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Divisors computing the minimal log discrepancy on a smooth surface

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Abstract

We study a divisor computing the minimal log discrepancy on a smooth surface. Such a divisor is obtained by a weighted blow-up. There exists an example of a pair such that any divisor computing the minimal log discrepancy computes no log canonical thresholds.

Let $P \in X$ be the germ of a smooth variety and $\mathfrak{a}$ be an $\mathbf{R}$-ideal on X. The minimal log discrepancy $\mathrm{mld}_P(X, \mathfrak{a})$ is an important invariant of singularities in view of Shokurov's reduction of the termination of flips [**7**], but we do not have a good understanding of it. Presumably, one of its reasons is that it is unclear which divisor E (over X) computes the minimal log discrepancy. That is, E has centre P and the log discrepancy of E equals $\mathrm{mld}_P(X, \mathfrak{a})$ or is negative. The purpose of this note is to study such E when X is a surface. We provide the description of E in terms of a weighted blow-up.

THEOREM 1. *Let $P \in X$ be the germ of a smooth surface and $\mathfrak{a}$ be an $\mathbf{R}$-ideal on X.*

(i) *If $(X, \mathfrak{a})$ is log canonical, then every divisor computing $\mathrm{mld}_P(X, \mathfrak{a})$ is obtained by a weighted blow-up.*
(ii) *If $(X, \mathfrak{a})$ is not log canonical, then some divisor computing $\mathrm{mld}_P(X, \mathfrak{a})$ is obtained by a weighted blow-up.*

The log canonical threshold is another invariant of singularities, roughly corresponding to the minimal log discrepancy divided by the multiplicity. The log canonical threshold is considered to be easier to handle, because the minimal model program extracts a divisor computing minimal log discrepancy zero. For example, the ACC for log canonical thresholds is proved completely by Hacon, McKernan and Xu [**3**]. Hence in treating positive $\mathrm{mld}_P(X, \mathfrak{a})$, it is a standard approach to find a suitable $\mathbf{R}$-ideal $\mathfrak{b}$ such that $\mathrm{mld}_P(X, \mathfrak{a}\mathfrak{b})$ equals zero. However even in dimension two, one can not expect the existence of the $\mathfrak{b}$ for which some divisor computes both $\mathrm{mld}_P(X, \mathfrak{a})$ and $\mathrm{mld}_P(X, \mathfrak{a}\mathfrak{b})$. We shall prove the following example later.

Example 2. Let $P \in \mathbf{A}^2 = \mathrm{Spec}\, k[x_1, x_2]$ where P is the origin. Consider the pair $(\mathbf{A}^2, \mathfrak{a}^{2/3})$ for $\mathfrak{a} = (x_1^2 + x_2^3, x_1x_2^2)$. There exists a unique divisor E computing $\mathrm{mld}_P(\mathbf{A}^2, \mathfrak{a}^{2/3}) = 2/3$, but E does not compute $\mathrm{mld}_P(\mathbf{A}^2, \mathfrak{a}^{2/3}\mathfrak{b})$ for any $\mathbf{R}$-ideal $\mathfrak{b}$ such that $\mathrm{mld}_P(\mathbf{A}^2, \mathfrak{a}^{2/3}\mathfrak{b}) = 0$.

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Remark 3. Blum [**2**] independently proved that every divisor computing $\mathrm{mld}_P(X, \mathfrak{a}) \geqslant 0$ on a smooth surface X computes $\mathrm{mld}_P(X, \mathfrak{b}) = 0$ for some $\mathfrak{b}$. This property is different from that discussed in Example 2, and follows from Theorem 1 immediately. Indeed, let E be the divisor obtained by the weighted blow-up of $P \in X$ with $\mathrm{wt}(x_1, x_2) = (w_1, w_2)$. Then $(X, \mathfrak{b})$ for $\mathfrak{b} = (x_1^{w_2}, x_2^{w_1})^{1/w_1+1/w_2}$ is lc since so is $(X, (x_1^{w_2})^{1/w_2}(x_2^{w_1})^{1/w_1})$. E computes $\mathrm{mld}_P(X, \mathfrak{b}) = 0$.

We will fix the terminology before proceeding to the proofs of Theorem 1 and Example 2. We work over an algebraically closed field k of characteristic zero. The germ is considered at a closed point.

An **R**-*ideal* on a variety X is a formal product $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$ of finitely many coherent ideal sheaves $\mathfrak{a}_j$ on X with positive real exponents r_j. The *order* of $\mathfrak{a}$ along a closed subvariety C of X is $\mathrm{ord}_C \mathfrak{a} = \sum_j r_j \mathrm{ord}_C \mathfrak{a}_j$, where $\mathrm{ord}_C \mathfrak{a}_j$ is the maximal $\nu \in \mathbf{N} \cup \{+\infty\}$ satisfying $\mathfrak{a}_{j,\eta} \subset \mathcal{I}_\eta^\nu$ for the ideal sheaf $\mathcal{I}$ of C and the generic point η of C. The *pull-back* of $\mathfrak{a}$ by a morphism $Z \to X$ is $\mathfrak{a}\mathcal{O}_Z = \prod_j (\mathfrak{a}_j \mathcal{O}_Z)^{r_j}$. If $Z \to X$ is birational and Z is normal, then we set $\mathrm{ord}_E \mathfrak{a} = \mathrm{ord}_E \mathfrak{a}\mathcal{O}_Z$ for a prime divisor E on Z. If $Z \to X$ is a birational morphism from a smooth variety Z whose exceptional locus is a divisor $\sum_i E_i$, then the *weak transform* on Z of $\mathfrak{a}$ is the **R**-ideal $\mathfrak{a}_Z = \prod_j (\mathfrak{a}_{jZ})^{r_j}$ defined by $\mathfrak{a}_{jZ} = \mathfrak{a}_j \mathcal{O}_Z(\sum_i (\mathrm{ord}_{E_i} \mathfrak{a}_j) E_i)$. This is different from the strict transform (see [**4**, III, definition 5]).

A prime divisor E on a normal variety Y equipped with a birational morphism $Y \to X$ is called a divisor *over* X, and the closure of the image in X of E is called the *centre* of E on X and denoted by $c_X(E)$. We write $\mathcal{D}_X$ for the set of all divisors over X. Two elements in $\mathcal{D}_X$ are often identified if they define the same valuation on the function field of X.

Suppose X to be smooth. The *log discrepancy* of E with respect to the pair $(X, \mathfrak{a})$ is

$$a_E(X, \mathfrak{a}) = 1 + \mathrm{ord}_E K_{Y/X} - \mathrm{ord}_E \mathfrak{a}.$$

We say that $(X, \mathfrak{a})$ is *log canonical* (*lc*) if $a_E(X, \mathfrak{a}) \geqslant 0$ for all $E \in \mathcal{D}_X$, and is *klt* (resp. *plt*) if $a_E(X, \mathfrak{a}) > 0$ for all $E \in \mathcal{D}_X$ (resp. all $E \in \mathcal{D}_X$ exceptional over X). The *minimal log discrepancy* of $(X, \mathfrak{a})$ at a closed point P in X is

$$\mathrm{mld}_P(X, \mathfrak{a}) = \inf\{a_E(X, \mathfrak{a}) \mid E \in \mathcal{D}_X,\ c_X(E) = P\}.$$

The $\mathrm{mld}_P(X, \mathfrak{a})$ is either a non-negative real number or $-\infty$, and $(X, \mathfrak{a})$ is lc about P iff $\mathrm{mld}_P(X, \mathfrak{a}) \geqslant 0$. If $E \in \mathcal{D}_X$ satisfies that $c_X(E) = P$ and that $a_E(X, \mathfrak{a}) = \mathrm{mld}_P(X, \mathfrak{a})$ (or is negative when $\mathrm{mld}_P(X, \mathfrak{a}) = -\infty$), then we say that E *computes* $\mathrm{mld}_P(X, \mathfrak{a})$.

Let $P \in X$ be the germ of a smooth variety. Let $x_1, \ldots, x_c$ be a part of a regular system of parameters in $\mathcal{O}_{X,P}$ and $w_1, \ldots, w_c$ be positive integers. For $w \in \mathbf{N}$, let $\mathcal{I}_w$ be the ideal in $\mathcal{O}_X$ generated by all monomials $x_1^{s_1} \cdots x_c^{s_c}$ such that $\sum_{i=1}^c s_i w_i \geqslant w$. The *weighted blow-up* of X with $\mathrm{wt}(x_1, \ldots, x_c) = (w_1, \ldots, w_c)$ is $\mathrm{Proj}_X(\bigoplus_{w \in \mathbf{N}} \mathcal{I}_w)$. See [**6**, 6·38] for its explicit description.

Proof of Theorem 1. If $(X, \mathfrak{a})$ is not lc, then after approximating a, any divisor computing $\mathrm{mld}_P(X, \mathfrak{a}^t) = 0$ for $t < 1$ computes $\mathrm{mld}_P(X, \mathfrak{a}) = -\infty$. Thus (ii) follows from (i).

We assume that $(X, \mathfrak{a})$ is lc to show (i). Let E be a divisor over X which computes $\mathrm{mld}_P(X, \mathfrak{a})$. For the maximal ideal $\mathfrak{m}$ in $\mathcal{O}_{X,P}$, we set

$$w_1 = \max_{x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2} \mathrm{ord}_E x_1, \qquad w_2 = \min_{x_2 \in \mathfrak{m} \setminus \mathfrak{m}^2} \mathrm{ord}_E x_2 = \mathrm{ord}_E \mathfrak{m}.$$

In order to verify the existence of the maximum w_1, let $Z \to X$ be a birational morphism

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from a smooth surface Z on which E appears as a divisor. Applying Zariski's subspace theorem [**1**, (10·6)] to $\mathcal{O}_{X,P} \subset \mathcal{O}_{Z,Q}$ for a closed point Q in E, one has an integer w such that $\mathcal{O}_Z(-wE)_Q \bigcap \mathcal{O}_{X,P} \subset \mathfrak{m}^2$. Then $\operatorname{ord}_E x_1 < w$ for any $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$, so $\operatorname{ord}_E x_1$ belongs to a finite subset of the positive integers and w_1 exists.

Take a regular system x_1, x_2 of parameters in $\mathcal{O}_{X,P}$ such that $w_i = \operatorname{ord}_E x_i$ for $i = 1, 2$. Let Y be the weighted blow-up of X with $\operatorname{wt}(x_1, x_2) = (w_1, w_2)$ and F be its exceptional divisor. We write $w_i = gw_i'$ by the greatest common divisor g of w_1 and w_2. We write $a_E = a_E(X, \mathfrak{a})$ and $a_F = a_F(X, \mathfrak{a})$ for simplicity. Then the inequality

$$a_E \leqslant a_F \tag{1}$$

holds because E computes $\operatorname{mld}_P(X, \mathfrak{a})$. It is enough to show that $E = F$.

For $i = 1, 2$, we let C_i denote the strict transform in Y of the curve defined on X by (x_i). Then one computes that

$$w_i = \operatorname{ord}_E x_i = \operatorname{ord}_F x_i \cdot \operatorname{ord}_E F + \operatorname{ord}_E C_i = w_i' \operatorname{ord}_E F + \operatorname{ord}_E C_i.$$

Here the order of a $\mathbf{Q}$-Cartier divisor such as F and C_i is defined in the way that $\operatorname{ord}_E A = r^{-1} \operatorname{ord}_E \mathcal{O}_Y(-rA)$ if A is a $\mathbf{Q}$-Cartier effective divisor on Y and r is a positive integer such that rA is Cartier. The $\operatorname{ord}_E C_i$ is positive iff the centre $c_Y(E)$ lies on C_i. Since C_1 and C_2 are disjoint about F by the definition of the weighted blow-up, at least one of $\operatorname{ord}_E C_1$ and $\operatorname{ord}_E C_2$ is zero. By

$$\frac{w_1'}{w_2'} = \frac{w_1}{w_2} = \frac{w_1' \operatorname{ord}_E F + \operatorname{ord}_E C_1}{w_2' \operatorname{ord}_E F + \operatorname{ord}_E C_2},$$

one concludes that $\operatorname{ord}_E C_1 = \operatorname{ord}_E C_2 = 0$. Hence if $E \neq F$, then the centre $c_Y(E)$ must be a closed point Q in $F \setminus (C_1 + C_2)$.

Assuming that $c_Y(E) = Q \in F \setminus (C_1 + C_2)$, we shall derive a contradiction. Restricting to the smooth locus of Y, one has the weak transform $\mathfrak{a}_Y$ of $\mathfrak{a}$ on the germ $Q \in Y$. Recall that if $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$ then $\mathfrak{a}_Y = \prod_j (\mathfrak{a}_{jY})^{r_j}$ where $\mathfrak{a}_{jY} = \mathfrak{a}_j \mathcal{O}_Y((\operatorname{ord}_F \mathfrak{a}_j)F)$. Then $a_E = a_E(Y, \mathfrak{a}_Y) + (a_F - 1) \operatorname{ord}_E F$ and

$$a_E(Y, \mathfrak{a}_Y \mathcal{O}_Y(-F)) = a_E(Y, \mathfrak{a}_Y) - \operatorname{ord}_E F = a_E - a_F \operatorname{ord}_E F \leqslant 0$$

by (1), so $(Y, \mathfrak{a}_Y \mathcal{O}_Y(-F))$ is not plt about Q. Then $(F, \mathfrak{a}_Y \mathcal{O}_F)$ is not klt about Q by inversion of adjunction [**5**, section 4·1]. This means that

$$\operatorname{ord}_Q(\mathfrak{a}_Y \mathcal{O}_F) \geqslant 1. \tag{2}$$

Write $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$. We take a general member f_j in $\mathfrak{a}_j$ and express

$$f_j = c_j x_1^{s_{1j}} x_2^{s_{2j}} \prod_{\lambda \in k^\times} (x_1^{w_2'} + \lambda x_2^{w_1'})^{t_{\lambda j}} + h_j$$

with $s_{ij}, t_{\lambda j} \in \mathbf{N}$, $c_j \in k^\times$ and $\operatorname{ord}_F h_j > \operatorname{ord}_F f_j$, so that

$$\operatorname{ord}_F \mathfrak{a}_j = \operatorname{ord}_F f_j = s_{1j} w_1' + s_{2j} w_2' + \sum_{\lambda \in k^\times} t_{\lambda j} w_1' w_2'. \tag{3}$$

Then by the identification of F with the weighted projective space $\mathbf{P}(w_1', w_2')$, the general member in $\mathfrak{a}_Y \mathcal{O}_F$ is

$$\prod_j \Big(x_1^{s_{1j}} x_2^{s_{2j}} \prod_{\lambda \in k^\times} (x_1^{w_2'} + \lambda x_2^{w_1'})^{t_{\lambda j}} \Big)^{r_j}.$$

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Let $\lambda_0 \in k^\times$ be the unique unit such that Q lies on the strict transform D in Y of the curve defined on X by $(x_1^{w_2'} + \lambda_0 x_2^{w_1'})$. Then

$$\begin{aligned}\operatorname{ord}_Q(\mathfrak{a}_Y\mathcal{O}_F) &= \operatorname{ord}_Q \prod_j \Big(x_1^{s_{1j}} x_2^{s_{2j}} \prod_{\lambda\in k^\times} (x_1^{w_2'} + \lambda x_2^{w_1'})^{t_{\lambda j}}\Big)^{r_j} \\ &= \sum_j r_j t_{\lambda_0 j} \leqslant \sum_j r_j (w_1'w_2')^{-1} \operatorname{ord}_F \mathfrak{a}_j = (w_1'w_2')^{-1} \operatorname{ord}_F \mathfrak{a},\end{aligned}$$

where the inequality follows from (3). Since $0 \leqslant a_F = a_F(X) - \operatorname{ord}_F \mathfrak{a} = w_1' + w_2' - \operatorname{ord}_F \mathfrak{a}$, one has that $\operatorname{ord}_F \mathfrak{a} \leqslant w_1' + w_2'$. Thus,

$$\operatorname{ord}_Q(\mathfrak{a}_Y\mathcal{O}_F) \leqslant (w_1'w_2')^{-1}(w_1' + w_2') = 1/w_1' + 1/w_2'.$$

Combining (2) and this, we obtain that $1 \leqslant 1/w_1' + 1/w_2'$, whence $w_2' = 1$ by the coprimeness of w_1', w_2'. But then

$$\operatorname{ord}_E(x_1 + \lambda_0 x_2^{w_1'}) = w_1' \operatorname{ord}_E F + \operatorname{ord}_E D > w_1' \operatorname{ord}_E F = w_1,$$

which contradicts the definition of w_1. The proof of Theorem 1 is completed.

Remark 4. It is proved in [**6**, theorem 6·40] after Varčenko that if $P \in X$ is the germ of a smooth complex analytic surface and $(X, \mathcal{I}^t)$ is lc but not klt for the ideal $\mathcal{I}$ defining a curve on X and a rational number t, then some divisor E satisfying $a_E(X, \mathcal{I}^t) = 0$ is obtained by a weighted blow-up (E may be a curve on X).

Proof of Example 2. Let C be the curve on $\mathbf{A}^2$ defined by $(x_1^2 + x_2^3)$. Let X_1 be the blow-up of X at P and E_1 be its exceptional divisor. For $i = 2, 3, 4$, let X_i be the blow-up of X_{i-1} at $E_{i-1} \bigcap C_{i-1}$ for the strict transform C_{i-1} of C, and E_i be its exceptional divisor. Then

$$\mathfrak{a}\mathcal{O}_{X_4} = \mathcal{O}_{X_4}(-2E_1 - 3E_2 - 6E_3 - 7E_4)$$

by the same notation E_i for its strict transform, and X_4 is a log resolution of $(\mathbf{A}^2, \mathfrak{a})$. One computes $a_{E_1}(\mathbf{A}^2, \mathfrak{a}^{2/3}) = 2/3$, $a_{E_2}(\mathbf{A}^2, \mathfrak{a}^{2/3}) = 1$, $a_{E_3}(\mathbf{A}^2, \mathfrak{a}^{2/3}) = 1$ and $a_{E_4}(\mathbf{A}^2, \mathfrak{a}^{2/3}) = 4/3$. Thus $\operatorname{mld}_P(\mathbf{A}^2, \mathfrak{a}^{2/3}) = 2/3$ and it is computed only by E_1.

Suppose that $\operatorname{mld}_P(\mathbf{A}^2, \mathfrak{a}^{2/3}\mathfrak{b}) = 0$ for an $\mathbf{R}$-ideal $\mathfrak{b} = \prod_j \mathfrak{b}_j^{r_j}$ and that it is computed by E_1. Then,

$$\operatorname{ord}_{E_1} \mathfrak{b} = a_{E_1}(\mathbf{A}^2, \mathfrak{a}^{2/3}) = 2/3.$$

On the other hand, since $\mathfrak{b}_j \subset \mathfrak{m}^{\operatorname{ord}_{E_1} \mathfrak{b}_j}$ for the maximal ideal $\mathfrak{m}$ defining P, one has that $\operatorname{ord}_{E_3} \mathfrak{b}_j \geqslant \operatorname{ord}_{E_1} \mathfrak{b}_j \cdot \operatorname{ord}_{E_3} \mathfrak{m} = 2 \operatorname{ord}_{E_1} \mathfrak{b}_j$, so

$$\operatorname{ord}_{E_3} \mathfrak{b} = \sum_j r_j \operatorname{ord}_{E_3} \mathfrak{b}_j \geqslant 2 \sum_j r_j \operatorname{ord}_{E_1} \mathfrak{b}_j = 2 \operatorname{ord}_{E_1} \mathfrak{b} = 4/3.$$

But then

$$a_{E_3}(\mathbf{A}^2, \mathfrak{a}^{2/3}\mathfrak{b}) = a_{E_3}(\mathbf{A}^2, \mathfrak{a}^{2/3}) - \operatorname{ord}_{E_3} \mathfrak{b} \leqslant -1/3,$$

which contradicts $\operatorname{mld}_P(\mathbf{A}^2, \mathfrak{a}^{2/3}\mathfrak{b}) = 0$.

The description in Theorem 1 holds only in dimension two.

Example 5. Let $P \in \mathbf{A}^3 = \operatorname{Spec} k[x_1, x_2, x_3]$ where P is the origin. Consider the pair $(\mathbf{A}^3, \mathfrak{a}^{4/3})$ for $\mathfrak{a} = (x_1x_2 + x_3^2) + (x_1, x_2, x_3)^3$, which has appeared substantially in [**6**, exercise

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6·45]. There exists a unique divisor E computing $\mathrm{mld}_P(\mathbf{A}^3, \mathfrak{a}^{4/3}) = 0$, but E is not obtained by a weighted blow-up.

Indeed, let X_1 be the blow-up of $\mathbf{A}^3$ at P and E_1 be its exceptional divisor. Let S_1 be the strict transform in X_1 of the surface defined on $\mathbf{A}^3$ by $(x_1x_2 + x_3^2)$. Let X_2 be the blow-up of X_1 along the curve $S_1 \cap E_1$ and E_2 be its exceptional divisor. Then

$$\mathfrak{a}\mathcal{O}_{X_2} = \mathcal{O}_{X_2}(-2E_1 - 3E_2)$$

by the same notation E_1 for its strict transform, and X_2 is a log resolution of $(\mathbf{A}^3, \mathfrak{a})$. One computes $a_{E_1}(\mathbf{A}^3, \mathfrak{a}^{4/3}) = 1/3$ and $a_{E_2}(\mathbf{A}^3, \mathfrak{a}^{4/3}) = 0$. Thus $\mathrm{mld}_P(\mathbf{A}^3, \mathfrak{a}^{4/3}) = 0$ and it is computed only by E_2. However, E_2 is not obtained by a weighted blow-up.

We remark a supplement to Theorem 1 which holds in any dimension.

PROPOSITION 6. *Let $P \in X$ be the germ of a smooth variety and $\mathfrak{a}$ be an $\mathbf{R}$-ideal on X. Let E be the divisor obtained by the blow-up of X at P.*

(i) *If* $\mathrm{ord}_P \mathfrak{a} \leqslant 1$, *then E computes* $\mathrm{mld}_P(X, \mathfrak{a})$.

(ii) *If* $\mathrm{ord}_P \mathfrak{a} < 1$, *then E is the unique divisor computing* $\mathrm{mld}_P(X, \mathfrak{a})$.

Proof. The $a_E(X, \mathfrak{a})$ is the limit of $a_E(X, \mathfrak{a}^{1-\epsilon})$ and $\mathrm{mld}_P(X, \mathfrak{a})$ is that of $\mathrm{mld}_P(X, \mathfrak{a}^{1-\epsilon})$ when ϵ goes to zero from above. Thus (i) follows from (ii).

Assume $\mathrm{ord}_P \mathfrak{a} < 1$ to show (ii). Let F be an arbitrary divisor over X other than E such that $c_X(F) = P$. Setting $(X_0, C_0) = (X, P)$, we build a tower of finitely many birational morphisms

$$X_n \to \cdots \to X_1 \to X_0 = X$$

such that;

(a) $X_i \to X_{i-1}$ is the composition of an open immersion $X_i \to Y_i$ and the blow-up $Y_i \to X_{i-1}$ along C_{i-1};

(b) $C_i = c_{X_i}(F)$ is smooth and non-empty; and

(c) C_n is a divisor (n is taken smallest).

Note that $n \geqslant 2$ since $E \neq F$. Write $\mathfrak{a}_i$ for the weak transform on X_i of $\mathfrak{a}$ and E_i for the exceptional divisor of $X_i \to X_{i-1}$.

Fixing a subvariety D_i of C_i such that $D_i \to C_{i-1}$ is quasi-finite and dominant, by [**4**, III, lemmata 7 and 8] one has that

$$\mathrm{ord}_{C_i} \mathfrak{a}_i \leqslant \mathrm{ord}_{D_i} \mathfrak{a}_i \leqslant \mathrm{ord}_{C_{i-1}} \mathfrak{a}_{i-1}.$$

Thus $\mathrm{ord}_{C_i} \mathfrak{a}_i < 1$ for any i by our assumption $\mathrm{ord}_{C_0} \mathfrak{a}_0 < 1$. On the other hand, one computes that

$$a_{E_i}(X, \mathfrak{a}) = \sum_{j \in I_{i-1}} (a_{E_j}(X, \mathfrak{a}) - 1) + \mathrm{codim}_{X_{i-1}} C_{i-1} - \mathrm{ord}_{C_{i-1}} \mathfrak{a}_{i-1},$$

where I_{i-1} denotes the set of $1 \leqslant j \leqslant i-1$ such that C_{i-1} lies on the strict transform of E_j. Using $\mathrm{ord}_{C_{i-1}} \mathfrak{a}_{i-1} < 1$, one concludes inductively that

$$a_{E_i}(X, \mathfrak{a}) > 1, \qquad a_{E_i}(X, \mathfrak{a}) > a_{E_{i-1}}(X, \mathfrak{a})$$

for any i. In particular,

$$a_F(X, \mathfrak{a}) = a_{E_n}(X, \mathfrak{a}) > a_{E_1}(X, \mathfrak{a}) = a_E(X, \mathfrak{a}),$$

and (ii) follows.

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