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Universal curves in the center problem for Abel differential equations

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Abstract. We study the center problem for the class $\mathcal{E}_\Gamma$ of Abel differential equations $dv/dt = a_1 v^2 + a_2 v^3$, $a_1, a_2 \in L^\infty([0, T])$, such that images of Lipschitz paths $\tilde{A} := (\int_0^{\cdot} a_1(s)\, ds, \int_0^{\cdot} a_2(s)\, ds) : [0, T] \to \mathbb{R}^2$ belong to a fixed compact rectifiable curve Γ. Such a curve is said to be universal if whenever an equation in $\mathcal{E}_\Gamma$ has center on $[0, T]$, this center must be universal, i.e. all iterated integrals in coefficients a_1, a_2 of this equation must vanish. We investigate some basic properties of universal curves. Our main results include an algebraic description of a universal curve in terms of a certain homomorphism of its fundamental group into the group of locally convergent invertible power series with the product being the composition of series, explicit examples of universal curves, and approximation of Lipschitz triangulable curves by universal ones.

1. *Introduction*

In this paper we study the center problem for the Abel differential equation

$$\frac{dv}{dt} = a_1 v^2 + a_2 v^3 \tag{1.1}$$

with coefficients a_1, a_2 in the space $L^\infty([0, T])$ of bounded Lebesgue measurable real functions on an interval $[0, T] \Subset \mathbb{R}$. Recall that equation (1.1) has a *center* on $[0, T]$ if for all sufficiently small initial values v_0 the corresponding solution satisfies $v(T) = v(0) := v_0$. The center problem is *to describe explicitly the set of coefficients* a_1, a_2 *for which the corresponding equations (1.1) have centers on* $[0, T]$. The problem is closely related to the classical Poincaré Center-Focus problem for planar polynomial vector fields (see e.g. [**BRY**, **I**] and references therein). Recently there has been an intensive study of the center problem for Abel differential equations focused on some composition conjectures for equations with piecewise smooth coefficients (see [**AL**, **A1**, **A2**, **A3**, **A4**, **C**] for some necessary and sufficient conditions for the existence of 'composition centers' and

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bounds on maximal multiplicity of periodic solutions of cubic and quartic non-autonomous differential equations with continuous piecewise linear coefficients, [**B2**, **B3**, **B4**, **B5**, **B6**, **B7**] for a general approach to the center problem for differential equations with bounded measurable coefficients and [**BlRY**, **BFY1**, **BFY2**, **BRY**, **BY**, **CGM1**, **CGM2**, **CGM3**, **GGL**, **GGS**, **P1**, **P2**, **P3**, **PRY**] for recent results on the 'composition centers' for Abel differential equations with polynomial and trigonometric coefficients).

It is known (see e.g. [**B1**]) that the set of centers of equation (1.1) consists of pairs $a = (a_1, a_2)$ satisfying an infinite system of equations $c_i = 0$, $i \in \mathbb{N}$, where

$$\begin{aligned} c_i &:= \sum_{i_1+\cdots+i_k=i} c_{i_1,\dots,i_k}(i) \cdot I_{i_1,\dots,i_k}(a) \quad \text{all } i_1, \dots, i_k \in \{1, 2\}, \\ c_{i_1,\dots,i_k}(i) &:= (i - i_1 + 1)(i - i_1 - i_2 + 1)(i - i_1 - i_2 - i_3 + 1) \cdots 1, \\ I_{i_1,\dots,i_k}(a) &:= \int \cdots \int_{0 \le s_1 \le \cdots \le s_k \le T} a_{i_k}(s_k) \cdots a_{i_1}(s_1)\, ds_k \cdots ds_1. \end{aligned} \tag{1.2}$$

In general, qualitative analysis of this system is highly arduous because of the complexity of the involved equations and the absence of apparent recursive relations between them. The simplest and, in a certain statistical sense, the most frequently occurring centers, so-called *universal centers*, are defined by vanishing of all iterated integrals $I_{i_1,\dots,i_k}(a)$. Such centers admit an equivalent characterization in terms of the *tree composition condition* (see [**B2**, Corollary 1.12, Theorem 1.14]) as follows.

Suppose that image $\Gamma_{\tilde{A}}$ of the map

$$\tilde{A} := (\tilde{a}_1, \tilde{a}_2) : [0, T] \to \mathbb{R}^2 \quad \text{where } \tilde{a}_i(t) := \int_0^t a_i(s)\, ds, \quad i = 1, 2, \tag{1.3}$$

is Lipschitz triangulable. Then equation (1.1) has universal center on $[0, T]$ if and only if there are a finite metric tree $\mathcal{T}$ and continuous maps $\hat{A} : [0, T] \to \mathcal{T}$, $\hat{A}(0) = \hat{A}(T)$, and $p : \mathcal{T} \to \Gamma_{\tilde{A}}$ such that $\tilde{A} = p \circ \hat{A}$.

Recall that $\Gamma \Subset \mathbb{R}^2$ is a Lipschitz triangulable curve if there exist a finite subset $S \subset \Gamma$ and Lipschitz and locally bi-Lipschitz arcs $h_1, \dots, h_k : (0, 1) \to \mathbb{R}^2$ such that $\Gamma \setminus S = \bigsqcup_{1 \le i \le k} h_i(0, 1)$. The class of Lipschitz triangulable curves includes, in particular, compact curves in $\mathbb{R}^2$ admitting piecewise C^1 parameterization or images of non-constant analytic maps $[0, T] \to \mathbb{R}^2$ (see e.g. [**BY**, Example 5.1]).

It was proved in [**B6**, §3.1] that if $\tilde{A}$ is non-constant analytic, then tree $\mathcal{T}$ in the above characterization of universal centers of equation (1.1) is a closed interval in $\mathbb{R}$. In this case we say that $\tilde{A}$ satisfies a *composition condition*. In general, tree composition and composition conditions are not equivalent, i.e. there exist Lipschitz maps $[0, T] \to \mathbb{R}^2$ which factor through continuous maps into non-interval trees but do not factor through continuous maps into intervals (see [**BY**, §3]).

The famous *composition conjecture* states that all centers of equation (1.1) with polynomial coefficients are universal (equivalently, for all centers of such equations the corresponding maps $\tilde{A}$ satisfy composition conditions); see [**BlRY**, **BFY1**, **BFY2**, **BRY**, **CGM2**, **CGM3**, **GGL**, **MP**, **P1**, **P2**, **P3**, **PRY**] for recent developments in this area. This is false already for Abel equations with coefficients being trigonometric polynomials of degree three (see [**GGS**], while it is still true if the degree of such polynomials is at most two, see [**Bl**]).

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One associates with each connected Lipschitz triangulable curve $\Gamma \Subset \mathbb{R}^2$ containing the origin an (uncountable) family of equations (1.1) with $\Gamma_{\tilde{A}} \subset \Gamma$. Universal centers of this family admit a simple topological description. Specifically, equation (1.1) with $\Gamma_{\tilde{A}} \subset \Gamma$ has universal center on $[0, T]$ if and only if *path* $\tilde{A} : [0, T] \to \Gamma$ *is closed and contractible in* Γ. This is equivalent to the above formulated tree composition condition for $\tilde{A}$ (see [**B2**] for details). At present, there is no way to explicitly describe non-universal centers of Abel differential equations. However, there exist Lipschitz triangulable curves $\Gamma \Subset \mathbb{R}^2$ satisfying the property that whenever an equation (1.1) with $\Gamma_{\tilde{A}} \subset \Gamma$ has a center on $[0, T]$, this center must be universal. We call such curves Γ *universal*. The basic examples of universal curves are (containing $0 \in \mathbb{R}^2$) connected *rectangular curves* composed of finitely many intervals each parallel to one of the coordinate axes (see [**B4**]) or connected Lipschitz triangulable curves whose fundamental groups are either trivial or isomorphic to $\mathbb{Z}$ (see [**B2**]).

In the present paper we study analytic and geometric properties of universal curves. In particular, we show that the set of 'sufficiently smooth' universal curves is dense with respect to the Hausdorff metric on compact subsets of $\mathbb{R}^2$ in the space of all connected Lipschitz triangulable curves in $\mathbb{R}^2$ containing the origin.

The paper is organized as follows. In §2 we formulate the main results of the paper accompanied by some important open problems in the area. These results include an algebraic description of a universal curve in terms of a certain homomorphism of its fundamental group into the group of locally convergent invertible power series with product being the composition of series, explicit examples of universal curves, and approximation of Lipschitz triangulable curves by universal ones. Section 3 is devoted to proofs of our results.

2. *Formulation of the main results*

2.1. *A characterization of universal curves.* We recall some results presented in [**B2**]. Let us introduce operations $*$ and $^{-1}$ on the set $\mathcal{A}$ of pairs $a = (a_1, a_2)$ of coefficients of equations (1.1) by the formulas

$$((a_1, a_2) * (b_1, b_2))(t) := \begin{cases} (2b_1(2t), 2b_2(2t)) & \text{if } 0 \le t \le \dfrac{T}{2}, \\ (2a_1(2t - T), 2a_2(2t - T)) & \text{if } \dfrac{T}{2} \le t \le T, \end{cases}$$

$$((a_1, a_2)^{-1})(t) := (-a_1(T - t), -a_2(T - t)), \quad 0 \le t \le T.$$

It is well known that for $a, b \in \mathcal{A}$ iterated integrals $I_{i_1,\dots,i_k}$ (see (1.2)) satisfy

$$I_{i_1,\dots,i_k}(a * b) = I_{i_1,\dots,i_k}(a) + \sum_{j=1}^{k-1} I_{i_1,\dots,i_j}(a) \cdot I_{i_{j+1},\dots,i_k}(b) + I_{i_1,\dots,i_k}(b). \tag{2.1}$$

$$I_{i_1,\dots,i_k}(a^{-1}) = (-1)^k I_{i_1,\dots,i_k}(a). \tag{2.2}$$

For $a \in \mathcal{A}$ by $P(a)$ we denote the first return map of the corresponding equation (1.1) (associating with a sufficiently small initial value r the value of the corresponding solution of (1.1) at T). It is represented as a convergent in a neighborhood of 0 power series in the

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initial value r of the Abel equation

$$P(a)(r) = r + \sum_{i=1}^{\infty} c_i r^{i+1} \tag{2.3}$$

where c_i is the weighted sum of iterated integrals in (1.2).

By $G_c[[r]]$ we denote the group of convergent in neighborhoods of 0 power series of form (2.3) with product $\circ$ defined by the composition of series. Then for $a, b \in \mathcal{A}$ we have

$$P(a * b) = P(a) \circ P(b), \quad P(a^{-1}) = P(a)^{-1}. \tag{2.4}$$

An important question is as follows (cf. [**B3**, Question 2, p. 481]).

Problem 1. Is it true that the first return map $P : \mathcal{A} \to G_c[[r]]$ is an epimorphism (i.e. each series in $G_c[[r]]$ is the first return map of an Abel equation (1.1))?

Next, let $\Gamma \Subset \mathbb{R}^2$ be a connected Lipschitz triangulable curve containing the origin. According to the definition, Γ is a finite connected one-dimensional *CW*-complex and so it is homotopically equivalent to the wedge sum of finitely many circles. In particular, the fundamental group $\pi_1(\Gamma)$ of Γ with base point $0 \in \mathbb{R}^2$ is isomorphic, for some $m \in \mathbb{Z}_+$, to the free group with m generators F_m. (Here F_0 stands for the trivial group.)

By $\mathcal{A}_0(\Gamma)$ we denote the subset of elements $a \in \mathcal{A}$ such that $\tilde{A}(T) = 0$ and $\tilde{A}([0, T]) := \Gamma_{\tilde{A}} \subset \Gamma$ (recall that $\tilde{A}(t) := \int_0^t a(s)\, ds$, $t \in [0, T]$, see (1.3)). Since Γ admits a Lipschitz triangulation, the set $\mathcal{A}_0(\Gamma)$ is uncountable. (For instance, the derivative of a closed Lipschitz path passing through the origin and determined as the product (in the sense of algebraic topology, see e.g. [**Hu**]) of Lipschitz arcs forming the triangulation of Γ belongs to $\mathcal{A}_0(\Gamma)$.) Clearly, $\mathcal{A}_0(\Gamma)$ is closed with respect to operations $*$ and $^{-1}$. Next, for $a \in \mathcal{A}_0(\Gamma)$ the closed Lipschitz path $\tilde{A}$ represents an element $[\tilde{A}]$ of $\pi_1(\Gamma)$, and the correspondence $\mathcal{A}_0(\Gamma) \ni a \mapsto [\tilde{A}] \in \pi_1(\Gamma)$ determines an epimorphism of monoids $\Psi_\Gamma : \mathcal{A}_0(\Gamma) \to \pi_1(\Gamma)$:

$$\Psi_\Gamma(a * b) = \Psi_\Gamma(a) \cdot \Psi_\Gamma(b), \quad \Psi_\Gamma(a^{-1}) = \Psi_\Gamma(a)^{-1}, \quad a, b \in \mathcal{A}_0(\Gamma). \tag{2.5}$$

Since for each $\gamma \in \pi_1(\Gamma)$ the first return map P is constant on $\Psi_\Gamma^{-1}(\gamma)$ (see [**B2**]) there exists a homomorphism $\hat{P}_\Gamma : \pi_1(\Gamma) \to G_c[[r]]$ such that

$$P = \hat{P}_\Gamma \circ \Psi_\Gamma \quad \text{on } \mathcal{A}_0(\Gamma). \tag{2.6}$$

THEOREM 2.1. *Curve Γ is universal if and only if $\hat{P}_\Gamma$ is a monomorphism.*

In particular, this implies that the group $\hat{P}_\Gamma(\pi_1(\Gamma)) \subset G_c[[r]]$ is isomorphic to F_m.

Problem 2. Describe all possible finitely generated subgroups of $G_c[[r]]$.

For instance, one can show that a 'generic' finitely generated subgroup of $G_c[[r]]$ is free and each finitely generated solvable subgroup of $G_c[[r]]$ is isomorphic to the free abelian group $\mathbb{Z}^m$, where m can be any natural number. However, it is not even known whether the fundamental group of a compact Riemann surface of genus ≥ 2 can be realized as a subgroup of $G_c[[r]]$ (see e.g. [**B4**, **B5**] and references therein).

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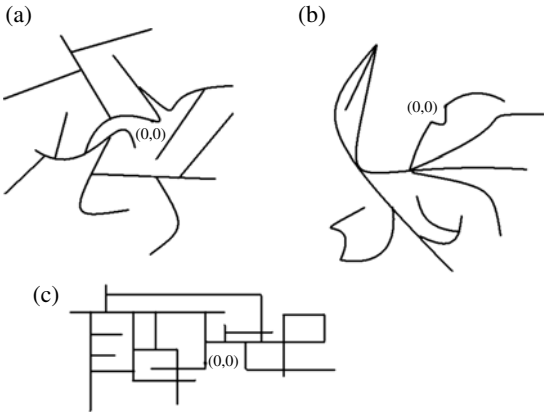


FIGURE 1. Examples of universal curves Γ: (a) $\pi_1(\Gamma)$ is trivial; (b) $\pi_1(\Gamma) \cong \mathbb{Z}$; (c) a rectangular curve.

2.2. *Explicit examples of universal curves.*

PROPOSITION 2.2. *Lipschitz triangulable curves $\Gamma \Subset \mathbb{R}^2$ containing the origin whose fundamental groups $\pi_1(\Gamma)$ are either isomorphic to F_0 (i.e. trivial) or isomorphic to F_1 ($\cong \mathbb{Z}$) (see Figure 1(a), (b)) are universal.*

A curve $\Gamma \Subset \mathbb{R}^2$ is said to be *rectangular* if it is the union of finitely many intervals each parallel to one of the coordinate axes (see Figure 1(c)). In [**B4**, Theorem 2.1] it was proved that *every connected rectangular curve Γ containing the origin is universal*. The proof is heavily reliant upon a deep result of [**Co**] on the structure of a certain subgroup of automorphisms of $\mathbb{C}$. We exploit the result of [**B4**] in the proof of Theorem 2.7 but now we use its corollary [**B5**, Theorem 2.4] to describe other explicit examples of universal curves.

Suppose P_1, P_2, Q_1, Q_2 are real polynomials without constant terms such that coefficients of all of them together are algebraically independent over $\mathbb{Q}$. Consider polynomial paths $A_i := (P_i, Q_i) : [0, 1] \to \mathbb{R}^2$, $i = 1, 2$. Next, define the set of points

$$S := \{(m \cdot A_1(1), n \cdot A_2(1)) \in \mathbb{R}^2 : (m, n) \in \mathbb{Z}^2\}.$$

We join each pair of points $v_1, v_2 \in S$ such that $v_2 - v_1 = A_1(1)$ by curve $X_{v_1,v_2} := \{v_1 + A_1(t) : t \in [0, 1]\}$ and each pair of points $w_1, w_2 \in S$ such that $w_2 - w_1 = A_2(1)$ by curve $Y_{w_1,w_2} := \{w_1 + A_2(t) : t \in [0, 1]\}$ thus obtaining a curvilinear lattice $\mathcal{L}$ in $\mathbb{R}^2$. Assume that polynomials P_i, Q_i were chosen so that maps $A_i : [0, 1] \to \mathbb{R}^2$ are embeddings and all possible curves X_{v_1,v_2} and Y_{w_1,w_2} (referred to as *edges* of $\mathcal{L}$) are either disjoint or intersect by one of the points v_1, v_2, w_1, w_2 only (see Figure 2). Then the following result holds.

PROPOSITION 2.3. *Each connected curve $\Gamma \Subset \mathcal{L}$ containing the origin composed of finitely many edges of $\mathcal{L}$ is universal.*

The following problem is a particular case of [**B3**, Question 3, p. 484].

Problem 3. Is it true that any connected piecewise linear curve $\Gamma \Subset \mathbb{R}^2$ containing the origin is universal?

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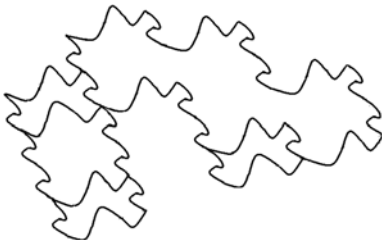


FIGURE 2. An example of a curve Γ composed of finitely many edges of a lattice $\mathcal{L}$.

Remark 2.4. Some piecewise linear curves are universal due to Proposition 2.3. In general, one can easily show that the first return map of the Abel equation $dv/dt = av^2 + bv^3$, $t \in [0, 1]$, $a, b \in \mathbb{R}$, $|a| + |b| \neq 0$, is

$$P_{a,b}(r) := \begin{cases} \dfrac{1}{\psi_{a,b}^{-1}(a^2 + \psi_{a,b}(\frac{1}{r}))} & \text{if } a \neq 0, \\ \dfrac{r}{\sqrt{1 - 2br^2}} & \text{if } a = 0, \end{cases}$$

where $\psi_{a,b}(x) := -ax + b \ln |ax + b|$, $x \in \mathbb{R}$. (Note that $\psi_{a,b}$ is invertible near $\pm\infty$ and $P_{a,b}$ admits a series expansion near 0.)

Then Problem 3 is equivalent to the following one (cf. [**B4**] and the main result of [**Co**]).

Problem 3′. Is it true that a composition $P_{a_1,b_1} \circ P_{a_2,b_2} \circ \cdots \circ P_{a_n,b_n}$ such that all vectors (a_i, b_i) and (a_{i+1}, b_{i+1}), $1 \leq i \leq n - 1$, $n \in \mathbb{N}$, are non-collinear is never the identity map?

2.3. *Lipschitz embeddings of universal curves.* By $\|\cdot\|_2$ we denote the Euclidean norm on $\mathbb{R}^2$. Let $K \Subset \mathbb{R}^2$ be a compact subset containing the origin. A continuous map $F : K \to \mathbb{R}^2$ is called a *Lipschitz embedding* of K if there exist constants $c_1, c_2 > 0$ such that

$$c_1\|x - y\|_2 \leq \|F(x) - F(y)\|_2 \leq c_2\|x - y\|_2 \quad \text{for all } x, y \in K. \tag{2.7}$$

The set of Lipschitz embeddings of K sending 0 to 0 will be denoted by $\mathcal{EL}_0(K)$. Clearly, if $F \in \mathcal{EL}_0(K)$, then F^{-1} exists and belongs to $\mathcal{EL}_0(F(K))$. By $I \in \mathcal{EL}_0(K)$ we denote the identity map. One easily shows that if $F \in \mathcal{EL}_0(K)$ satisfies (2.7), then $F + \lambda I \in \mathcal{EL}_0(K)$ for all $\lambda \in (-c_1, c_1)$.

THEOREM 2.5. *Let $\Gamma \Subset \mathbb{R}^2$ be a universal curve and $F \in \mathcal{EL}_0(\Gamma)$ satisfy (2.7) for some $c_1, c_2 > 0$. Then there is an at most countable subset S of the open interval $(-c_1, c_1)$ such that for all $\lambda \in (-c_1, c_1)\backslash S$ the Lipschitz triangulable curves $(F + \lambda I)(\Gamma)$ are universal.*

In Remark 3.1 we give an explicit description of set S.

Let $\mathcal{L}_0(K)$ be the Banach space of Lipschitz maps $F : K \to \mathbb{R}^2$ such that $F(0) = 0$ equipped with norm

$$\|F\| := \sup_{x \neq y} \frac{\|F(x) - F(y)\|_2}{\|x - y\|_2}.$$

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One easily shows that if $F \in \mathcal{EL}_0(K)$ satisfies (2.7), then $F + G \in \mathcal{EL}_0(K)$ for all $G \in \mathcal{L}_0(K)$ such that $\|G\| < c_1$. In particular, $\mathcal{EL}_0(K)$ is a non-empty open subset of $\mathcal{L}_0(K)$.

Suppose $\Gamma \Subset \mathbb{R}^2$ is a universal curve. Let $U(\Gamma) \subset \mathcal{EL}_0(\Gamma)$ be the subset of Lipschitz embeddings $F : \Gamma \hookrightarrow \mathbb{R}^2$, $F(0) = 0$, such that curves $F(\Gamma)$ are universal. We equip $U(\Gamma)$ with topology induced from $\mathcal{EL}_0(\Gamma)$. The following result shows that $U(\Gamma)$ is a 'massive' dense subset of $\mathcal{EL}_0(\Gamma)$.

THEOREM 2.6. *There exists an at most countable family of closed nowhere dense subsets $S_i \subset \mathcal{EL}_0(\Gamma)$ such that $U(\Gamma) = \mathcal{EL}_0(\Gamma)\backslash(\bigcup_i S_i)$.*

In view of this result, the following problem seems to be plausible.

Problem 4. Let $\Gamma \Subset \mathbb{R}^2$ be a connected Lipschitz triangulable curve containing the origin. Is it true that there exists an $F \in \mathcal{EL}_0(\Gamma)$ such that curve $F(\Gamma)$ is universal?

2.4. *Approximation of Lipschitz triangulable curves by universal ones.* In what follows, $\mathcal{H}^1$ denotes the Hausdorff 1-measure on $\mathbb{R}^2$ and $d_{\mathcal{H}}$ the Hausdorff metric on the set of compact subsets of $\mathbb{R}^2$. Let $\Gamma \Subset \mathbb{R}^2$ be a Lipschitz triangulable curve containing the origin. We say that $x \in \Gamma$ has *order* k (written $\mathrm{ord}(x) = k$) if for all sufficiently small open disks D_x centered at x, $\Gamma\backslash\{x\}$ has k connected components in $D_x\backslash\{x\}$. (Note that since Γ is triangulable, $\mathrm{ord}(x)$ is correctly defined.) By $S_\Gamma \subset \Gamma$ we denote the (finite) set of points of order $\neq 2$. Our main result is as follows.

THEOREM 2.7. *Suppose Γ is piecewise linear. Then there exists a sequence $\{\Gamma_i\}_{i\in\mathbb{N}}$ of Lipschitz triangulable curves containing the origin such that:*

(a) *each Γ_i is a universal curve with $\pi_1(\Gamma_i) \cong \pi_1(\Gamma)$;*

(b) $\mathrm{ord}(x) \leq 3$ *for all* $x \in \Gamma_i$, $i \in \mathbb{N}$;

(c) *each $\Gamma_i\backslash S_{\Gamma_i}$ is a closed C^∞ submanifold of $\mathbb{R}^2\backslash S_{\Gamma_i}$;*

(d) $\lim_{i\to\infty} \mathcal{H}^1(\Gamma_i) = \mathcal{H}^1(\Gamma)$ *and* $\lim_{i\to\infty} d_{\mathcal{H}}(\Gamma_i, \Gamma) = 0$.

If (d) is valid for a sequence of curves $\{\Gamma_i\}_{i\in\mathbb{N}}$ and a curve Γ, then we say that $\{\Gamma_i\}_{i\in\mathbb{N}}$ converges to Γ with respect to $\mathcal{H}^1$-measure and metric $d_{\mathcal{H}}$. Since each Lipschitz triangulable curve can be approximated with respect to $\mathcal{H}^1$-measure and metric $d_{\mathcal{H}}$ by a sequence of piecewise linear curves (because each Lipschitz arc $[0, 1] \to \mathbb{R}^2$ admits such approximation), as an immediate corollary of Theorem 2.7 we obtain the following.

COROLLARY 2.8. *Suppose $\Gamma \Subset \mathbb{R}^2$ is a Lipschitz triangulable curve containing the origin. Then there exists a sequence $\{\Gamma_i\}_{i\in\mathbb{N}}$ of universal curves converging to Γ with respect to $\mathcal{H}^1$-measure and metric $d_{\mathcal{H}}$ such that each $\Gamma_i\backslash S_{\Gamma_i}$ is a closed C^∞ submanifold of $\mathbb{R}^2\backslash S_{\Gamma_i}$.*

The following problem is related to the composition conjecture for Abel equations with polynomial coefficients (see §1).

Problem 5. Do there exist universal curves $\Gamma \Subset \mathbb{R}^2$ with $\pi_1(\Gamma) \cong F_m$, $m \geq 2$, which are images of analytic maps $[0, T] \to \mathbb{R}^2$? The same question applies for images of polynomial maps.

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With regard to this problem we mention that universal centers of Abel equations with real analytic coefficients can be easily described in terms of vanishing of finitely many moments in the coefficients of the equations; see the algorithm in the Introduction of [**B7**].

2.5. *Remarks on non-universal curves.* While the structure of universal centers of equation (1.1) is well understood, non-universal centers of this equation are of obscure nature requiring further scrutiny. If equation (1.1) has a non-universal center on $[0, T]$ and the image $\Gamma_{\tilde{A}}$ of the corresponding map $\tilde{A}$ is Lipschitz triangulable, then $\Gamma_{\tilde{A}}$ is a non-universal curve; in particular, homomorphism $\hat{P}_{\Gamma_{\tilde{A}}} : \pi_1(\Gamma_{\tilde{A}}) \to G_c[[r]]$ (see (2.6)) has a non-trivial kernel (enclosing the minimal normal subgroup containing the element $[\tilde{A}] \in \pi_1(\Gamma_{\tilde{A}})$ represented by path $\tilde{A}$). This implies that for every $b = (b_1, b_2) \in \mathcal{A}_0(\Gamma_{\tilde{A}})$ such that the closed path $\tilde{B} := \int_0^{\cdot} b(s)\, ds : [0, T] \to \Gamma_{\tilde{A}}$ is homotopic to a non-trivial element of Ker $\hat{P}_{\Gamma_{\tilde{A}}}$ the corresponding Abel equation $dv/dt = b_1 v^2 + b_2 v^3$ has a non-universal center on $[0, T]$. Abel equations having non-universal centers can be obtained, e.g. by the application of the Cherkas transformation to certain elements of the Lotka–Volterra or Darboux components of the set of centers of planar polynomial vector fields of degree two, see [**Bl**, §5.3]. In these cases, coefficients a_1 and a_2 of the obtained Abel equations are trigonometric polynomials of degrees two and four, respectively; hence the corresponding curves $\Gamma_{\tilde{A}}$ are Lipschitz triangulable (and non-universal). Recently, in [**GGS**] the authors presented an example of an Abel equation with coefficients being trigonometric polynomials of degree three having non-universal center (while each center of an Abel trigonometric equation with coefficients of degree at most two is universal, see [**Bl**]). We mention also the following simple property: if $\Gamma \subset \tilde{\Gamma} \Subset \mathbb{R}^2$ is a pair of connected Lipschitz triangulable curves containing the origin and Γ is non-universal, then $\tilde{\Gamma}$ is non-universal as well.

3. *Proofs*

Proof of Theorem 2.1. First, assume that Γ is universal. Suppose $\hat{P}(g) = 1 \in G_c[[r]]$ for some $g \in \pi_1(\Gamma)$. Let $a_g \in \mathcal{A}_0(\Gamma)$ be such that $\Psi_\Gamma(a_g) = g$. Then due to (2.6), $P(a_g) = 1$, i.e. equation (1.1) corresponding to a_g has center on $[0, T]$. Since Γ is universal, this center is universal. Then, by [**B2**, Theorem 1.14], the closed Lipschitz path $\tilde{A}_g := \int_0^{\cdot} a_g(s)\, ds : [0, T] \to \Gamma$ is contractible, i.e. $g = \Psi_\Gamma(a_g) = 1 \in \pi_1(\Gamma)$. This shows that $\hat{P}_\Gamma$ is a monomorphism.

Conversely, suppose that $\hat{P}_\Gamma$ is a monomorphism. Let $a \in \mathcal{A}_0(\Gamma)$ be such that the corresponding equation (1.1) has center on $[0, T]$. Due to (2.6) and our assumption, this implies that $\Psi_\Gamma(a) = 1 \in \pi_1(\Gamma)$. Then the closed Lipschitz path $\tilde{A} : [0, T]$ is contractible. In turn, by the results of [**B2**], the corresponding center is universal. So Γ is universal by our definition. □

Proof of Proposition 2.2. If $\pi_1(\Gamma) = F_0$, then $\hat{P}_\Gamma : F_0 \to G_c[[r]]$ is obviously a monomorphism; thus such curve Γ is universal by Theorem 2.1.

Next, if $\pi_1(\Gamma) \cong F_1$ but Γ is not universal, then due to Theorem 2.1 $\hat{P}_\Gamma$ maps $\pi_1(\Gamma)$ to $1 \in G_c[[r]]$. Let g be a generator of $\pi_1(\Gamma)$ and $a_g = (a_{g1}, a_{g2}) \in \mathcal{A}_0(\Gamma)$ be such that $\Psi(a_g) = g$. Then, according to our hypothesis and (2.6), equation (1.1) corresponding to

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a_g has center on $[0, T]$. In particular, from (1.2) with $i = 3$ (see also notation (1.3)) we obtain

$$3 \cdot \int_0^T \tilde{a}_{g1}(s) \cdot a_{g2}(s)\, ds + 2 \cdot \int_0^T \tilde{a}_{g2}(s) \cdot a_{g1}(s)\, ds = 0. \tag{3.1}$$

Also, since $\tilde{a}_{gi}(T) = 0$, $i = 1, 2$, we have

$$\int_0^T \tilde{a}_{g1}(s) \cdot a_{g2}(s)\, ds + \int_0^T \tilde{a}_{g2}(s) \cdot a_{g1}(s)\, ds = (\tilde{a}_1(t) \cdot \tilde{a}_2(t))|_0^T = 0.$$

Thus (3.1) is equivalent to

$$\int_0^T \tilde{a}_{g1}(s) \cdot a_{g2}(s)\, ds = 0. \tag{3.2}$$

The latter can be rewritten as the contour integral of 1-form $x\, dy$ over the closed path $\hat{A}_g := \int_0^{\cdot} a_g(s)\, ds : [0, T] \to \Gamma$:

$$\int_{\hat{A}_g} x\, dy = 0.$$

Since $\pi_1(\Gamma) \cong \mathbb{Z}$ and Γ is Lipschitz triangulable, by the Jordan theorem $\mathbb{R}^2 \backslash \Gamma$ contains only one bounded component D_Γ homeomorphic to the open unit disk. In turn, by our definition $\hat{A}_g$ represents the generator $g \in \pi_1(\Gamma)$ and so by the Green formula we obtain

$$0 = \left| \int_{\hat{A}_g} x\, dy \right| = \left| \int_{D_\Gamma} dx\, dy \right| = \text{Area}\,(D) \neq 0,$$

a contradiction which shows that $\hat{P}_\Gamma(g) \neq 1$, i.e., by Theorem 2.1, Γ is a universal curve. □

Proof of Proposition 2.3. Let $\Gamma \Subset \mathcal{L}$ be a curve satisfying conditions of the proposition. According to Theorem 2.1 we have to prove that the homomorphism $\hat{P}_\Gamma : \pi_1(\Gamma) \to G_c[[r]]$ is injective. Consider elements $a_i := A_1' \in \mathcal{A}$, $i = 1, 2$ (here $'$ stands for the derivative of a path). According to [**B5**, Theorem 2.4, Example 2.5] the first return maps $P(a_i) \in G_c[[r]]$, $i = 1, 2$, generate a subgroup isomorphic to F_2. Further, each element $g \in \pi_1(\Gamma)$ is represented by a path $A_g : [0, 1] \to \Gamma$ which is the product (in the sense of algebraic topology) of finitely many paths A_i and their inverses. Assume that $\hat{P}_\Gamma(g) = 1 \in G_c[[r]]$. Then due to (2.6) $P(A_g') = 1 \in G_c[[r]]$. Thus, the Abel equation (1.1) corresponding to the pair of coefficients A_g' has center on $[0, 1]$. But by our definition A_g' is a finite $*$-product of elements a_i and their inverses. Then due to [**B5**, Theorem 2.4] this center is universal. This implies (see [**B2**]) that path A_g is closed and contractible inside its image. But A_g has image in Γ; hence A_g is contractible in Γ. This is equivalent to $g = 1 \in \pi_1(\Gamma)$. Therefore, $\hat{P}_\Gamma$ is a monomorphism and, by Theorem 2.1, Γ is a universal curve. □

Proof of Theorem 2.5. Let $\ell_g : [0, T] \to \Gamma$ be a closed Lipschitz path representing an element $g \in \pi_1(\Gamma)$. Then for each $\lambda \in (-c_1, c_1)$ the countable family of paths $\{\ell_{g\lambda}\}_{g \in \pi_1(\Gamma)}$, $\ell_{g\lambda} := (F + \lambda I)(\ell_g)$, represents all elements of the fundamental group

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$\pi_1(\Gamma_\lambda)$ ($\cong \pi_1(\Gamma)$), $\Gamma_\lambda := (F + \lambda I)(\Gamma)$. According to formula (1.2) the first return map of the Abel equation corresponding to the pair of coefficients $\ell'_{g\lambda} \in \mathcal{A}$ has the form

$$P(\ell'_{g\lambda})(r) = r + \sum_{j=1}^{\infty} c_{gj}(\lambda) r^{j+1},$$

where c_{gj} is a real polynomial in λ of degree at most j ($1 \le j < \infty$).

Since Γ is universal, the first return map $P(\ell'_g) \subset G_c[[r]]$ is not the identity map (see Theorem 2.1). Thus there exists $j_g \in \mathbb{N}$ such that $c_{gj_g}(0) \neq 0$. This implies that c_{gj_g} is a not identically zero polynomial in λ of degree at most j_g. By S_{j_g} we denote the set of its zeros in $(-c_1, c_1)$. Clearly, card $S_{j_g} \le j_g$. We define

$$S := \bigcup_{g \in \pi_1(\Gamma)} S_{j_g}.$$

Since $\pi_1(\Gamma)$ is countable, the set $S \subset (-c_1, c_1)$ is at most countable. By definition, for each $t \in (-c_1, c_1) \backslash S$ and $g \in \pi_1(\Gamma)$ we have $P(\ell'_{g\lambda}) \neq \mathrm{id}$. This shows that the kernel of the corresponding homomorphism $\hat{P}_{\Gamma_\lambda} : \pi_1(\Gamma_\lambda) \to G_c[[r]]$ (see (2.6)) is trivial. Therefore, by Theorem 2.1, Γ_λ is a universal curve.

The proof of the theorem is complete. □

Remark 3.1. It is easily seen that coefficients of all polynomials c_{gj} belong to the minimal numerical field $\mathbb{F} \subset \mathbb{R}$ containing all iterated integrals in ℓ'_g and ℓ'_{g0}. Therefore, to make sure that all $c_{gj}(\lambda_0) \neq 0$ it suffices to choose λ_0 to be a transcendental number over $\mathbb{F}$. Since $\mathbb{F}$ is a countable set, its algebraic closure $\mathrm{cl}_a(\mathbb{F})$ is countable as well. Thus we may define the required set S of the theorem as $S := (-c_1, c_1) \cap \mathrm{cl}_a(\mathbb{F})$.

Proof of Theorem 2.6. We retain notation from the proof of Theorem 2.5. For each $\ell_g \in [0, T] \to \Gamma$, $g \in \pi_1(\Gamma)$, and $G \in \mathcal{L}_0(\Gamma)$ consider the first return map

$$P((G \circ \ell_g)') = r + \sum_{j=1}^{\infty} c_{gj}(G) r^{j+1}.$$

Due to equation (1.2) and the Rademacher theorem (on almost everywhere differentiability of Lipschitz maps), the coefficients c_{gj} are real polynomials of degrees at most j on the Banach space $\mathcal{L}_0(\Gamma)$. Since Γ is universal, $P((I \circ \ell_g)') \neq \mathrm{id}$. Therefore, there exists $j_g \in \mathbb{N}$ such that $c_{gj_g} \not\equiv 0$. In turn, the set of zeros $\mathcal{Z}_{gj_g}$ of c_{gj_g} is a closed nowhere dense subset of $\mathcal{L}_0(\Gamma)$. We have

$$U(\Gamma) = \mathcal{E}\mathcal{L}_0(\Gamma) \Bigg\backslash \Bigg(\bigcup_{g \in \pi_1(\Gamma)} \Bigg(\bigcap_{\{j : c_{gj} \not\equiv 0\}} \mathcal{Z}_{gj} \Bigg) \Bigg).$$

Since $\pi_1(\Gamma)$ is countable and $\mathcal{E}\mathcal{L}_0(\Gamma)$ is an open subset of $\mathcal{L}_0(\Gamma)$, the previous identity yields the required statement. □

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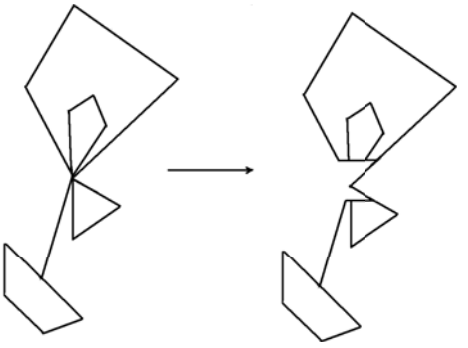


FIGURE 3. Example of curves Γ and Γ_v.

Proof of Theorem 2.7. The proof consists of two parts. In the first part we approximate Γ by a sequence of piecewise linear curves satisfying all the above conditions but (c). Then we 'smooth the corners' of curves of the constructed sequence to get the required sequence $\{\Gamma_i\}$.

(I) We consider Γ as a connected finite plane graph whose vertex set contains S_Γ and the origin. By definition, each vertex not in S_Γ has degree two. First, let us show how to reduce the approximation problem to certain plane graphs whose vertices have degrees ≤ 3.

Making a rotation of Γ about $0 \in \mathbb{R}^2$ by a small angle, without loss of generality we may assume that no edges of Γ are parallel to the x-axis. Suppose $v \in S_\Gamma$ has degree ≥ 3. Let $E(v)$ be the set of edges of Γ emanating from v. Let $\ell_\pm := \{(x, y) \in \mathbb{R}^2 : y = v_\pm, \operatorname{sgn}(v_\pm - v_y) = \pm 1\}$ be two lines parallel to the x-axis with y coordinates v_+, v_- larger and smaller than that of v (denoted by v_y). For sufficiently small $|v_\pm - v_y|$ the union of these lines intersect all edges in $E(v)$. Let $I_\pm \subset \ell_\pm$ be closed intervals with endpoints in $E(v)$ whose union contains all points in $(\ell_+ \sqcup \ell_-) \cap E(v)$. Consider a piecewise linear curve Γ_v obtained from Γ by removing all parts of edges of $E(v)$ between ℓ_- and ℓ_+ except for one part for each choice of sign $+$ or $-$ and then adding intervals I_+ and I_- instead (cf. Figure 3).

One can easily check that, for all sufficiently small $|v_\pm - v_y|$, $\pi_1(\Gamma_v) \cong \pi_1(\Gamma)$. (Indeed, in this case, according to the Seifert–van Kampen theorem the fundamental group of the union U of Γ and of the minimal convex set containing I_+, I_- and all parts of edges of $E(v)$ between ℓ_- and ℓ_+ is isomorphic to $\pi_1(\Gamma)$. Also, Γ_v is the deformation retract of U, i.e. $\pi_1(\Gamma_v) \cong \pi_1(U) \cong \pi_1(\Gamma)$ as required.) Moreover, in a neighborhood of v all vertices of Γ_v have degrees ≤ 3, and if a vertex there has degree three, then two of three edges adjacent to it lie on a straight line; all other vertices of Γ_v are the same as for Γ, and the family of such curves Γ_v converges to Γ with respect to $\mathcal{H}^1$-measure and metric $d_{\mathcal{H}}$ (cf. (d) of the theorem) as $|v_\pm - v_y|$ and the above angle of rotation of Γ about the origin tend to zero.

Subsequently applying a similar procedure to other vertices of Γ with degrees ≥ 3, after a finite number of steps we obtain a piecewise linear curve of the same homotopy type as Γ with vertices of degrees ≤ 3 satisfying the same property as Γ_v in a neighborhood of v. Moreover, there exists a sequence of such curves converging to Γ with respect to $\mathcal{H}^1$-measure and metric $d_{\mathcal{H}}$.

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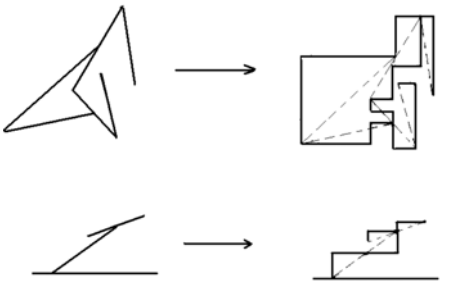


FIGURE 4. Examples of curves Γ and Γ_r.

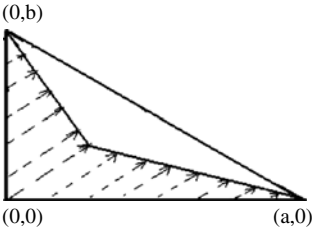


FIGURE 5. Isotopy between the union of catheti and the hypotenuse.

Thus, from now on, without loss of generality we will assume that *all vertices of* Γ *have degrees* ≤ 3*, and if a vertex has degree three, then two of three edges adjacent to it lie on a straight line*.

As a next step, we construct a rectangular piecewise linear curve Γ_r by replacing edges of Γ by unions of catheti of right triangles with pairwise non-intersecting interiors of their sides and with hypotenuses lying on edges of Γ (see Figure 4). Also, we transfer to Γ_r edges of Γ parallel to one of the coordinate axes. To show that the required Γ_r exists, it suffices to construct the corresponding right triangles locally in neighborhoods of vertices of Γ. This is possible since, due to the above property of Γ, each closed quadrant with respect to the coordinate axes centered at a vertex of Γ contains in a small open neighborhood of this vertex no more than two edges of Γ.

Since in a right triangle the union of catheti can be isotopically deformed to the hypotenuse, the curve Γ_r can be isotopically deformed to Γ and, in particular, $\pi_1(\Gamma_r) \cong \pi_1(\Gamma)$. We make use of a specific isotopy between Γ_r and Γ defined in each right triangle T with hypotenuse lying on an edge of Γ as follows. Using a suitable affine transformation L we map T to the right triangle with coordinates $(0, 0)$, $(a, 0)$ and $(0, b)$, $a, b > 0$. The hypotenuse in this triangle is given by equation $y = (b/a)(a - x)$ for $x \in [0, a]$. We deform the union of catheti $K_1 := \{(x, y) : x \in [0, a],\ y = 0\}$ and $K_2 := \{(x, y) : x = 0,\ y \in [0, b]\}$ along the direction of the line $y = (b/a)x$ so that at time $t \in [0, 1]$ cathetus K_1 transfers to the line $y = (tb(a - x)/a(2 - t))$, $x \in [(a/2)t, a]$, and K_2 to the line $x = (ta(b - y)/b(2 - t))$, $y \in [(b/2)t, b]$ (see Figure 5). Then we apply the inverse affine transformation L^{-1} to place the deformed break line back inside the original triangle T.

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Without loss of generality, we will assume that the vertex set of Γ is a subset of the vertex set of Γ_r. By $\Phi : \Gamma_r \times [0, 1] \to \mathbb{R}^2$ we denote the above described isotopy between Γ_r and Γ which deforms catheti of each right triangle with hypotenuse an edge of Γ toward the hypotenuse by the above formulas (modulo an affine transformation). We set $\Gamma_t := \Phi(t, \Gamma_r)$.

Further, each *piecewise linear path* $\gamma : [0, T] \to \Gamma_r$ (i.e. a path composed of finitely many paths moving along each edge with constant velocities) can be transformed using Φ to a piecewise linear path $\gamma_1 : [0, T] \to \Gamma$. We set $\gamma_t := \Phi(\gamma, t)$, $t \in (0, 1)$. Each γ_t is a piecewise linear path in Γ_t. Let $\gamma_t' \in \mathcal{A}$ (see §2) be the velocity of path γ_t. We require the following result.

LEMMA 3.2. *Each iterated integral $I_{i_1,\dots,i_k}(\gamma_t')$ is a real polynomial in t of degree $\le k$.*

Proof. By definition, γ_t is a product (in the sense of algebraic topology) of finitely many linear paths with images in edges of γ_t. In turn, γ_t' is a $*$-product of velocities of these linear paths. Due to formulas for iterated integrals of $*$-products, see (2.1), (2.2), it suffices to prove the result for velocities of linear paths appearing as factors of γ_t in the above product. Then by the definition of γ_t we must consider the following cases.

(1) γ_t does not depend on t. This means that the image of γ_1 belongs to a vertical or a horizontal interval. In this case the statement of the lemma is obvious.

(2) The image of γ_t belongs to one of the intervals inside a straight triangle with hypotenuse being an edge of Γ (cf. Figure 5). Making an affine transformation of this triangle (which clearly does not affect the final result) without loss of generality we may assume that we are in the position of Figure 5. Suppose for definiteness that the image of γ_t belongs to the line $y = (tb(a - x)/a(2 - t))$, $x \in [(a/2)t, a]$ (the lower line inside the triangle in Figure 5) and the endpoints of γ_t have x-coordinates $(a/2)t \le s_0 < s_1 \le a$. By the definition of isotopy Φ in this case we have

$$\gamma_t(s) := \Phi(\gamma_0(s), t) = \left(\frac{at}{2} + \frac{(2 - t)\gamma_0(s)}{2}, \frac{bt}{2} - \frac{bt\gamma_0(s)}{2a} \right), \quad s \in [0, T].$$

Here $\gamma_0 : [0, T] \to K_1 \subset \mathbb{R}$ is a linear path with image in the cathetus K_1 and endpoints $s_0, s_1 \in K_1$. Assuming that $\gamma_0' = v_0$ ($\in \mathbb{R}$), we obtain

$$\gamma_t'(s) := \left(\frac{(2 - t)v_0}{2}, -\frac{btv_0}{2a} \right), \quad s \in [0, T].$$

Since γ_t' depends linearly on t, we have (see (1.2)) that each iterated integral $I_{i_1,\dots,i_k}(\gamma_t')$ is a real polynomial in t of degree at most k as required.

All other choices of γ_t are treated similarly. □

Using Lemma 3.2 we prove the following.

PROPOSITION 3.3. *There exists an at most countable subset $Z \subset [0, 1]$ such that for each $t \in [0, 1] \setminus Z$ the piecewise linear curve Γ_t is universal.*

Proof. Since $\Gamma_r := \Gamma_0$ is a rectangular curve, it is universal, see [**B4**]. Let $\ell_1, \dots, \ell_m$ be closed rectangular paths in Γ_r containing the origin representing generators of $\pi_1(\Gamma_r) \cong F_m$ (without loss of generality we assume that $m \ge 1$). Each element $g \in \pi_1(\Gamma_r)$

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is represented by a closed rectangular path ℓ_g which is the finite product of paths ℓ_i or their inverses. Further, the isotopy Φ induces isomorphisms $\pi_1(\Gamma_t) \cong \pi_1(\Gamma_r)$. In particular, closed paths $(\ell_g)_t : [0, T] \to \Gamma_t$, $g \in \pi_1(\Gamma_r)$, are mutually distinct and represent all elements of $\pi_1(\Gamma_t)$. According to Lemma 3.2 and formula (1.2) (containing expressions for coefficients of series expansions of the first return maps of Abel equations),

$$P((\ell_g)'_t)(r) = r + \sum_{j=1}^{\infty} c_{gj}(t) r^{j+1},$$

where c_{gj} is a real polynomial in t of degree at most j $(1 \le j < \infty)$.

Since Γ_r is universal, the first return map $P(\ell'_g) \subset G_c[[r]]$ is not the identity map (see Theorem 2.1). Thus there exists $j_g \in \mathbb{N}$ such that $c_{gj_g}(0) \neq 0$. This implies that c_{gj_g} is a not identically zero polynomial in t of degree at most j_g. By Z_{j_g} we denote the set of its zeros in [0, 1]. Clearly, card $Z_{j_g} \le j_g$. We define

$$Z := \bigcup_{g \in \pi_1(\Gamma_r)} Z_{j_g}.$$

Since $\pi_1(\Gamma_r)$ is countable, the set $Z \subset [0, 1]$ is at most countable. By definition, for each $t \in [0, 1] \backslash Z$ and $g \in \pi_1(\Gamma_r)$ we have $P((\ell_g)'_t) \neq \mathrm{id}$. This shows that the kernel of the corresponding homomorphism $\hat{P}_{\Gamma_t} : \pi_1(\Gamma_t) \to G_c[[r]]$ (see (2.6)) is trivial. Therefore, by Theorem 2.1, Γ_t is a universal curve.

The proof of the proposition is complete. □

Now, we choose a sequence $\{t_i\}_{i \in \mathbb{N}} \subset [0, 1] \backslash Z$ converging to 1. Then, according to our construction and the previous proposition, piecewise linear curves Γ_{t_i} are universal and isotopic to Γ and the sequence of such curves converges to Γ as $t_i \to 1$ with respect to $\mathcal{H}^1$-measure and metric $d_{\mathcal{H}}$ (see (d) of Theorem 2.7). This finishes part (I) of the proof.

(II) In this part, given a piecewise linear universal curve Γ we approximate it (with respect to $\mathcal{H}^1$-measure and metric $d_{\mathcal{H}}$) by a sequence of isotopic universal curves having the same sets of points of order $\neq 2$ and C^∞ outside of these sets.

As in part (I), we isotopically deform Γ to smooth its corners. Specifically, let $\mathcal{V}_2$ be the set of vertices of Γ of degree two whose adjacent edges do not lie on a straight line. Fix a family of mutually disjoint open connected neighborhoods U_v of $v \in \mathcal{V}_2$ such that each U_v contains no vertices other than v. We simultaneously deform parts of Γ in all U_v, $v \in \mathcal{V}_2$, preserving points of Γ outside of these neighborhoods. To this end, we map each U_v by a suitable affine transformation L_v onto the angle $R_0 := \{(x, y) \in \mathbb{R}^2 : y = |x|,\ x \in [-2, 2]\}$ (see Figure 6). We construct an isotopy for this angle which preserves parts of the edges situated outside the open disk of radius $\sqrt{2}$ centered at the origin and then, applying the inverse affine transformation L_v^{-1}, attach the deformed piece of the curve to Γ.

For this purpose consider an auxiliary C^∞ function

$$\rho(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

Then the isotopy $\phi : R_0 \times [0, 1] \to \mathbb{R}^2$ is given by the formula $\phi((x, y), t) := (x, th(x/t))$, $y := |x|$, $x \in \mathbb{R}$, where

$$h(x) := \frac{1}{2} + \frac{(|x| - 1/2)\rho(2|x| - 1)}{\rho(2|x| - 1) + \rho(2 - 2|x|)}. \tag{3.3}$$

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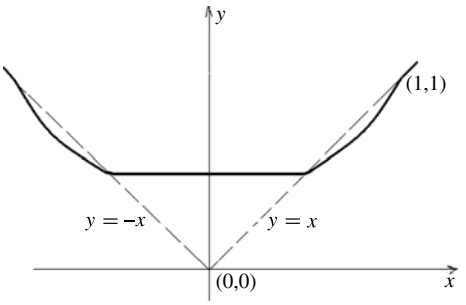


FIGURE 6. Smoothing the corner of the angle formed by the graph $y = |x|$.

(By the definition, h is a C^∞ function on $\mathbb{R}$ equal to $\frac{1}{2}$ if $|x| \le \frac{1}{2}$ and $|x|$ if $|x| \ge 1$.)

By $\Phi : \Gamma \times [0, 1] \to \mathbb{R}^2$ we denote the (above described) isotopy given by the formula $\Phi(z, t) := L_v^{-1}(\phi(L_v(z), t))$, $(z, t) \in U_v \times [0, 1]$, for all $v \in \mathcal{V}_2$. (Without loss of generality we assume that Φ is correctly defined, i.e. that U_v were chosen so small that, for each $t \in [0, 1]$, images $\Phi(U_v, t)$ for distinct v are pairwise disjoint.) We set $\Gamma_t := \Phi(\Gamma, t)$ and follow the lines of the proof given in step (I). Specifically, for a closed piecewise linear path $\gamma : [0, T] \to \Gamma$ representing a non-trivial element of $\pi_1(\Gamma)$ we define $\gamma_t := \Phi(\gamma, t) : [0, T] \to \Gamma_t$. Then, for this choice of γ_t, we require an analog of Lemma 3.2.

LEMMA 3.4. *Each iterated integral $I_{i_1,\dots,i_k}(\gamma_t')$ is a real polynomial in t of degree $\le k$.*

Proof. Adding additional vertices if necessary, without loss of generality we may assume that each edge of Γ contains at most one vertex from $\mathcal{V}_2$.

Since iterated integrals are constant on the set of derivatives of closed paths representing the same element of $\pi_1(\Gamma)$, one can choose γ so that it is a finite product of linear paths whose images are edges of Γ. Hence, due to formulas (2.1) and (2.2), it suffices to prove the result for a linear path γ whose image is an edge e of Γ. If e joins two vertices not in $\mathcal{V}_2$, then $\gamma_t = \gamma$ for all t and the required result is obvious. Otherwise, making a suitable transformation L_v, without loss of generality we may assume that γ is a linear map onto a subinterval of the line $y = x$ with endpoints $(0, 0)$ and (a, a) for some $a > 2$. Applying anew formulas (2.1) and (2.2) to a proper factorization of γ, we see that it suffices to prove the lemma for γ whose image is interval $[(0, 0), (1, 1)]$. In this case making a change of parameter we may assume that $\gamma(s) := (s, s)$, $s \in [0, 1]$. Hence, $\gamma_t(s) = (s, th(s/t))$, $s \in [0, 1]$. We present γ_t as product of paths $\gamma_{1t}(s) := \gamma_t(ts)$, $s \in [0, 1]$, and $\gamma_{2t}(s) := \gamma_t(t + (1 - t)s)$, $s \in [0, 1]$. Therefore, due to formulas (2.1), (2.2) it suffices to prove the result for iterated integrals of derivatives of paths γ_{it}, $i = 1, 2$. We have

$$\gamma_{1t}'(s) := (t, th'(s)), \quad s \in [0, 1].$$

Thus (see (1.2)),

$$I_{i_1,\dots,i_k}(\gamma_{1t}') = t^k \cdot I_{i_1,\dots,i_k}(\gamma_{11}') \quad (= \text{const} \cdot t^k)$$

is a polynomial in t of degree at most k.

In turn,

$$\gamma_{2t}'(s) := \left(1 - t, (1 - t) \cdot h'\left(1 + \frac{1 - t}{t}s\right)\right) = (1 - t, 1 - t), \quad s \in [0, 1],$$

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since $h(x) = x$ for $x \geq 1$. Hence,

$$I_{i_1,\dots,i_k}(\gamma'_{2t}) = \frac{(1-t)^k}{k!}$$

is also a polynomial in t of degree at most k.

This completes the proof of the lemma. □

Repeating word-for-word the arguments of the proof of Proposition 3.3 (now based on Lemma 3.4) we get in our case the following proposition.

PROPOSITION 3.5. *There exists an at most countable subset $Z \subset [0, 1]$ such that for each $t \in [0, 1]\backslash Z$ the curve Γ_t (which is C^∞ outside the set S_Γ) is universal.*

Choosing a sequence $\{t_i\}_{i\in\mathbb{N}} \subset [0, 1]\backslash Z$ converging to 0 we obtain that curves Γ_{t_i}, having the same sets S_{Γ_i} and C^∞ outside of these sets, are universal and isotopic to Γ and that the sequence of such curves converges to Γ as $t_i \to 0$ with respect to $\mathcal{H}^1$-measure and metric $d_{\mathcal{H}}$.

This finishes the proof of part (II) and hence of the theorem. □

Remark 3.6. According to our construction, the sequence of universal curves $\{\Gamma_i\}_{i\in\mathbb{N}}$ converging to Γ has the property that for each $x \in S_{\Gamma_i}$ of order three there is an open disk D_x centered at x such that $\Gamma_i \cap (D_x\backslash\{x\})$ is the union of three open intervals. Using the method of part (II) of the proof of Theorem 2.7 one can additionally smooth curves Γ_i to obtain new curves $\widetilde{\Gamma}_i$ satisfying conditions (a)–(d) of the theorem and such that for each $x \in S_{\widetilde{\Gamma}_i}$, $\mathrm{ord}(x) = 3$, in an open disk D_x centered at x, $\widetilde{\Gamma}_i$ is the union of a connected C^∞ curve and its tangent at x having infinite order contact at x. Moreover, one can easily show that such a curve $\widetilde{\Gamma}_i$ can be obtained as the image of a C^∞ map $\tilde{A} : [0, T] \to \mathbb{R}^2$ which corresponds to an Abel equation (1.1) with C^∞ coefficients. We leave the details to the reader.

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REFERENCES

[**A1**] M. A. M. Alwash. On the composition conjectures. *Electron. J. Differential Equations* **2003**(69) (2003), 1–4.

[**A2**] M. A. M. Alwash. The composition conjecture for Abel equation. *Expo. Math.* **27** (2009), 241–250.

[**A3**] M. A. M. Alwash. Polynomial differential equations with piecewise linear coefficients. *Differ. Equ. Dyn. Syst.* **19**(3) (2011), 267–281.

[**A4**] M. A. M. Alwash. Composition conditions for two-dimensional polynomial systems. *Differ. Equ. Appl.* **5**(1) (2013), 1–12.

[**AL**] M. A. M. Alwash and N. G. Lloyd. Non-autonomous equations related to polynomial two-dimensional systems. *Proc. Roy. Soc. Edinburgh Sect. A* **105** (1987), 129–152.

[**B1**] A. Brudnyi. An explicit expression for the first return map in the center problem. *J. Differential Equations* **206** (2004), 306–314.

https://doi.org/10.1017/etds.2014.104 Published online by Cambridge University Press

[**B2**] A. Brudnyi. On the center problem for ordinary differential equations. *Amer. J. Math.* **128**(2) (2006), 419–451.
[**B3**] A. Brudnyi. Formal paths, iterated integrals and the center problem for ordinary differential equations. *Bull. Sci. Math.* **132** (2008), 455–485.
[**B4**] A. Brudnyi. Center problem for ODEs with coefficients generating the group of rectangular paths. *C. R. Math. Acad. Sci. Soc. R. Can.* **31**(2) (2009), 33–44.
[**B5**] A. Brudnyi. Free subgroups of the group of formal power series and the center problem for ODEs. *C. R. Math. Acad. Sci. Soc. R. Can.* **31**(4) (2009), 97–106.
[**B6**] A. Brudnyi. Composition conditions for classes of analytic functions. *Nonlinearity* **25** (2012), 3197–3209.
[**B7**] A. Brudnyi. Moments finiteness problem and characterization of universal centers of ODEs with analytic coefficients. *Nonlinearity* **27** (2014), 1611–1631.
[**BFY1**] M. Briskin, J.-P. Francoise and Y. Yomdin. The Bautin ideal of the Abel equation. *Nonlinearity* **11** (1998), 41–53.
[**BFY2**] M. Briskin, J.-P. Francoise and Y. Yomdin. Center conditions, compositions of polynomials and moments on algebraic curves. *Ergod. Th. & Dynam. Sys.* **19** (1999), 1201–1220.
[**Bl**] M. Blinov. Center and composition conditions for Abel equation. *PhD Thesis*, Weizmann Institute of Science, 2002.
[**BlRY**] M. Blinov, N. Roytvarf and Y. Yomdin. Center and moment conditions for Abel equation with rational coefficients. *Funct. Differ. Equ.* **10**(1–2) (2003), 95–106.
[**BRY**] M. Briskin, N. Roytvarf and Y. Yomdin. Center conditions at infinity for Abel differential equations. *Ann. of Math. (2)* **172**(1) (2010), 437–483.
[**BY**] A. Brudnyi and Y. Yomdin. Tree composition condition and moments vanishing. *Nonlinearity* **23** (2010), 1651–1673.
[**C**] C. Christopher. Abel equations: composition conjectures and the model problem. *Bull. Lond. Math. Soc.* **32** (2000), 332–338.
[**CGM1**] A. Cima, A. Gasull and F. Mãnosas. Centers for trigonometric Abel equations. *Qual. Theory Dyn. Syst.* **11** (2012), 19–37.
[**CGM2**] A. Cima, A. Gasull and F. Mãnosas. A simple solution of some composition conjectures for Abel equations. *J. Math. Anal. Appl.* **398** (2013), 477–486.
[**CGM3**] A. Cima, A. Gasull and F. Mãnosas. An explicit bound of the number of vanishing double moments forcing composition. *J. Differential Equations* **255**(3) (2013), 339–350.
[**Co**] S. D. Cohen. The group of translations and positive rational powers is free. *Q. J. Math.* **46**(1) (1995), 21–93.
[**GGL**] J. Giné, M. Grau and J. Llibre. Universal centers and composition conditions. *Proc. Lond. Math. Soc.* **106**(3) (2013), 481–507.
[**GGS**] J. Giné, M. Grau and X. Santallusia. Universal centers in the cubic trigonometric Abel equation. *Electron. J. Qual. Theory Differ. Equ.*(1) (2014), 1–7.
[**Hu**] S.-T. Hu. *Homotopy Theory*. Academic Press, New York, 1959.
[**I**] Yu. Il'yashenko. Centennial history of Hilbert's 16th problem. *Bull. Amer. Math. Soc. (N.S.)* **39**(3) (2002), 301–354.
[**MP**] M. Muzychuk and F. Pakovich. Solution of the polynomial moment problem. *Proc. Lond. Math. Soc.* **99**(3) (2009), 633–657.
[**P1**] F. Pakovich. On the polynomial moment problem. *Math. Res. Lett.* **10**(2–3) (2003), 401–410.
[**P2**] F. Pakovich. On rational functions orthogonal to all powers of a given rational function on a curve. *Moscow Math. J.* **13**(4) (2013), 693–731.
[**P3**] F. Pakovich. Solution of the parametric center problem for the Abel differential equation. *Preprint*, 2014, arXiv:1407.0150.
[**PRY**] F. Pakovich, N. Roytvarf and Y. Yomdin. Cauchy type integrals of algebraic functions. *Israel J. Math.* **144** (2004), 221–291.

https://doi.org/10.1017/etds.2014.104 Published online by Cambridge University Press