\cdot)),$$

and that the same line of inequalities holds with n replaced by t.

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It follows from the definition of o_i in (6.23) and (R_0.1) that $\mathscr{R}_0$ is o_i-bounded. Moreover, (R_0.1) and (6.26) imply the required equitability of $\mathscr{R}_0$, which yields (RAL.b).

Finally, (RAL.c) follows immediately from (P.6.7.i) and (P.6.7.ii).

Consequently, we can apply Lemma 4.2 to V, $\mathscr{O} = \mathscr{R}_0$, and $\mathscr{H}^{(k)} = \mathscr{X}^{(k)}$, with constants o_i, s_i, η_i, ν_i, and $\varepsilon_i : \mathbb{N}^{k-1} \to (0,1]$. Lemma 4.2 then asserts that there exist a family of partitions $\mathscr{S} = \mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$, and a partition $\mathscr{Y}^{(k)} = \{\mathcal{Y}_1^{(k)}, \dots, \mathcal{Y}_{s'}^{(k)}\}$ of $[V]^k$ so that

(RAL.S1) $\mathscr{S}$ is $(\eta_i, \varepsilon_i(\boldsymbol{a}^{\mathscr{S}}), \boldsymbol{a}^{\mathscr{S}})$-equitable and t_{i+1}-bounded family of partitions (since $t_{i+1} \geqslant t_{\mathrm{RAL}}(o_i, s_i, \eta_i, \nu_i, \varepsilon_i(\cdot, \dots, \cdot))$ by (6.25)),

(RAL.S2) $\mathscr{S} \prec \mathscr{R}_0(k-1) = \{\mathscr{R}_0^{(j)}\}_{j=1}^{k-1}$,

(RAL.Y1) $\mathcal{Y}_\ell^{(k)}$ is perfectly $(\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}))$-regular w.r.t. $\mathscr{S}$ for every $\ell \in [s']$,

(RAL.Y2) $|\mathcal{Y}_\ell^{(k)} \triangle \mathcal{X}_\ell^{(k)}| \leqslant \nu_i n^k$ for every $\ell \in [s']$, and

(RAL.Y3) if $\mathcal{X}_\ell^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ then $\mathcal{Y}_\ell^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ for every $\ell \in [s']$ and $\mathscr{Y}^{(k)} \prec \mathscr{R}_0^{(k)} \prec \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$.

In particular, (P.6.7.i) and (RAL.Y3) show that $\mathscr{X}^{(k)}$ and $\mathscr{Y}^{(k)}$ refine $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$, respectively. Hence, due to (RAL.Y2) and the first part of (RAL.Y3) the assumptions of Proposition 6.3 are satisfied for V, $\mathscr{R}_0^{(1)}$, $\mathscr{H}^{(k+1)}$, $\mathscr{X}^{(k)}$, $\mathscr{Y}^{(k)}$, $s_{\mathrm{P.6.3}} = s' \leqslant s_i$, and $\nu_{\mathrm{P.6.3}} = \nu_i$ and, consequently, Proposition 6.3 yields

$$\begin{aligned}\mathrm{ind}(\mathscr{Y}^{(k)}) &\geqslant \mathrm{ind}(\mathscr{X}^{(k)}) - 3(k+1)s_{\mathrm{RL}}s_i^{k+1}\nu_i \\ &\overset{(6.23)}{\geqslant} \mathrm{ind}(\mathscr{X}^{(k)}) - \tfrac{\delta^4_{k+1,\mathrm{RL}}}{4} \overset{\text{(P.6.7.iii)}}{\geqslant} \mathrm{ind}(\mathscr{R}_i^{(k)}) + \tfrac{\delta^4_{k+1,\mathrm{RL}}}{4}. \quad (6.32)\end{aligned}$$

Our next temporary goal is to construct a partition $\mathscr{Z}^{(k)}$ of $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ which forms a family of partitions together with $\mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$. This means that such an $\mathscr{Z}^{(k)}$ has to satisfy two conditions: it must partition $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ and it must refine $\{\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) : \hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$. The partition $\mathscr{Y}^{(k)}$ fails to satisfy any of these two requirements. It partitions all of $[V]^k$ (rather than only $\mathrm{Cross}_k(\mathscr{S}^{(1)})$) and, more importantly, we cannot ensure that it refines $\{\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) : \hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$. However, we easily 'fix' these shortcomings of $\mathscr{Y}^{(k)}$ and define $\mathscr{Z}^{(k)}$ as follows:

$$\mathscr{Z}^{(k)} = \{\mathcal{Y}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) : \mathcal{Y}^{(k)} \in \mathscr{Y}^{(k)} \text{ and } \hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}. \quad (6.33)$$

For convenience we set for every $\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$

$$\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)}) = \{\mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)} : \mathcal{Z}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \neq \varnothing\}. \quad (6.34)$$

The partition $\mathscr{Z}^{(k)}$ has the following properties which we verify below.

(Z1) $\mathscr{Z}^{(k)}$ partitions $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ and $\mathscr{Z}^{(k)} \prec \{\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) : \hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$,

(Z2) $|\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)})| \leqslant s_i$ for every $\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$, and

(Z3) for every $\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$ and $\mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)})$ we have that $\mathcal{Z}^{(k)}$ is $(\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}))$-regular w.r.t. $\hat{\mathcal{S}}^{(k-1)}$,

(Z4) $\mathscr{Z}^{(k)} \prec \mathscr{R}_0^{(k)} \prec \mathrm{Cross}_k(\mathscr{R}_0^{(1)})$, and

(Z5) $\mathrm{ind}(\mathscr{Z}^{(k)}) \geqslant \mathrm{ind}(\mathscr{R}_i^{(k)}) + \delta^4_{k+1,\mathrm{RL}}/4$.

Property (Z1) follows from the fact that $\mathscr{Y}^{(k)}$ partitions all of $[V]^k$ and the definition of $\mathscr{Z}^{(k)}$ in (6.33). Then (Z2) is an immediate consequence of (6.34) and $|\mathscr{Y}^{(k)}| = s' \leqslant s_i$ (*cf.* (RAL.c)). We also note that (Z3) is simply a reformulation of (RAL.Y1).

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Hence, it is only left to verify (Z4) and (Z5). First we consider (Z4). For that we first note that $\{\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\colon \hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$ partitions a superset of $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ due to (RAL.S2). Consequently, (Z4) follows from the definition of $\mathscr{Z}^{(k)}$ in (6.33) and (RAL.Y3).

Finally we focus on (Z5). For that we consider the restriction of $\mathscr{Y}^{(k)}$ on $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$, *i.e.*,

$$\mathscr{Y}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})} = \{\mathcal{Y}^{(k)} \in \mathscr{Y}^{(k)}\colon \mathcal{Y}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})\}.$$

It follows from Definition 13 that the index of $\mathscr{Y}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}$ w.r.t. $\mathscr{R}_0^{(1)}$ and $\mathscr{H}^{(k+1)}$ satisfies

$$\mathrm{ind}\big(\mathscr{Y}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}\big) = \mathrm{ind}(\mathscr{Y}^{(k)}) \overset{(6.32)}{\geqslant} \mathrm{ind}(\mathscr{R}_i^{(k)}) + \tfrac{\delta^4_{k+1,\mathrm{RL}}}{4}. \tag{6.35}$$

On the other hand, in view of (Z4) the restriction of $\mathscr{Z}^{(k)}$ on $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$,

$$\mathscr{Z}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})} = \{\mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}\colon \mathcal{Z}^{(k)} \subseteq \mathrm{Cross}_k(\mathscr{R}_0^{(1)})\},$$

is a partition of $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$, and therefore $\mathrm{ind}(\mathscr{Z}^{(k)}) = \mathrm{ind}\big(\mathscr{Z}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}\big)$. Moreover, we observe that $\mathscr{Z}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})} \prec \mathscr{Y}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}$ due to (6.33).

Finally, Proposition 6.5 then yields (Z5):

$$\mathrm{ind}(\mathscr{Z}^{(k)}) = \mathrm{ind}\big(\mathscr{Z}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}\big) \geqslant \mathrm{ind}\big(\mathscr{Y}^{(k)}|_{\mathrm{Cross}_k(\mathscr{R}_0^{(1)})}\big) \overset{(6.35)}{\geqslant} \mathrm{ind}(\mathscr{R}_i^{(k)}) + \tfrac{\delta^4_{k+1,\mathrm{RL}}}{4}.$$

Having verified (Z1)–(Z5) we come to the last part of the proof and define the family of partitions $\mathscr{R}_{i+1}$. The careful reader (who has managed not to get lost in details) will note that due to (Z1) the partition $\mathscr{Z}^{(k)}$ together with $\mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$ forms a family of partitions on V. Moreover, due to (RAL.S2) and (Z4) it satisfies (R_{i+1}.2) and due to (RAL.S1), (Z2), and (Z3) it 'almost' satisfies (R_{i+1}.1). But unfortunately, the densities of the $\mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}$ vary and thus this family of partitions $\mathscr{Z}^{(k)} \cup \mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$ is not equitable. In the final step of this proof we derive $\mathscr{R}_{i+1}$ from $\mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}}) \cup \mathscr{Z}^{(k)}$ by 'cleaning the imperfections' of $\mathscr{Z}^{(k)}$ mentioned above. For that we will use the following claim, which somewhat dry proof is based on repeated applications of Proposition 3.5.

Claim 6.10. *There exist a partition $\mathscr{T}^{(k)}$ of $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ such that*

(T1) $\mathscr{T}^{(k)} \prec \{\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\colon \hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}\}$,

(T2) $|\{\mathcal{T}^{(k)} \in \mathscr{T}^{(k)}\colon \mathcal{T}^{(k)} \subseteq \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})\}| = f(s_i)$ *for every fixed* $\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$,

(T3) *every* $\mathcal{T}^{(k)} \in \mathscr{T}^{(k)}$ *is* $\big(\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), 1/f(s_i)\big)$*-regular w.r.t. the unique* $\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$ *which satisfies* $\mathcal{T}^{(k)} \subseteq \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})$,

(T4) $\mathscr{T}^{(k)} \prec \mathscr{R}_0^{(k)}$, *and*

(T5) $\mathscr{T}^{(k)}$ *is a* $(\delta^4_{k+1,\mathrm{RL}}/8)$*-refinement of* $\mathscr{Z}^{(k)}$.

We first finish the proof of Lemma 6.9 and give the proof of Claim 6.10, which makes use of (Z1)–(Z4), afterwards. In order to conclude the proof of Lemma 6.9 we have to define a family of partitions $\mathscr{R}_{i+1}$ on V, which satisfies (R_{i+1}.1), (R_{i+1}.2), and (6.30). With this in mind

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we set

$$\boldsymbol{a}^{\mathscr{R}_{i+1}} = \big(a_1^{\mathscr{S}}, \dots, a_{k-1}^{\mathscr{S}}, f(s_i)\big),$$
$$\mathscr{R}_{i+1}^{(j)} = \begin{cases} \mathscr{S}^{(j)} & \text{for } j \in [k-1] \\ \mathscr{T}^{(k)} & \text{for } j = k \end{cases} \quad \text{and} \quad \mathscr{R}_{i+1}(k, \boldsymbol{a}^{\mathscr{R}_{i+1}}) = \{\mathscr{R}_{i+1}^{(j)}\}_{j=1}^{k}.$$

We now show first that $\mathscr{R}_{i+1} = \mathscr{R}_{i+1}(k, \boldsymbol{a}^{\mathscr{R}_{i+1}})$ is a family of partitions on V. Because $\mathscr{S}(k-1, \boldsymbol{a}^{\mathscr{S}})$ is a family of partitions on V, we only have to verify that $\mathscr{R}_{i+1}^{(k)} = \mathscr{T}^{(k)}$ fulfils both requirements of part (ii) of Definition 7. However, this is immediate from (T1) and (T2).

Next we consider (R_{i+1}.1). Note that (RAL.S1) (combined with (6.24)) and (T3) show that $\mathscr{R}_{i+1}$ is $(\eta_{\mathrm{RL}}, \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{R}_{i+1}}), \boldsymbol{a}^{\mathscr{R}_{i+1}})$-equitable. Moreover, (RAL.S1) and the choice of $t_{i+1} \geqslant f(s_i)$ in (6.25) imply that $\max_{j\in[k]} a_j^{\mathscr{R}_{i+1}} \leqslant t_{i+1}$. In other words, $\mathscr{R}_{i+1}$ is t_{i+1}-bounded and (R_{i+1}.1) holds.

The property (R_{i+1}.2) follows from (RAL.S2) and (T4) and (6.30) is a consequence of (Z5) and (T5), combined with Proposition 6.6.

Hence $\mathscr{R}_{i+1}$ has the desired properties and we conclude the proof of Lemma 6.9 based on Claim 6.10. □

Proof of Claim 6.10. We have to show that there is a partition $\mathscr{T}^{(k)}$ of $\mathrm{Cross}_k(\mathscr{S}^{(1)})$ satisfying (T1)–(T5). For technical reasons we first extend the partition $\mathscr{R}_0^{(k)}$ from a partition of $\mathrm{Cross}_k(\mathscr{R}_0^{(1)})$ to a partition of $[V]^k$ and we set

$$\widetilde{\mathcal{R}}^{(k)} = [V]^k \setminus \mathrm{Cross}_k(\mathscr{R}_0^{(1)}) \quad \text{and} \quad \widetilde{\mathscr{R}}_0^{(k)} = \mathscr{R}_0^{(k)} \cup \widetilde{\mathcal{R}}^{(k)}. \tag{6.36}$$

In view of (T1) and (T4) it seems natural to define $\mathscr{T}^{(k)}$ separately for every pair

$$\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}, \quad \mathcal{R}^{(k)} \in \widetilde{\mathscr{R}}_0^{(k)} \quad \text{satisfying} \quad \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)} \neq \varnothing. \tag{6.37}$$

In fact, we will prove the following claim.

Claim 6.10′. *For every pair $\hat{\mathcal{S}}^{(k-1)}$, $\mathcal{R}^{(k)}$ satisfying* (6.37) *there exists a partition $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ of $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}$ satisfying the following properties:*

(T2′)

$$|\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})| = \begin{cases} \frac{f(s_i)}{a_k^{\mathscr{R}_0}} & \text{if } \mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)}, \\ f(s_i) & \text{if } \mathcal{R}^{(k)} = \widetilde{\mathcal{R}}^{(k)}, \end{cases}$$

[4]

(T3′) *every $\mathcal{T}^{(k)} \in \mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ is $\big(\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), 1/f(s_i)\big)$-regular w.r.t. $\hat{\mathcal{S}}^{(k-1)}$,*

(T5′) *and*

$$\Big|\bigcup\{\mathcal{T}^{(k)} : \mathcal{T}^{(k)} \in \mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)}) \text{ and } \mathcal{T}^{(k)} \nsubseteq \mathcal{Z}^{(k)}\ \forall \mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}\}\Big| \leqslant \tfrac{\delta_{k+1,\mathrm{RL}}^4}{8} |\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}|.$$

[4] Note that $f(s_i)/a_k^{\mathscr{R}_0}$ is an integer since $a_k^{\mathscr{R}_0} \leqslant t_0$ (*cf.* (R_0.1)) and because the definition of the function $f(\cdot)$ in (6.22) ensures that $f(s_i)$ is a multiple of $(t_0)!$.

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Before we verify Claim 6.10′, we deduce Claim 6.10 from it. So let $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ be given for every $\hat{\mathcal{S}}^{(k-1)}$, $\mathcal{R}^{(k)}$ satisfying (6.37). We then set

$$\mathscr{T}^{(k)} = \bigcup \{ \mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)}) \colon \hat{\mathcal{S}}^{(k-1)} \text{ and } \mathcal{R}^{(k)} \text{ satisfy (6.37)} \}.$$

Clearly, $\mathscr{T}^{(k)}$ is a partition of $\mathrm{Cross}_k(\mathscr{S}^{(1)})$, since $\widetilde{\mathscr{R}}_0^{(k)}$ is a partition of $[V]^k$ (*cf.* (6.36)). Furthermore, (T1) and (T4) are immediate since we constructed $\mathscr{T}^{(k)}$ separately on $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}$. Moreover, it is easy to see that (T2), (T3), and (T5) are implied by its 'prime' counterpart.

This finishes the reduction of Claim 6.10 to Claim 6.10′, which is the last missing piece in the proof of the implication $\mathrm{RAL}(k) \Longrightarrow \mathrm{RL}(k+1)$. □

Below we prove Claim 6.10′. The proof resembles some ideas from [20, Section 5]. The main tool in that proof is the somewhat technical slicing lemma, Proposition 3.5, and we first give an informal outline to convey the idea.

Suppose $\hat{\mathcal{S}}^{(k-1)}$ and $\mathcal{R}^{(k)}$ satisfy (6.37). Let $\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ be the collection of those partition classes $\mathcal{Z}^{(k)}$ of $\mathscr{Z}^{(k)}$ which are contained in $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}$, *i.e.*,

$$\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)}) = \{ \mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)} \colon \mathcal{Z}^{(k)} \subseteq \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)} \}. \tag{6.38}$$

Note that, due to (Z1) and (Z4),

$$\{ \mathcal{Z}^{(k)} \colon \mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)}) \} \text{ partitions } \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}. \tag{6.39}$$

Indeed, by (Z1), $\mathscr{Z}^{(k)}$ has each of its partition classes completely within or outside $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})$ and by (Z4) the same is true for $\mathcal{R}^{(k)}$.

We will use the slicing lemma twice. In the first round we apply the slicing lemma separately to each $\mathcal{Z}^{(k)} \in \mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ to slice it in such a way that all but at most one slice ('left-over' part) has density $1/f(s_i)$ w.r.t. $\hat{\mathcal{S}}^{(k-1)}$. On the other hand, we infer from the choice of $\mu_{i+1} \geqslant \mu_{\lceil 8/\delta_{k+1,\mathrm{RL}}^4 \rceil}$ and $(\mathrm{R}_0.1)$ that $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}$ is still $(\delta', 1/a_k^{\mathscr{R}_0})$-regular with $\delta' \ll \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))$ (*cf.* (6.26)). (In the special case $\mathcal{R}^{(k)} = \widetilde{\mathcal{R}}^{(k)}$ we have $(\delta', 1)$-regularity for any $\delta' > 0$.) Consequently, the union of the earlier 'left-overs' must have a density very close to a multiple of $1/f(s_i)$, since it is $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}$ minus regular pieces of density $1/f(s_i)$. Therefore, we can use the slicing lemma again (second round) to 'recycle' the 'left-overs', splitting it into regular pieces of density $1/f(s_i)$, and the 'recycled' partition will satisfy (T2′) and (T3′). Finally, we will show that it also exhibits (T5′) since we chose $f(\cdot)$ in (6.22) in such a way that $|\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})| \times 1/f(s_i) \leqslant s_i/f(s_i) \ll \delta_{k+1,\mathrm{RL}}/8$, which is an upper bound on the density of the union of the 'left-overs'.

Below we give the technical details of the plan outlined above.

Proof of Claim 6.10′. Let $\hat{\mathcal{S}}^{(k-1)} \in \hat{\mathscr{S}}^{(k-1)}$ and $\mathcal{R}^{(k)} \in \widetilde{\mathscr{R}}_0^{(k)}$ satisfying (6.37) be given. We start with a few observations. From the choice of the function ε_i in (6.24) and n_i in (6.27) combined with (S1), we infer by Theorem 3.1 that

$$|\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})| \geqslant \frac{1}{2} \prod_{j=2}^{k-1} \left(\frac{1}{a_j^{\mathscr{S}}} \right)^{\binom{k}{j}} \times \left(\frac{n}{a_1^{\mathscr{S}}} \right)^k \geqslant \frac{n^k}{2t_{i+1}^{2^k}}. \tag{6.40}$$

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Suppose $\mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)}$. Let $\hat{\mathcal{R}}^{(k-1)}$ be the polyad in $\hat{\mathscr{R}}_0^{(k-1)}$ such that $\mathcal{R}^{(k)} \subseteq \mathcal{K}_k(\hat{\mathcal{R}}^{(k-1)})$. Since $\mathscr{S}^{(k-1)} \prec \mathscr{R}_0^{(k-1)}$ (*cf.* (RAL.S2)) and $\mathcal{R}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \neq \varnothing$ (*cf.* (6.37)), we have that $\hat{\mathcal{S}}^{(k-1)} \subseteq \hat{\mathcal{R}}^{(k-1)}$. Consequently, we infer from (6.40) and (R_0.1) that if $\mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)}$ then

$$\mathcal{R}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \text{ is } \big(2t_{i+1}^{2^k}\mu_{\lceil 8/\delta_{k+1,\mathrm{RL}}^4\rceil}, 1/a_k^{\mathscr{R}_0}\big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)}.$$

Moreover, if $\mathcal{R}^{(k)} = \widetilde{\mathcal{R}}^{(k)}$, then assumption (6.37) yields that $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \subseteq \mathcal{R}^{(k)}$ and $\mathcal{R}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})$ is $(\delta', 1)$-regular w.r.t. $\hat{\mathcal{S}}^{(k-1)}$ for every $\delta' > 0$. Therefore, in view of

$$\mu_{\lceil 8/\delta_{k+1,\mathrm{RL}}^4\rceil} \leqslant \mu_{i+1} \overset{(6.26)}{\leqslant} \frac{\delta_{\mathrm{RL}}(t_{i+1},\dots,t_{i+1},f(s_i))}{12t_{i+1}^{2^k}} \overset{(\mathrm{RAL.S1})}{\leqslant} \frac{\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}},f(s_i))}{12t_{i+1}^{2^k}},$$

we have for every $\mathcal{R}^{(k)} \in \widetilde{\mathscr{R}}_0^{(k)}$ that

$$\mathcal{R}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \text{ is } \big(\tfrac{1}{6}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}},f(s_i)), d_{\mathcal{R}^{(k)}}\big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)}, \tag{6.41}$$

where

$$d_{\mathcal{R}^{(k)}} = \begin{cases} 1/a_k^{\mathscr{R}_0} & \text{if } \mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)}, \\ 1 & \text{if } \mathcal{R}^{(k)} = \widetilde{\mathcal{R}}^{(k)}. \end{cases} \tag{6.42}$$

Furthermore, we infer from (6.41) combined with $d_{\mathcal{R}^{(k)}} \geqslant 1/a_k^{\mathscr{R}_0} \geqslant 1/t_0$ (*cf.* (R_0.1)) that

$$d\big(\mathcal{R}^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}\big) \geqslant \frac{1}{t_0} - \frac{1}{6}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}},f(s_i)) \overset{(6.21)}{>} \max\Big\{\frac{1}{3}\delta_{\mathrm{RL}}\big(\boldsymbol{a}^{\mathscr{S}},f(s_i)\big), \frac{2}{3t_0}\Big\}. \tag{6.43}$$

Recall the definition $\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})$ from (6.38) and let $\{\mathcal{Z}_1^{(k)},\dots,\mathcal{Z}_z^{(k)}\}$ be an enumeration of its members. Clearly, $\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)}) \subseteq \mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)})$ (*cf.* (6.34)), and due to (Z2) we have

$$z \leqslant s_i. \tag{6.44}$$

Our plan is first to apply the slicing lemma to every member $\mathcal{Z}_j^{(k)}$ of $\mathscr{Z}^{(k)}(\hat{\mathcal{S}}^{(k-1)},\mathcal{R}^{(k)})$. For this we have to satisfy the assumptions of the slicing lemma among which we have to ensure that $d(\mathcal{Z}_j^{(k)} \mid \hat{\mathcal{S}}^{(k-1)})$ is not 'too small'. However, since the $\mathcal{Z}_j^{(k)}$ arose from an application of RAL(k), we only have limited control over their densities, which leads to the following definition:

$$Z_{\mathrm{THIN}} = \Big\{j \in [z] \colon d\big(\mathcal{Z}_j^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}\big) < \tfrac{1}{f(s_i)}\Big\}. \tag{6.45}$$

Moreover, for every $j \in [z]$ we set

$$\zeta_j = \big\lfloor f(s_i) \times d\big(\mathcal{Z}_j^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}\big)\big\rfloor. \tag{6.46}$$

Clearly, $\zeta_j > 0$ if and only if $j \notin Z_{\mathrm{THIN}}$. We now apply the slicing lemma, Proposition 3.5, for every $j \in [z] \setminus Z_{\mathrm{THIN}}$ separately with

$$j_{\mathrm{SL}} = k, \quad s_{0,\mathrm{SL}} = s_{5.1}, \quad r_{\mathrm{SL}} = 1, \quad \delta_{0,\mathrm{SL}} = \varepsilon_i\big(\overbrace{t_{i+1},\dots,t_{i+1}}^{(k-1)\text{-times}}\big), \quad \varrho_{0,\mathrm{SL}} = \tfrac{1}{f(s_i)},$$
$$\text{and} \quad p_{0,\mathrm{SL}} = \tfrac{1}{f(s_i)}, \quad \text{to } \hat{\mathcal{P}}_{\mathrm{SL}}^{(k-1)} = \hat{\mathcal{S}}^{(k-1)}, \quad \text{and} \quad \mathcal{P}_{\mathrm{SL}}^{(k)} = \mathcal{Z}_j^{(k)} \quad \text{with } s_{\mathrm{SL}} = \zeta_j,$$
$$\delta_{\mathrm{SL}} = \varepsilon_i(\boldsymbol{a}^{\mathscr{S}}), \quad \varrho_{\mathrm{SL}} = d(\mathcal{Z}_j^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}), \quad \text{and} \quad \Big\{p_{\xi,\mathrm{SL}} = \frac{1/f(s_i)}{d(\mathcal{Z}_j^{(k)} \mid \hat{\mathcal{S}}^{(k-1)})} \colon \xi \in [\zeta_j]\Big\}.$$

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It follows from (6.40) that the assumption (i) of Proposition 3.5 is satisfied for $\hat{\mathcal{P}}_{\mathrm{SL}}^{(k-1)} = \hat{\mathcal{S}}^{(k-1)}$. Moreover, (ii) is a consequence of (Z3) (yielding the $(\delta_{\mathrm{SL}}, \varrho_{\mathrm{SL}}, r_{\mathrm{SL}})$-regularity of $\mathcal{P}_{\mathrm{SL}}^{(k)} = \mathcal{Z}_j^{(k)}$), the definition of Z_{THIN} (yielding $\varrho_{\mathrm{SL}} \geqslant \varrho_{0,\mathrm{SL}}$), (RAL.S1) and the monotonicity of the function ε_i (yielding $\delta_{\mathrm{SL}} \geqslant \delta_{0,\mathrm{SL}}$), and the choice of ε_i in (6.24) (yielding $\varrho_{0,\mathrm{SL}} \geqslant 2\delta_{\mathrm{SL}}$). Furthermore, assumption (iii) of Proposition 3.5 is a consequence of the fact that $d(\mathcal{Z}_j^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}) \leqslant 1$ (yielding $p_{\xi,\mathrm{SL}} \geqslant p_{0,\mathrm{SL}}$ for $\xi \in [\zeta_j] = \{1, \ldots, \zeta_j\}$ and the choice of the integer parameter ζ_j in (6.46) (yielding $\sum_{\xi \in [\zeta_j]} p_{\xi,\mathrm{SL}} \leqslant 1$).

Having verified the assumptions of Proposition 3.5 for every $j \in [z] \setminus Z_{\mathrm{THIN}}$, we infer that for every such j there exists a family $\{\mathcal{T}_{j,0}^{(k)}, \mathcal{T}_{j,1}^{(k)}, \ldots, \mathcal{T}_{j,\zeta_j}^{(k)}\}$ such that

$$\left\{\mathcal{T}_{j,0}^{(k)}, \mathcal{T}_{j,1}^{(k)}, \ldots, \mathcal{T}_{j,\zeta_j}^{(k)}\right\} \text{ partitions } \mathcal{Z}_j^{(k)}, \tag{6.47}$$

$$\mathcal{T}_{j,\xi}^{(k)} \text{ is } \big(3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}), 1/f(s_i)\big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)} \text{ for every } \xi = 1, \ldots, \zeta_j, \quad \text{and} \tag{6.48}$$

$$\mathcal{T}_{j,0}^{(k)} \text{ is } (3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}), d_{j,0})\text{-regular for some } 0 \leqslant d_{j,0} \leqslant d(\mathcal{Z}_j^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}) - \tfrac{\zeta_j}{f(s_i)} \overset{(6.46)}{\leqslant} \tfrac{1}{f(s_i)}. \tag{6.49}$$

Unfortunately, the 'left-over' hypergraph $\mathcal{T}_{j,0}^{(k)}$ might not be empty and has a density differing from $1/f(s_i)$. Moreover, in general Z_{THIN} is not empty and we have to recycle the 'left-overs' $\mathcal{T}_{j,0}^{(k)}$ with $j \notin Z_{\mathrm{THIN}}$ and the hypergraphs $\mathcal{Z}_j^{(k)}$ with $j \in Z_{\mathrm{THIN}}$. For that we consider their union

$$\mathcal{U}^{(k)} = \bigcup_{j \in [z] \setminus Z_{\mathrm{THIN}}} \mathcal{T}_{j,0}^{(k)} \cup \bigcup_{j \in Z_{\mathrm{THIN}}} \mathcal{Z}_j^{(k)}. \tag{6.50}$$

Clearly, $\mathcal{U}^{(k)}$ is the complement of

$$\bigcup_{j \in [z] \setminus Z_{\mathrm{THIN}}} \bigcup_{\xi \in [\zeta_j]} \mathcal{T}_{j,\xi}^{(k)}$$

in $\mathcal{R}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)})$. Consequently, in view of (6.48), $|Z_{\mathrm{THIN}}| \leqslant z \leqslant s_i$ (*cf.* (6.44)), and (6.41), an application of Proposition 3.2 yields that

$$\mathcal{U}^{(k)} \text{ is } \Big(\tfrac{1}{6}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)) + 3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}})s_i, d_{\mathcal{R}^{(k)}} - \tfrac{\sum_{j \notin Z_{\mathrm{THIN}}} \zeta_j}{f(s_i)}\Big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)}. \tag{6.51}$$

Recall that $d_{\mathcal{R}^{(k)}}$ is an integer multiple of $1/f(s_i)$. This is obvious if $\mathcal{R}^{(k)} = \widetilde{\mathcal{R}}^{(k)}$ since $f(s_i)$ is an integer-valued function. Moreover, if $\mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)}$, then $d_{\mathcal{R}^{(k)}} = 1/a_k^{\mathscr{R}_0}$ which is a multiple of $1/f(s_i)$ since $(t_0)!$ divides $f(s_i)$ (*cf.* (6.22)) and $t_0 \geqslant a_k^{\mathscr{R}_0}$ (*cf.* (R_0.1)). Consequently,

$$d_{\mathcal{R}^{(k)}} - \frac{\sum_{j \notin Z_{\mathrm{THIN}}} \zeta_j}{f(s_i)} = \frac{u}{f(s_i)} \quad \text{for some integer } 0 \leqslant u \leqslant f(s_i). \tag{6.52}$$

This observation and the choice of the function ε_i in (6.24) allow us to rewrite (6.51):

$$\mathcal{U}^{(k)} \text{ is } \Big(\tfrac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), \tfrac{u}{f(s_i)}\Big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)}. \tag{6.53}$$

The further treatment of $\mathcal{U}^{(k)}$ depends on its value and we consider two cases.

Case 1 ($u = 0$)**.** Note that the assumption $u = 0$ and (6.53) does not necessarily imply that $\mathcal{U}^{(k)} = \varnothing$. Instead, it yields

$$d\big(\mathcal{U}^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}\big) \leqslant \tfrac{1}{3}\delta_{\mathrm{RL}}\big(\boldsymbol{a}^{\mathscr{S}}, f(s_i)\big).$$

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On the other hand, by (6.43),

$$d\big(\mathcal{R}^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}\big) > \tfrac{1}{3}\delta_{\mathrm{RL}}\big(\boldsymbol{a}^{\mathscr{S}}, f(s_i)\big).$$

Therefore, from (6.39), we infer that $Z_{\mathrm{THIN}} \neq [z]$ and there exist some $j_0 \in [z] \setminus Z_{\mathrm{THIN}}$ with $\zeta_{j_0} \geqslant 1$ and hence $\mathcal{T}^{(k)}_{j_0,1}$ exists. We then define the promised partition $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ as follows:

$$\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)}) = \{\mathcal{T}^{(k)}_{j,\xi} \colon\ j \in [z] \setminus Z_{\mathrm{THIN}},\ j \neq j_0,\ \text{ and }\ \xi \in [\zeta_j]\} \cup \{\mathcal{T}^{(k)}_{j_0,\xi} \colon\ \xi = 2, \dots, \zeta_j\} \cup \{\mathcal{T}^{(k)}_{j_0,1} \cup \mathcal{U}^{(k)}\}.$$

From the definition of $\mathcal{U}^{(k)}$ in (6.50), and by (6.47) and (6.39), it follows that $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ defined above indeed partitions $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}$. We conclude this case with the verification of properties (T2′), (T3′), and (T5′).

First we consider (T2′). Clearly, $|\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})| = \sum_{j \in [z] \setminus Z_{\mathrm{THIN}}} \zeta_j$. So, in view of (6.52), we infer from the assumption $u = 0$, in this case, that

$$|\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})| = \sum_{j \in [z] \setminus Z_{\mathrm{THIN}}} \zeta_j = d_{\mathcal{R}^{(k)}} \times f(s_i) \overset{(6.42)}{=} \begin{cases} \frac{f(s_i)}{a_k^{\mathscr{R}_0}} & \text{if } \mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)}, \\ f(s_i) & \text{if } \mathcal{R}^{(k)} = \widetilde{\mathcal{R}}^{(k)}, \end{cases}$$

which is (T2′).

Since $\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}) \leqslant \frac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))$ (*cf.* (6.24)), (6.48) guarantees (T3′) for every member of $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ with exception $\mathcal{T}^{(k)}_{j_0} \cup \mathcal{U}^{(k)}$. Consequently, verifying (T3′) reduces to showing that

$$\mathcal{T}^{(k)}_{j_0,1} \cup \mathcal{U}^{(k)} \ \text{ is } \ \big(\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), 1/f(s_i)\big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)}. \tag{6.54}$$

However, this follows from (6.48) and (6.53) by Proposition 3.2, since $u = 0$ and since by the choice in (6.24) we have $3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}) \leqslant \frac{2}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))$.

Finally, we consider (T5′). Due to the definition of the partition $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ it suffices to show that

$$d\big(\mathcal{T}^{(k)}_{j_0,1} \cup \mathcal{U}^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}\big) \leqslant \frac{\delta^4_{k+1,\mathrm{RL}}}{8} d\big(\mathcal{R}^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}\big). \tag{6.55}$$

For that we first derive from (6.54) combined with the definition of the function $f(\cdot)$ in (6.22) and the bound from (6.21) that

$$d\big(\mathcal{T}^{(k)}_{j_0,1} \cup \mathcal{U}^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}\big) \overset{(6.54)}{\leqslant} \frac{1}{f(s_i)} + \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)) \leqslant \frac{\delta^4_{k+1,\mathrm{RL}}}{12t_0}. \tag{6.56}$$

Therefore, (6.55) follows from (6.56) and (6.43), which concludes the proof of Claim 6.10′ in this case. ◊

Case 2 ($u > 0$). Recall that by (6.53) the hypergraph $\mathcal{U}^{(k)}$ is $\big(\frac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), \frac{u}{f(s_i)}\big)$-regular w.r.t. $\hat{\mathcal{S}}^{(k-1)}$. In this case we are going to apply the slicing lemma to 'recycle' the edges of $\mathcal{U}^{(k)}$, *i.e.*, to partition it into regular pieces of density $1/f(s_i)$. More precisely, we apply Proposition 3.5

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with

$$j_{\mathrm{SL}} = k, \quad s_{0,\mathrm{SL}} = f(s_i), \quad r_{\mathrm{SL}} = 1, \quad \delta_{0,\mathrm{SL}} = \tfrac{1}{3}\delta_{\mathrm{RL}}(\overbrace{t_{i+1},\dots,t_{i+1}}^{(k-1)\text{-times}}, f(s_i)), \quad \varrho_{0,\mathrm{SL}} = \tfrac{1}{f(s_i)},$$
$$\text{and} \quad p_{0,\mathrm{SL}} = \tfrac{1}{f(s_i)}, \quad \text{to} \quad \hat{\mathcal{P}}^{(k-1)}_{\mathrm{SL}} = \hat{\mathcal{S}}^{(k-1)}, \quad \text{and} \quad \mathcal{P}^{(k)}_{\mathrm{SL}} = \mathcal{U}^{(k)} \quad \text{with}$$
$$s_{\mathrm{SL}} = u, \quad \delta_{\mathrm{SL}} = \frac{1}{3}\delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), \quad \varrho_{\mathrm{SL}} = \frac{u}{f(s_i)}, \quad \text{and} \quad \left\{p_{\xi,\mathrm{SL}} = \frac{1}{u} : \xi \in [u]\right\}.$$

It follows from (6.40) that the assumption (i) of Proposition 3.5 is satisfied for $\hat{\mathcal{P}}^{(k-1)}_{\mathrm{SL}} = \hat{\mathcal{S}}^{(k-1)}$. Moreover, (ii) is a consequence of (6.53) (yielding the $(\delta_{\mathrm{SL}}, \varrho_{\mathrm{SL}}, r_{\mathrm{SL}})$-regularity of $\mathcal{P}^{(k)}_{\mathrm{SL}} = \mathcal{U}^{(k)}_j$), the assumption of the case $u \geqslant 1$ (yielding $\varrho_{\mathrm{SL}} \geqslant \varrho_{0,\mathrm{SL}}$), (RAL.S1) and the monotonicity (*cf.* (6.19)) of the function δ_{RL} (yielding $\delta_{\mathrm{SL}} \geqslant \delta_{0,\mathrm{SL}}$), and of (6.21) (yielding $\varrho_{0,\mathrm{SL}} \geqslant 2\delta_{\mathrm{SL}}$). Furthermore, note that $p_{\xi,\mathrm{SL}} \geqslant p_{0,\mathrm{SL}}$, since $u \leqslant f(s_i)$ (*cf.* (6.52)) and that that

$$\sum_{\xi\in[u]} p_{\xi,\mathrm{SL}} = 1. \tag{6.57}$$

Consequently, assumption (iii) of Proposition 3.5 holds for the choice of parameters above.

Having verified the assumptions of Proposition 3.5, we infer that there exists a family

$$\{\mathcal{U}^{(k)}_1,\dots,\mathcal{U}^{(k)}_u\}$$

(note that due to (6.57) there is a 'left-over' class $\mathcal{U}^{(k)}_0$) such that

$$\{\mathcal{U}^{(k)}_1,\dots,\mathcal{U}^{(k)}_u\} \text{ partitions } \mathcal{U}^{(k)} \tag{6.58}$$
$$\mathcal{U}^{(k)}_\xi \text{ is } \big(\delta_{k,\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i)), 1/f(s_i)\big)\text{-regular w.r.t. } \hat{\mathcal{S}}^{(k-1)} \text{ for every } \xi \in [u]. \tag{6.59}$$

We finally define the required family $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ in a straightforward manner:

$$\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)}) = \{\mathcal{T}^{(k)}_{j,\xi} : j \in [z] \setminus Z_{\mathrm{THIN}}, \text{ and } \xi \in [\zeta_j]\} \cup \{\mathcal{U}^{(k)}_1,\dots,\mathcal{U}^{(k)}_u\}. \tag{6.60}$$

Again it directly follows from the definition of $\mathcal{U}^{(k)}$ in (6.50), together with (6.58) and (6.39), that $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ partitions $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}$, and it is left to verify (T2′), (T3′), and (T5′) for this partition.

First we consider (T2′). By the definition of $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ we have

$$|\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})| = \sum_{j\in[z]\setminus Z_{\mathrm{THIN}}} \zeta_j + u \overset{(6.52)}{=} d_{\mathcal{R}^{(k)}} f(s_i) \overset{(6.42)}{=} \begin{cases} \frac{f(s_i)}{a_k^{\mathscr{P}_0}} & \text{if } \mathcal{R}^{(k)} \neq \widetilde{\mathcal{R}}^{(k)}, \\ f(s_i) & \text{if } \mathcal{R}^{(k)} = \widetilde{\mathcal{R}}^{(k)}, \end{cases}$$

which is (T2′).

Property (T3′) is immediate from (6.59) and (6.48) combined with the choice of the function ε_i in (6.24), which easily ensures $3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}}) \leqslant \delta_{\mathrm{RL}}(\boldsymbol{a}^{\mathscr{S}}, f(s_i))$.

Finally, we discuss (T5′). By (6.47) and due to the definition of $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ in (6.60), the left-hand side of (T5′) is bounded by $|\mathcal{U}^{(k)}|$ and thus it suffices to show that

$$d\big(\mathcal{U}^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}\big) \leqslant \frac{\delta^4_{k+1,\mathrm{RL}}}{8} d\big(\mathcal{R}^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}\big). \tag{6.61}$$

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From the definition of $\mathcal{U}^{(k)}$ in (6.50), combined with (6.49) and the definition of Z_{THIN} in (6.45) we infer that

$$d\big(\mathcal{U}^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}\big) \leqslant |[z] \setminus Z_{\mathrm{THIN}}|\left(\frac{1}{f(s_i)} + 3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}})\right) + |Z_{\mathrm{THIN}}|\frac{1}{f(s_i)} \overset{(6.44)}{\leqslant} \frac{s_i}{f(s_i)} + 3\varepsilon_i(\boldsymbol{a}^{\mathscr{S}})s_i,$$

which by definition of $f(\cdot)$ in (6.22) and (6.24) gives $d\big(\mathcal{U}^{(k)} \mid \hat{\mathcal{S}}^{(k-1)}\big) \leqslant \delta^4_{k+1,\mathrm{RL}}/(12t_0)$.

Therefore, (6.61) follows from the last inequality and (6.43). This verifies (Z5′), which finishes the proof of Claim 6.10′ in this case. ◊

In both cases we constructed a partition $\mathscr{T}^{(k)}(\hat{\mathcal{S}}^{(k-1)}, \mathcal{R}^{(k)})$ of $\mathcal{K}_k(\hat{\mathcal{S}}^{(k-1)}) \cap \mathcal{R}^{(k)}$, which exhibits properties (T2′), (T3′), and (T5′), as required in Claim 6.10′. □

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