ON COUNTERS WITH RANDOM DEAD TIME. I

By J. M. HAMMERSLEY

Received 15 January 1953; *and in revised form* 30 May 1953

1. *Introduction.* The following work was undertaken in connexion with a device for counting blood cells electronically, which has been developed in the Clinical Pathology Department of the Radcliffe Infirmary with financial assistance from the Medical Research Council and the Nuffield Foundation. Although the results will apply to other types of counter, it will help to fix the ideas if we consider the problem for this specific device. A large number of blood cells, contained in a shallow chamber, are scanned by a photoelectric cell. The depth of the chamber and the concentration of blood cells in solution therein allow blood cells (supposed distributed at random throughout the chamber) to overlap when viewed from above by the scanner. The field of view of the scanner at any instant is somewhat larger than the size of a blood cell, but is, nevertheless, of much the same order of magnitude. With passage of time the chamber moves underneath the photocell so that the field of view traces out a long narrow path not crossing or overlapping itself and only embracing a portion of the whole chamber. The blood cells have no motion relative to the chamber. As each blood cell comes under the photocell it produces an electrical impulse, whose duration depends upon the size and shape and orientation of the blood cell. These impulses go to a counter, which counts them except that it will not count any impulse which is overlapped by a previous impulse. The problem is to determine the number of blood cells in the chamber from a knowledge of the recorded count and the distribution of the lengths of individual impulses. A further complication arises because it is inadvisable to count two impulses separated only by a very short interval of time, and therefore the counter itself generates some self-paralysing impulses additional to the blood-cell impulses and in a manner which is correlated with them. Until § 3, however, we shall neglect paralysis, because a proper understanding of its effects cannot be reached unless we first know how the system behaves in the absence of paralysis.

Without paralysis the problem is equivalent to the distribution of the number of gaps between intervals of random length distributed at random on a line.† This problem has been considered in various guises by various authors (see Domb (2, 3) and Feller (5) for references); and Domb (2) gave a solution for the particular case when the intervals are all of the same fixed length. Of the general case Domb (3) writes: '...the problem is no longer a simple generalisation....The possibility of one interval completely covering another which follows it causes considerable mathematical difficulties.'

† It might at first sight appear that the two-dimensional problem is essentially harder than the one-dimensional, in that every one-dimensional interval (e.g. photocell impulse) is wholly scanned (e.g. counted) if it is scanned at all, whereas a two-dimensional interval (e.g. shadow of a blood cell) may be partially scanned if only a fraction of its area enters the field of view. Closer scrutiny shows the difference more apparent than real; the issue is partly a question of mechanical design and will be discussed more appropriately in the sequel to the present paper.

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

I am very much indebted to Domb, who showed me how to surmount these difficulties by a brilliant application of the elementary theorem that the expectation of the sum of several (possibly interdependent) quantities is the sum of their expectations. A second key technique, due to Domb and Feller, is the use of Laplace transforms.

Following a technique due to Stevens (9), it is convenient to consider a closed scan, by replacing the line by a circle and the intervals by arcs of random length distributed at random on this circle. This makes no appreciable difference when y, the circumference of the circle, is large in comparison with τ, the maximum length of an interval; but it simplifies the analysis in that we do not have to consider what happens at the ends of the scan. Suppose that $U(t)$ is the cumulative distribution function of the lengths t of the intervals, that T is their mean length, and that λ is the mean number of intervals per unit length of the scan. Then we shall show that the number of recorded intervals is asymptotically normal as $y \to \infty$, with mean

$$\mu_1 = \lambda y e^{-\lambda T} \tag{1}$$

and variance

$$\mu_2 = \lambda y e^{-\lambda T}\left\{1 - 2\int_0^\tau [\lambda e^{-\lambda T} - W(z)]\,dz\right\}, \tag{2}$$

where

$$W(z) = \lambda U(z) \exp\left\{-\lambda z + \lambda \int_0^z U(t)\,dt\right\}. \tag{3}$$

These expressions are complicated (and the corresponding results, given in § 4, allowing for paralysis as well are worse); but in practical applications it may be enough to know that the recorded count is asymptotically normal with parameters which are proportional to the length of scan. Equations (1) and (2) are exact for all $y > \tau$. Exact higher moments can be derived from equation (16). The results can be extended without much difficulty to the case in which $y \leqslant \tau \leqslant \infty$.

Our results are rather more complete than those of recent Hungarian work along similar lines (see (7), (8), (10), (11)).

The discussion of this paper is confined wholly to determining the distribution of the recorded count, when the expected number λ of intervals per unit length of scan is known. This is a necessary preliminary to the inverse inductive problem,† to be discussed in a second paper, of estimating λ when the recorded count is known.

I am indebted to Drs Macfarlane and Poole and Mr Tomlinson for helpful discussion of their electronic blood-cell counter.

2. *Detailed theory for random dead time without paralysis.* Let x $(0 \leqslant x < y)$ be a current coordinate measured clockwise on the circle from a fixed origin. Write PQ for the arc taken clockwise from a point P to a point Q. If we wish to emphasize the status of end-points of PQ we shall use one of the forms (PQ), $(PQ]$, $[PQ)$, $[PQ]$, in which a round bracket denotes the arc is open at that end and a square bracket denotes closure. We shall say that P is a distance x' before (after) Q if the clockwise (anticlockwise) distance from P to Q is x'. Distribute at random on to this circle a number of intervals. These intervals are to be *closed* arcs of the same radius as that of the circle; hereafter we refer to them exclusively as 'intervals' to distinguish them from 'arcs' of the circle

† That is to say, the type of problem with which Albert and Nelson (1) deal under simpler initial assumptions.

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

on to which they are distributed. The extreme anticlockwise point of closure of an interval is called its 'start'. The length of an interval is the distance between its points of closure measured clockwise from its start along an arc of the circle. The intervals are drawn at random from an infinite population having $U(t)$ for the cumulative distribution function of interval length t. It will be important to remember that $U(t)$, being a distribution function, is continuous on the right, whilst $U(t-0)$ is continuous on the left. We write τ for the greatest lower bound of all numbers t satisfying $U(t) = 1$; so that τ is the maximum possible interval length. For convenience only,† we shall suppose τ is finite and less than y. The intervals are placed at random on the circle in such a way that the positions of all start points are mutually independent (in the stochastic sense), that the position of the start of any particular interval is independent of the length of that interval, and that there is a probability $\lambda\delta x + o(\delta x)$ as $\delta x \to 0$ of a specified arc of length δx containing a start.

It follows that the probability, that the specified arc of length δx will contain the start of an interval of length t satisfying $\xi < t \leqslant \eta$, is $\lambda \Delta U(\xi, \eta)\, \delta x + o(\delta x)$, where $\Delta U(\xi, \eta) = U(\eta) - U(\xi)$. Hence, for $p = 0, 1, 2, \ldots$,

$$f_p(x; \xi, \eta) = \{\lambda x \Delta U(\xi, \eta)\}^p e^{-\lambda x \Delta U(\xi, \eta)}/p! \tag{4}$$

is the probability that precisely p intervals of lengths $\xi < t \leqslant \eta$ will start in a specified arc of length x (whatever other intervals may start there).

We shall say that a point of the circle is 'free' if it does not belong to any interval after completing the distribution of intervals. We shall say that a start is 'recorded' if it is the limit point of a set of free points, i.e. if it does not belong to any other interval except perhaps as a start of that interval.

Let O be the origin and P be the point with coordinate $z > 0$. Let $\delta > 0$ be prescribed. Choose points P_i' with coordinates z_i' satisfying

$$O = z_1' < z_2' < \ldots < z_{m+1}' = z, \quad m = m(\delta), \quad \max (z_{i+1}' - z_i') \leqslant \delta.$$

Consider the three statements

S_1: for every $i = 1, 2, \ldots, m$, the arc $(P_i' P_{i+1}']$ does not contain the start of an interval of length $t \geqslant z - z_{i+1}'$.

S_2: P is not covered by any interval whose start lies in $(OP]$.

S_3: for every $i = 1, 2, \ldots, m$, the arc $(P_i' P_{i+1}']$ does not contain the start of an interval of length $t \geqslant z - z_i'$.

We are going to determine $V(z)$, the probability that S_2 holds. Since S_1 implies S_2, which in turn implies S_3, we have from (4) with $p = 0$

$$\exp\left\{-\lambda \sum_{i=1}^{m} [1 - U(z - z_{i+1}' - 0)]\,(z_{i+1}' - z_i')\right\} = \text{prob}(S_1) \leqslant V(z)$$
$$\leqslant \text{prob}(S_3) = \exp\left\{-\lambda \sum_{i=1}^{m} [1 - U(z - z_i' - 0)]\,(z_{i+1}' - z_i')\right\}.$$

Now $U(t)$ is a Riemann integrable function (being bounded and non-decreasing) with at most an enumerable set of discontinuities. Hence as $\delta \to 0+$, each of the above sums tends to

$$\int_0^z 1 - U(z - t - 0)\, dt = \int_0^z 1 - U(t - 0)\, dt = z - \int_0^z U(t)\, dt;$$

† The effect of dispensing with this supposition appears in equation (41).

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

so that
$$V(z) = \exp\left\{-\lambda z + \lambda \int_0^z U(t)\,dt\right\}. \tag{5}$$

We notice for future reference that, when $z \geqslant \tau$,

$$V(z) = \exp\left\{-\lambda z + \lambda[tU(t)]_0^z - \lambda \int_0^z t\,dU(t)\right\} = e^{-\lambda T}, \tag{6}$$

where T is the mean length of the population of intervals.

Next let Q be the point with coordinate ζ where $0 < \zeta \leqslant z$. Consider the three statements

S_4: precisely p intervals start in $(OQ]$, and all these intervals are of length $t < z - \zeta$.

S_5: precisely p intervals start in $(OQ]$, and none of these intervals contain P.

S_6: precisely p intervals start in $(OQ]$, and all these intervals are of length $t < z$.

Notice, for instance, that S_4 is quite different from the statement 'precisely p intervals of length $t < z - \zeta$ start in $(OQ]$' because this latter statement allows intervals of length $t \geqslant z - \zeta$ to start in $(OQ]$. Write $V'_p(z, \zeta) = \text{prob}(S_5)$. The probability of S_4 is the probability that precisely p intervals start in $(OQ]$ multiplied by the probability that each of these intervals is of length $t < z - \zeta$. Since S_4 implies S_5, which in turn implies S_6, we have by putting $\Delta U(\xi, \eta) = 1$ in (4)

$$\begin{aligned}\{(\lambda\zeta)^p e^{-\lambda\zeta}/p!\}\{U(z-\zeta-0)\}^p &= \text{prob}(S_4) \leqslant V'_p(z,\zeta) \leqslant \text{prob}(S_6) \\ &= \{(\lambda\zeta)^p e^{-\lambda\zeta}/p!\}\{U(z-0)\}^p. \end{aligned} \tag{7}$$

We also have
$$g_p(z, \zeta) = V(z)\, V'_p(z, \zeta) \tag{8}$$

for the probability that precisely p intervals start in $(OQ]$ and that P is not covered by any interval which starts in $(OP]$; for this probability is the probability that P is not covered by any interval which starts in $(OP]$ multiplied by the probability that precisely p intervals start in $(OQ]$ conditional upon the information that none of these p intervals contain P.

Now let P_j, Q_j $(j = 1, 2, \ldots, k)$ be a fixed set of points arranged around the circle in clockwise order $OP_1Q_1P_2Q_2 \ldots P_kQ_kO$. Write P_{k+1} as an alternative name for P_1, and let z_j denote the length of P_jP_{j+1} and ζ_j the length of P_jQ_j. Suppose that $0 < \zeta_j \leqslant z_j$ and that $\zeta_j \leqslant \zeta$ for all j, where $\zeta > 0$ is prescribed. Evidently

$$\sum_{j=1}^{k} z_j = y. \tag{9}$$

We have
$$g = \prod_{j=1}^{k} g_{p_j}(z_j, \zeta_j) \tag{10}$$

for the probability that all the P_j are free points whilst precisely p_j intervals start in $(P_jQ_j]$; for this event will occur if and only if P_{j+1} is not covered by any interval which starts in $(P_jP_{j+1}]$, whilst precisely p_j intervals start in $(P_jQ_j]$; and the sets of events, which contribute to the separate terms $g_{p_j}(z_j, \zeta_j)$ for any two distinct values of j, must be independent of each other since they necessarily refer to two sets of intervals having no interval common to both sets. Let α_j be a positive integer; and let ν_j be the number of recorded starts in $(P_jQ_j]$. Then $\nu_j = p_j$ if $p_j = 0$ or 1, whilst $\nu_j \leqslant p_j$

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

otherwise. Consequently $\nu_j^{\alpha_j} \leqslant p_j^{\alpha_j}$, with equality holding when $p_j = 0$ or 1. Equations (7), (8), (10) yield

$$\begin{aligned}
&\prod_{j=1}^{k} V(z_j)\,\lambda\zeta_j\,U(z_j-\zeta_j-0) \leqslant \prod_{j=1}^{k} V(z_j)\,V'(z_j,\zeta_j) = \prod_{j=1}^{k} g_1(z_j,\zeta_j)\\
&\leqslant \mathscr{E}\left(\prod_{j=1}^{k}\nu_j^{\alpha_j}\right) \leqslant \mathscr{E}\left(\prod_{j=1}^{k} p_j^{\alpha_j}\right) = \sum_{p_1\geqslant 1,\,p_2\geqslant 1,\,\ldots,\,p_k\geqslant 1} \prod_{j=1}^{k} p_j^{\alpha_j} g_{p_j}(z_j,\zeta_j)\\
&= \prod_{j=1}^{k} V(z_j) \sum_{p_j=1}^{\infty} p_j^{\alpha_j} V'_{p_j}(z_j,\zeta_j) \leqslant \prod_{j=1}^{k} V(z_j) \sum_{p_j=1}^{\infty} p_j^{\alpha_j}\{\lambda\zeta_j\,U(z_j-0)\}^{p_j} e^{-\lambda\zeta_j}/p_j!\\
&\leqslant \prod_{j=1}^{k}\left\{V(z_j)\,\lambda\zeta_j\,U(z_j-0)\sum_{p_j=1}^{\infty} p_j^{\alpha_j}(\lambda\zeta_j)^{p_j-1}/p_j!\right\}. \qquad (11)
\end{aligned}$$

The series in the last term of this expression converges uniformly in $0 \leqslant \zeta_j \leqslant \zeta$, and hence is $1+o(1)$ as $\zeta_j \to 0+$.

Also $U(z_j-\zeta_j-0) = U(z_j-0)\{1+o(1)\}$ as $\zeta_j \to 0+$. Therefore (11) yields

$$\mathscr{E}\left(\prod_{j=1}^{k}\nu_j^{\alpha_j}\right) = \left\{\prod_{j=1}^{k} V(z_i)\,U(z_j-0)\,\lambda\zeta_j\right\}\{1+o(1)\} \quad \text{as} \quad \zeta\to 0+. \qquad (12)$$

For prescribed $\zeta > 0$ let P_j^* $(j = 1, 2, \ldots, N;\ N = N(\zeta))$ be a given set of *distinct* points arranged in clockwise order around the circle such that the length of every arc $P_j^* P_{j+1}^*$ is less than ζ, where P_{N+1}^* is an alternative name for P_1^*. Consider any given subset of k of the points P^* $(k \leqslant N)$ and, moving clockwise from O, name these points $P_1, P_2, \ldots, P_k$ as in the previous paragraph. Take Q_j to be point of the P^* set which follows immediately after P_j. It may happen that Q_j coincides with P_{j+1}. In any particular random arrangement of the intervals upon the circle, let n denote the total number of recorded starts, let n_j $(j = 1, 2, \ldots, N)$ denote the number of recorded starts in $(P_j^* P_{j+1}^*]$, and let ν_j $(j = 1, 2, \ldots, k)$ denote the number of recorded starts in $(P_j Q_j]$. The number of terms in the expansion of

$$n^m = \left(\sum_{j=1}^{N} n_j\right)^m \quad (m = \text{positive integer})$$

which take the form $\nu_1^{\alpha_1}\nu_2^{\alpha_2}\ldots\nu_k^{\alpha_k}$ for fixed k, fixed $\nu_1, \nu_2, \ldots, \nu_k$ and arbitrary $\alpha_j \geqslant 1$, is the number of ways of selecting one element from each of the m identical rows

$$\begin{array}{cccc} n_1 & n_2 & \ldots & n_N \\ n_1 & n_2 & \ldots & n_N \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots} \\ n_1 & n_2 & \ldots & n_N \end{array}$$

in such a way that only elements from the columns $\nu_1, \nu_2, \ldots, \nu_k$ are selected and that at least one element is selected from each such column. It is therefore the number of ways of distributing m dissimilar objects into k dissimilar boxes, permutations within a box being disregarded. This number is $\Delta^k(0^m)$, the kth initial difference of the mth powers $0^m, 1^m, 2^m, \ldots$. Since the expected value of the sum of a finite number of quantities is the sum of their expected values (whether or not the quantities are independent), we have for the mth moment (about the origin) of the number of recorded starts n

$$\mu'_m = \mathscr{E}(n^m) = \sum_{k=1}^{m} \Delta^k(0^m) \sum_{P\in P^*}^{(k)} \mathscr{E}\left(\prod_{j=1}^{k}\nu_j^{\alpha_j}\right), \qquad (13)$$

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

where the second sum is taken over all k-pointed subsets P of the set P^*, permutations within a subset being disregarded. Let x_j denote the coordinate of P_j; so that by (9)

$$z_j = x_{j+1} - x_j \quad (j = 1, 2, \ldots, k-1), \quad z_k = y + x_1 - x_k. \tag{14}$$

Notice that $U(z-0)\, V(z)$ is Riemann integrable and that μ'_m is independent of ζ. Let $\zeta \to 0+$, and in the limit replace ζ_j by dx_j; whereupon from (9), (3) and (5)

$$\lim_{\zeta \to 0+} \sum_{P \in P^*}^{(k)} \mathscr{E}\left(\prod_{j=1}^{k} \nu_j^{\alpha_j}\right) = \int\cdots\int_{0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_k \leqslant y} \left\{\prod_{j=1}^{k} \lambda U(z_j - 0)\, V(z_j)\, dx_j\right\}$$

$$= \int\cdots\int_{0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_k \leqslant y} \left\{\prod_{j=1}^{k} W(z_j)\, dx_j\right\} = I_k(y), \quad \text{say}, \tag{15}$$

the replacement of $U(z_j - 0)$ by $U(z_j)$ being permissible for $k > 1$ because these two functions only differ at an enumerable set of points, and for $k = 1$ because by (14) they do not differ when $z_1 = y > \tau$. Equations (13) and (15) provide

$$\mu'_m = \sum_{k=1}^{m} \Delta^k(0^m)\, I_k(y). \tag{16}$$

From (6), an easy calculation verifies that (1) and (2) hold for all $y > \tau$.

It remains to show that the distribution is asymptotically normal. We begin by deriving a recurrence relation between the integrals I_k. First transform the integrand of I_k by means of

$$z_0 = x_1, \quad z_1 = x_2 - x_1, \quad z_2 = x_3 - x_2, \quad \ldots, \quad z_{k-1} = x_k - x_{k-1}.$$

This yields

$$I_k = \int\cdots\int W(z_1)\, W(z_2) \ldots W(z_{k-1})\, W\left(y - \sum_{j=1}^{k-1} z_j\right) dz_0 dz_1 \ldots dz_{k-1},$$

where the range of integration is given by the inequalities

$$0 \leqslant x_{j+1} - x_j = z_j \quad (j = 1, 2, \ldots, k-1), \quad 0 \leqslant z_0 = x_k - \sum_{j=1}^{k-1} z_j \leqslant y - \sum_{j=1}^{k-1} z_j.$$

Perform the integration with respect to z_0 and deduce

$$I_k = \int\cdots\int_{D_{k-1}(y)} \left(y - \sum_{j=1}^{k-1} z_j\right) W(z_1)\, W(z_2) \ldots W(z_{k-1})\, W\left(y - \sum_{j=1}^{k-1} z_j\right) dz_1 dz_2 \ldots dz_{k-1},$$

in which $D_{k-1}(y)$ is the generalized tetrahedron $0 \leqslant z_j, \sum_{j=1}^{k-1} z_j \leqslant y$. In this integral make the transformation $z'_i = y - \sum_{j=1}^{k-1} z_j$; $z'_j = z_j$ $(j \neq i)$, and obtain

$$I_k = \int\cdots\int_{D_{k-1}(y)} z'_i W(z'_1)\, W(z'_2) \ldots W(z'_{k-1})\, W\left(y - \sum_{j=1}^{k-1} z'_j\right) dz'_1 dz'_2 \ldots dz'_{k-1}.$$

Add the sum of these integrals for $i = 1, 2, \ldots, k-1$ to the previous integral and deduce

$$I_k(y) = \frac{y}{k} \int\cdots\int_{D_{k-1}(y)} W(z_1)\, W(z_2) \ldots W(z_{k-1})\, W\left(y - \sum_{j=1}^{k-1} z_j\right) dz_1 dz_2 \ldots dz_{k-1}. \tag{17}$$

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

Multiply (17) by $kW(Y-y)/y$ and integrate from $y=0$ to $y=Y$. In the resulting multiple integral on the right-hand side make the substitution $z_k = y - \sum_{j=1}^{k-1} z_j$. This yields the required recurrence relation

$$\int_0^Y \frac{kI_k(y)}{y} W(Y-y)\,dy = \frac{(k+1)\,I_{k+1}(Y)}{Y} \quad (k \geqslant 1), \tag{18}$$

which starts from the initial condition

$$I_1(y)/y = W(y). \tag{19}$$

The recurrence relations (18) and (19) will define $I_k(y)$ for all $y \geqslant 0$. We shall also need the inequality

$$0 \leqslant I_k(y) \leqslant (My)^k/k!, \tag{20}$$

where M is the least upper bound of $W(z)$ for $0 \leqslant z \leqslant \tau$. This result (20) follows from (6) and (17).

Define an auxiliary function

$$\psi(\theta, y) = \sum_{k=1}^{\infty} kI_k(y)\,\theta^{k-1}/y. \tag{21}$$

The inequality (20) shows that $|\psi(\theta, y)| \leqslant Me^{My|\theta|}$; and hence ψ is an integral function of θ. This remark justifies the subsequent term-by-term integrations of ψ and the rearrangement of series. Writing ϑ for the operator $\partial/\partial x$, we have the identity

$$(e^\theta - 1)^k = [(e^\vartheta - 1)^k e^{x\theta}]_{x=0} = \left[\Delta^k \sum_{m=0}^{\infty} (x\theta)^m/m!\right]_{x=0} = \sum_{m=k}^{\infty} \Delta^k(0^m)\,\theta^m/m!;$$

whence

$$\phi(\theta, y) = 1 + y\int_0^\theta e^z \psi(e^z - 1, y)\,dz = 1 + \sum_{k=1}^{\infty} (e^\theta - 1)^k I_k(y)$$

$$= 1 + \sum_{m=1}^{\infty} \sum_{k=1}^{m} \Delta^k(0^m)\,I_k(y)\,\theta^m/m! \tag{22}$$

is the moment-generating function of n. From (18), (19) and (21)

$$\psi(\theta, Y) = W(Y) + \sum_{k=1}^{\infty} \theta^k(k+1)\,I_{k+1}(Y)/Y = W(Y) + \theta\int_0^Y \psi(\theta, y)\,W(Y-y)\,dy. \tag{23}$$

Take the Laplace transform of this integral equation. With

$$\Psi(\theta, p) = \int_0^\infty e^{-py}\,\psi(\theta, y)\,dy, \quad \Omega(p) = \int_0^\infty e^{-py}\,W(y)\,dy,$$

we find

$$\Psi(\theta, p) = \Omega(p) + \theta\Psi(\theta, p)\,\Omega(p) \quad (\Re(p) > M\,|\,\theta\,|);$$

whence the inverse Laplace transform yields

$$\psi(\theta, y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{py}\,\Omega(p)}{1 - \theta\Omega(p)}\,dp \quad (c > M\,|\,\theta\,|). \tag{24}$$

For the next stage of the analysis it is convenient to write $\kappa_1 y$, $\kappa_2 y$, $\kappa_3 y$ for the first three cumulants of the distribution of n. It may be verified with the aid of (16) that

$$\left.\begin{aligned} \kappa_1 &= \lambda e^{-\lambda T}, \quad \kappa_2 = \kappa_1\left\{1 + 2\int_0^\tau M(z)\,dz\right\}, \\ \kappa_3 &= \kappa_1\left\{\frac{3}{2}\left(\frac{\kappa_2}{\kappa_1}\right)^2 - \tfrac{1}{2} - 6\kappa_1\int_0^\tau zM(z)\,dz\right\}, \end{aligned}\right\} \tag{25}$$

where

$$M(z) = W(z) - \kappa_1. \tag{26}$$

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

Further, since $U(t)$ is a cumulative distribution function,

(i) $|M(z)| \leqslant M \geqslant \kappa_1 > 0$; (27)

(ii) $M(z)$ has total limited fluctuation.

Let $\gamma > 0$ be prescribed. Let $q = q(\theta)$ denote any solution of

$$\frac{1}{\theta} = \Omega(q) = \frac{\kappa_1}{q} + \int_0^\tau M(z)\, e^{-qz}\, dz, \quad \Re(q) \geqslant -\gamma \tag{28}$$

for any sufficiently small non-zero complex θ. When $|\theta| < e^{-\gamma\tau}/M\tau$, (27) and (28) yield

$$\frac{\kappa_1}{|q|} - M\tau e^{\gamma\tau} \leqslant \frac{1}{|\theta|} \leqslant \frac{\kappa_1}{|q|} + M\tau e^{\gamma\tau};$$

whence there exists a real function $\vartheta = \vartheta(\theta)$ such that

$$q = \kappa_1 \theta e^{i\vartheta} + O(\theta^2) \quad \text{as} \quad \theta \to 0.$$

Substitution into (28) shows that $\vartheta = O(\theta)$ as $\theta \to 0$; and this justifies the solution of (28) by successive approximation. Using (25) we get

$$q = \kappa_1\theta + \tfrac{1}{2}(\kappa_2 - \kappa_1)\,\theta^2 + \tfrac{1}{6}(\kappa_3 - 3\kappa_2 + 2\kappa_1)\,\theta^3 + O(\theta^4) \quad \text{as} \quad \theta \to 0; \tag{29}$$

and thence

$$\begin{aligned}\Omega'(q) = \frac{\partial\Omega(q)}{\partial q} &= -\frac{\kappa_1}{q^2} - \int_0^\tau zM(z)\, e^{-qz}\, dz \\ &= \frac{-1}{\kappa_1^2\theta^2}\left\{\kappa_1 - (\kappa_2 - \kappa_1)\,\theta + \left(\frac{\kappa_2^2}{\kappa_1} - \tfrac{1}{2}\kappa_2 - \tfrac{1}{2}\kappa_3\right)\theta^2\right\} + O(\theta) \quad \text{as} \quad \theta \to 0. \end{aligned} \tag{30}$$

Suppose that (28) has two distinct solutions q and $q + q'$ for small θ. Then (29) shows that $q' = O(\theta^4) \to 0$ as $\theta \to 0$; and so

$$0 = \frac{\Omega(q+q') - \Omega(q)}{q'} = \Omega'(q) + O(1) = -\frac{1}{\kappa_1\theta^2} + O(\theta^{-1}) \quad \text{as} \quad \theta \to 0,$$

which is a contradiction. Hence, for sufficiently small complex θ, q is the unique root of (28), and furthermore it is simple by virtue of (30). Let $p = \rho + i\sigma$, where ρ and σ are real. For any fixed value of ρ, $M(z)\, e^{-\rho z}$ has total limited fluctuation in $0 \leqslant z \leqslant \tau$ in view of (27); and therefore the Riemann-Lebesgue lemma (Whittaker and Watson (12), p. 172) applied to (28) demonstrates

$$\Omega(\rho + i\sigma) = O(\sigma^{-1}) \quad \text{as} \quad \sigma \to \pm\infty. \tag{31}$$

Further, by (28), $\Omega(p)$ is analytic for all finite p except $p = 0$, where it has a simple pole with residue κ_1. By the inverse Laplace transform and (6) we have, for $y > \tau$,

$$-\frac{1}{2\pi i}\int_{-\gamma-i\infty}^{-\gamma+i\infty} e^{py}\,\Omega(p)\,dp = \kappa_1 - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{py}\,\Omega(p)\,dp = \kappa_1 - W(y) = 0.$$

Hence, by a further use of (31), (24) yields for small θ and large y

$$\begin{aligned}\psi(\theta, y) &= \frac{-e^{qy}}{\theta^2\Omega'(q)} + \frac{1}{2\pi i}\int_{-\gamma-i\infty}^{-\gamma+i\infty} e^{py}\left[\frac{\Omega(p)}{1-\theta\Omega(p)} - \Omega(p)\right]dp \\ &= \frac{-e^{qy}}{\theta^2\Omega'(q)} + \frac{\theta\, e^{-\gamma y}}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\sigma y}\{\Omega(-\gamma+i\sigma)\}^2}{1-\theta\Omega(-\gamma+i\sigma)}\,d\sigma \\ &= \frac{-e^{qy}}{\theta^2\Omega'(q)} + O(\theta\, e^{-\gamma y}) \quad \text{as} \quad \theta \to 0,\ y \to +\infty. \end{aligned} \tag{32}$$

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

It should be noticed that (32) has been established in such a way that it is valid whether or not θ and y are dependent, provided only $\theta \neq 0$. We now choose any fixed real number $v \neq 0$ and then any real number $u \neq 0$ satisfying $|u| \leqslant |v|$. Set

$$\theta = e^{iu/\sqrt{y}} - 1. \tag{33}$$

From (29), (30) and (32), it is a matter of algebra to verify that

$$iy^{\frac{1}{2}}\, e^{iu/\sqrt{y}}\, \psi(e^{iu/\sqrt{y}} - 1, y) = \left\{\frac{\partial K(u)}{\partial u} + O(y^{-1})\right\} e^{K(u)+O(y^{-1})} \quad \text{as} \quad y \to +\infty, \tag{34}$$

where $$K(u) = i\kappa_1 y^{\frac{1}{2}} u - \tfrac{1}{2}\kappa_2 u^2 - \tfrac{1}{6} i\kappa_3 u^3 y^{-\frac{1}{2}},$$

and the O-symbols in (34) depend only upon v and y for all $u \neq 0$ in $|u| \leqslant |v|$. Since $\frac{\partial K(u)}{\partial u} = O(y^{\frac{1}{2}})$ and $\Re\{K(u)\} = O(1)$ as $y \to +\infty$, we deduce

$$iy^{\frac{1}{2}}\, e^{iu/\sqrt{y}}\, \psi\,(e^{iu/\sqrt{y}} - 1, y) = \frac{\partial K(u)}{\partial u}\, e^{K(u)} + O(y^{-\frac{1}{2}}) \quad \text{as} \quad y \to +\infty, \tag{35}$$

the O-symbol being uniform in $0 < |u| \leqslant |v|$. According to (21), $\psi(0, y) = \kappa_1$; and hence (35) holds uniformly in $0 \leqslant |u| \leqslant |v|$ even when $v = 0$. From (22) the characteristic function of $(n - \kappa_1 y)/\sqrt{y}$ is for all real finite v

$$e^{-i\kappa_1 v\sqrt{y}}\, \phi(ivy^{-\frac{1}{2}}, y) = e^{-i\kappa_1 v\sqrt{y}} \left\{1 + y\int_0^{iv/\sqrt{y}} e^z \psi(e^z - 1, y)\, dz\right\}$$

$$= e^{-i\kappa_1 v\sqrt{y}} \left\{1 + \int_0^v iy^{\frac{1}{2}}\, e^{iu/\sqrt{y}}\, \psi(e^{iu/\sqrt{y}} - 1, y)\, du\right\} \to e^{-\frac{1}{2}\kappa_2 v^2} \quad \text{as} \quad y \to +\infty;$$

and the asymptotic normality of the cumulative distribution function of n follows, since the right-hand side is a continuous function of v.

Since $\phi(\theta, y)$ is an integral function of θ, it determines the frequency function of n uniquely; and this frequency function must therefore be the coefficient of $e^{n\theta}$ in $\phi(\theta, y)$, namely,

$$f(n, y) = \sum_{k=n}^{\infty} (-)^{k-n} \binom{k}{n} I_k(y) \quad (n = 1, 2, \ldots). \tag{36}$$

Equation (36) is also valid for $n = 0$ provided we interpret $I_0(y) = 1$. If $J_k(p)$ is the Laplace transform of $kI_k(y)/y$, (18) and (19) yield

$$J_k(p) = \{\Omega(p)\}^k \quad (k = 1, 2, \ldots). \tag{37}$$

Also (19) and (20) show that $|\Omega(p)| < 1$ for $\Re(p) > M$. Suppose this inequality holds, divide (36) by y, take the Laplace transform, and deduce

$$\int_0^\infty e^{-py} \frac{f(n, y)}{y}\, dy = \sum_{k=n}^{\infty} (-)^{k-n} \binom{k}{n} \frac{\{\Omega(p)\}^k}{k}$$

$$= \frac{1}{n} \left\{\frac{\Omega(p)}{1 + \Omega(p)}\right\}^n \quad (\Re(p) > M,\ n \geqslant 1). \tag{38}$$

Similarly, with $n = 0$ in (36),

$$\int_0^\infty e^{-py} \frac{f(0, y) - 1}{y}\, dy = -\log\{1 + \Omega(p)\} \quad (\Re(p) > M). \tag{39}$$

The inverse Laplace transform of (38) yields

$$f(n, y) = \Sigma_j \frac{\{-\eta_j y^{\eta_j} / \Omega^{(\eta_j)}(r_j)\}^n}{(n\eta_j)!}\, e^{yr_j} \quad (n \geqslant 1), \tag{40}$$

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

where r_j are the roots of $\Omega(p) = -1$ with respective multiplicities η_j, and

$$\Omega^{(\eta)}(p) = \partial^\eta \Omega(p)/\partial p^\eta.$$

Equation (40) seems rather intractable; but it may be simpler than (36) for special functional forms $\Omega(p)$.

In discussing the solution of (28) we *proved* that the equation $\Omega(p) = 1/\theta$ has a unique dominant root (i.e. a root whose real part exceeds the real part of any other root) when θ is sufficiently small, and we deduced that the cumulative distribution function of n was asymptotically normal. If we *assume* that a unique simple dominant root also exists when $\theta = -1$ (and I can as yet find no theoretical proof to justify this assumption), then we can establish the stronger conclusion that the frequency function of n is asymptotically normal. For with this assumption, there will be a dominant term

$$\{-y/\Omega'(r)\}^n e^{yr}/n!$$

in (40) for sufficiently large y; and this will establish that the frequency function of n is unimodal for large enough y. Then asymptotic normality of the distribution function will imply asymptotic normality of the frequency function (Hammersley (6)).

It is not difficult to extend the results to the case when $y \leqslant \tau \leqslant \infty$. For the probability that at least one interval of length $t \geqslant y$ will fall on the circle is

$$1 - \exp\{-\lambda y + \lambda y U(y-0)\}. \tag{41}$$

In this event the recorded count will be zero; whilst otherwise the previous theory will apply with probability $\exp\{-\lambda y + \lambda y U(y-0)\}$, $U(t)$ being replaced by $U(t)/U(y-0)$.

Stevens (9) was concerned with the probability of complete coverage of the circle when a fixed number of intervals of fixed length were distributed at random on to it. Equations (36) and (39), the former with $n = 0$, provide the corresponding solution when the number of intervals is also random (with given expectation λ per unit arc) and their lengths are random (with given cumulative distribution function U). For the probability of complete coverage is the probability that there is at least one interval on the circle and no observed starts; and is therefore $f(0, y) - e^{-\lambda y}$.

3. *Introductory remarks for random dead time with paralysis.* We now consider the effects of artificially introducing paralysing impulses in addition to the photocell impulses. To begin with, we shall suppose that these paralyses are impulses of constant length $\varpi < y$; only finally shall we generalize the results to the case in which the paralyses are of random duration. To specify the instants at which paralyses commence we generalize the previous uses of the words 'free' and 'recorded'. Now we shall say a point of the scan is free if it does not belong to any photocell impulse or to any paralysis; a photocell impulse is 'recorded' if its start is the limit point of a set of free points (in the new generalized sense); and finally every start of a recorded (photocell) impulse is the start of a paralysis and every paralysis starts simultaneously with a recorded (photocell) impulse. These generalizations of the words 'free' and 'recorded' are to carry with them the convention that, whenever as in § 4 we temporarily disregard paralysis, these words shall automatically and temporarily revert to their previous senses.

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

If ϖ is less than the minimum length of photocell impulse, the introduction of paralysis will not affect the total count. [This statement is true from the point of view of theory; but it ignores the main practical reason for introducing paralysis, namely, that artificial paralysis will override unwanted electrical noise on the leading edge of a photocell impulse and so will reduce the tendency for an impulse of ragged profile to be counted more than once, although it will not entirely eliminate spurious counts because it will not override noise on a trailing edge.] On the other hand, if $\varpi \geqslant \tau$, it is neither theoretically nor practically correct to suppose that the count will be the same as if there were no paralysis and the duration of every photocell impulse were ϖ [because a photocell impulse will not be lengthened to ϖ if it is not recorded, and so it will not prevent the recording of a later impulse, which it would have otherwise overlapped]. This misconception is worth mentioning, since it has been responsible for the introduction of unwarranted or excessive paralysis into certain electronic counters. As will be shown in a subsequent paper, paralysis should be kept to the minimum that is necessary for noise elimination, as excess paralysis will needlessly inflate the coefficient of variation for the estimate of λ.

We shall see that, even when paralysis is taken into account, the distribution of recorded impulses remains asymptotically normal as $y \to \infty$; and that the mean and variance remain proportional to y for all $y > \varpi + \tau$. On the other hand, the constants of proportionality depend upon the evaluation of some complicated integrals; and, in the absence of simple approximations, these constants are probably beyond the reach of anyone who has not fairly extensive facilities for numerical computation. This is another reason for avoiding paralysis if possible.

4. *Detailed theory for random dead time with paralysis.* For brevity we shall relax the rigour previously advisable in view of the possible discontinuities of $U(t)$, remarking merely that all subsequent steps can indeed be justified by exactly similar arguments.

Until further notice let us suppose that there is no paralysis. Let O be the origin on the circle and P a point with coordinate $x > \varpi$. Let H denote the information that O and P are both free points and that the arc $(0, dx]$ contains a recorded start. Let $0 < x_1 < x_2 < \ldots < x_k < x$. The conditional probability, given H, that the arcs $(x_j, x_j + dx_j]$ $(j = 1, 2, \ldots, k)$ each contain one recorded start is (to the kth order of small quantities)

$$W(x_1)\, W(x_2 - x_1)\, W(x_3 - x_2) \ldots W(x_k - x_{k-1})\, U(x - x_k - 0)\, dx_1 dx_2 \ldots dx_k, \tag{42}$$

where $W(z)$ is given by (3). This result follows from the same argument that led to (12), except that the known freedom of O and P now replaces the consideration of the circumference of the circle being closed. If $n(x, \varpi)$ denotes the number of recorded starts in the arc $[x - \varpi, x)$, the mth moment (about the origin) of the conditional distribution of $n(x, \varpi)$ given H is

$$\mu'_m(x, \varpi) = \sum_{k=1}^{m} \Delta^k(0^m)\, I^*_k(x, \varpi), \tag{43}$$

where

$$I^*_k(x, \varpi) = \int\int \ldots \int_{x - \varpi \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_k \leqslant x} W(x_1)\, W(x_2 - x_1) \ldots W(x_k - x_{k-1})\, U(x - x_k - 0)\, dx_1 dx_2 \ldots dx_k. \tag{44}$$

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

The proof of (43) is identical with the proof of (16). Hence the moment-generating function of $n(x, \varpi)$ given H is

$$\phi^*(\theta, x, \varpi) = 1 + \sum_{k=1}^{\infty} (e^\theta - 1)^k I_k^*(x, \varpi), \tag{45}$$

by the same argument as was used to develop (22); and therefore, as in the proof of (36), the conditional probability given H that $[x-\varpi, x)$ contains no recorded start is

$$f_0^*(x) = f_0^*(x, \varpi) = 1 + \sum_{k=1}^{\infty} (-)^k I_k^*(x, \varpi) \quad (x > \varpi). \tag{46}$$

The convergence of the various series and the validity of the sundry manipulations involved in the foregoing arguments are secured by the observation

$$0 \leqslant I_k^*(x, \varpi) \leqslant (Mx)^k/k!,$$

which is analogous to (20). Let us also define

$$f_0^*(x) = 0 \quad (0 \leqslant x \leqslant \varpi). \tag{47}$$

With these preliminaries settled, we may now take account of paralysis.† The point P will be free if and only if (a) it would be free were paralysis not taken into account, and (b) the arc $[x-\varpi, x)$ would contain no recorded starts were paralysis not taken into account. This statement remains true when $0 \leqslant x \leqslant \varpi$. Hence, taking paralysis into account, the probability, given that O is free, that one start occurs in $(0, dx]$ and that P is free is by virtue of (46) and (47)

$$W^*(x)\,dx = W(x)\,f_0^*(x)\,dx \quad (0 \leqslant x < y), \tag{48}$$

to the first order of small quantities. This, however, is the same as the probability that one impulse causing paralysis starts in $(0, dx]$ and that with this and any other attendant paralysis no impulse which starts in $(OP]$ overlaps P. Thus the required solution in the case of paralysis will now result from carrying through the whole of the previous analysis with $W^*(z) = W^*(z, \varpi)$ in place of $W(z)$, provided that $W^*(z)$ is constant for all sufficiently large z. To verify this last proviso we make the substitution

$$x - x_k = z_1, \quad x_k - x_{k-1} = z_2, \quad \ldots, \quad x_2 - x_1 = z_k$$

in (44), which gives

$$I_k^*(x, \varpi) = \int\!\!\ldots\!\!\int_{D_k(\varpi)} U(z_1)\,W(z_2)\ldots W(z_k)\,W\Big(x - \sum_{j=1}^{k} z_j\Big)\,dz_1 dz_2 \ldots dz_k$$

$$= \int_0^{\varpi} W(x-\zeta)\,\Big[\int\!\!\ldots\!\!\int_{D_{k-1}(\zeta)} U(z_1)\,W(z_2)\ldots W(z_{k-1})\,W\Big(\zeta - \sum_{j=1}^{k-1} z_j\Big)\,dz_1 \ldots dz_{k-1}\Big]\,d\zeta.$$

Write

$$Q(\zeta) = U(\zeta) + \sum_{k=1}^{\infty} (-)^k \int\!\!\ldots\!\!\int_{D_k(\zeta)} U(z_1)\,W(z_2)\ldots W(z_k)\,W\Big(\zeta - \sum_{j=1}^{k} z_j\Big)\,dz_1 \ldots dz_k. \tag{49}$$

† The ensuing analysis may be profitably viewed in the light of the techniques used by Domb (4).

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

It is easy to verify that $Q(\zeta)$ satisfies the integral equation

$$Q(\zeta) = U(\zeta) - \int_0^{\zeta} W(\zeta - z)\, Q(z)\, dz, \tag{50}$$

which may be compared with the somewhat similar integral equation (23) satisfied by $\psi(-1, \zeta)$. From (46), (47) and (48) we find

$$W^*(z, \varpi) = \begin{cases} 0 & (0 \leqslant z \leqslant \varpi), \\ W(z)\left\{1 - \int_0^{\varpi} W(z-\zeta)\, Q(\zeta)\, d\zeta\right\} & (\varpi < z \leqslant \varpi + \tau), \\ \lambda\, e^{-\lambda T}\left\{1 - \lambda\, e^{-\lambda T} \int_0^{\varpi} Q(\zeta)\, d\zeta\right\} & (z > \varpi + \tau). \end{cases} \tag{51}$$

Thus $W^*(z, \varpi)$ is constant, as required, for all sufficiently large z. It is also easy to verify that $W(z) = W^*(z, \varpi)$ in case ϖ is less than the minimum length of photocell impulse; for then $U(z) = W(z) = 0$ for $0 \leqslant z \leqslant \varpi$, whereupon (49) shows that $Q(\zeta) = 0$ for $0 \leqslant \zeta \leqslant \varpi$.

We can now apply our previous methods to show that, when $y > \varpi + \tau$, the recorded count n^*, allowing for paralysis, will have mean

$$\mu_1^* = yC(\lambda, \varpi) \tag{52}$$

and variance
$$\mu_2^* = yC(\lambda, \varpi)\left\{1 - 2\int_0^{\varpi+\tau} [C(\lambda, \varpi) - W^*(z)]\, dz\right\}, \tag{53}$$

where
$$C(\lambda, \varpi) = W^*(y) = \lambda\, e^{-\lambda T}\left\{1 - \lambda\, e^{-\lambda T} \int_0^{\varpi} Q(\zeta)\, d\zeta\right\}; \tag{54}$$

and that the cumulative distribution function of n^* will be asymptotically normal as $y \to \infty$.

The foregoing three expressions will not be analytically tractable except for special forms of $U(t)$. The simplest of such exceptions is the one in which all photocell impulses have the same constant length τ. In this case $T = \tau$, and it is readily verified by induction that

$$\int\cdots\int_{D_k(\zeta)} U(z_1)\, W(z_2) \ldots W(z_k)\, W\left(\zeta - \sum_{j=1}^{k} z_j\right) dz_1 \ldots dz_k = \begin{cases} 0 & (0 \leqslant \zeta < k\tau + \tau), \\ \dfrac{[(\zeta - k\tau - \tau)\, \lambda\, e^{-\lambda\tau}]^k}{k!} & (\zeta \geqslant k\tau + \tau). \end{cases} \tag{55}$$

To deal with the consequences of (55) we introduce the 'tapered exponential function' defined by

$$e^*(a, b) = 1 + \sum_{k=1}^{\infty} \{\tfrac{1}{2} a(b - k + |\, b - k\,|)\}^k / k!, \tag{56}$$

which enjoys the property

$$e^*(a, B+1) = 1 + a\int_0^B e^*(a, b)\, db \quad (B \geqslant 0). \tag{57}$$

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

We find that
$$Q(\zeta) = \begin{cases} 0 & (\zeta < \tau), \\ e^*(-\lambda\tau e^{-\lambda\tau}, \zeta/\tau - 1) & (\zeta \geqslant \tau); \end{cases} \tag{58}$$

and thence that

$$\mu_1^* = (y/\tau)\, a e^*(-a, b), \tag{59}$$

$$\mu_2^* = (y/\tau)\, a e^*(-a, b)\{3 - 2e^*(-a, b+1) - 2a(b+1)\, e^*(-a, b)\}, \tag{60}$$

where
$$a = \lambda\tau e^{-\lambda\tau}, \quad b = \varpi/\tau. \tag{61}$$

Although these results apply strictly to the case of photocell impulses of constant length alone, I should not be surprised were they to prove fairly good approximations in the general case, provided T takes the place of τ in (61).

For the exact solution with some arbitrary function $U(t)$ for photocell impulse length, there are three methods worth considering for the evaluation of $Q(\zeta)$ and the determination of μ_1^* and μ_2^* therefrom. First, we may evaluate the multiple integrals on the right-hand side of (49) and then sum the resulting infinite† series; this was the method adopted above for (55) and (58). Secondly, we may solve the integral equation (50); this is probably the best approach if $U(t)$ is given empirically and one has to resort to numerical methods. Thirdly, we may solve (50) by means of Laplace transforms to get

$$Q(\zeta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Upsilon(p)\, e^{p\zeta}\, dp}{1 + \Omega(p)},$$

where Υ is the Laplace transform of U; this will be appropriate for certain special functional forms U.

We deal finally with the case in which paralysis is random, having a cumulative distribution function $F(\varpi)$ for the length of paralysis. If we suppose that each particular paralysis is drawn at random from this population independently of every other paralysis and independently of the position and duration of the photocell impulse to which it is attached, then it suffices to work with the function

$$W_F^*(z) = \int_0^\infty W^*(z, \varpi)\, dF(\varpi)$$

in place of the function $W^*(z)$ in (52), (53) and (54). Naturally the final results will be extremely complicated unless F has a fortunate form.

The tapered exponential function, defined in (56), crops up in various compound Poisson processes besides the present application; and also, in view of (57), it is likely to be useful in the solution of integro-difference equations. There seems to be a prima facie case for tabulating it extensively‡; but this is not the place to do so, and here we content ourselves with a short table designed merely for use with (59) and (60), and thus after the first relationship of (61) restricted to the range $-e^{-1} \leqslant -a \leqslant 0$. I am indebted to Mrs M. Parke who computed this table.

† The series terminates after a finite number of terms if $U(\epsilon) = 0$ for some $\epsilon > 0$.

‡ Mr F. J. Anscombe has kindly informed me that such a table was prepared by the Scientific Computing Service, but was never published. He believes there is a copy deposited at the National Physical Laboratory.

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press

Short table of the tapered exponential function $e^*(-a, b)$

$a=$	0	0·05	0·10	0·15	0·20	0·25	0·30	0·35	0·3679
$b=$ **0·0**	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
0·5	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
1·0	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
1·5	1·0000	0·9750	0·9500	0·9250	0·9000	0·8750	0·8500	0·8250	0·8161
2·0	1·0000	0·9500	0·9000	0·8500	0·8000	0·7500	0·7000	0·6500	0·6321
2·5	1·0000	0·9253	0·8512	0·7778	0·7050	0·6328	0·5612	0·4903	0·4651
3·0	1·0000	0·9012	0·8050	0·7112	0·6200	0·5312	0·4450	0·3612	0·3319
3·5	1·0000	0·8778	0·7612	0·6502	0·5448	0·4450	0·3507	0·2619	0·2315
4·0	1·0000	0·8550	0·7198	0·5944	0·4787	0·3724	0·2755	0·1878	0·1588
4·5	1·0000	0·8327	0·6807	0·5434	0·4205	0·3115	0·2161	0·1337	0·1074
5·0	1·0000	0·8111	0·6437	0·4968	0·3694	0·2606	0·1693	0·0947	0·0719
6·0	1·0000	0·7694	0·5756	0·4152	0·2851	0·1823	0·1039	0·0470	0·0314
7·0	1·0000	0·7299	0·5147	0·3469	0·2200	0·1275	0·0637	0·0232	0·0134
8·0	1·0000	0·6925	0·4602	0·2899	0·1698	0·0892	0·0390	0·0114	0·0056
9·0	1·0000	0·6569	0·4115	0·2423	0·1310	0·0624	0·0239	0·0056	0·0023
10·0	1·0000	0·6232	0·3680	0·2025	0·1011	0·0436	0·0147	0·0027	0·0009

REFERENCES

(1) Albert, G. E. and Nelson, L. Contributions to the statistical theory of counter data. *Ann. math. Statist.* 24 (1953), 9–22.

(2) Domb, C. The problem of random intervals on a line. *Proc. Camb. phil. Soc.* 43 (1947), 329–41.

(3) Domb, C. Some probability distributions connected with recording apparatus. II. *Proc. Camb. phil. Soc.* 46 (1950), 429–35.

(4) Domb, C. The statistics of correlated events. I. *Phil. Mag.* (7), 41 (1950), 969–82.

(5) Feller, W. On probability problems in the theory of counters. *Courant Anniversary Volume* (1948), pp. 105–15.

(6) Hammersley, J. M. Tauberian theory for the asymptotic forms of statistical frequency functions. *Proc. Camb. phil. Soc.* 48 (1952), 592–9.

(7) Rényi, A. On some problems concerning Poisson distributions. *Publ. math., Debrecen*, 2 (1951), 66–73.

(8) Rényi, A. On composed Poisson distributions. II. *Acta math. hung.* 2 (1951), 83–98.

(9) Stevens, W. L. Solution to a geometrical problem in probability. *Ann. Eugen., Lond.*, 9 (1939), 315–20.

(10) Takács, L. Discussion of phenomena of occurrence and coincidence in case the distribution of the duration of happenings is arbitrary. *Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei*, 1 (1951), 371–86.

(11) Takács, L. Occurrence and coincidence phenomena in case of happenings with arbitrary distribution law of duration. *Acta math. hung.* 2 (1951), 275–98.

(12) Whittaker, E. T. and Watson, G. N. *Modern analysis*, 4th ed. (Cambridge, 1940).

Lecturership in the Design and Analysis of Scientific Experiment
University of Oxford

https://doi.org/10.1017/S0305004100028826 Published online by Cambridge University Press