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Minimal spacing of eigenvalues on fractals

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Abstract. On the unit interval (I) and the Sierpinski gasket (SG), the spectral decimation function of the Laplacian has similar properties that result in a positive minimum spacing of eigenvalues. Other fractals, for example, the level-3 SG, SG_3, may not enjoy these properties. Our goal is to obtain a simple sufficient criterion for positive infimum spacing between eigenvalues in the spectrum, based on the properties of the corresponding spectral decimation function of the fractal. We also give a sufficient condition for zero infimum spacing between eigenvalues.

1 Introduction

The study of the spectrum of the Laplacian on fractals dates back to the work of Rammal and Toulouse [11, 12], who discovered that the Weyl scaling exponent, or the spectral dimension d_S, for the spectrum of the Sierpinski gasket (SG) does not correspond to the Hausdorff dimension. Their work also showed that, unlike the situation described by the classical Weyl law, this spectrum has large gaps and high multiplicities, so that the renormalized eigenvalue counting function converges to a log-periodic function (see, e.g., [6, 16]). Later work has extended these ideas to a much wider class of fractals (see, e.g., [9, 8, 13]).

Following Strichartz [14], we say that the spectrum of the Laplacian on a compact fractal, which consists of an increasing discrete set of eigenvalues λ_n, has large gaps if

$$\mathrm{Gap}_\Delta := \limsup_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} > 1.$$

Strichartz found that the existence of these large gaps has interesting consequences for Fourier analysis on these sets; for example, he showed that large gaps and heat kernel estimates together imply that there are subsequences of eigenvalues along which the Dirichlet kernel forms an approximation of the identity, and hence that the fractal analog of Fourier series of a continuous function converges uniformly along the subsequence, which is unusual in classical analysis. Other papers, including [5, 7], have discussed large gaps in the spectrum of the Laplacian.

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A natural and important quantity to consider in this context is the infimum spacing between the ordered sequence of distinct eigenvalues given by

$$\text{Spacing}_\Delta := \inf_{\lambda \neq \lambda', \lambda, \lambda' \in \sigma(\Delta)} |\lambda - \lambda'|.$$

The infimum spacing is useful for estimating the error that results from approximating the Laplace operator on a given space, as seen, for example, in [2, 17]. In the present work, we give sufficient conditions on classes of fractals for this measurement of the spacing of eigenvalues to be strictly positive and a sufficient condition for it to be zero.

It may be technically difficult to determine the exact value of this infimum. For example, the case of the SG was treated in the recent work of Alonso Ruiz [1], where using the technique of spectral decimation found in [6], it was shown that this minimum is attained and coincides with the spectral gap – the spacing between the first two eigenvalues.

The structure of this article is as follows. In Section 2, we consider some preliminaries, including graph approximations, and briefly discuss spectral decimation.

Section 3 is the main part of this text. It is divided into two sections. In Section 3.1, we prove Theorem 3.1, which shows that if the multiplier of a nonzero fixed point of the spectral decimation function dominates that of the zero fixed point, then the infimum spacing must be zero. This is a partial converse to Theorem 3.5 in Section 3.2, where we give a condition under which the infimum spacing is positive. The main idea is to apply Lemma 3.4 to the discrete case and then pass to the continuous case using spectral decimation. In each section, we provide simple examples that illustrate the conclusions of the main theorems. Examples 3.2 and 3.7 are particularly interesting in that they also show that certain features that may hold on SG may not necessarily apply to SG_3.

2 Preliminaries

2.1 Graph approximations

We begin with a discussion of graph approximations following Strichartz [15].

A self-similar set K is defined to be the unique nonempty compact set satisfying:

$$K = \bigcup_{j=1}^{N} \varphi_j(K), N \geq 2 \tag{2.1}$$

for some distinct injective contraction mappings φ_j on a complete metric space. Equation (2.1) is known as a self-similar identity. Here, K is self-similar in the sense that it is a union of copies of itself.

To illustrate, we take the unit interval I as an example. The mappings

$$F_0(x) = \frac{x}{2}, F_1(x) = \frac{x}{2} + \frac{1}{2} \tag{2.2}$$

on I, with fixed points 0 and 1, respectively, give the self-similar identity

$$I = F_0 I \cup F_1 I. \tag{2.3}$$

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More generally, by composing the maps F_i, we obtain

$$I = \bigcup_{|w|=m} F_w I, w = (w_1, \ldots, w_m), w_j \in \{0,1\}, \tag{2.4}$$

where $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$. We call w a word of length $m = |w|$, $F_w I$ an m-level cell, and write $x \underset{m}{\sim} y$ if there is an m-level cell with x and y as its boundary points. Let $V_0 = \{q_0, q_1\}$, $q_0 = 0$, $q_1 = 1$, and define inductively

$$V_m := \bigcup_{i\in\{0,1\}} F_i V_{m-1}. \tag{2.5}$$

We regard the sets V_m as the vertices of the graph Γ_m, with edge relation $x \underset{m}{\sim} y$, built inductively from Γ_{m-1} by taking the two images $F_0\Gamma_{m-1}$ and $F_1\Gamma_{m-1}$. Observe that V_m increases to V_*, the set of dyadic rationals in I, which is dense in I. Hence, if we are interested only in continuous functions on I, then it suffices to know their values on V_*.

The case of SG may be viewed as a generalization of the case of the unit interval (I) with

$$F_i(x) = \frac{1}{2}(x - q_i) + q_i, i = 0,1,2, \tag{2.6}$$

where the points q_i are the vertices of a nondegenerate triangle in the plane. SG satisfies the self-similar identity:

$$SG = \bigcup_{i=0}^{2} F_i(SG). \tag{2.7}$$

In the case of SG, we take $V_0 = \{q_0, q_1, q_2\}$, the vertices of the triangle, and construct a graph Γ_m with vertices V_m by defining the edge relation $x \underset{m}{\sim} y$ if there is a cell of level m containing both x and y. Γ_m is obtained by taking three copies $F_i\Gamma_{m-1}$ of Γ_{m-1} and identifying the points $\{F_i q_j = F_j q_i\}_{i\neq j}$, $i, j = 0, 1, 2$.

2.2 Spectral decimation

The precise definition of spectral decimation can be found in [4]. For the purpose of this article, it suffices to describe it as follows.

Let V_m denote the vertices of the m-level approximation of K. We define the discrete Laplacian Δ_m on V_m by

$$\Delta_m u(x) := \sum_{y\in V_m, x\underset{m}{\sim} y} (u(y) - u(x)), x \in V_m. \tag{2.8}$$

It is known, for example, from [7] that for a fully symmetric self-similar structure on a finitely ramified fractal K, there is a function R, called the spectral decimation function relating the eigenvalues of Δ_m and those of Δ_{m-1}. In [13, Proposition 2.18], Shima shows that $R(0) = 0$ and $c_\Delta := R'(0) > 1$. Thus, 0 is a repulsive fixed point of the spectral decimation function.

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Definition 2.1 Define the standard Laplacian on K as

$$\Delta u(x) = \lim_{n\to\infty} c_\Delta^n \Delta_n u(x), u \in C(K) \tag{2.9}$$

if the limit exists.

Let $\phi_0, \ldots, \phi_L$ be the partial inverses of R with ϕ_0 satisfying $\phi_0(0) = 0$.
If all the eigenvalues of Δ have the form

$$\lambda_k = \lim_{n\to\infty} c_\Delta^n \lambda_k^{(n)} = \lim_{n\to\infty} c_\Delta^{n_0+n} \phi_0^n \lambda_k^{(n_0)}, k \in \mathbb{N}, \tag{2.10}$$

where $\lambda_k^{(n)}$ is the k-th eigenvalue of Δ_n, then Δ is said to admit spectral decimation [7].

The class of fully symmetric self-similar finitely ramified fractals admits spectral decimation.

Note that Definition 2.1 describes a phenomenon where a suitable sequence of eigenvalues in the finite-level approximation of a self-similar set is produced by an orbit of a particular dynamical system. This phenomenon is what is termed spectral decimation [6].

3 Minimal spacing of eigenvalues

The main results of this article are derived in this section. We consider a condition for zero infimum spacing and then a condition for positive infimum spacing. Following this, we prove Lemma 3.4, which plays a key role in proving Theorem 3.5 by passing from the discrete case to the continuous case using spectral decimation.

3.1 Zero infimum spacing

The next theorem gives a condition under which the infimum spacing is zero. Here, our main result is Theorem 3.1, where we show that the existence of a nonzero repulsive fixed point of the spectral decimation function with a multiplier larger than the zero fixed point leads to a zero infimum spacing in the spectrum.

Theorem 3.1 *Assume Δ admits spectral decimation with spectral decimation function R and suppose $0, \zeta > 0$ are fixed points of R with $|R'(\zeta)| > |R'(0)| > 1$. Then,* $\inf\{|\lambda - \lambda'| : \lambda \neq \lambda', \lambda, \lambda' \in \sigma(\Delta)\} = 0$.

Proof Let ϕ_ζ be the inverse branch of R with ζ in its range. Note that given $\delta > 0$, there is a neighborhood U_δ of ζ, such that ϕ_ζ^m is a contraction on U_δ with the contraction rate bounded by $(|R'(\zeta)| - \delta)^{-m}$. Moreover, since ζ is in the Julia set of R and is an attracting fixed point of ϕ_ζ, we may choose n such that $x_1, x_2 \in \sigma(\Delta_n) \cap U_\delta$. Now, by [13, Proposition 3.1],

$$c_\Delta^{n+j+m} \phi_0^j(\phi_\zeta^m(x_1)), \quad c_\Delta^{n+j+m} \phi_0^j(\phi_\zeta^m(x_2))$$

converge to eigenvalues of Δ say $\lambda^{(m)}, (\lambda^{(m)})'$, respectively, as $j \to \infty$. And by the Kœnigs linearization theorem (see, e.g., [10, Theorem 6.1]), there is a local holomorphic change of coordinates $\mathscr{R}$, called the Kœnigs function, such that

$$\lambda^{(m)} = c^{n+m} \mathscr{R}(\phi_\zeta^m(x_1)), \quad (\lambda^{(m)})' = c_\Delta^{n+m} \mathscr{R}(\phi_\zeta^m(x_2)).$$

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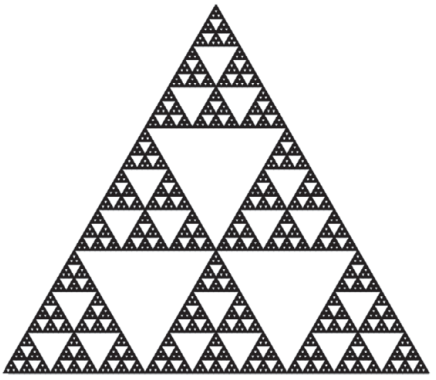


Figure 1: SG_3.

Hence, for some γ_m lying between $\phi_\zeta^m(x_1)$ and $\phi_\zeta^m(x_2)$, the spacing between these two eigenvalues is bounded by

$$\begin{aligned}|\lambda^{(m)} - (\lambda^{(m)})'| &= c_\Delta^n |\mathscr{R}'(\gamma_m)| c_\Delta^m |(\phi_\zeta^m(x_1) - \phi_\zeta^m(x_2))| \\ &\leq c_\Delta^n |\mathscr{R}'(\gamma_m)| \left(\frac{c_\Delta}{|R'(\zeta)| - \delta}\right)^m |x_1 - x_2|,\end{aligned}$$

which gets arbitrarily small as $m \to \infty$ for small δ. ∎

Example 3.2 (Level- 3 Sierpinski gasket, SG_3)

In [3], Bajorin et al. have described the spectrum of the discrete Laplacian on SG_3 (Figure 1), which is given by

$$\sigma(\Delta_n) = (R^{-n}(0)) \bigcup \left(R^{-(n-1)}\left(\frac{3 \pm \sqrt{5}}{4}\right)\right) \bigcup \left\{\frac{3}{2}\right\}, n \geq 0$$

and whose spectral decimation function is

$$R(x) = \frac{6x(x-1)(4x-5)(4x-3)}{6x-7}$$

with derivative,

$$R'(x) = \frac{6(288x^4 - 1024x^3 + 1290x^2 - 658x + 105)}{(6x-7)^2}.$$

Note that $0, 1, \frac{3}{4}, \frac{5}{4} \in R^{-n}(0) \subseteq \sigma(\Delta_n)$. R also has a fixed point between 1.30 and 1.31. A direct computation shows $R'(0) = \frac{90}{7} < 23.542849 \approx \inf_{y \in [1.2, 1.5]} R'(y)$. Hence, by Theorem 3.1, we must have that the infimum spacing in $\sigma(\Delta)$ is zero.

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Theorem 3.3 *Let $\{\lambda_{n,k}\}_k \subseteq \sigma(\Delta_n)$ be the complete list of distinct finite-level eigenvalues of Δ_n in increasing order. Then,*

$$\inf\{|\lambda - \lambda'| : \lambda \neq \lambda', \lambda, \lambda' \in \sigma(\Delta)\} = 0$$

if and only if

$$\lim_n \min_k \frac{|\lambda_{n,k} - \lambda_{n,k+1}|}{|\lambda_{n,0} - \lambda_{n,1}|} = 0.$$

Proof Using the notation in the previous theorem, note that for each n and k, we have that $c_\Delta^n \mathscr{R}(\lambda_{n,k})$ and $c_\Delta^n \mathscr{R}(\lambda_{n,k+1})$ are eigenvalues of Δ. Moreover,

$$\begin{aligned}\min_k |c_\Delta^n \mathscr{R}(\lambda_{n,k}) - c_\Delta^n \mathscr{R}(\lambda_{n,k+1})| &= \min_k |\mathscr{R}'(\gamma_{n,k})| c_\Delta^n |\lambda_{n,k} - \lambda_{n,k+1}| \\ &= c_\Delta^n |\lambda_{n,0} - \lambda_{n,1}| \min_k |\mathscr{R}'(\gamma_{n,k})| \frac{|\lambda_{n,k} - \lambda_{n,k+1}|}{|\lambda_{n,0} - \lambda_{n,1}|}.\end{aligned}$$

Since $\mathscr{R}'$ is bounded and the factor $c_\Delta^n |\lambda_{n,0} - \lambda_{n,1}|$ converges to $\mathscr{R}(x) - \mathscr{R}(y)$ for some x, y, the reverse implication follows by taking the limit as $n \to \infty$.

On the other hand, if $\inf_{\lambda \neq \lambda', \lambda, \lambda' \in \sigma(\Delta)} |\lambda - \lambda'| = 0$, then for each $m \in \mathbb{N}$, there exist $\lambda, \lambda' \in \sigma(\Delta)$ such that $|\lambda - \lambda'| \leq \frac{1}{m}$. Now note that for any two distinct eigenvalues, $\lambda, \lambda' \in \sigma(\Delta)$, we may choose $n \geq m$ sufficiently large such that $\lambda = c_\Delta^n \mathscr{R}(\lambda_{n,j})$, $\lambda' = c_\Delta^n \mathscr{R}(\lambda_{n,j'})$, and we have

$$\begin{aligned}\frac{1}{m} &\geq |\lambda - \lambda'| \\ &= |c_\Delta^n \mathscr{R}(\lambda_{n,j}) - c_\Delta^n \mathscr{R}(\lambda_{n,j'})| \\ &= |\mathscr{R}'(\gamma_{n,j})| c_\Delta^n |\lambda_{n,j} - \lambda_{n,j'}| \\ &\geq |\mathscr{R}'(\gamma_{n,j})| c_\Delta^n |\lambda_{n,0} - \lambda_{n,1}| \min_k \frac{|\lambda_{n,k} - \lambda_{n,k+1}|}{|\lambda_{n,0} - \lambda_{n,1}|} \\ &\geq C \min_k \frac{|\lambda_{n,k} - \lambda_{n,k+1}|}{|\lambda_{n,0} - \lambda_{n,1}|}.\end{aligned}$$

The conclusion follows by letting n go to infinity. ∎

3.2 Positive infimum spacing

The proof of the next result relies on the mean value theorem and induction. It is crucial for proving the main Theorem 3.5, which guarantees that the infimum spacing in the spectrum is positive under certain conditions. While it has many technical assumptions, to the author's knowledge, all of them – except that $R'(0)$ is maximal – are satisfied in all examples of spectral decimation considered in [3, 13].

Lemma 3.4 *Assume Δ admits spectral decimation with spectral decimation function R, and suppose 0 is a fixed point of R with $|R'(0)| > 1$.*

Let $D_0 = \{0, x_1, \ldots, x_r\}$ be a finite set with $0 < x_1 < \cdots < x_r$. Suppose the interval $[0, x_r]$ is a backward invariant under R and the inverse branches ϕ_j of R are defined on all of this interval.

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Assume further that

- $R^{-1}(0, x_r) \cap D_0 = \varnothing$, $R^{-1}\{0, x_r\} \subseteq D_0$ *and*
- $R'(0) = \max_{x \in R^{-1}[0,x_r]} |R'(x)|$,

and set $D_n := \cup_{m=0}^{n} R^{-m}(D_0)$. *Then, there is a constant* C_0, *depending on* D_0, *such that*

$$min_{x,y \in D_n, x \neq y} |x - y| \geq \frac{C_0}{(R'(0))^n}.$$

Proof Let $C_0 = min_k |x_k - x_{k-1}|, x_j \in D_0$. We know $D_1 = D_0 \cup R^{-1}(D_0)$. If $x, y \in D_0, x \neq y$, then we obtain

$$|x - y| \geq C_0 \geq \frac{C_0}{R'(0)}.$$

If $x \in D_0$ and $y \in R^{-1}(D_0)$, then $y \in R^{-1}[0, x_r]$. Thus, $y \in \phi_k[0, x_r]$, for some k. But $R^{-1}(0, x_r) \cap D_0 = \varnothing, R^{-1}\{0, x_r\} \subseteq D_0$ implies $|x - y| \geq \min_{z \in \{0,x_r\}} |y - \phi_k(z)|$. For $z \in \{0, x_r\}$, we have $\left|\frac{R(y) - R(\phi_k(z))}{y - \phi_k(z)}\right| = |R'(c)| \leq R'(0)$, for some $c \in \phi_k[0, x_r]$. Thus, $|y - \phi_k(z)| \geq \frac{|R(y) - R(\phi_k(z))|}{R'(0)} \geq \frac{C_0}{R'(0)}$.

For $x, y \in R^{-1}(D_0), x \neq y$: It suffices to assume $x, y \in \phi_k[0, x_r]$, for some k, since $\phi_j\{0, x_r\} \subseteq D_0$ for all j. R is a bijection on this interval, which implies $R(x) \neq R(y)$. We obtain

$$\frac{|R(x) - R(y)|}{|x - y|} = R'(c) \leq R'(0) \qquad \text{for some } c \in \phi_k[0, x_r].$$

Thus,

$$|x - y| \geq \frac{|R(x) - R(y)|}{R'(0)} \geq \frac{C_0}{R'(0)} \qquad \text{since } R(x), R(y) \in D_0.$$

The general case follows by induction and this completes the proof. ∎

Theorem 3.5 *Suppose there is* D_0, *so that* $\sigma(\Delta_n) \subseteq D_n$ *for all* n *and* R, D_n *satisfy the conditions of Lemma 3.4. Then,*

$$\inf\{|\lambda - \lambda'| : \lambda \neq \lambda' \textit{ are Dirichlet eigenvalues of } \Delta\} > 0.$$

Proof It follows from Lemma 3.4 that the spacings in $\sigma(c_\Delta^n \Delta_n)$ are bounded below uniformly in n, where $c_\Delta = R'(0)$. The conclusion follows. ∎

Remark 3.6 Theorem 3.5 again holds for the Neumann eigenvalues of Δ.

Although the following examples are elementary, they are useful for understanding the results obtained in this section and how they apply to other complex scenarios.

Example 3.7 *(SG)*

Alonso Ruiz shows in [1, Theorem 3] that

$$\min\{|\lambda - \lambda'| : \lambda \neq \lambda', \text{Dirichlet eigenvalues of } \Delta\} = \lambda_0^{(5)} - \lambda_0^{(2)} > 0$$

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and

$$\min\{|\lambda - \lambda'| : \lambda \neq \lambda', \text{Neumann eigenvalues of } \Delta\} = \lambda_0^{(6)} > 0.$$

Thus, the infimum spacing between eigenvalues coincides with the spectral gap, which is positive.

Example 3.8 (I)

Since the spectrum of the Laplacian on I is explicit, a direct computation shows that the infimum spacing is again the spectral gap.

Note that the Dirichlet spectrum is given by $\{\pi^2 k^2\}_{k=1}^{\infty}$. Hence, the spacing between successive eigenvalues is given by

$$\pi^2(k+1)^2 - \pi^2 k^2 = \pi^2((k+1)^2 - k^2) = \pi^2(2k+1),$$

which is increasing in k; hence, the infimum spacing is again the spectral gap, which occurs when $k = 1$. Thus,

$$\min\{|\lambda - \lambda'| : \text{ Dirichlet eigenvalues of } \Delta\} = 3\pi^2 > 0.$$

Recall that the Neumann spectrum is given by $\{\pi^2 k^2\}_{k=0}^{\infty}$. Thus, the spacing between successive eigenvalues is again given by $\pi^2(2k+1)$, which coincides with the Dirichlet case, with minimum occurring when $k = 0$. Thus,

$$\min\{|\lambda - \lambda'| : \text{Neumann eigenvalues of } \Delta\} = \pi^2 > 0.$$

Remark 3.9 Typically, to choose D_0, one fixes a large n and sets $D_0 = R^n \sigma(\Delta_n)$.

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