On the product $\prod_{n=2}^{\infty} \zeta(n)$

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The Euler-Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \tag{1}$$

where $\mathrm{Re}(s) > 1$ and the product is taken over all primes p.

The above identity, discovered by L. Euler (1707-1783) for a real variable s and extended by B. Riemann (1826-1866) to the complex field, plays a fundamental part in the analytic theory of prime numbers.

The values of the zeta function for integers $n \geqslant 2$ especially occur in the series representations

$$\log \Gamma(1 - z) = \gamma z + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} z^n \qquad (|z| < 1) \tag{2}$$

and

$$-\Psi(1 - z) = \gamma + \sum_{n=2}^{\infty} \zeta(n)\, z^{n-1} \qquad (|z| < 1) \tag{2'}$$

for the logarithm of the gamma function and its derivative, respectively [1, p. 45]. Here γ stands for the Euler constant:

$$\gamma = \lim_{n \to \infty} \gamma_n, \qquad \gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \log n.$$

In [2] Euler gives, among other formulae, the following expressions for γ:

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n},$$

$$1 - \sum_{n=2}^{\infty} \frac{\zeta(n) - 1}{n}$$

$$1 - \log\frac{3}{2} - \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \frac{\zeta(2n+1) - 1}{2n+1}.$$

If $z = -\frac{1}{2}$, the formula (10) in [1, p. 46] yields the last expression above. The other series relating to $\zeta(n)$ are also given in [3]. The series in (2) and (2′) diverge at $z = 1$, i.e. the sum of the coefficients is infinite. But the infinite product formed with the coefficients $\zeta(n)$ in (2′) is convergent and satisfies certain relations involving partition numbers, primes and Mersenne numbers.

https://doi.org/10.1017/mag.2015.78 Published online by Cambridge University Press

Theorem 1: The infinite product

$$A = \prod_{n=2}^{\infty} \zeta(n) \tag{3}$$

is convergent and

$$2 \leqslant A \leqslant e. \tag{4}$$

Proof: The following is a well-known theorem about infinite products: If $a_n \geqslant 0$ for all $n \geqslant 1$, then $\sum_{n=1}^{\infty} a_n$ and $\prod_{n=1}^{\infty} (1 + a_n)$ converge simultaneously.

If S and P denote their respective values, then $1 + S \leqslant P \leqslant e^S$. Here the lower estimate follows from $\prod_{i=1}^{n} (1 + a_i) \geqslant 1 + \sum_{i=1}^{n} a_i$ and the upper estimate from $1 + a_n \leqslant e^{a_n}$. If $a_n = \zeta(n+1) - 1$ the relation $\sum_{n=1}^{\infty} a_n = 1$ (for example see [3]) then implies (4).

Finally let us add that we compute the constant A using Table 23.3 in [4] which gives the numerical estimate $A \approx 2.2948566$.

Theorem 2: We have

$$A = \prod_{p} \sum_{n=0}^{\infty} \frac{A_n - A_{n-1}}{p^n}, \tag{5}$$

where $A_0 = 1, A_{-1} = 0$ and A_n is the number of partitions of n for $n \geqslant 1$.

Proof: From (1),

$$\zeta(n) = \prod_{p} \left(1 - \frac{1}{p^n}\right)^{-1} \qquad (n \in \mathbb{N}, n \geqslant 2). \tag{6}$$

Then

$$A = \prod_{n=2}^{\infty} \zeta(n) = \prod_{n=2}^{\infty} \prod_{p} \left(1 - \frac{1}{p^n}\right)^{-1}. \tag{7}$$

Noting that the infinite products in (1) and (3) are absolutely convergent we can express (7) as

$$A = \prod_{p} \prod_{n=2}^{\infty} \left(1 - \frac{1}{p^n}\right)^{-1}. \tag{8}$$

Now Euler's identity [5, p. 308]

$$\prod_{n=1}^{\infty} (1 - x^n)^{-1} = \sum_{n=0}^{\infty} A_n x^n \qquad (|x| < 1),$$

when multiplied by $(1 - x)$, gives

$$\prod_{n=2}^{\infty} (1 - x^n)^{-1} = \sum_{n=0}^{\infty} (A_n - A_{n-1}) x^n.$$

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Putting $x = 1/p$ and comparing with (8) yields (5).

Theorem 3:

$$\log A = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p} \frac{1}{p^k (p^k - 1)}. \tag{9}$$

Proof: If we take logarithms in (6) and use the series expansion

$$-\log(1 - x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \qquad (|x| < 1)$$

it follows that

$$\log \zeta(n) = \sum_{p} \sum_{k=1}^{\infty} \frac{p^{-nk}}{k}.$$

We then have

$$\log A = \sum_{n=2}^{\infty} \log \zeta(n) = \sum_{n=2}^{\infty} \sum_{p} \sum_{k=1}^{\infty} \frac{p^{-nk}}{k}.$$

Since the terms of this convergent multiple series are positive, the order of summations can be changed:

$$\log A = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p} \sum_{n=2}^{\infty} \left(p^{-k}\right)^n = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p} \frac{1}{p^k (p^k - 1)}.$$

The next theorem rests on the following lemma and its corollary:

Lemma: If $s \geqslant 2$, the zeta function satisfies the inequalities

$$1 + \frac{\frac{1}{2}}{2^s - 1} < \zeta(s + 1) < 1 + \frac{\frac{2}{3}}{2^s - 1}. \tag{10}$$

Proof: Let

$$\lambda(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \qquad (s > 1). \tag{11}$$

This function is related to the zeta function by the formula [1, p. 32]

$$\left(1 - 2^{1-s}\right) \zeta(s) = \lambda(s) = 1 - 2^{-s} + 3^{-s} - \ldots. \tag{12}$$

The series representing $\lambda(s)$ being alternating we infer from (12) that

$$1 - 2^{-s} < \left(1 - 2^{1-s}\right) \zeta(s) < 1 - 2^{-s} + 3^{-s}.$$

Dividing through by $\left(1 - 2^{1-s}\right)$ and simplifying,

$$1 + \frac{1}{2^s - 2} < \zeta(s) < 1 + \frac{1}{2^s - 2}\left(1 + \left(\tfrac{2}{3}\right)^s\right) \qquad (s > 1). \tag{13}$$

Incidentally (13) gives the asymptotic expression

$$\zeta(s) - 1 \sim 2^{-s} \sim \left(2^s - 1\right)^{-1} \qquad (s \to \infty).$$

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Let us rewrite (13) with $(s + 1)$ in place of s:

$$1 + \frac{\frac{1}{2}}{2^s - 1} < \zeta(s + 1) < 1 + \frac{\frac{1}{2}}{2^s - 1} g(s) \qquad (s > 0), \tag{14}$$

where

$$g(s) = 1 + \left(\tfrac{2}{3}\right)^{s+1} \qquad (s > 0).$$

If $s \geqslant 2$ then $g(s) < \frac{4}{3}$. Therefore (14) now implies (10).

Corollary:

$$1 + \frac{\frac{1}{2}}{M_n} < \zeta(n + 1) < 1 + \frac{\frac{2}{3}}{M_n} \qquad (n \in \mathbb{N}, n \geqslant 1), \tag{15}$$

where

$$M_n = 2^n - 1 \tag{16}$$

is the nth Mersenne number.

Proof: By the above lemma it only remains to verify the case $n = 1$:

$$1 + \frac{1}{2} < \frac{\pi^2}{6} < 1 + \frac{2}{3}.$$

Theorem 4: There exists one, and only one, number $a \in \left(\frac{1}{2}, \frac{2}{3}\right)$ such that

$$A = \prod_{n=1}^{\infty} \left(1 + \frac{a}{M_n}\right). \tag{17}$$

Proof: From (3) and (15),

$$\prod_{n=1}^{\infty} \left(1 + \frac{\frac{1}{2}}{M_n}\right) < A < \prod_{n=1}^{\infty} \left(1 + \frac{\frac{2}{3}}{M_n}\right). \tag{18}$$

Let

$$F(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{M_n}\right) \qquad (z \in \mathbb{C}).$$

If $|z| \leqslant R < \infty$ this product converges absolutely and uniformly. Therefore F is an entire function. In particular, F is continuous on $\mathbb{R}$. Now (18) takes the form

$$F\left(\tfrac{1}{2}\right) < A < F\left(\tfrac{2}{3}\right).$$

Hence we conclude that there exists an $a \in \left(\frac{1}{2}, \frac{2}{3}\right)$ such that $F(a) = A$. Uniqueness follows from the fact that $F(x)$ is strictly increasing if x is real and positive. A method of iteration leads to $a \approx 0.616266$ if we take $A \approx 2.2948566$.

Theorem 5: Let n be a positive integer and $i, k = 1, \ldots, n$. If

$$a_{ik} = k^{-\frac{1}{2}(i+1)} \tag{19}$$

then the determinant, D_n, of the matrix $(a_{ik})_{1 \leqslant i,k \leqslant n}$ satisfies the inequality

$$D_n^2 < A. \tag{20}$$

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Proof: From (3) and (19) we have

$$A = \prod_{i=1}^{\infty} \zeta(i+1) = \prod_{i=1}^{\infty} \sum_{k=1}^{\infty} k^{-(i+1)} = \prod_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2.$$

As

$$A = \prod_{i=1}^{\infty} (1+u_i);\ u_i = \sum_{k=2}^{\infty} a_{ik}^2 > \sum_{k=2}^{n} a_{ik}^2 \text{ and } A > \prod_{i=1}^{n} (1+u_i)$$

we get

$$A > \prod_{i=1}^{n} \sum_{k=1}^{n} a_{ik}^2. \tag{21}$$

The inequality (20) then follows from (21) and Hadamard's inequality [6, p. 36], which states that the determinant Δ_n of an $n \times n$ matrix with complex elements c_{ik} satisfies the inequality

$$|\Delta_n|^2 \leqslant \prod_{i=1}^{n} \sum_{k=1}^{n} |c_{ik}|^2.$$

Remark: We have $|D_n| < A^{1/2}$ for all $n > 1$. But it is not known if the sequence $(|D_n|)_{n \geqslant 1}$ tends to a limit as $n \to \infty$.

Acknowledgement: The author is grateful to Professor Cem Tezer for a critical reading of the manuscript.

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doi:10.1017/mag.2015.78

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https://doi.org/10.1017/mag.2015.78 Published online by Cambridge University Press