is conceptually inferior to vector algebra. One is left with the impression that somehow (2) intrinsically evades direct and short proofs. However, that need not be the case.

I hope to have shown in this Note that viewing the problem as a planar one does lead to a short and fairly straightforward derivation of (2), provided that the validity of the planar proposition is first demonstrated. Since that proposition is significant in itself, it should be established anyway if planar vector algebra is properly developed.

Drawing attention to the need for a more expressive planar vector algebra is a goal of this Note, apart from proving (2). It is particularly important to stress that by putting to use $\wedge$ and $^{\perp}$, a wealth of properties, such as orientation, relative location, and related phenomena receive due consideration that is impossible when restricted to only linear operations and the scalar product. Making use of $\wedge$ and $^{\perp}$ brings to the calculations in planar geometry many benefits, here only glimpsed at.

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100.18 Newton's semicircle and the probability integral

Students who learn probability or statistics are frequently surprised by the fact that the standard normal density function, $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}w^2}$ where $-\infty < w < \infty$, contains a factor of $\sqrt{2\pi}$. Since π is the ratio of the circumference of a circle to the diameter, where is the circle? Textbooks usually start with

$$\left(\int_{-\infty}^{\infty} e^{-u^2}du\right)\left(\int_{-\infty}^{\infty} e^{-v^2}dv\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)}du\,dv$$

using Fubini's theorem. The transformation of the above double integral from rectangular coordinates to polar coordinates gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)}du\,dv = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}r\,dr\,d\theta = \pi.$$

Consequently, the probability integral or Gaussian integral is evaluated as

$$\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}. \tag{1}$$

The substitution $u = \frac{w}{\sqrt{2}}$ leads to

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{\frac{-w^2}{2}}dw = 1,$$

which yields the standard normal density function. Since a complete revolution is 2π radians in the coordinate transformation, with the circle formula $C = 2\pi r$, it is reasonable to argue that the transformation is a

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standard lens used for observing where the π comes from or where the circle is. In this Note, we shall look at this issue from a historical viewpoint using neither the Fubini argument nor the coordinate transformation involving the Jacobian.

One of the calculations Newton performed during the plague years 1665-1666 in Woolsthorpe was to employ an upper semicircle with radius $\frac{1}{2}$ and centre at $\left(\frac{1}{2},\ 0\right)$ for estimating the value of π [1]. The purpose of this Note is to link Newton's semicircle to the probability integral $\int_{-\infty}^{\infty} e^{-u^2} du$ using Euler's beta and gamma functions. Such a link implies that the factor $\sqrt{\pi}$ in the standard normal density function can be traced back to a circle with an amazing history of the approximation of π. The article [2] is a useful compendium of different proofs for (1). The survey article [3] puts various methods for evaluating the probability integral into historical context.

The equation of Newton's upper semicircle is

$$\left(x - \tfrac{1}{2}\right)^2 + y^2 = \tfrac{1}{4}, \qquad 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant \tfrac{1}{2}.$$

Further simplification gives the function $y = \sqrt{x(1 - x)}$.

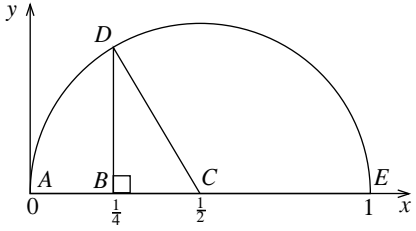


FIGURE 1

In Figure 1, A is the origin, B is the point $\left(\frac{1}{4},\ 0\right)$, C is the centre of the semicircle, $\left(\frac{1}{2},\ 0\right)$, and BD is perpendicular to AE. The Pythagorean theorem yields

$$BD = \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{3}}{4}.$$

Hence, the area of the right-angled triangle BCD is $\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{32}$. Given $\angle BCD = \frac{\pi}{3}$, it implies that the area of sector ACD is

$$\frac{1}{3}\left[\frac{1}{2}\pi\left(\frac{1}{2}\right)^2\right] = \frac{\pi}{24}.$$

The area of the circular segment ABD is then equal to $\left(\dfrac{\pi}{24} - \dfrac{\sqrt{3}}{32}\right)$.

Furthermore, the area of the circular segment ABD is computed by the definite integral $\int_0^{1/4} \sqrt{x(1 - x)}\, dx$. Newton evaluated the integral by first applying his discovery of the generalised binomial theorem to expand

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$\sqrt{x(1-x)}$ in terms of the form $x^{k/2}$ where k is a positive integer, and then performing term-by-term integration for the first twenty-two terms. He wrote, "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time." (See [1].)

An example of computation for the first five terms gives

$$\sqrt{x(1-x)} = x^{\frac{1}{2}}(1-x)^{\frac{1}{2}} = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}} - \frac{5}{128}x^{\frac{9}{2}} - \ldots$$

$$\text{and } \int_0^{1/4} \sqrt{x(1-x)}\,dx \approx \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}} - \frac{5}{704}x^{\frac{11}{2}}\right]_0^{1/4}$$

$$\approx 0.07677372.$$

Consequently, $\left(\frac{\pi}{24} - \frac{\sqrt{3}}{32}\right) \approx 0.07677372$. Solving for π gives an estimate

$$\pi \approx 24\left(0.07677372 + \frac{\sqrt{3}}{32}\right) \approx 3.141607.$$

It is worth noting that, because all the terms in the binomial expansion after the first one are negative, we will always get over-estimates of the value of π. Using this approach with more than 20 terms of the expansion, Newton correctly estimated the value of π to 15 decimal places.

When we move on to evaluate the area of the whole upper semicircle with radius $\frac{1}{2}$, we obtain

$$\frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{8} = \int_0^1 \sqrt{x(1-x)}\,dx. \tag{2}$$

The second equality is due to the definite integral which describes the area under the curve $y = \sqrt{x(1-x)}$ from $x = 0$ to $x = 1$. The right-hand side of (2) simultaneously equals $B\left(\frac{3}{2}, \frac{3}{2}\right)$, where $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$ is the beta function or Euler's first integral.

Euler was the first to identify the relation

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ is the gamma function or Euler's second integral. Note that most proofs of the beta-gamma relation (for example, see [4]) use a similar but slightly more involved Fubini argument to that mentioned at the beginning of this Note, which defeats our object. Fortunately, Artin [5] avoids the Fubini argument by using instead the Bohr-Mollerup characterisation of the gamma function to establish the beta-gamma relation. It follows that

$$B\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} = \frac{1}{8}\Gamma\left(\frac{1}{2}\right)^2, \tag{3}$$

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where the second equality comes from the recursive property of the gamma function, $\Gamma(z+1) = z\Gamma(z)$ for $z > 0$ from integration by parts, and $\Gamma(1) = 1$. Combining (2) and (3) yields $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Moreover, we get

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = 2\int_0^\infty e^{-u^2} du = \int_{-\infty}^\infty e^{-u^2} du$$

by substituting $t = u^2$ for the second equality and using the symmetry of e^{-u^2} for the third equality. The probability integral $\int_{-\infty}^{\infty} e^{-u^2} du$ is thus equal to $\sqrt{\pi}$ using Newton's semicircle and Euler's beta-gamma relation. Note that the recent book [4] applies Newton's semicircle to calculate the factorial

$$\left(\frac{1}{2}\right)! = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2},$$

with a different focus to that of this Note.

The key point in the above proof is to show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ via $B\left(\frac{3}{2}, \frac{3}{2}\right)$. As pointed out by the referee, another popular route is via

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \Gamma\left(\frac{1}{2}\right)^2,$$

where $B\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 \frac{1}{\sqrt{x(1-x)}} dx$. This integral may be evaluated with the substitution $x = \sin^2\theta$, but it may be of interest to note that it can be evaluated via the area of the unit circle as follows:

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}} dx = \int_0^1 \left(\frac{\sqrt{1-x}}{\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{1-x}}\right) dx$$

$$= 2\int_0^1 \frac{\sqrt{1-x}}{\sqrt{x}} dx = 4\int_0^1 \sqrt{1-t^2} dt = \pi,$$

where the second equality comes from symmetry and the third equality is from the substitution $x = t^2$. This route gives a direct connection with the unit circle, but without the Newton connection.

Finally, there are two anecdotes related to the subject of this Note. The first one is an excerpt from [6]:

> De Morgan was explaining to an actuary what was the chance that a certain proportion of some group of people would at the end of a given time be alive; and quoted the actuarial formula, involving π, which, in answer to a question, he explained stood for the ratio of the circumference of a circle to its diameter. His acquaintance, who had so far listened to the explanation with interest, interrupted him and exclaimed, "My dear friend, that must be a delusion, what can a circle have to do with the number of people alive at the end of a given time?"

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The second one is Feynman's address to science teachers in 1966 [7], coincidentally three centuries after Newton's pursuit for approximating π. Feynman mentioned,

> … Pi was a thing with circles, and here is pi coming out of …
> Where was the circle?

Echoing and responding to the questions of De Morgan's acquaintance and Feynman, we justify π in the standard normal density function to come out of a semicircle, namely, Newton's semicircle in the history of the approximation of π.

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100.19 A quick evaluation of $\lim_{m\to\infty}\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m$

Just as there are ‘earworms’ – catchy tunes that lodge in the brain and refuse to budge – so, I believe, there are ‘mathematical earworms’ – problems that niggle us and will not let go. One such for me is the limit in the title which I first encountered some ten years ago in the articles [1, 2, 3]. Michael Spivey's original proof in [1, 2] used the Euler-Maclaurin summation formula to estimate $\sum_{k=0}^{m}\left(\frac{k}{m}\right)^m$ and then evaluate the limit; Finbarr Holland in [3] used monotone convergence arguments to find the limit via

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