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An example of a factor map without a saturated compensation function

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Abstract. Let (X, S), (Y, T) be topological dynamical systems and $\pi : X \to Y$ a factor map. A function $F \in C(X)$ is a compensation function if $P(F + \phi \circ \pi) = P(\phi)$ for all $\phi \in C(Y)$. We present an example of a factor map $\pi : X \to Y$ between subshifts of finite type X, Y that does not have a saturated compensation function and an example of a non-Markovian factor map with a saturated compensation function. Also we provide a necessary and sufficient condition for a certain type of factor map to have a saturated compensation function.

1. *Introduction*

Let (X, σ_X) and (Y, σ_Y) be subshifts of finite type and $\pi : X \to Y$ a factor map. Consider, for $\alpha > 0$, the weighted entropy function

$$\phi_\alpha(\mu) = h_\mu(\sigma_X) + \alpha h_{\pi\mu}(\sigma_Y)$$

and the set K_α of all ergodic shift-invariant measures μ on X which maximize the function ϕ_α. Gatzouras and Peres [**3**] asked whether there is always a unique measure which achieves $\sup_\mu \phi_\alpha(\mu)$ (which will be called Question 1), and, if so, what the dynamical properties of any such maximizing measure would be. In [**5**] a partial answer under certain conditions was given, that is, the existence of a certain kind of compensation function was confined (see later). First the following result characterizes K_α, $\alpha \geq 0$, in terms of equilibrium states, in the case where there is a certain type of compensation function.

THEOREM 1.1. [**5**] *Let X, Y be subshifts and $\pi : X \to Y$ a factor map. Suppose $G \in C(Y)$ and $G \circ \pi \in C(X)$ is a compensation function for π. Then for each $\alpha \geq 0$, K_α is the set of equilibrium states of $[\alpha/(\alpha + 1)]G \circ \pi \in C(X)$.*

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In answering Question 1, an interesting question would be whether or not every factor map has a compensation function of type $G \circ \pi \in C(X)$ with $G \in C(Y)$, in which case we say $G \circ \pi \in C(X)$ is a *saturated compensation function* (or a compensation function in $C(Y)$). When X, Y are irreducible subshifts of finite type, there always exists a compensation function in $C(X)$ [**6**]. However, it is not necessarily true in the case of $C(Y)$. We provide an example of a factor map $\pi : X \to Y$ between subshifts of finite type X, Y that does not have a saturated compensation function (Example 2.2). We also show a non-Markovian factor map with a saturated compensation function (Example 2.1). An infinite-to-one factor map $\pi : X \to Y$ between two subshifts of finite type X, Y is said to be *Markovian* if it lifts each Markov measure on Y to a Markov measure on X. Despite the fact that the concept of a compensation function originated from the study of Markovian maps [**1**], in order to have a saturated compensation function, it is not necessary for a factor map to be Markovian. We generalize the ideas used for these examples to provide a necessary and sufficient condition for a certain type of factor map between subshifts of finite type to have a saturated compensation function (Theorems 3.3 and 3.4).

When X is a (one-step) topologically mixing subshift of finite type, a (linear dense) subspace $\mathcal{F}(X)$ of $C(X)$ is defined (Definition 4.1) and if $f \in \mathcal{F}(X)$, then f has a unique equilibrium state [**6**]. When combined with these facts, Theorem 1.1 yields a partial answer to Question 1 as follows.

COROLLARY 1.2. [**5**] *Let X, Y be topologically mixing subshifts of finite type and $\pi : X \to Y$ a factor map. Suppose $G \in \mathcal{F}(Y)$ and $G \circ \pi \in \mathcal{F}(X)$ is a compensation function for π. Then the following hold:*

(1) *for each $\alpha \geq 0$, K_α consists of a unique element which is a (unique) equilibrium state of $[\alpha/(\alpha+1)]G \circ \pi \in C(X)$ and it is mixing;*
(2) *in particular, if $G \in \mathcal{F}(Y)$ is locally constant, then the unique measure in K_α, $\alpha \geq 0$, is a Markov measure, hence it is mixing and Bernoulli.*

Therefore when looking for a complete answer to Question 1, we might be interested in studying saturated compensation functions whose constant multiples have a unique equilibrium state. In fact we extend Corollary 1.2 to the case where $G \in H(Y)$, i.e. $G \in C(Y)$ has a homogeneous measure. By presenting an example of a non-coboundary function $f \in H(X)$ (i.e. $f \not\sim 0$) for which there exists $K > 0$ such that for any $n \geq 1$ and for any $x \in X$, $|\sigma_n f(x)| \leq K$ (Example 4.1), we show that $\mathcal{F}(X) \subsetneq H(X)$.

We now collect some standard definitions and notation. If (X, S) is a topological dynamical system, i.e. X is a compact metric space and $S : X \to X$ is a homeomorphism, then the set of all S-invariant Borel probability measures on X is denoted by $M(X)$, and for $f \in C(X)$, $(S_n f)(x)$ denotes $\sum_{i=0}^{n-1} f(S^i x)$ for each $n \geq 1$. For each $n \geq 1$, $P_n(X) = \{x \in X \mid S^n x = x\}$. Two functions $f, g \in C(X)$ are said to be *cohomologous* if there exists $\phi \in C(X)$ with $f - g = \phi \circ S - \phi$, in which case we write $f \sim g$. The *pressure function* $P_X : C(X) \to \mathbb{R}$ may be defined by $P_X(f) = \sup\{h(\mu) + \int f d\mu \mid \mu \in M(X)\}$ for $f \in C(X)$. A measure $\mu \in M(X)$ is called an *equilibrium state* of a 'potential function' $f \in C(X)$ if $h(\mu) + \int f d\mu = P_X(f)$. We are mainly considering the cases where X is a (two-sided) subshift, that is, a shift-invariant closed subset of $\{1, 2, \ldots, k\}^{\mathbb{Z}}$ for some $k \geq 1$, and S is the shift transformation on X, which will be denoted by σ_X or simply by σ.

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A metric d on a subshift X is given by $d(x, y) = 1/(n+1)$ where $n = \inf\{|i| \mid i \in \mathbb{Z}$ and $x_i \neq y_i\}$. A subshift X is called a subshift of finite type (SFT) if there exists a finite forbidden block system $\mathcal{F}$ for X, i.e. $X = \{x \in \{1, 2, \ldots, k\}^{\mathbb{Z}} \mid$ no block $\omega \in \mathcal{F}$ occurs in $x\}$. We say that X is an N-step SFT if there exists a finite forbidden block system $\mathcal{F}$ for X which only contains blocks of length less than or equal to $N+1$. It is known that any SFT is topologically conjugate to a one-step SFT (see [**4**]). Throughout this work we assume that an SFT is a one-step irreducible subshift of finite type. When (X, σ_X) is a subshift of finite type, a *Markov measure* for σ_X is defined to be a measure $\mu \in M(X)$ for which the information function I_μ has a continuous version depending only on a fixed finite number of coordinates of $x \in X$ or, alternatively, a measure which is a (unique) equilibrium state of a locally constant function on X. Then a Markov measure on X is fully supported on X.

Let (X, S) and (Y, T) be topological dynamical systems. A *factor map* $\pi : X \to Y$ is a continuous surjective map such that $\pi \circ S = T \circ \pi$. Then there is a natural induced (onto) map (which will also be denoted by π) from $M(X)$ to $M(Y)$. In [**6**], Walters defines explicitly the concept of a compensation function, which was first considered by Boyle and Tuncel [**1**], and he relates compensation functions to lifting T-invariant measures to S-invariant measures. A function $F \in C(X)$ is called a *compensation function* for (S, T, π) if

$$P_X(F + \phi \circ \pi) = P_Y(\phi) \quad \text{for all } \phi \in C(Y).$$

For the various results and the details, see [**1**] and [**6**]. In particular, among the many results related to saturated compensation functions (which mainly concern measures of maximal relative entropy), we mention the following which is a key to this work and in general is related to the existence of a saturated compensation function.

THEOREM 1.3. [**6**] *Let X, Y be subshifts and $\pi : X \to Y$ a factor map. For each $n \geq 1$ and $y \in Y$, let $D_n(y)$ consist of one point from each non-empty set $\pi^{-1}\{y\} \cap {}_0[x_0 x_1 \cdots x_{n-1}]_{n-1}$. Let $G \in C(Y)$. Then $G \circ \pi \in C(X)$ is a compensation function for π if and only if*

$$\int \limsup_{n\to\infty} \frac{1}{n}[\ln(\exp(\sigma_n G(y)) \cdot |D_n(y)|)]\, dv = 0 \quad \textit{for all } v \in M(Y). \tag{1.1}$$

2. *Factor maps with a singleton clump*

Let X and Y be subshifts of finite type. A factor map $\pi : X \to Y$ will be called a *factor map with a singleton clump* if there is a symbol a of Y such that $|\pi^{-1}\{a\}| = 1$. We start with an example of a non-Markovian factor map which falls within this class.

Example 2.1. Let $X \subset \{1, 2, 3, 4\}^{\mathbb{Z}}$ and $Y \subset \{1, 2\}^{\mathbb{Z}}$ be the subshifts of finite type determined by allowing the transitions marked on Figure 2.1. The map π is the one-block map determined by mapping 1 to 1, and 2, 3, 4 to 2. For each $y \in Y$, let $m(y) = \inf\{i - j \mid i \geq 0,\ j \leq -1$ and $y_i = y_j = 1\}$ if it exists, and ∞ otherwise. Define $G \in C(Y)$ by

$$G(y) = \begin{cases} -(\ln 2)/m(y), & \text{if } m(y) \text{ is odd} \\ 0, & \text{otherwise.} \end{cases}$$

Then $G \circ \pi \in C(X)$ is a saturated compensation function for π (see Remark 3.3).

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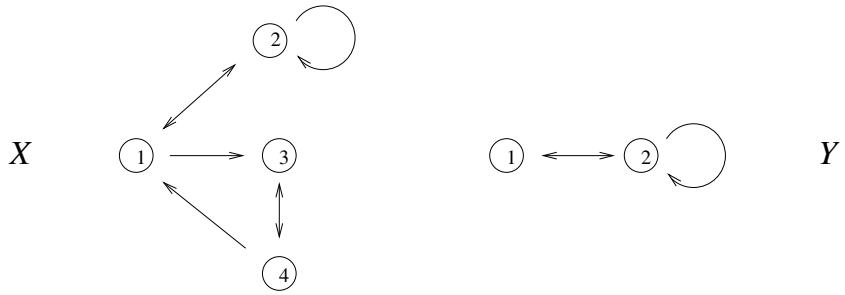


FIGURE 2.1.

In order to see that π is not Markovian, it suffices to show that there is a Markov measure on Y which does not lift to any Markov measure on X, because if one Markov measure on Y lifts to a Markov measure on X, then so must all other Markov measures on Y (and the map is Markovian) [**1**]. Let $0 < b < 1$. Let $Q = (Q_{ij})$ be a matrix defined by

$$Q = \begin{bmatrix} 0 & 1 \\ b & 1-b \end{bmatrix}$$

and ν the Markov measure on Y determined by Q. Suppose μ is an N-step Markov measure on X, $N \geq 1$ and $\pi\mu = \nu$. We may assume that N is even. Let $P = (P(B))$ be a stochastic matrix compatible with the transition matrix of the Nth higher block representation of X, where B represents each $(N+1)$-cylinder set in X, and P determine μ. Let $p = (p(B'))$ be the probability vector such that $pP = p$, where B' represents each N-cylinder set in X. Noticing that the vector $q = (b+1)^{-1}(b, 1)$ satisfies $qQ = q$, we have

$$\nu([12^n1]) = q_1 Q_{12}(Q_{22})^{n-1} Q_{21} = b^2(1-b)^{n-1}/(b+1) \tag{2.1}$$

for any $n \geq 1$. Also for any $n \geq N$,

$$\mu([12^n1]) = p(12^{N-1})P(12^N)(P(2^{N+1}))^{n-N} P(2^N1) \tag{2.2}$$

and $\mu([12^{N-1}1]) = p(12^{N-1})P(12^{N-1}1)$. Since for any odd integer $n \geq 1$, $\nu([12^n1]) = \mu(\pi^{-1}[12^n1]) = \mu([12^n1])$, it follows that $P(2^{N+1}) = 1 - b$ and so $P(2^N1) = 1 - P(2^{N+1}) = b$. Compare (2.1) and (2.2), getting

$$b(1-b)^{N-1}/(b+1) = p(12^{N-1})P(12^N) \tag{2.3}$$

and with $n = N - 1$,

$$b^2(1-b)^{N-2}/(b+1) = p(12^{N-1})P(12^{N-1}1). \tag{2.4}$$

Adding (2.3) to (2.4), we obtain $b(1-b)^{N-2}/(b+1) = p(12^{N-1})$, so that $\nu([12^{N-1}]) = \mu([12^{N-1}])$. Meanwhile $\pi^{-1}[12^{N-1}] = [12^{N-1}] \cup [1(34)^{N/2-1}3] \subset X$, which implies $\nu([12^{N-1}]) = \mu([12^{N-1}]) + \mu([1(34)^{N/2-1}3])$. So $\mu([1(34)^{N/2-1}3]) = 0$, which contradicts the fact that μ is a Markov measure and so fully supported on X. Thus ν lifts to no Markov measure on X.

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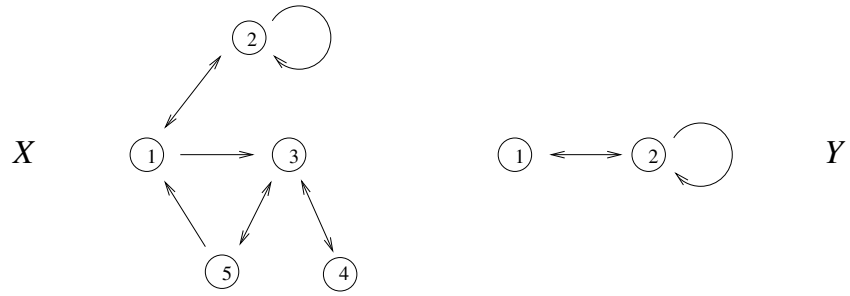


FIGURE 2.2.

Example 2.2. Let $X \subset \{1,2,3,4,5\}^{\mathbb{Z}}$ and $Y \subset \{1,2\}^{\mathbb{Z}}$ be the subshifts of finite type determined by allowing the transitions marked on Figure 2.2. The map π is the one-block map determined by mapping 1 to 1, and 2, 3, 4, 5 to 2. Then it will be shown later that there is no saturated compensation function for π (see Remark 3.1).

3. *Main theorems*

In this section we present a necessary and sufficient condition for a factor map with a singleton clump to have a saturated compensation function. Let $X \subset \{1,\dots,r\}^{\mathbb{Z}}$ and $Y \subset \{1,\dots,s\}^{\mathbb{Z}}$ be subshifts of finite type and $\pi : X \to Y$ a factor map such that $\pi^{-1}\{1\} = \{1\}$ and Y contains the golden mean shift Y_0 (i.e. the subshift of $\{1,2\}^{\mathbb{Z}}$ with a forbidden block system $\{11\}$). Throughout the work, we assume that π is a one-block code and for a given n-cylinder set C in Y, $n \geq 1$, we define $|\pi^{-1}C|$ to be the number of n-cylinder sets in X that project to C. For each $n \geq 1$ and for each $y \in Y$, let $D_n(y)$ consist of one point from each non-empty set $\pi^{-1}\{y\} \cap {}_0[x_0x_1\cdots x_{n-1}]_{n-1}$.

LEMMA 3.1. *For $2^\infty \in Y$, there is $C > 0$ such that for each $n \geq 1$,*

$$|D_n(2^\infty)| \geq C \cdot |\pi^{-1}[2^n]|.$$

LEMMA 3.2. *Suppose $G \in C(Y)$ and $G \circ \pi \in C(X)$ is a compensation function for π. Let $m \geq 1$ and $0 \leq p \leq m$. Define $y^* \in P_{m+1}(Y)$ by $y^* = \cdots 12^m 12^{m-p}.2^p 12^m 1\cdots$. Then $\sigma_m G(y^*) = -\ln|\pi^{-1}[12^m 1]|$. Also $G(2^\infty) = -S_0$ (see (3.1) for the notation).*

We assume these results and prove the following.

THEOREM 3.3. *Let X, Y and π defined as before. Suppose*

$$S_0 \equiv \limsup_{n\to\infty} \frac{1}{n} \ln|\pi^{-1}[2^n]| > \liminf_{n\to\infty} \frac{1}{n} \ln|\pi^{-1}[12^n 1]| \equiv Z_1. \tag{3.1}$$

Then there is no saturated compensation function for π.

Remark 3.1. It is always true that

$$S_0 \geq Z_0 \equiv \limsup_{n\to\infty} \frac{1}{n} \ln|\pi^{-1}[12^n 1]| \geq Z_1, \tag{3.2}$$

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since for each $n \geq 1$, $|\pi^{-1}[2^n]| \geq |\pi^{-1}[12^n1]|$. Suppose the limit

$$\lim_{n\to\infty} \frac{1}{n} \ln|\pi^{-1}[12^n1]|$$

does not exist. This, combined with (3.2), implies (3.1). In Example 2.2, it can be shown that for each $n \geq 1$, $|\pi^{-1}[12^n1]| = 2^{n/2-1} + 1$ if n is even and 1 if n is odd. Thus there is no saturated compensation function for π.

Proof of Theorem 3.3. Suppose $G \circ \pi \in C(X)$ is a saturated compensation function for π. To show a contradiction, first note that there exist $\{m_k\}_{k=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$ such that $m_k \to \infty$, $n_k \to \infty$ as $k \to \infty$, and

$$Z_0 = \lim_{n_k\to\infty} \frac{\ln|\pi^{-1}[12^{n_k}1]|}{n_k} \geq \lim_{m_k\to\infty} \frac{\ln|\pi^{-1}[12^{m_k}1]|}{m_k} = Z_1. \tag{3.3}$$

By taking a subsequence of $\{m_k\}_{k=1}^{\infty}$, we may assume that there is $C > 0$ such that for all $k \geq 1$, $m_k \geq C \cdot n_k$ and $C > \max\{7, 4S_0/(S_0 - Z_1)\}$. For a given $k \geq 1$, let $n_k^* = [(n_k + 1)/2]$ and define $x^k \in P_{m_k+1}(Y)$ and $y^k \in P_{n_k+1}(Y)$ by

$$x^k = \ldots 12^{m_k} 12^{m_k - n_k^*}.2^{n_k^*} 12^{m_k} 1 \ldots, \quad \text{and} \quad y^k = \ldots 12^{n_k} 12^{n_k - n_k^*}.2^{n_k^*} 12^{n_k} 1 \ldots .$$

Note that for each $i = 0, 1, \ldots, n_k$, $d(\sigma^i x^k, \sigma^i y^k) \leq 1/n_k^* \leq 2/n_k$. Since G is uniformly continuous, for a given $\epsilon > 0$, there is $\delta > 0$ such that whenever $d(x, y) < \delta$ for $x, y \in Y$, $|G(x) - G(y)| < \epsilon$. Choose ϵ so that $0 < \epsilon < (S_0 - Z_1)/3$. Let $N > 2/\delta$. Fix $k \geq 1$ with $n_k \geq N$. For each $i = 0, 1, \ldots, n_k$, we have $d(\sigma^i x^k, \sigma^i y^k) \leq 2/n_k \leq 2/N < \delta$, so that $|G(\sigma^i x^k) - G(\sigma^i y^k)| < \epsilon$. For each $i = n_k + 1, \ldots, m_k$, we have $d(\sigma^i x^k, 2^{\infty}) \leq 2/n_k < \delta$, so that $|G(\sigma^i x^k) - G(2^{\infty})| = |G(\sigma^i x^k) + S_0| < \epsilon$. Let $D = |\sigma_{m_k+1} G(x^k) - \sigma_{n_k+1} G(y^k)|$. Then

$$\begin{aligned} D &\geq \left| \sum_{i=n_k+1}^{m_k} G(\sigma^i x^k) \right| - \sum_{i=0}^{n_k} |G(\sigma^i x^k) - G(\sigma^i y^k)| \\ &\geq (m_k - n_k)(S_0 - \epsilon) - (n_k + 1)\epsilon. \end{aligned} \tag{3.4}$$

Meanwhile using (3.3), we have $M > 0$ such that for any $k \geq M$, $n_k \geq N$ and

$$\begin{aligned} (Z_1 - \epsilon)m_k &< \ln|\pi^{-1}[12^{m_k}1]| < (Z_1 + \epsilon)m_k, \\ (Z_0 - \epsilon)n_k &< \ln|\pi^{-1}[12^{n_k}1]| < (Z_0 + \epsilon)n_k. \end{aligned}$$

Choose $k \geq M$. Since from Lemma 3.2, $D = |\ln|\pi^{-1}[12^{m_k}1]| - \ln|\pi^{-1}[12^{n_k}1]||$, we have

$$D < \max\{(Z_1 + \epsilon)m_k - (Z_0 - \epsilon)n_k, (Z_0 + \epsilon)n_k - (Z_1 - \epsilon)m_k\}. \tag{3.5}$$

To compare (3.4) and (3.5), first note that

$$\begin{aligned} &(m_k - n_k)(S_0 - \epsilon) - (n_k + 1)\epsilon - (Z_1 + \epsilon)m_k + (Z_0 - \epsilon)n_k \\ &\quad = (S_0 - Z_1 - 2\epsilon)m_k - (S_0 - Z_0 + \epsilon)n_k - \epsilon \geq P \cdot n_k - \epsilon \end{aligned} \tag{3.6}$$

where $P = C \cdot (S_0 - Z_1 - 2\epsilon) - (S_0 - Z_0 + \epsilon) > 0$, since

$$\frac{S_0 - Z_0 + \epsilon}{S_0 - Z_1 - 2\epsilon} \leq \frac{4S_0/3}{(S_0 - Z_1)/3} < C.$$

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Similarly

$$(m_k - n_k)(S_0 - \epsilon) - (n_k + 1)\epsilon - (Z_0 + \epsilon)n_k + (Z_1 - \epsilon)m_k \geq Q \cdot n_k - \epsilon \tag{3.7}$$

where $Q = C \cdot (S_0 + Z_1 - \epsilon) - (S_0 + Z_0 + \epsilon) > 0$, since

$$\frac{S_0 + Z_0 + \epsilon}{S_0 + Z_1 - 2\epsilon} \leq \frac{2S_0 + \epsilon}{S_0 - 2\epsilon} \leq \frac{7S_0/3}{S_0/3} < C.$$

For any $k \geq M$ with $n_k > \max\{\epsilon/P, \epsilon/Q\}$, the first terms in (3.6) and (3.7) both are positive, which contradicts (3.4) and (3.5). So there is no saturated compensation function for π. □

Now we give the proofs of the lemmas.

Proof of Lemma 3.1. We may assume that $U \equiv \pi^{-1}\{2\} = \{2, \ldots, p+1\}$ for some $p < r$. For each $n \geq 1$, for each $i, j = 2, \ldots, p+1$, let $C_{ij}^n = \#\{w = w_1 \cdots w_n \mid w \text{ is a block in } X \text{ such that } w_1 = i, w_n = j, \text{ and } \pi(w) = 2^n\}$. Let $E, F \subset U$ be defined by

$$E = \{j \in U \mid \text{there exists } M_j \geq 1 \text{ such that for all } n \geq M_j, i \in U, C_{ij}^n = 0\}$$

and

$$F = \{i \in U \mid \text{there exists } N_i \geq 1 \text{ such that for all } n \geq N_i, j \in U, C_{ij}^n = 0\}.$$

Let $A = U \setminus (E \cup F)$. Let $M = \sup\{M_j \mid j \in E\}$ if $E \neq \emptyset$ and $M = 0$ otherwise, and let $N = \sup\{N_i \mid i \in F\}$ if $F \neq \emptyset$ and $N = 0$ otherwise. Fix $n \geq M + N + 1$ and let $l = n - M - N \geq 1$. Let $w = uw_1 \cdots w_l v \in \pi^{-1}[2^n]$, where $|u| = M$ and $|v| = N$. Then for each $j = 1, \ldots, l$, $w_j \in A$. So for each $k \geq 1$, there exist y^k and z^k such that $|y^k| = |z^k| = k$, $\pi(y^k) = \pi(z^k) = 2^k$ and $y^k w_1$, $w_l z^k$ are allowable in X. Thus $\bar{w} \equiv w_1 \cdots w_l \in D_l(2^\infty)$. For such an $\bar{w}$, there exist at most p^M u's and p^N v's such that $w = u\bar{w}v \in \pi^{-1}[2^n]$. Hence

$$p^M \cdot p^N \cdot |D_{n-M-N}(2^\infty)| \geq |\pi^{-1}[2^n]|.$$

Note that $|\pi^{-1}[2^n]| \geq |\pi^{-1}[2^{n-M-N}]|$. Thus for any n large enough,

$$|D_n(2^\infty)| \geq p^{M+N} \cdot |\pi^{-1}[2^n]|,$$

which completes the proof. □

Remark 3.2. In the case where X is a transitive subshift of finite type, it can be similarly shown that with $y^+ = \cdots 1.2^\infty \in Y$, there is $C^+ > 0$ such that for any $i \geq 1$ and $n \geq 1$,

$$|D_n(\sigma^i y^+)| \geq C^+ \cdot |\pi^{-1}[2^n]|.$$

Proof of Lemma 3.2. Let $\nu = \delta_{2^\infty} \in M(Y)$. Using Lemma 3.1 and the fact that for each $n \geq 1$, $|\pi^{-1}[2^n]| \geq |D_n(2^\infty)|$, we get

$$\limsup_{n\to\infty} \frac{1}{n} \ln|D_n(2^\infty)| = \limsup_{n\to\infty} \frac{1}{n} \ln|\pi^{-1}[2^n]| = S_0.$$

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Since for each $n \geq 1$, $\sigma_n G(2^\infty) = n \cdot G(2^\infty)$, it follows from (1.1) that $G(2^\infty) + S_0 = 0$. To show $\sigma_m G(y^*) = -\ln|\pi^{-1}[12^m 1]|$, define $\nu \in M(Y)$ by

$$\nu = \frac{1}{m+1} \sum_{j=0}^{m} \delta_{\sigma^j y^*}.$$

It similarly follows from (1.1) that

$$0 = \frac{1}{m+1} \sum_{j=0}^{m} \left[\limsup_{n\to\infty} \frac{1}{n} (\sigma_n G(\sigma^j y^*) + \ln|D_n(\sigma^j y^*)|) \right].$$

Fix j, $0 \leq j \leq m$. Let $M \geq 0$, $0 \leq l \leq m$ and $n = M(m+1) + l$. Since $\sigma^{m+1} y^* = y^*$,

$$\sigma_n G(\sigma^j y^*) = \sum_{s=0}^{M-1} \sigma_{m+1} G(y^*) + \sum_{i=0}^{l-1} G(\sigma^{i+j} y^*)$$

and hence

$$\lim_{n\to\infty} \frac{1}{n} \cdot \sigma_n G(\sigma^j y^*) = \lim_{n\to\infty} \frac{1}{n} \cdot M \cdot \sigma_{m+1} G(y^*) = \frac{1}{m+1} \cdot \sigma_{m+1} G(y^*).$$

Thus it is enough to show

$$\lim_{n\to\infty} \frac{1}{n} \ln|D_n(\sigma^j y^*)| = \frac{1}{m+1} \ln|\pi^{-1}[12^m 1]|. \tag{3.8}$$

Let $b_0 = 1$ and for each $n \geq 1$, let $a_n = |\pi^{-1}[12^n 1]|$ and $b_n = |\pi^{-1}[2^n]|$. Let $M \geq 0$, $1 \leq l \leq m+1$, and $n = p + M(m+1) + l$. Since $[y_0^* \cdots y_{n-1}^*] = [2^p 12^m 1 \cdots 12^m 12^{l-1}]$,

$$|D_n(y^*)| = |D_p(y^*)| \cdot |\pi^{-1}[12^m 1]|^M \cdot |D_l(\sigma^{n-l} y^*)| \leq b_p \cdot (a_m)^M \cdot b_{l-1}.$$

Note that b_p, b_{l-1} and a_m are bounded by a constant that is independent of n. Also $|D_n(y^*)| \geq |\pi^{-1}[12^m 1]|^M = (a_m)^M$. Thus

$$\frac{\ln a_m}{m+1} \leq \liminf_{n\to\infty} \frac{1}{n} \ln|D_n(y^*)| \leq \limsup_{n\to\infty} \frac{1}{n} \ln|D_n(y^*)| \leq \frac{\ln a_m}{m+1}.$$

So (3.8) holds for $j = 0$. It similarly follows that for each $j = 1, \ldots, m$, (3.8) holds. □

The following, together with Theorem 3.3, gives a necessary and sufficient condition for a certain type of factor map to have a saturated compensation function.

THEOREM 3.4. *Let $X \subset \{1, \ldots, r\}^{\mathbb{Z}}$ be a subshift of finite type, Y either $\{1, 2\}^{\mathbb{Z}}$ or Y_0 (defined as before), and $\pi : X \to Y$ a factor map such that $\pi^{-1}\{1\} = \{1\}$. If*

$$S_0 = \lim_{n\to\infty} \frac{1}{n} \ln|\pi^{-1}[12^n 1]|, \tag{3.9}$$

then there exists a saturated compensation function for π.

Proof. For notation, let $a_0 = b_0 = 1$ and for each $n \geq 1$, let $a_n = |\pi^{-1}[12^n 1]|$ and $b_n = |\pi^{-1}[2^n]|$. For each $y \in Y$, let $m(y) = \inf\{i - j \mid i \geq 0, j \leq -1, \text{and } y_i = y_j = 1\}$ if it exists and ∞ otherwise, and let

$$G(y) = -(\ln a_{m(y)-1})/m(y)$$

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(throughout this work we will use the notation such as $\ln a_k/(k+1)$, which means S_0, when $k=\infty$). For each $n \geq 1$ and for each $y \in Y$, let $D_n(y)$ consist of one point from each non-empty set $\pi^{-1}\{y\} \cap {}_0[x_0x_1\cdots x_{n-1}]_{n-1}$, and $|D_0(y)| = 1$. We show that, for each $y \in Y$,

$$\limsup_{n\to\infty} \frac{1}{n} \ln(\exp(\sigma_n G(y)) \cdot |D_n(y)|) = 0,$$

in which case by Theorem 1.3, $G \circ \pi \in C(X)$ is a compensation function. We consider two cases. First let $y \in Y$ with $y_i = 2$ for all $i \geq 0$. For a given $n \geq 1$, $\sigma_n G(y) = -n \cdot S_0$ and clearly $|D_n(y)| \leq b_n$. Using either Lemma 3.1 or Remark 3.2, we have a constant $C > 0$ such that for each $n \geq 1$, $|D_n(y)| \geq C \cdot b_n$. Thus

$$\limsup_{n\to\infty} \frac{1}{n} \ln(\exp(\sigma_n G(y)) \cdot |D_n(y)|) = -S_0 + \limsup_{n\to\infty} \frac{1}{n} \ln b_n = 0.$$

We now consider the other case. Fix $n \geq 1$. Let $k \geq 1$, $n_0 \geq 0, n_k \geq 0$, $0 \leq n_*$, $n^* \leq \infty$, $m_i \geq 1$ for $i = 1, \ldots, k$, $n_i \geq 1$ for $i = 1, \ldots, k-1$, and $n_0 + \sum_{i=1}^{k}(m_i + n_i) = n$. Let $y = \cdots 12^{n_*}.2^{n_0}1^{m_1}2^{n_1}1^{m_2}\cdots 1^{m_k}2^{n_k}2^{n^*}1\cdots \in Y$. Repeating a routine process, we get

$$\sigma_n G(y) = -(n_0+1) \cdot \frac{\ln(a_{n_*+n_0})}{n_*+n_0+1} - \sum_{i=1}^{k-1} \ln a_{n_i} - n_k \cdot \frac{\ln(a_{n_k+n^*})}{n_k+n^*+1}.$$

For given l, m, $0 \leq l, m \leq \infty$, and for each $j = 0, 1, \ldots, m$, let $d_{l,m}^{j} = |D_j(y)|$, where $y = \cdots 12^l.2^m1\cdots \in Y$. Using this notation, we have

$$|D_n(y)| = |D_{n_0}(y)| \cdot \left(\prod_{i=1}^{k-1} a_{n_i}\right) \cdot |D_{n_k+1}(\sigma^{n-n_k-1}y)| = d_{n_*,n_0}^{n_0} \cdot \left(\prod_{i=1}^{k-1} a_{n_i}\right) \cdot d_{0,n_k+n^*}^{n_k}.$$

Let $A = (a_{n_k+n^*})^{-n_k/(n_k+n^*+1)} \cdot d_{0,n_k+n^*}^{n_k}$ and $B = (a_{n_*+n_0})^{-(n_0+1)/(n_*+n_0+1)} \cdot d_{n_*,n_0}^{n_0}$, so that $\exp(\sigma_n G(y)) \cdot |D_n(y)| = A \cdot B$. Since $n_0 < \infty$, it follows that (for a given $y \in Y$) for any $n > n_0$, B is fixed, i.e. a constant. So $\lim_{n\to\infty}(\ln B)/n = 0$. Thus it is enough to show

$$\limsup_{n\to\infty} \frac{1}{n} \ln A = 0. \tag{3.10}$$

For each $n \geq 1$, let $\gamma(n) = e^{-S_0(n+1)} \cdot b_n$ and $\delta(n) = e^{S_0(n+1)}/a_n$. By (3.9), $\lim_{n\to\infty}(\ln \gamma(n))/n = \lim_{n\to\infty}(\ln \delta(n))/n = 0$. Fix $\epsilon > 0$. Since $d_{0,n_k+n^*}^{n_k} \leq b_{n_k}$, it follows that

$$A \leq (a_{n_k+n^*})^{-n_k/(n_k+n^*+1)} \cdot b_{n_k} = e^{S_0} \cdot \gamma(n_k) \cdot \delta(n_k+n^*)^{n_k/(n_k+n^*+1)}.$$

First for $A_2 \equiv \gamma(n_k)$, choose $N_0 = N_0(\epsilon) > 0$ such that for any $n \geq N_0$, $|\ln \gamma(n)|/n < \epsilon/3$. Let $K_0 = K_0(\epsilon) = \sup\{|\ln \gamma(n)|/n \mid 1 \leq n \leq N_0\}$. Then for $n > \max\{N_0, 3N_0K_0/\epsilon\}$,

$$\left|\frac{1}{n} \ln A_2\right| = \frac{n_k}{n} \cdot \left|\frac{\ln \gamma(n_k)}{n_k}\right| < \frac{\epsilon}{3}.$$

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For $A_3 \equiv \delta(n_k + n^*)^{n_k/(n_k+n^*+1)}$, choose $N_1 = N_1(\epsilon) > 0$ so that for any $n \geq N_1$, $|\ln \delta(n)|/n < \epsilon/3$. Let $K_1 = K_1(\epsilon) = \sup\{|\ln \delta(n)|/n \mid 1 \leq n \leq N_1\}$. Then for $n > \max\{N_1, 3N_1K_1/\epsilon\}$,

$$\left|\frac{1}{n}\ln A_3\right| \leq \frac{n_k}{n} \cdot \left|\frac{\ln \delta(n_k + n^*)}{n_k + n^*}\right| < \frac{\epsilon}{3}.$$

Thus letting $A_1 \equiv e^{S_0}$, for any $n > \max\{3S_0/\epsilon, N_0, 3N_0K_0/\epsilon, N_1, 3N_1K_1/\epsilon\}$,

$$\frac{1}{n}\ln A \leq \left|\frac{1}{n}\ln A_1\right| + \left|\frac{1}{n}\ln A_2\right| + \left|\frac{1}{n}\ln A_3\right| < \epsilon. \tag{3.11}$$

Let $Q = \{i \geq 1 \mid y_i = 1\}$. Assume Q is finite. Let $M = \sup\{i \mid i \in Q\}$. Fix $n > M$. Since $n^* = \infty$, it follows from Remark 3.2 that for a constant $C > 0$, $A = e^{-S_0 \cdot n_k} \cdot d_{0,\infty}^{n_k} \geq e^{-S_0 \cdot n_k} \cdot C \cdot b_{n_k} \geq C \cdot \gamma(n_k)$. Thus for any $n > \max\{M, |3 \ln C|/\epsilon, N_0, 3N_0K_0/\epsilon\}$,

$$\frac{1}{n}\ln A \geq -\left|\frac{\ln C}{n}\right| - \left|\frac{\ln \gamma(n_k)}{n}\right| > -\epsilon$$

which, together with (3.11), implies (3.10). Now assume $Q = \{m_s\}_{s=1}^{\infty}$, where $m_s < m_{s+1}$ for all $s \geq 1$. For a given $s \geq 1$, let $n = m_s$. Since $n^* = 0$, it follows that $A = a_{n_k}^{-n_k/(n_k+1)} \cdot d_{0,n_k}^{n_k} = a_{n_k}^{1/(n_k+1)}$. Thus in this case also (3.10) holds, since

$$\limsup_{n\to\infty} \frac{1}{n}\ln A \geq \limsup_{s\to\infty} \frac{1}{m_s}\ln A = 0. \qquad \square$$

Remark 3.3. It follows from the proof of Theorem 3.4 that the function $G \in C(Y)$ constructed in Example 2.1 provides a saturated compensation function for the map π.

4. *Compensation functions with a homogeneous measure*

Throughout this section we assume that Y is a topologically mixing subshift of finite type. A (linear dense) subset $\mathcal{F}(Y)$ of $C(Y)$ is defined as follows [**6**].

Definition 4.1. We say $f \in \mathcal{F}(Y)$ if $f \in C(Y)$ and f is cohomologous to a one-sided function f_0 which satisfies the following condition: (as x^+, z^+ vary on Y) if $x_0 = z_0$, then

$$C_{f_0}(x^+, z^+) = \sup_{n\geq 1}\left[\sup\left\{\sum_{i=1}^{n}(f_0(a_{-i}, \ldots, a_{-1}, x^+) - f_0(a_{-i}, \ldots, a_{-1}, z^+)) \;\middle|\; (a_{-n}, \ldots, a_{-1}, x_0) \text{ is an allowable block in } Y\right\}\right]$$

is finite and bounded above by a constant C_{f_0}, and $C_{f_0}(x^+, z^+) \to 0$ as $d(x^+, z^+) \to 0$.

Definition 4.2. Let $\phi \in C(Y)$. A measure $\mu \in M(Y)$ is said to be a *homogeneous measure* for ϕ if there exist constants $C > 0$ and $P > 0$ such that for any $n \geq 1$ and for any $x \in Y$,

$$\frac{1}{C} < \frac{\mu[x_0 \cdots x_{n-1}]}{e^{-nP+\sigma_n\phi(x)}} < C.$$

We say $\phi \in H(Y)$ if $\phi \in C(Y)$ has a homogeneous measure in $M(Y)$.

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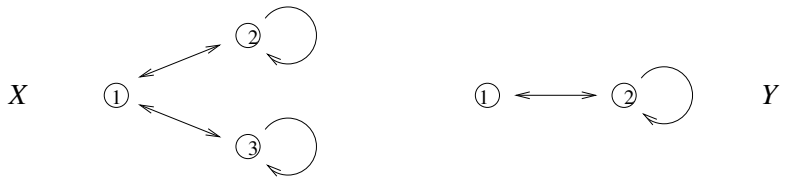


FIGURE 4.1.

For a given $\phi \in H(Y)$, a constant P is uniquely determined and $P = P_Y(\phi)$. It is known that $\phi \in H(Y)$ if and only if there exists $K > 0$ such that for each $n \geq 1$ and for any $x, y \in Y$ with ${}_0[x_0x_1 \cdots x_{n-1}]_{n-1} = {}_0[y_0y_1 \cdots y_{n-1}]_{n-1}$, $|\sigma_n\phi(x) - \sigma_n\phi(y)| \leq K$. If $f \in \mathcal{F}(Y)$, then f has a unique equilibrium state which is a homogeneous measure for f [**6**]. Thus $\mathcal{F}(Y) \subset H(Y)$. To show $\mathcal{F}(Y) \subsetneq H(Y)$, we present a function $f \in C(Y)$ for which there is $K > 0$ such that for any $n \geq 1$ and $y \in Y$, $|\sigma_n f(y)| \leq K$, but $f \not\sim 0$.

Example 4.1. Let $X \subset \{1, 2, 3\}^{\mathbb{Z}}$ and $Y \subset \{1, 2\}^{\mathbb{Z}}$ be the subshifts of finite type determined by allowing the transitions marked on Figure 4.1. The map π is the one-block map determined by mapping 1 to 1, and 2, 3 to 2. Define $\psi \in C(Y)$ by $\psi(y) = \ln 2$ if $y \in [21]$, and 0 otherwise. For a given $y \in Y$, let $m(y) = \inf\{i - j \mid i \geq 0, j \leq -1, \text{ and } y_i = y_j = 1\}$, and define $\phi \in C(Y)$ by $\phi(y) = \ln 2/m(y)$. Let $f = \psi - \phi \in C(Y)$. It is not difficult to see that $f \not\sim 0$. We now show that there is $K > 0$ such that for any $n \geq 1$ and $y \in Y$, $|\sigma_n f(y)| \leq K$. In fact $-\psi \circ \pi \in C(X)$ is a compensation function for π [**5**]. Also it follows from the proof of Theorem 3.4 that $-\phi \circ \pi \in C(X)$ is a compensation function for π. So ψ and ϕ have a unique equilibrium state $\pi \max_X \in M(Y)$ and $P_Y(\psi) = P_Y(\phi) = P_X(0)$ [**5**]. Since ψ is locally constant, $\psi \in H(Y)$. Also $\phi \in H(Y)$ (see Remark 4.1). Then it easily follows that there is $K > 0$ such that for any $n \geq 1$ and $y \in Y$, $|\sigma_n\psi(y) - \sigma_n\phi(y)| \leq K$, i.e. $|\sigma_n f(y)| \leq K$. It is known that for a given $\phi \in \mathcal{F}(Y)$, if there is such a $K > 0$, then $\phi \sim 0$. Thus $f \notin \mathcal{F}(Y)$. Since $f \in H(Y)$ ($H(Y)$ is a linear subspace of $C(Y)$), $\mathcal{F}(Y) \subsetneq H(Y)$.

Let X and Y be topologically mixing subshifts of finite type and $\pi : X \to Y$ a factor map. Note that if $\phi \in H(Y)$, then $\phi \circ \pi \in H(X)$. Since $H(X)$ is a linear subspace in $C(X)$, Corollary 1.2 expands as follows: If $G \in H(Y)$ and $G \circ \pi \in H(X)$ is a compensation function for π, then for each $\alpha \geq 0$, K_α consists of a unique element which is a (unique) equilibrium state of $[\alpha/(\alpha+1)]G \circ \pi \in H(X)$. We provide the following necessary and/or sufficient conditions for π to have a saturated compensation function in $H(X)$. This can be proved easily using arguments similar to the ones used in the case where $\phi, \psi \in \mathcal{F}(Y)$ [**6**]. If $\phi \in H(Y)$, then $\mu_\phi \in M(Y)$ denotes the unique equilibrium state of ϕ.

THEOREM 4.1.

(1) *If $\phi \circ \pi \in H(X)$, $\psi \in H(Y)$ and $\pi\mu_{\phi\circ\pi} = \mu_\psi$, then for some $c \in \mathbb{R}$, $c + (\phi - \psi) \circ \pi \in H(X)$ is a saturated compensation function for π.*

(2) *Let $\phi \in H(Y)$. Then $\phi \circ \pi \in H(X)$ is a compensation function for π if and only if there exists $C > 0$ such that for each $n \geq 1$ and for any $y \in Y$,*

$$1/C < \exp(\sigma_n\phi(y))|\pi^{-1}[y_0 \cdots y_{n-1}]| < C.$$

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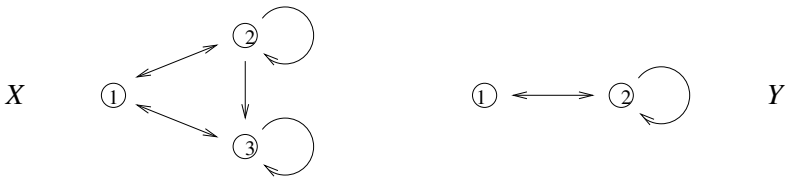


FIGURE 4.2.

Remark 4.1. Let X, Y and π be defined as in Theorem 3.4 and (3.9) hold. Let a_n, $n \geq 0$, $m(y)$, $y \in Y$, and $G \in C(Y)$ be defined as in the proof of Theorem 3.4, i.e. $G(y) = -\ln a_{m(y)-1}/m(y)$ and $G \circ \pi \in C(X)$ is a compensation function for π. It can be shown that $G \in H(Y)$ (so $G \circ \pi \in H(X)$) if and only if there is $M > 0$ such that

$$\left| S_0 - \frac{\ln a_n}{n+1} \right| \leq \frac{M}{n} \quad \text{for any } n \geq 1. \tag{4.1}$$

In Example 2.1, $a_n = 2$ if n is even, and 1 if n is odd. Clearly $M = 1$ satisfies (4.1). Thus $G \in H(Y)$ (but an argument similar to the one used in Example 4.1 shows that $G \notin \mathcal{F}(Y)$). So for each $\alpha \geq 0$ there exists a unique measure of maximal α-weighted entropy for the factor map π. Now consider the case where $a_n = n + 1$, $n \geq 0$ (see Figure 4.2). Since $S_0 = 0$ and no $M > 0$ satisfies (4.1), it follows that $G \notin H(Y)$ in this case.

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