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PRICING FORMULA FOR EXCHANGE OPTION BASED ON STOCHASTIC DELAY DIFFERENTIAL EQUATION WITH JUMPS

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This paper deals with pricing formulae for a European call option and an exchange option in the case where underlying asset price processes are represented by stochastic delay differential equations with jumps (hereafter "SDDEJ"). We introduce a new model in which Poisson jumps are added in stochastic delay differential equations to capture behaviors of an underlying asset process more precisely. We derive explicit pricing formulae for the European call option and the exchange option by proving a Lemma on the conditional expectation. Finally, we show that our "SDDEJ" model is meaningful through some numerical experiments and discussions.

Keywords: black–scholes model, delay differential equations with jumps, european call option, exchange option, pricing formula

1. INTRODUCTION

Investors analyze past prices of an underlying asset to decide an investment strategy since the historic price information is known. So it was pointed out by Arriojas *et al.* [1], Kazmerchuk *et al.* [4], McWilliams [9] that the volatility in the price movement should depend on "past states $S(t-b) := S(t-b_1), S(t-b_2), \ldots, S(t-b_n)$" of underlying asset prices. Therefore, the underlying asset prices are often represented by a stochastic delay differential equation (hereafter "SDDE"), where the volatility is a function $g(S(t-b))$of the past price processes. The "SDDE" model is as follows:

$$dS(t) = S(t)\{\mu S(t-a)dt + g(S(t-b))dW(t)\},$$

where $S(t-b_j)$denotes the underlying asset price at time $t-b_j$ and $\{b_j\}_{j=1}^n$ are delay parameters. The definition of $S(t-a)$ is similar to that of $S(t-b)$.

In the case where the underlying asset process is represented by the "SDDE" model, pricing formulae for various options have been derived by Lin *et al.* [7], Arriojas *et al.* [1], Kazmerchuk *et al.* [4], McWilliams [9], and Swishchuk [15].

In Arriojas *et al.* [1], authors have derived an explicit delayed Black–Scholes formula for a European call option and proved that the corresponding market is complete by using a martingale method, where the underlying asset prices were assumed to follow the "SDDE" model. Lin *et al.* [7] derived the pricing formula for an exchange option on two underlying

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assets with such volatility as a function of past asset prices. McWilliams [9] studied pricing for an Asian option and a basket option in the "SDDE" model; however, no explicit formula but only the lower and upper bounds were researched.

The Black–Scholes model with jumps is based on the assumption that the price process of underlying asset is represented by two stochastic processes: (1) small, continuous price movements are generated by a Wiener process and (2) large, infrequent price jumps are generated by a Poisson process. These two stochastic processes reflect an efficient market in which major information arrives infrequently and randomly in the intuitive view (see Merton [10], Kou [6], and Ma [8]). The empirical study on underlying asset return shows that "discontinuity" or "jump" is an essential component of financial asset prices. Fortunately, a jump-diffusion process with Poisson jumps was proposed and option pricing has been studied in this environment.

When the underlying asset price process was generated by a jump-diffusion process, an explicit pricing formula for a European foreign exchange option in the presence of Levy jumps was also obtained in Ma [8]. Moreover, Xiao and Zhang [17], Shokrollahi and Kilgman [13], and Kim *et al.* [5] combined a fractional Brownian motion, a mixed fractional Brownian motion, and a generalized mixed fractional Brownian motion with Poisson jumps, respectively, and applied these models to currency option pricing.

Recently, regime-switching models was introduced to explain the volatility clustering effect which means that there are periods of high and low volatility and a new exact pricing formula for a European option was derived by Zhu *et al.* [18]; moreover, pricing of both European-style and American-style barrier options was considered by Elliott *et al.* [3]. Costabile *et al.* [2] embedded jumps to the regime-switching model and derived an explicit pricing formula of a European call option in this model.

In a financial market, an investor makes a strategy based the careful investigation of the past movements of an underlying asset, according to the change of the investment strategy, the volatility in the underlying asset prices will be undergone; therefore, it has been pointed out more recently that the volatility should depend on "past states" of underlying asset prices. Moreover, since the impulsive information arrives infrequently in an efficient market, the underlying asset prices will be influenced by large, infrequent jumps. Therefore, discontinuities are not only undeniable but also essential components of underlying asset prices.

In this paper, we propose a new combination of "SDDE" model and Poisson jumps. The choices of this framework are motivated by two main aspects.

"SDDE" models allow to capture the stochastic volatility depending on the "past states $S(t-b)$" of an underlying asset; the addition of jumps to the "SDDE" model contributes to capture infrequent jumps unexplainable by the classical log-normal model.

The paper is organized as follows. In Section 2, we concretely introduce the underlying "SDDEJ" model to use in option pricing. In Section 3, pricing formulae of a European call option and an exchange option are presented. In Section 4, we test the performance of the proposed formulae via numerical experiments.

2. "SDDEJ" MODEL

We consider a riskless asset and two risky assets being traded in a finance market. The model is as follows:

$$\begin{aligned} &dB(t) = rB(t)dt, \quad B(0) = 1 \\ &dS_i(t) = S_i(t)\{\mu_i S_i(t-a_i)dt + g_i(S_i(t-b_i))dW_i(t) + (e^{\xi_i(t)} - 1)d\bar{N}(t)\}, \\ &S_i(t) = \phi_i(t), \quad t \in [-L_i, 0], \quad i = 1, 2 \end{aligned} \tag{2.1}$$

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where $B(t)$, $S_i(t)$, $i = 1, 2$ are the prices of the riskless asset and the risky assets at time t, r is a riskless interest rate, μ, a_i, b_i are positive constants, $L_i = \max\{a_i, b_i\}$ and $g_i(\cdot) : R^+ \to R^+$ is a continuous function. The price processes $S_1(t), S_2(t)$ of underlying assets are defined on $(\Omega, \Im, P)$. $\Im = \{\Im_t^1, \Im_t^2\}_{t\geq 0}$ ($\{\Im_t^i\}$, $i = 1, 2$) is a σ-field. $S_i(t - a_i)$ and $S_i(t - b_i)$ represent the prices of delayed underlying assets. For $i = 1, 2$, $W_i(t)$ is a standard Wiener process which is adapted to the filtration $\{\Im_t^i\}_{t\geq 0}$ and $dW_1(t)dW_2(t) = \rho dt$ with a correlation coefficient ρ. $N(t)$ is a Poisson process with rate λ. $\bar{N}(t) = N(t) - \lambda t$ is a centered Poisson process. $\xi_i(t)$ is the jump size percent at time t which is a sequence of independent identically distributed random variables with $E[e^{\xi_i(t)} - 1] = \mu_i$, $V[e^{\xi_i(t)} - 1] = \delta_i^2$. Eq. (2.1) is called a Stochastic Delay Differential Equation with jumps (denoted by "SDDEJ"). From the Girsanov theorem, there exists a risk-neutral measure Q (see Section 2.1.3 of [12] and Chapter 19 of Shreve [14]), under which Eq. (2.1) is transformed into the following equation:

$$\begin{aligned} &dS_i(t) = S_i(t)\{rdt + g_i(S_i(t - b_i))d\tilde{W}_i(t) + (e^{\xi_i(t)} - 1)d\bar{N}(t)\}, \\ &S_i(t) = \varphi_i(t), \quad t \in [-L_i, 0], \quad i = 1, 2, \end{aligned} \tag{2.2}$$

where $\tilde{W}_i(t)$ is a standard Wiener process with $d\tilde{W}_1(t)d\tilde{W}_2(t) = \rho dt$ under the risk-neutral measure Q. After measure transformation, Eq. (2.2) involves the delay term only in the volatility component. However, the delay term in the drift component still influences on the measurability of contingent claims.

Theorem 1 in Arriojas *et al.* [1] has assured existence and uniqueness of a solution of "SDDE" without jumps. Also by Theorem 1.19 in Øksendal and Sulem [11], the existence and uniqueness of a solution of Levy SDE with only jump terms has been guaranteed. Hence, for "SDDEJ", multiplication of these two solutions allows us to confirm the existence and uniqueness of a solution of Eq. (2.4) (or not the more Eqs. (2.1) and (2.2)) under some conditions like linear growth and Lipschitz continuity. Throughout this paper, we assume that all the conditions for the existence and uniqueness are satisfied and derive option pricing formulae in the environment.

In the next section, we derive pricing formulae for a European call option and an exchange option in the more general model (2.2) than the standard Black–Sholes model with jumps (case of constant volatility).

3. MAIN RESULT

LEMMA 3.1: *Let X, Y be random variables on a probability space $(\Omega, \Im, P)$ and $\mathcal{G}$ be a sub σ-field of $\Im$. Suppose that Y is independent of $\sigma(\mathcal{G}, \sigma(X))$, $f(x, y)$ is a function of two variables and $g(x) := E f(x, Y)$. Then, $E[f(X, Y)|\mathcal{G}] = E[g(X)|\mathcal{G}]$.*

3.1. Pricing Formula for European Call Option

At first, we consider a special case of $i = 1$ in model (2.1) to derive a pricing formula for a European call option$F = (S(T) - K)^+$. The stock price process is represented by the following equation with $\bar{N}(t) = N(t) - \lambda t$ and $\mu = \mu_\xi$ under the market measure P,

$$\begin{aligned} &dB(t) = rB(t)dt, \quad B(0) = 1 \\ &dS(t) = S(t)[\mu_\xi S(t - a)dt + g(S(t - b))dW(t) + (e^{\xi(t)} - 1)d\bar{N}(t)], \\ &S(t) = \varphi(t), \quad t \in [-L, 0], \end{aligned} \tag{2.3}$$

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while if we define a stochastic process $\theta(t)$ and a new measure Q as

$$\theta(t) := (\mu_\xi S(t-a) - r)/g(S(t-b)),$$
$$\frac{dQ}{dP} := \exp\left\{-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\right\},$$

then, by Girsanov theorems (Theorem 2 of Arriojas *et al.* [1], or Theorem 1.30 and Theorem 1.32 of Øksendal and Sulem [11]), $\tilde{W}(t) = W(t) + \int_0^t \theta(s)ds$ is a standard Wiener process under the measure Q and Eq. (2.3) is replaced by

$$dS(t) = S(t)[rdt + g(S(t-b))d\tilde{W}(t) + (e^{\xi(t)} - 1)d\bar{N}(t)], \quad S(t) = \varphi(t), \quad t \in [-L, 0]. \tag{2.4}$$

Therefore, the discounted stock price process become a jump martingale under the risk-neutral measure Q and by Theorem 1.14 and Example 1.15 of Øksendal and Sulem [11], the solution of above "SDDEJ" under the risk-neutral measure Q is

$$S(t) = S(0)exp\left\{(r - \lambda\mu_\xi)t - \frac{1}{2}\int_0^t g^2(S(u-b))du + \int_0^t g(S(u-b))d\tilde{W}(u) + \sum_{j=1}^{N(t)} \xi_j\right\},$$

since in comparison with the results of Øksendal and Sulem [11], this is the special case with

$$\gamma(t,z) \equiv \gamma(t) = e^{\xi(t)} - 1,\ R = \infty,\ \int_0^t \int_{\mathbf{R}} \gamma(t,z)\nu(dz)dt = \int_0^t \int_{\mathbf{R}} (e^{\xi(t)} - 1)\lambda P(dz)dt = \lambda\mu_\xi t.$$

Also, this is coincident with the solution of Xiao [17] and Shokrollahi and Kiligman [13], except the drift replaced by $(r - \lambda\mu_\xi)t$ under the measure Q.

THEOREM 3.1: *Denote $V(S(t), t)$ by the price of a European call option in model (2.4). Then, the price at every $t \in [T-l, T]$ of the European call option with strike price K is given by*

$$V(S(t), t) = \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n}{n!} e^{-\lambda(T-t)} \varepsilon_n \times \left\{S(t)\exp\left[(-\lambda\mu_\xi(T-t) + \sum_{j=1}^{n}\xi_j\right]\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)\right\},$$

where $l = \min\{a, b\}$, $\Phi(\cdot)$ is a distribution function of a standard normal random variable,

$$d_\pm = \frac{1}{\sigma}\left[\ln\left(\frac{S(t)}{K}\right) + (r - \lambda\mu_\xi)(T-t) + \sum_{j=1}^{n}\xi_j\right] \pm \frac{1}{2}\sigma, \tag{2.5}$$

$\sigma^2 := \int_t^T g^2(S(u-b))du$ and ξ_j is the jth jump size percent in the interval $[t, T]$. ε_n denotes the expectation operator over the distribution of $\sum_{j=1}^{n}\xi_j$.

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THEOREM 3.2: *The price at $t \in [0, T-l]$ of the European call option is given by*

$$\begin{aligned} V(S(t), t) &= e^{-r(T-t)} \sum_{n=0}^{\infty} \frac{(\lambda l)^n}{n!} e^{-\lambda l} \cdot E_Q[(S_n(T) - K)^+ | \Im_t] \\ &= e^{-r(T-t)} \sum_{n=0}^{\infty} \frac{(\lambda l)^n}{n!} e^{-\lambda l} \\ &\quad \cdot E_Q\left[\varepsilon_n\{[S(T-l)\exp\{(r-\lambda\mu_\xi)l + \sum_{j=1}^{n} \xi_j]\Phi(d_+) - K\Phi(d_-)\}\right| \Im_t], \end{aligned}$$

where instead of σ^2, $S(t)$ in Eq. (2.5) of $d_\pm$, $\sigma^2 = \int_{T-l}^{T} g^2(S(u-b))du$ and $S(T-l)$ are substituted, respectively.

REMARK 3.1: *Suppose that the price process of an underlying asset is driven by an "SDDE" without jumps, that is,*

$$\begin{aligned} &dB(t) = rB(t)dt, \quad B(0) = 1 \\ &dS(t) = S(t)\{\mu_\xi S(t-a)dt + g(S(t-b))dW(t)\}, \quad S(t) = \varphi(t), \quad t \in [-L, 0]. \end{aligned} \tag{2.6}$$

Then, the price of the European call option in this model is obtained from Theorem 3.1 and Theorem 3.2 by deleting, respectively, some term of jumps, that is, $\sum_{n=0}^{\infty}((\lambda(T-t)^n)/(n!))e^{-\lambda(T-t)}$ and $(-\lambda\mu_\xi)(T-l) + \sum_{j=1}^{n}\xi_j$ in Theorem 3.1, $\sum_{n=0}^{\infty}((\lambda l)^n/n!)$ $e^{-\lambda l}$ and $(-\lambda\mu_\xi)l + \sum_{j=1}^{n}\xi_j$ in Theorem 3.2. This is the just result of Arriojas et al. [1].

3.2. Exchange Option

Next, we consider two underlying assets and derive pricing formulae for an exchange option.

Suppose that $l_i = \min\{a_i, b_i\}$ and a $\Im_0$-measurable initial value process $\varphi_i = \varphi_i(u) \in C([-L_i, 0], (0, \infty))$ is given. Then, the solution of Eq. (2.2) is given identically to that of Eq. (2.4) under the measure$Q$, that is,

$$S_i(t) = \varphi_i(0)\exp\left\{(r-\lambda\mu_i)t - \frac{1}{2}\int_0^t g_i^2(S_i(u-b_i))\,du + \int_0^t g_i(S_i(u-b_i))d\tilde{W}_i(u) + \sum_{j=1}^{N(t)} \xi_i^j\right\},$$

$$t \in [0, l_i]$$

where $\xi_i^j$is the jth jump size percent of ith asset in $[0, t]$.

Similarly, $S_i(t)$ can be computed explicitly for $t \in [l_i, 2l_i]$ after knowing solutions in $[0, l_i]$. Thus, we are able to compute the unique $S_i(t)$ for all $t \in [0, T]$ step by step.

If $\varphi_i = \varphi_i(0) \geq 0$, a.s., then $S_i(t) \geq 0$ for all $t \in [0, T]$.

THEOREM 3.3: *Denote the prices of the exchange option in the model (2.2) by $V(S_1(t), S_2(t), t)$ and define $\sigma_i(u) := g_i(S_i(u-b_i))$. Then, the price at every $t \in [T-l, T]$*

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of the exchange option is given by

$$V(S_1(t), S_2(t), t) = \sum_{n=0}^{\infty} \frac{\lambda^n (T-t)^n}{n!} e^{-\lambda(T-t)} \times \varepsilon_n \left\{ S_1(t) \exp\left(\lambda(\beta - \mu_1 + \mu_2)(T-t) + \sum_{j=1}^{n} (\xi_1^j - \xi_2^j) \right) \Phi(d_+) - S_2(t)\Phi(d_-) \right\},$$

where $l = \min\{a_1, a_2, b_1, b_2\}$, $\beta := E[(\xi_1(t) - \xi_2(t))(e^{\xi_2(t)} - 1)]$,

$$d_\pm = \frac{1}{\sigma} \left[\ln \frac{S_1(t)}{S_2(t)} + \lambda(\beta - \mu_1 + \mu_2)(T-t) + \sum_{j=1}^{n} (\xi_1^j - \xi_2^j) \right] \pm \frac{1}{2}\sigma, \tag{2.7}$$

$\sigma^2(u) := \sigma_1^2(u) - 2\rho\sigma_1(u)\sigma_2(u) + \sigma_2^2(u)$, $\sigma^2 := \int_t^T \sigma^2(u)du$ *and* ξ_i^j *is the* jth *jump size percent of* ith *asset in the interval* $[t, T]$.

THEOREM 3.4: *The price at every* $t \in [0, T-l]$ *of the exchange option is given by*

$$V(S_1(t), S_2(t), t) = \sum_{n=0}^{\infty} \frac{(\lambda l)^n}{n!} e^{-\lambda l} E_Q \times \left[\varepsilon_n \left\{ S_1(T-l) \exp\left(\lambda(\beta - \mu_1 + \mu_2)l + \sum_{j=1}^{n} (\xi_1^j - \xi_2^j) \right) \Phi(d_+) - S_2(T-l))\Phi(d_-) \right\} \middle| \Im_t \right],$$

where instead of σ, $S_i(t)$ *in Eq. (2.7) of* $d_\pm$, $\sigma^2 = \int_{T-l}^{T} \sigma^2(u)du$ *and* $S_i(T-l)$ *are substituted, respectively.*

REMARK 3.2: *Let us consider two asset processes described by the "SDDE" model without jumps, that is,*

$$\begin{aligned} dB(t) &= rB(t)dt, \quad B(0) = 1 \\ dS_i(t) &= S_i(t)\{\mu_i S_i(t - a_i)dt + g_i(S_i(t - b_i))dW_i(t)\}, \\ S_i(t) &= \varphi_i(t), \quad t \in [-L_i, 0], \quad i = 1, 2 \end{aligned} \tag{2.8}$$

Then, the price of the exchange option in this model is obtained from Theorem 3.3 and Theorem 3.4 as mentioned in Remark 3.1. This is the just result of Lin et al. [7].

4. NUMERICAL EXPERIMENTS

At first, we approximate underlying asset prices by using the Euler–Maruyama method.

Next, we calculate the prices of the European call option using the theoretical formulae for various models; the BM, the "SDDE", and the "SDDEJ" models.

Also, through numerical experiments, effects of jump and delay parameters are investigated. The code line is written in MATLAB.

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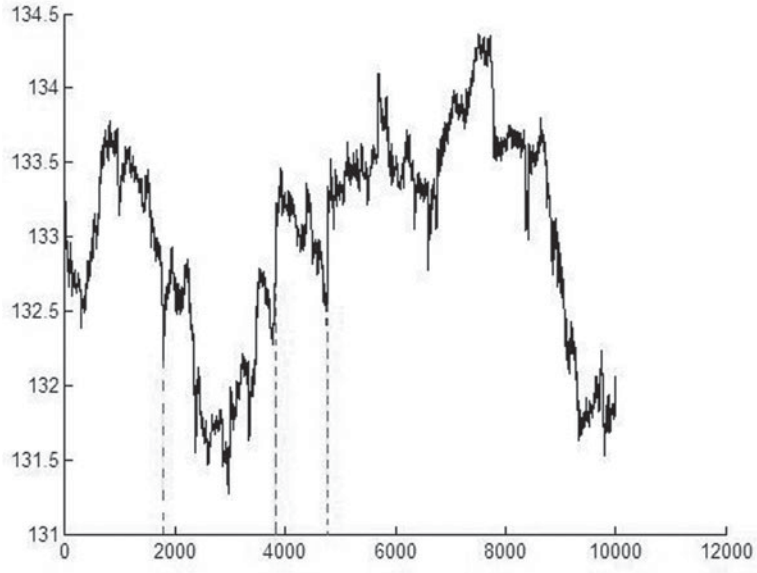


FIGURE **1.** EUR/JPY exchange rate from 1 January 2010 to 30 April 2010.

4.1. Approximation Scheme

Let $h = b/N$, where N is a sufficiently large positive integer.

Each mesh point is denoted by $t_n = n \cdot h, \quad n = -N, -(N-1), \ldots, \quad 0, 1, \ldots$.

Then, the jump Euler–Maruyama approximation scheme is represented by the following equation:

$$\hat{S}(t_n) = \hat{S}(t_{n-1}) \left[1 + rh + g(\hat{S}(t_{n-1-N}))\Delta W_n(t) + \sum_{n=0}^{N(t_n)} (e^{\xi(t_n)} - 1) \right]$$

$$\hat{S}(t_n) = \varphi(n \cdot h), \quad n = -N, -(N-1), \ldots, 0,$$

where $\Delta W_n(t) = W(t_n) - W(t_{n-1})$, $N(t_n)$ is the number of jumps occurred by t_n, and $(e^{\xi(t_n)} - 1)$ is the jump size at time t_n.

4.2. Comparison of Approximated Values with Real Data

The efficiency of Euler–Maruyama method are demonstrated by comparing the approximated underlying asset prices with real prices.

By comparing the underlying asset prices computed by the "SDDEJ" model with the prices computed by the "SDDE" model, we show that the "SDDEJ" model is better than the "SDDE" model in description of underlying asset prices that involve not only delayed response but also jumps.

Figure 1 shows the sample path of EUR/JPY exchange rate from 1 January 2010 to 30 April 2010. Through Figure 1, we can easily find that it has three jumps and they are estimated by the same way in Wang *et al.* [16]. Table 1 shows jump parameters. Also, we assume that the volatility function with delay response is $g(u) = u^{0.05}$, $b = 1$ as in Lin *et al.* [7].

The time-step size and the risk-free interest rate are fixed as 0.1 and 0.002, respectively.

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TABLE 1. The jump parameters in the SDDEJ model

Model type	ξ_1	ξ_2	ξ_3
P_{SDDEJ}	0.008	0.01	0.011

TABLE 2. Actual and simulated values by BM, SDDE, and SDDEJ models

$S_{\text{BM}}(S_0 = 120)$	S_{SDDE}	S_{SDDEJ}	S_{actual}
132.0131	–	–	133.118
132.1302	–	–	133.106
132.2033	–	–	133.122
132.2302	–	–	133.116
132.2411	–	–	133.115
132.2615	132.9954	133.3043	133.209
132.2931	133.7953	133.3102	133.208
132.3091	133.0083	133.3205	133.232
132.4143	133.1912	133.3634	133.238
132.6543	133.2434	133.3342	133.234
132.6953	133.1823	133.2363	133.175
132.7453	133.3423	133.2342	133.204
132.8330	133.2642	133.2061	133.177
132.9024	133.2454	133.1989	133.152
132.9487	133.2034	133.2074	133.139
133.2321	133.1253	133.1844	133.133
133.3844	133.0424	133.1474	133.062
133.4124	133.0023	133.0474	132.968
133.5313	132.8343	132.6473	132.991
133.5998	132.9034	132.9078	133.024
133.7672	133.0083	132.7924	132.987
133.8339	132.3534	132.6743	132.951
133.9149	132.6453	133.0066	132.968

Table 2 shows the actual values and the exchange rates simulated by the BM, the "SDDE", and the "SDDEJ" models. Table 3 presents the absolute deviations between the simulated and the actual values, which indicate that the three jumps 7, 12, and 22 exist. Figure 2 shows these values on the lag subtracted 5 from the number in Table 2. The averages of the final line in Table 3 are 0.152883 and 0.107417, respectively; thus the "SDDEJ" model is more efficient than the "SDDE" model in capturing the fluctuation of exchange rates.

Now, we compute the prices of the European call option with various delays and compare them with Black–Sholes prices.

Table 4 shows all parameters for computing the prices of the European call option.

Table 5 shows the "SDDEJ" prices, the "SDDE" prices, the differences between these and P_{BM}, respectively, according to various delays, moreover the Black–Sholes price P_{BM} for comparison.

As we can see in Table 5, when the delay b is increased from 0.5 to 1.25, $P_{\text{SDDE}} - P_{\text{BM}}$ and $P_{\text{SDDEJ}} - P_{\text{BM}}$ increase, but they decrease when the delay b is increased from 1.5 to 2.25. This implies the necessity of the delayed response. Moreover, all differences $P_{\text{SDDEJ}} - P_{\text{BM}}$ are considerably larger than $P_{\text{SDDE}} - P_{\text{BM}}$, although we assume that the jump number is only one. The results suggest that with the increase of the jump number $P_{\text{SDDEJ}} - P_{\text{BM}}$ might be larger and larger. At the same time, we have shown that the "SDDEJ" model

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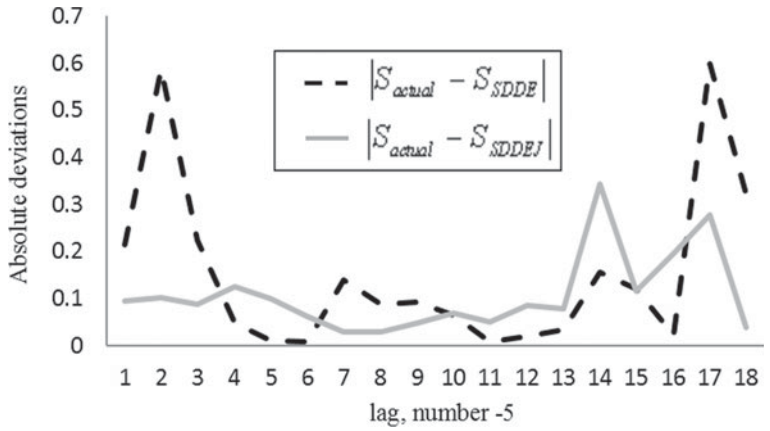


FIGURE **2.** Absolute deviations between actual and simulated by the SDDE and SDDEJ models.

TABLE **3.** Absolute deviation between the simulated and actual values

No.	$\lvert S_{\text{actual}} - S_{\text{BM}} \rvert$	$\lvert S_{\text{actual}} - S_{\text{SDDE}} \rvert$	$\lvert S_{\text{actual}} - S_{\text{SDDEJ}} \rvert$
1	1.1049		
2	0.9758		
3	0.9187		
4	0.8858		
5	0.8739		
6	0.9475	0.2136	0.0953
7	0.9149	0.5873	0.1022
8	0.9229	0.2237	0.0885
9	0.8237	0.0468	0.1254
10	0.5797	0.0094	0.1002
11	0.4797	0.0073	0.0613
12	0.4587	0.1383	0.0302
13	0.344	0.0872	0.0291
14	0.2496	0.0934	0.0469
15	0.1903	0.0644	0.0684
16	0.0991	0.0077	0.0514
17	0.3224	0.0196	0.0854
18	0.4444	0.0343	0.0794
19	0.5403	0.1567	0.3437
20	0.5758	0.1206	0.1162
21	0.7802	0.0213	0.1946
22	0.8829	0.5976	0.2767
23	0.9469	0.3227	0.0386
Sum	15.2621	2.7519	1.9335
Average	0.66357	0.152883	0.107417

TABLE **4.** The valuation of chosen parameters used in these models ($T = 30,\ t = 27$)

Model type	r			ξ	μ_ξ		a	b	$g(u)$
BM	0.002	0.105	100	–	–	–	–	–	
$SDDE$	0.002	–	100	–			0.2	1	$u^{0.05}$
$SDDEJ$	0.002	–	100	0.01	0.00065	0.0015	0.2	1	$u^{0.05}$

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TABLE 5. SDDEJ and SDDE prices with various delays, Black–Sholes price

b	0.5	0.75	1	1.25	1.5	1.75	2.0	2.25
P_{SDDE}	12.7438	12.7597	12.7804	12.7924	12.7678	12.7421	12.7183	12.6978
$P_{\mathrm{SDDE}} - P_{\mathrm{BM}}$	0.7286	0.7445	0.7652	0.7772	0.7526	0.7269	0.7031	0.6826
P_{SDDEJ}	12.7563	12.7950.	12.8032	12.8054	12.7934	12.7631	12.7332	12.7213
$P_{\mathrm{SDDEJ}} - P_{\mathrm{BM}}$	0.7411	0.7798	0.788	0.7902	0.7782	0.7479	0.718	0.7061
$P_{\mathrm{BM}}(S_0 = 80)$	12.0152							

is more efficient than the "SDDE" model to describe such underlying asset prices as the exchange rates. Therefore, we can conclude that it is significant to consider the jumps as well as the delays for option pricing.

The search of appropriate jump number and jump size involved in underlying asset prices is open problem.

5. CONCLUSION

Option pricing is very important both in theory and practice. In this paper, we introduce a more general model of underlying asset prices in which not only delays but also jumps are included. We obtained the pricing formulae for a European call option and an exchange option in this model by using a Lemma on conditional expectation. We show numerically that the "SDDEJ" model is more flexible in some degree than the "SDDE" model to describe fluctuations of underlying asset prices. We can find that the differences between the Black–Scholes option prices and the option prices simulated by the "SDDEJ" model are all larger than ones by the "SDDE" model even the case with only one jump. All these facts imply that not only delay response but also jumps are important factors in option pricing context.

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APPENDIX

PROOF: $\mathcal{G} \subset \sigma(\mathcal{G}, \sigma(X))$, *thus from the property of conditional expectation*

$$E[f(X, Y)|\mathcal{G}] = E[E[f(X, Y)|\sigma(\mathcal{G}, \sigma(X))]|\mathcal{G}].$$

X is $\sigma(\mathcal{G}, \sigma(X))$-measurable and the random variable Y and $\sigma(\mathcal{G}, \sigma(X))$ are independent, thus if we suppose $g(x) := E[f(x, Y)]$, then from Lemma 4.15 of Shreve [14], $E[f(X, Y)|\sigma(\mathcal{G}, \sigma(X))] = g(X)$, that is, $E[f(X, Y)|\mathcal{G}] = E[g(X)|\mathcal{G}]$. ■

PROOF: *The solution of* Eq. (2.4) *at everyt* $\in [T - l, T]$ *is*

$$S(T) = S(t) \exp\left\{(r - \lambda\mu_\xi)(T - t) - \frac{1}{2}\int_t^T g^2(S(u - b))du + \int_t^T g(S(u - b))d\tilde{W}(u) + \sum_{j=1}^{N(T-t)} \xi_j\right\}.$$

Thus, under the risk-neutral measure Q, the discounted price process is a martingale, so the price of $F = (S(T) - K)^+$ at time t is

$$V(S(t), t) = e^{-r(T-t)} E_Q[F|\Im_t]$$

(see Arriojas *et al.* [1] and Kazmerchuk *et al.* [4]). Now denote

$$S_n(T) := S(t) \exp\left\{(r - \lambda\mu_\xi)(T - t) - \frac{1}{2}\int_t^T g^2(S(u - b))du + \int_t^T g(S(u - b))d\tilde{W}(u) + \sum_{j=1}^{n} \xi_j\right\}. \tag{A.1}$$

Since $N(T - t)$ is independent with $\Im_t$ and the other random variables, if we put $Y := N(T - t)$, $\mathcal{G} := \Im_t$ and apply Lemma 3.1, then we have

$$\begin{aligned} V(S(t), t) &= e^{-r(T-t)} E_Q[F|\Im_t] = e^{-r(T-t)} E_Q[(S(T) - K)^+|\Im_t] \\ &= e^{-r(T-t)} E_Q\left[\sum_{n=0}^{\infty} \frac{\lambda^n (T - t)^n}{n!} e^{-\lambda(T-t)} (S_n(T) - K)^+|\Im_t\right] \\ &= e^{-r(T-t)} \sum_{n=0}^{\infty} \frac{\lambda^n (T - t)^n}{n!} e^{-\lambda(T-t)} E_Q[(S_n(T) - K)^+|\Im_t]. \end{aligned} \tag{A.2}$$

From Lin *et al.* [7], $\int_t^T g(S(u - b))d\tilde{W}(u)$ is the same as the distribution of $\sigma\xi$ under the measure Q, where ξ is a standard normal random variable and $\sigma^2 = \int_t^T g^2(S(u - b))du$.

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Now, we define a function $f(u, \sigma^2, \nu, \beta) := [u \exp\{(r - \lambda\mu_\xi)(T-t) - (1/2)\sigma^2 + \sigma\nu + \beta\} - K]^+$, then

$$[S_n(T) - K]^+ = f(S(t),\ \int_t^T g^2\left(S(u-b))du,\ \xi,\ \sum_{j=1}^n \xi_j\right).$$

Denote $x = (u, \sigma^2),\ y = (\nu, \beta),\ X = (S(t),\ \int_t^T g^2(S(u-b))du),\ Y = \left(\xi, \sum_{j=1}^n \xi_j\right),\ \mathcal{G} = \Im_t$ and then we apply Lemma 4.15 of Shreve [14]. Since X is $\Im_t$-measurable and Y is independent with $\Im_t$, if we assume that $h(x) := E_Q[f(x, Y)]$, then we have

$$E_Q[(S_n(T) - K)^+ | \Im_t] = E_Q[f(X, Y)|\Im_t] = h(X) = h(S(t),\ \int_t^T g^2(S(u-b))du). \tag{A.3}$$

Using the property of conditional expectation and the independence of ξ and $\sum_{j=1}^n \xi_j$, we have, from Lemma 4.15 of Shreve [14],

$$\begin{aligned} h(x) = h(u,\ \sigma^2) &= E_Q\left[u \exp\left\{(r - \lambda\mu_\xi)(T-t) - \frac{\sigma^2}{2} + \sigma\xi + \sum_{j=1}^n \xi_j\right\} - K\right]^+ \\ &= E_Q\left[E_Q\left[u \exp\left\{(r - \lambda\mu_\xi)(T-t) - \frac{\sigma^2}{2} + \sigma\xi + \sum_{j=1}^n \xi_j\right\} - K\right]^+ \middle| \sum_{j=1}^n \xi_j\right], \\ &= \varepsilon_n\left[\upsilon\left(u,\ \sigma^2,\ \sum_{j=1}^n \xi_j\right)\right], \end{aligned} \tag{A.4}$$

where $\upsilon(u,\ \sigma^2,\ y) = E_Q[u \exp\{(r - \lambda\mu_\xi)(T-t) - (\sigma^2/2) + \sigma\xi + y\} - K]^+$.

Since $\xi \sim N(0,\ 1)$, by the popular calculation of expectation, we have

$$\upsilon(u,\ \sigma^2,\ y) = u \exp\{(r - \lambda\mu_\xi)(T-t) + y\}\Phi(d_+) - K\Phi(d_-), \tag{A.5}$$

where $d_\pm = (1/\sigma)[\ln(u/K) + (r - \lambda\mu_\xi)(T-t) + y] \pm (\sigma/2)$.

Substituting (A.5) in (A.4) and using (A.3) and (A.2), we have for $\sigma^2 = \int_t^T g^2(S(u-b))du$

$$\begin{aligned} V(S(t)\,,\ t) &= e^{-r(T-t)} \sum_{n=0}^\infty \frac{\lambda^n (T-t)^n}{n!} e^{-\lambda(T-t)} \varepsilon_n \\ &\quad \times \left[\left\{S(t) \exp[(r - \lambda\mu_\xi)(T-t) + \sum_{j=1}^n \xi_j\right\}\Phi(d_+) - K\Phi(d_-)\right] \\ &= \sum_{n=0}^\infty \frac{\lambda^n (T-t)^n}{n!} e^{-\lambda(T-t)} \varepsilon_n \\ &\quad \times \left\{S(t) \exp\left[(-\lambda\mu_\xi(T-t) + \sum_{j=1}^n \xi_j\right]\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)\right\}. \end{aligned}$$

■

PROOF: *The solution of* Eq. (2.4) *can be represented as*

$$S(T) = S(T-l) \exp\left\{(r - \lambda\mu_\xi)l - \tfrac{1}{2}\int_{T-l}^v g^2(S(u-b))du + \int_{T-l}^T g(S(u-b))d\tilde{W}(u) + \sum_{j=1}^{N(l)} \xi_j\right\}.$$

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For $t \in [0, T-l]$, $S(T-l)$ and $\int_{T-l}^{T} g^2(S(u-b))du$ can be no $\Im_t$-measurable, therefore the conditional expectation under the measure Q_2 is remained in the pricing formula, except this symbol, the others are the same as proof of Theorem 3.1. ■

PROOF: *The solution of* Eq. (2.2) *at every* $t \in [T-l, T]$ *is as follows:*

$$S_i(T) = S_i(t)\exp\left\{(r-\lambda\mu_i)(T-t) - \frac{1}{2}\int_t^T g_i^2(S_i(u-b_i))du + \int_t^T g_i(S_i(u-b_i))d\tilde{W}_i(u) + \sum_{j=1}^{N(T-t)} \xi_i^j\right\}. \tag{A.6}$$

Denote

$$Z(t) = \frac{S_1(t)}{S_2(t)},$$

then from (A.6) we obtain

$$Z(T) = Z(t)exp\left\{-\lambda(\mu_1-\mu_2)(T-t) - \frac{1}{2}\int_t^T (\sigma_1^2(u) - \sigma_2^2(u))du + \int_t^T \sigma_1(u)d\tilde{W}_1(u) - \int_t^T \sigma_2(u)d\tilde{W}_2(u) + \sum_{j=1}^{N(T-t)} (\xi_1^j - \xi_2^j)\right\}. \tag{A.7}$$

Now, let define a new measureQ_2 on $\Im_T$ as follows:

$$\frac{dQ_2}{dQ} := \eta(T),$$

where

$$\eta(u) := exp\left\{\int_t^u \sigma_2(u)d\tilde{W}_2(u) - \frac{1}{2}\int_t^u \sigma_2^2(u)du + \sum_{j=1}^{N(u-t)} \xi_2^j - \lambda\mu_2(u-t)\right\}. \tag{A.8}$$

Then, $\eta(u)$ of Eq. (A.8) is the solution of the stochastic differential equation

$$d\eta(u) = \eta(u)[\sigma_2(u)d\tilde{W}_2(u) + (e^{\xi_2(u)} - 1)d\bar{N}(u)],\ u \in [t,T],\ \eta(t) = 1,$$

where $\bar{N}(t) = N(t) - \lambda t$. Since $\tilde{W}_1(u)$ and $\tilde{W}_2(u)$ are not independent, but correlated standard Wiener processes, we cannot apply the two-dimensional Girsanov theorem. So we apply the one-dimensional Girsanov theorem for Ito-diffusions of Øksendal and Sulem [11]. In fact, noting that from $d\tilde{W}_1(u)\, d\tilde{W}_2(u) = \rho du$,

$$\frac{d[\eta\,,\, \tilde{W}_1](u)}{\eta(u)} = \rho\sigma_2(u)\, du\ ,\quad \frac{d[\eta\,,\, \tilde{W}_2](u)}{\eta(u)} = \sigma_2(u)du,$$

then by Remark 1.31 and Theorem 1.32, also by Lemma 1.33 of Øksendal and Sulem [11] under the measure Q_2

$$\hat{W}_1(u) = \tilde{W}_1(u) - \rho\int_t^u \int \sigma_2(s)ds, \tag{A.9}$$

$$\hat{W}_2(u) = \tilde{W}_2(u) - \int_t^u \sigma_2(s)ds, \quad u \in [t, T],$$

are also standard Wiener processes and correlated with $d\hat{W}_1(u)\, d\hat{W}_2(u) = \rho du$. Also noting that by Eq. (A.8),$1 - \theta(s, z) := e^{\xi_2(s)}$, $\nu(dz) = \lambda Q(dz)$, $\int_0^u \int_R (1-\theta(s, z))\nu(dz)ds = \lambda E[e^{\xi_2(s)}]u =$

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$\lambda(\mu_2+1)u$ and applying Lemma 1.33 of Øksendal and Sulem [11], then under the measure Q_2, $\bar{N}_{Q_2}(u) = N(u) - \lambda(\mu_2+1)u$ is a centered Poisson process, thus $N_{Q_2}(u) = N(u) - \lambda\mu_2 u$ is a Poisson process. Substituting (A. 9) in (A.7), under Q_2 we have

$$\begin{aligned} Z(T) &= Z(t)\exp\left\{-\lambda(\mu_1-\mu_2)(T-t) - \frac{1}{2}\int_t^T(\sigma_1^2(u)-\sigma_2^2(u))du\right. \\ &\quad \left. + \int_t^T \sigma_1(u)[d\hat{W}_1(u) + \rho\sigma_2(u)du] - \int_t^T \sigma_2(u)[d\hat{W}_2(u)+\sigma_2(u)du] + \sum_{j=1}^{N(T-t)}(\xi_1^j - \xi_2^j)\right\}, \\ &= Z(t)\exp\left\{-\lambda(\mu_1-\mu_2)(T-t) + \int_t^T \sigma_1(u)d\hat{W}_1(u) - \int_t^T \sigma_2(u)d\hat{W}_2(u)\right. \\ &\quad \left. -\frac{1}{2}\int_t^T[\sigma_1^2(u) - 2\rho\sigma_1(u)\sigma_2(u) + \sigma_2^2(u)]du + \sum_{j=1}^{N(T-t)}(\xi_1^j-\xi_2^j)\right\}. \end{aligned} \tag{A.10}$$

There exists such a function $\sigma(u)$ that the term of standard Wiener processes in (A.10) can be represented as follows:

$$\int_t^T \sigma_1(u)d\hat{W}_1(u) - \int_t^T \sigma_2(u)d\hat{W}_2(u) = \int_0^t \sigma(u)d\hat{W}(u).$$

In fact, the function $\sigma(u)$ satisfies the following equation $\sigma^2(u) = \sigma_1^2(u) - 2\rho\sigma_1(u)\sigma_2(u) + \sigma_2^2(u)$.

Therefore, from the above-mentioned evidence, we have

$$Z(T) = Z(t)\exp\left\{-\lambda(\mu_1-\mu_2)(T-t) + \int_t^T \sigma(u)d\hat{W}(u) - \frac{1}{2}\int_t^T \sigma^2(u)du + \sum_{j=1}^{N(T-t)}(\xi_1^j-\xi_2^j)\right\}. \tag{A.11}$$

Now denote

$$\gamma(t,z) := e^{[\xi_1(t)-\xi_2(t)]} - 1,\ \alpha_{12} := E\gamma(t,z) = \int_R \gamma(t,z)\nu(dz).$$

then

$$\int_t^T\int_R \ln(1+\gamma(s,z))N_{Q_2}(ds,dz) = \sum_{j=1}^{N_{Q_2}(T-t)}(\xi_1^j-\xi_2^j)$$

and by Theorem 1.34 of Øksendal and Sulem [11] and the definition of β

$$\begin{aligned} \sum_{j=1}^{N(T-t)}(\xi_1^j-\xi_2^j) &= \sum_{j=1}^{N_{Q_2}(T-t)}(\xi_1^j-\xi_2^j) + \int_t^T\int_R(\xi_1(t,z)-\xi_2(t,z))(e^{\xi_2(t,z)}-1)\nu(dz))ds \\ &= \sum_{j=1}^{N_{Q_2}(T-t)}(\xi_1^j-\xi_2^j) + \lambda\beta(T-t). \end{aligned} \tag{A.12}$$

If we denote

$$B := \int_t^T \sigma(u)d\hat{W}(u) - \frac{1}{2}\int_t^T \sigma^2(u)du$$

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and substitute Eq. (A.12) to Eq. (A.11), then we have

$$\begin{aligned} Z(T) &= Z(t)\exp\left\{-\lambda(\mu_1-\mu_2)(T-t)+B+\sum_{j=1}^{N_{Q_2}(T-t)}(\xi_1^j-\xi_2^j)+\lambda\beta(T-t)\right\} \\ &= Z(t)exp\left\{B+\sum_{j=1}^{N_{Q_2}(T-t)}(\xi_1^j-\xi_2^j)-\lambda\alpha_{12}(T-t)+\lambda\delta(T-t)\right\} \\ &= Z(t)\exp\left\{\lambda\delta(T-t)\right\}\exp\{B+\sum_{j=1}^{N_{Q_2}(T-t)}(\xi_1^j-\xi_2^j)-\lambda\alpha_{12}(T-t)\Big\} \\ &= \exp\left\{\lambda\delta(T-t)\right\}Y(T), \end{aligned}$$

where $\delta := \alpha_{12}+\beta-\mu_1+\mu_2$, $Y(T) = Z(t)\exp\{B+\sum_{j=1}^{N_{Q_2}(T-t)}(\xi_1^j-\xi_2^j)-\lambda\alpha_{12}(T-t)\}$.

Moreover, $Y(T)$ is a solution of the following stochastic differential equation:

$$dY(u) = Y(u)[\sigma(u)d\hat{W}(u)+(e^{\xi_1(u)-\xi_2(tu)}-1)d\bar{N}_{Q_2}(u), \quad Y(t)=Z(t), \quad u\in[t,T]. \qquad \textbf{(A.13)}$$

Therefore, the price of the exchange option $F=(S_1(T)-S_2(T))^+$ at time t is

$$\begin{aligned} V(S_1(t),S_2(t),t) &= E_Q[e^{-r(T-t)}(S_1(T)-S_2(T)^+|\Im_t)] = e^{-r(T-t)}E_Q\left[S_2(T)\left[\frac{S_1(T)}{S_2(T)}-1\right]^+|\Im_t\right] \\ &= E_Q\left[S_2(t)\exp\left\{-\lambda\mu_2(T-t)-\frac{1}{2}\int_t^T\sigma_2^2du+\int_t^T\sigma_2d\tilde{W}_2(u)+\sum_{j=1}^{N(T-t)}\xi_2^j\right\}\right. \\ &\quad \left.\times\left[\frac{S_1(T)}{S_2(T)}-1\right]^+|\Im_t\right] \\ &= S_2(t)E_Q[\eta(T)(Z(T)-1)^+|\Im_t] \\ &= S_2(t)E_{Q_2}[(Z(T)-1)^+|\Im_t] = S_2(t)E_{Q_2}[(e^{\lambda\delta(T-t)}Y(T)-1)^+|\Im_t] \\ &= S_2(t)e^{\lambda\delta(T-t)}E_{Q_2}[Y(T)-e^{-\lambda\delta(T-t)}]^+|\Im_t] \end{aligned}$$

Since $Y(u)$, $u\in[t,T]$ is a martingale under measure Q_2 from (A.13), if we assign

$$S(t)=Y(t), \quad K=e^{-\lambda\delta(T-t)}, \quad \mu_\xi=\alpha_{12}, \quad \sum_{j=1}^n\xi_j=\sum_{j=1}^n(\xi_1^j-\xi_2^j), \quad r=0,$$

then, Theorem 3.1 for the European call option can be applied. Therefore, the price of the exchange option is as follows:

$$V(S_1(t),S_2(t),t) = S_2(t)e^{\lambda\delta(T-t)}E_{Q_2}[(Y(T)-e^{-\lambda\delta(T-t)})^+|\Im_t]$$

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$$
\begin{aligned}
&= S_2(t)e^{\lambda\delta(T-t)}\sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n}{n!}e^{-\lambda(T-t)}\varepsilon_n \\
&\quad\times\left\{Y(t)\exp\left(-\lambda\alpha_{12}(T-t)+\sum_{j=1}^{n}(\xi_1^j-\xi_2^j)\right)\Phi(d_+)-e^{-\lambda\delta(T-t)}\Phi(d_-)\right\} \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n}{n!}e^{-\lambda(T-t)}S_2(t)\varepsilon_n \\
&\quad\times\left\{Z(t)e^{\lambda\delta(T-t)}\exp\left(-\lambda\alpha_{12}(T-t)+\sum_{j=1}^{n}(\xi_1^j-\xi_2^j)\right)\Phi(d_+)-\Phi(d_-)\right\} \\
&= \sum_{n=0}^{\infty}\frac{\lambda^n(T-t)^n}{n!}e^{-\lambda(T-t)}\varepsilon_n \\
&\quad\times\left\{S_1(t)\exp\left(\lambda(\beta-\mu_1+\mu_2)(T-t)+\sum_{j=1}^{n}(\xi_1^j-\xi_2^j)\right)\Phi(d_+)-S_2(t)\Phi(d_-)\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
d_\pm &= \frac{1}{\sigma}\left[\ln\frac{Y(t)}{K}-\lambda\alpha_{12}(T-t)+\sum_{j=1}^{n}(\xi_1^j-\xi_2^j)\right]\pm\frac{\sigma}{2} \\
&= \frac{1}{\sigma}\left[\ln\frac{S_1(t)}{S_2(t)}+\lambda(\beta-\mu_1+\mu_2)(T-t)+\sum_{j=1}^{n}(\xi_1^j-\xi_2^j)\right]\pm\frac{\sigma}{2}, \\
\sigma &:= \int_t^T\sigma^2(u)du
\end{aligned}
$$

■

PROOF: *Just as using of Theorem 3.1 to prove Theorem 3.3, we can prove Theorem 3.4 by applying Theorem 3.2.* ■

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