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Generalized adjunction and applications

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Introduction

The linear system $|K+C|$ 'adjoint' to a curve C on a projective surface was studied by the classical Italian geometers. The adjoint system to a hyperplane section H of a smooth projective surface was investigated systematically, in modern terms, by Sommese [**22**] and Van de Ven [**26**]. The map associated to the linear system $|K+(r-1)H|$, where H is a hyperplane section of a smooth variety of arbitrary dimension r, was used to classify submanifolds of $\mathbb{P}^n$ with 'small invariants' (e.g. degree, sectional genus, etc.); see [**10**]. On the other hand, Sommese [**23**, **24**, **25**] studied adjoint systems to a smooth ample divisor H on a smooth threefold X and obtained, as applications, many interesting results about the pair (X, H). As noticed independently by several authors (see e.g. [**17**], [**4**], [**11**]) the appearance of Mori's deep contribution [**20**] (see also [**21**]) put the subject of adjunction in a new perspective. Accordingly, the present paper – which relies heavily on Mori's results and on the contraction theorem due to Kawamata–Shokurov (see [**14**]) – contains a systematical study of various adjoint systems to an ample (possibly non-effective) divisor on a manifold of arbitrary dimension. More precisely, the main result (which is contained in Section 1) gives the precise description of polarized pairs (X, H), where X is a complex projective manifold of dimension r and H an ample divisor on it (not necessarily effective), such that K_X+iH is not semiample (respectively ample) for $1 \leqslant i = r+1, r, r-1, r-2$ (respectively $i = r+1, r, r-1$). This theorem was first proved, for surfaces, by Lanteri and Palleschi [**17**] using Mori's results, and by Beltrametti and Palleschi [**4**] and Lanteri and Palleschi [**18**] for threefolds, using Mori's results and the Kawamata–Shokurov contraction theorem. Our approach does not make use of the precise description of varieties whose canonical bundles are not numerically effective, which is available only for dimensions $\leqslant 3$. Inspired by [**19**], theorem 4, we prove in Section 0 a useful estimation for the size of the locus of curves belonging to an extremal ray (Theorem 0·4). This is used at a critical step in the proof of the main result. The following applications of the main theorem are given in Section 2. First we recover and sometimes slightly improve the main results in Sommese's paper [**24**] (or [**25**]); we also give an alternative proof of a theorem due to L. Bădescu (see [**1**, **2**, **3**]) classifying smooth projective threefolds which support a geometrically ruled surface as an ample divisor; more generally, we describe all (smooth, projective) threefolds containing a smooth surface which is not of general type as an ample divisor. Finally, using the idea of [**4**], we classify the polarized pairs (X, H) of arbitrary dimension r whose sectional genus is 'small' with respect to the 'degree' (H^r) (see also [**10**] and [**11**], where we considered the case when H is a very ample divisor).

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this paper. Working independently, M. Beltrametti and M. Palleschi also obtained partial results of the same kind.

When writing the present version of this paper, we received a note by T. Fujita entitled 'Generalized adjunction mappings' announcing, without proofs, results similar to ours. However, some of the exceptional varieties are missing in his list. We also received a manuscript by A. J. Sommese entitled 'On the adjunction theoretic structure of projective varieties', dealing with similar questions about varieties admiting certain kinds of singularities, but working with spanned ample line bundles. His techniques are quite different from ours.

Section 0

In this section we fix our terminology and notation, recall some results needed in the sequel and prove a useful estimation for the dimension of the locus of curves belonging to an extremal ray.

We shall work over the field of *complex* numbers unless otherwise specified. A smooth connected algebraic variety is called simply a *manifold*. All manifolds are assumed to be *projective*, unless otherwise stated. A *polarized pair*, denoted (X, H), means a (projective) manifold X together with an *ample* divisor H not necessarily effective. A divisor D on X is said to be *nef* if $(D.C) \geqslant 0$ for any effective curve C. A nef divisor D is '*big*' if $(D^r) > 0, r = \dim X$. A divisor D is called *semi-ample* if the linear system $|mD|$ is base-point-free, for $m \gg 0$. K_X will denote a canonical divisor of the manifold X. We write '$\sim$' (respectively '$\approx$') for the linear (respectively numerical) equivalence of divisors. If Z is a closed subscheme of X and D is a divisor (class) on X we denote by $D|_Z$ its restriction to Z. A rational curve is an irreducible reduced curve whose normalization is $\mathbb{P}^1$.

We refer the reader to [**20**] for definitions and properties of extremal rays, extremal rational curves, etc. This paper relies on the following two fundamental results:

(0·1) MORI'S CONE THEOREM (see [**20**], theorem 1·5). *Let X be a manifold. Then the closed cone of effective curves, denoted $NE(X)$, is the smallest closed convex cone containing the set $\overline{NE}_-(X) = \{C \in \overline{NE}(X) | (C.K_X) \geqslant 0\}$ and all the extremal rays. For any open convex cone U containing $\overline{NE}_-(X) - \{0\}$ there are only finitely many extremal rays that do not lie in $U \cup \{0\}$. Every extremal ray is spanned by an extremal rational curve.*

(0·2) KAWAMATA–SHOKUROV CONTRACTION THEOREM (see [**14**], theorem 2·6). *Let X be a manifold and D a nef divisor on it. Assume that $aD - K_X$ is nef and big for some $a \geqslant 1$. Then D is semi-ample.*

Using [**21**], proposition 3·1, the next corollary follows from (0·2).

(0·3) COROLLARY. *Let X be a manifold of dimension r and let R be an extremal ray. Then there exists a normal projective variety Y and a surjective morphism $f = \mathrm{cont}_R: X \to Y$ with connected fibres such that, for any integral curve C on X, $\dim f(C) = 0$ is equivalent to $[C] \in R$.*

The next theorem, which seems to be interesting in itself, will play a key role in the proof of the main result. Its proof was largely inspired by Mori's proof of theorem 4 in [**19**].

(0·4) THEOREM. *Let X be a manifold of dimension r defined over any algebraically closed field and let R be an extremal ray of X. Denote by k the codimension in X of the locus*

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of points of curves belonging to R. Put $b = \min\{-(K_X.C) | [C] \in R, C \text{ a rational curve}\}$. *Then we have* $2k \leqslant r+1-b$.

Proof. Let C_0 be a rational curve in R such that $b = -(K_X.C_0)$ and let $f: \mathbb{P}^1 \to C_0$ be the normalization morphism. Let $\mathrm{Hom}(\mathbb{P}^1, X)$ denote the scheme representing the functor of morphisms from $\mathbb{P}^1$ to X and let U be an irreducible component of maximal dimension containing $[f]$, given the reduced structure. The universal family over $\mathrm{Hom}(\mathbb{P}^1, X)$ induces the commutative diagram:

$$\begin{array}{ccccc} Z & \xhookrightarrow{i} & \mathbb{P}^1 \times X \times U & \xrightarrow{m} & X \times U \\ & \searrow^{v} & \downarrow{\scriptstyle n} & & \\ & & U & & \end{array}$$

where m and n are the projections and i is a closed immersion. Consider the closed (reduced) subscheme Z' of $X \times U$ given by $Z' = m(i(Z))$. By generic flatness we get an open nonempty subset U_0 of U such that the restriction of Z' to $X \times U_0$ is flat over U_0. Thus, by the universal property of the Hilbert scheme of X, we get a morphism $\alpha: U_0 \to \mathrm{Hilb}_X$. So, for any closed point $t \in \alpha(U_0)$, the corresponding curve is got by taking the common image of the maps associated to points in $\alpha^{-1}(t)$. Denote by T the closure of $\alpha(U_0)$ and let C be any curve corresponding to some closed point in T. We claim:

(1) $C \approx C_0$ and, moreover, C is irreducible and generically reduced.

Indeed, to prove the first part, we may assume that C corresponds to a point in $\alpha(U_0)$ associated to some map, say $g: \mathbb{P}^1 \to X$. Then, for any $L \in \mathrm{Pic}(X)$, we get

$$(C_0.L) = \deg f^*L = \deg g^*L = d(C.L),$$

where d is the degree of g. This shows that $C_0 \approx dC$. But then $[C] \in R$ since R is an extremal ray. As C is a rational curve, the minimality of C_0 gives $d = 1$. On the other hand, it is not difficult to see that any irreducible component of a curve C corresponding to a closed point in T is rational. Since $C_0 \approx C$, C must be irreducible and generically reduced again by minimality of C_0.

Returning now to the map α, we remark that the above argument shows that, for any closed point $t \in \alpha(U_0)$, $\alpha^{-1}(t)$ identifies to a part of $\mathrm{Aut}(\mathbb{P}^1)$. Therefore we get

(2) $\dim T \geqslant \dim U_0 - 3$.

By [19], proposition 3, and the Riemann–Roch theorem, we get

(3) $\dim U_0 = \dim_{[f]} \mathrm{Hom}(\mathbb{P}^1, X) \geqslant \chi(f^*T_X) = b+r$ (where T_X is the tangent bundle of X).

Consider now the commutative diagram

$$\begin{array}{ccccc} Y & \xhookrightarrow{j} & X \times T & \xrightarrow{p} & X \\ & \searrow^{w} & \downarrow{\scriptstyle q} & & \\ & & T & & \end{array}$$

induced by base-change from the universal family over Hilb_X, p and q being the projections and j a closed immersion. By (2) and (3) we have

(4) $\dim Y \geqslant b+r-2$.

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By (1) it follows that $p(Y)$ is contained in the locus of points of curves belonging to R. Therefore we get

(5) $\dim p(Y) \leqslant r-k$.

If we put $Y_x = p^{-1}(x) \cap Y$, $T_x = w(Y_x) \simeq Y_x$, we get by (4) and (5)

(6) $\dim T_x = \dim Y_x \geqslant b+k-2$, for any $x \in p(Y)$.

We get from (6)

$$\dim w^{-1}(T_x) \geqslant b+k-1.$$

Since we have *a fortiori* $\dim p(w^{-1}(T_x)) \leqslant r-k$, we may once again apply the same reasoning with $w^{-1}(T_x)$ replacing Y. Thus, if we let $T_{x,x'} = w(Y_x) \cap w(Y_{x'})$ for $x \in p(Y)$, $x' \in p(w^{-1}(T_x))$, we get

(7) $\dim T_{x,x'} \geqslant b-r+2k-1$.

Next we claim:

(8) $\dim T_{x,x'} \leqslant 0$, for $x \in p(Y)$, $x' \in p(w^{-1}(T_x))$, $x' \neq x$.

The conclusion of the theorem follows from (7) and (8).

Assume that (8) is false. Then we may find a complete curve D contained in T, such that the corresponding curves in X all pass through the points x, x'. Moreover, we may choose x, x' to be smooth distinct points of some curve of the family parametrized by D. Let $\tilde{D}$ be the normalization of D. Let S be the reduced scheme structure of the surface got by base-change over $\tilde{D}$ from the map w: $Y \to T$ and let $\tilde{S}$ be the normalization of S. By (1) all the fibres of the map $\tilde{w}$: $S \to \tilde{D}$ deduced from w are irreducible rational curves. It follows easily that the same holds for the fibres of the map π: $\tilde{S} \to \tilde{D}$ obtained by composing $\tilde{w}$ with the normalization morphism. Therefore π is a $\mathbb{P}^1$-bundle and, in particular, $\tilde{S}$ is smooth. But we have at least two disjoint sections E, E' for π, which are mapped to the points x and x' respectively. We get

$$(E^2) < 0, \quad (E'^2) < 0, \quad (E.E') = 0 \quad \text{and} \quad (E-E')^2 = 0,$$

which is absurd. This contradiction gives (8) and thereby completes the proof of the theorem.

We also need the following statement, which follows from the explicit description of extremal rays in case of threefolds; see [**20**] or [**21**], theorem 2·3 and theorem 2·5.

(0·5) COROLLARY. *If C is an extremal rational curve on a (smooth, projective) threefold X such that $(K_X.C) = -4$, it follows that X is a Fano threefold with* $\mathrm{Pic}(X) \simeq \mathbb{Z}$.

We shall also use the following simple lemma.

(0·6) LEMMA. *Let (X, H) be a polarized pair. Assume that $K_X + iH \approx 0$ for some integer $i > 0$. Then $K_X + iH \sim 0$.*

Proof. We have $\chi(\mathcal{O}_X(K_X+iH)) = \chi(\mathcal{O}_X)$ (see [**15**], chapter II, § 2, theorem 1). The hypothesis implies that $-K_X$ is ample. Therefore $\chi(\mathcal{O}_X(K_X+iH)) = h^0(\mathcal{O}_X(K_X+iH))$ and $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) = 1$ by Kodaira vanishing. Thus we get $h^0(\mathcal{O}_X(K_X+iH)) = 1$, which gives $K_X + iH \sim 0$.

The following useful characterizations of projective spaces and hyperquadrics will be needed several times.

THEOREM (see [**16**] and [**6**]). *Let X be an integral projective scheme of dimension r and H an ample divisor on it.*

(0·7) *If $(H^r) = 1$ and $h^0(\mathcal{O}_X(H)) \geqslant r+1$, then $X \simeq \mathbb{P}^r$ and $H \in |\mathcal{O}(1)|$;*

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(0·8) *If* $(H^r) = 2$ *and* $h^0(\mathcal{O}_X(H)) \geqslant r+2$, *then* X *is isomorphic to a hyperquadric in* $\mathbb{P}^{r+1}$ (*denoted* Q^r) *and* H *is its hyperplane section* (*we shall write* $H \in |\mathcal{O}(1)|$). *Assume moreover that* X *is a manifold.*

(0·9) *If* $K_X + (r+1)H \sim 0$, *then* $X \simeq \mathbb{P}^r$ *and* $H \in |\mathcal{O}(1)|$;

(0·10) *If* $K_X + rH \sim 0$, *then* $X \simeq Q^r$ *and* $H \in |\mathcal{O}(1)|$.

(0·11) Let (X, H) be a polarized pair with $\dim X = r$. An effective divisor $E \subset X$ is called *exceptional* if $E \simeq \mathbb{P}^{r-1}$, $\mathcal{O}_X(H) \otimes \mathcal{O}_E \simeq \mathcal{O}(1)$ and $\mathcal{O}_X(E) \otimes \mathcal{O}_E \simeq \mathcal{O}(-1)$. Note that the set of exceptional divisors contained in X is finite, and, if $r \geqslant 3$, any two such exceptional divisors are disjoint. If (X, H) is a polarized pair and E an exceptional divisor on X, consider the morphism $\sigma\colon X \to X'$ which contracts E (to a smooth point) and the unique divisor H' on the manifold X' such that

$$\mathcal{O}_X(H) \simeq \sigma^*(\mathcal{O}_{X'}(H')) \otimes \mathcal{O}_X(-E).$$

By [7], lemma 5·7, it follows that H' is ample on X'. Continuing in this way, we find a new polarized pair (X', H') such that $u\colon X \to X'$ is the blowing-up of n distinct points $P_1, \ldots, P_n \in X'$; $u^{-1}(P_i) = E_i$ is an exceptional divisor for $i = 1, \ldots, n$,

$$\mathcal{O}_X(H) \simeq u^*(\mathcal{O}_{X'}(H')) \otimes \mathcal{O}_X(-E_1 - \ldots - E_n)$$

and (X', H') does not contain exceptional divisors. Such a pair (X', H') will be called a *reduction* of (X, H) (see [**24**]). In case $r \geqslant 3$, the contraction u is uniquely determined by (X, H). This is no longer true if $r = 2$, but all we shall need is the existence of a reduction.

Let (X, H) be a polarized pair with $\dim X = r \geqslant 2$. (X, H) is called a *scroll* if there is a morphism $f\colon X \to Y$ onto some manifold Y with $\dim Y = s > 0$, which is a $\mathbb{P}^{r-s}$-bundle, such that H induces $\mathcal{O}(1)$ on each fibre. (X, H) is called a *hyperquadric fibration* if there is a morphism $f\colon X \to C$ onto some smooth curve C such that each (closed) fibre of f is isomorphic to a hyperquadric and H induces $\mathcal{O}(1)$ on it. As we shall see in Section 1, any fibre of f is reduced and, if $r \geqslant 3$, it is also irreducible.

We shall introduce now several notations for the isomorphism classes of polarized pairs which will appear frequently in what follows.

(0·12) $(X, H) \in \mathscr{A}$ will mean $X \simeq \mathbb{P}^r$, $H \in |\mathcal{O}(1)|$;

(0·13) $(X, H) \in \mathscr{B}$ will mean $X \simeq Q^r$, $H \in |\mathcal{O}(1)|$ or (X, H) is a scroll over a curve;

(0·14) $(X, H) \in \mathscr{B}'$ will mean $(X, H) \in \mathscr{B}$ or $X \simeq \mathbb{P}^2$, $H \in |\mathcal{O}(2)|$;

(0·15) $(X, H) \in \mathscr{C}$ will mean that either

(1) $K_X + (r-1)H \sim 0$ ($\dim X = r$); these are called *Del Pezzo manifolds*, see [**8**], and their classification is completely known, see [**12**] and [**8**];

(2) (X, H) is a hyperquadric fibration; or

(3) (X, H) is a scroll over a surface.

(0·16) $(X, H) \in \mathscr{D}$ will mean that either $X \simeq \mathbb{P}^4$, $H \in |\mathcal{O}(2)|$; $X \simeq \mathbb{P}^3$, $H \in |\mathcal{O}(3)|$; $X \simeq Q^3$, $H \in |\mathcal{O}(2)|$, or X is a $\mathbb{P}^2$-bundle over a smooth curve and H induces $\mathcal{O}(2)$ on each fibre.

Finally, we shall need the following simple fact.

(0·17) Lemma. *Let* (X, H) *be a polarized pair with reduction* (X', H'). *If*

$$(X', H') \in \mathscr{A} \cup \mathscr{B}' \cup \mathscr{C},$$

then

$$(X, H) \in \mathscr{A} \cup \mathscr{B}' \cup \mathscr{C}.$$

Proof. If $X' \simeq \mathbb{P}^r$, $H' \in |\mathcal{O}(1)|$ or $X' \simeq Q^r$, $H' \in |\mathcal{O}(1)|$, or (X', H') is a scroll, or $r \geqslant 3$ and (X', H') is a hyperquadric fibration, it follows easily that we must have $X = X'$.

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If (X', H') is a Del Pezzo manifold, the same holds for (X, H). If (X', H') is a two-dimensional hyperquadric fibration, the same holds for (X, H). Finally, if $X' \simeq \mathbb{P}^2$, $H' \in |\mathcal{O}(2)|$, (X, H) has to be a scroll over $\mathbb{P}^1$ (of degree 3), unless $X = X'$.

Section 1

This section is devoted to the statement and proof of the main result. Using the definitions and notations given in (0·11)–(0·16), our main theorem is the following:

THEOREM. *Let (X, H) be a polarized pair, with* $\dim X = r \geqslant 1$. *Then we have:*

(1·1) *$K_X + (r+1)H$ is semi-ample;*

(1·2) *if $(X, H) \notin \mathscr{A}$, $K_X + (r+1)H$ is ample;*

(1·3) *if $(X, H) \notin \mathscr{A}$, $K_X + rH$ is semi-ample;*

(1·4) *if $(X, H) \notin \mathscr{A} \cup \mathscr{B}$, $K_X + rH$ is ample.*

If $r \geqslant 2$, then

(1·5) *if $(X, H) \notin \mathscr{A} \cup \mathscr{B}'$, $K_X + (r-1)H$ is semi-ample;*

(1·6) *if $(X, H) \notin \mathscr{A} \cup \mathscr{B}' \cup \mathscr{C}$, there is a reduction (X', H') such that $K' + (r-1)H'$ is ample (where $K' =: K_{X'}$).*

If furthermore $r \geqslant 3$, then

(1·7) *if $(X, H) \notin \mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$, there is a reduction (X', H') such that either $(X', H') \in \mathscr{D}$ or $K' + (r-2)H'$ is semi-ample.*

Remarks. (1) Clearly (1·1) and (1·2) follow from (1·3); we stated them separately since we shall prove them in this order, and moreover, this way the result looks more symmetrical.

(2) If $|H|$ is moreover assumed to be base-point-free, (1·2) was proved (even if not stated) by L. Ein in [5].

(3) Using (0·2), we may restate (1·3) as follows: if $K_X + rH$ is not nef, then $X \simeq \mathbb{P}^r$, $H \in |\mathcal{O}(1)|$. This characterization of $\mathbb{P}^r$ should be compared with J. Wahl's result in [28].

(4) The above results were first proved by Lanteri and Palleschi [17] in the case of surfaces, and by Beltrametti and Palleschi [4] and Lanteri and Palleschi [18] in the case of threefolds.

(5) In case H is very ample (1·5) can be made more precise: if $(X, H) \notin \mathscr{A} \cup \mathscr{B}'$, $|K_X + (r-1)H|$ is base-point-free (see [22], Proposition 1·5, [26], theorem II, for $r = 2$, and [10], theorem 1·4, for $r \geqslant 3$). Is it true that, with the hypotheses of (1·7) and assuming moreover H to be very ample, $|K' + (r-2)H'|$ is base-point-free if $(X', H') \notin \mathscr{D}$?

The proof of the Theorem is based on a lemma which will be stated below. Let (X, H) be a polarized pair with $\dim X = r \geqslant 1$, let $i \geqslant 1$ be an integer and assume that $K_X + iH$ is semi-ample. Therefore, if $m \gg 0$, the morphism

$$\phi_i =: \phi_{|m(K_X + iH)|} : X \to \mathbb{P}^N$$

has connected fibres and maps X onto some normal variety Y. With these assumptions and notations we have the following:

LEMMA. (*a*) *One has either* $\dim Y = r$, *or* $\dim Y \leqslant r + 1 - i$.

(*b*) *Assume that* $\dim Y = r$ *and* $i \geqslant r - 1 \geqslant 1$. *If ϕ_i is not a finite morphism, then $i = r - 1$ and X contains an exceptional divisor* (*see* (0·11)).

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(*c*) *Assume that* $\dim Y < r$. *Then*

$$\begin{aligned} &\text{if} \quad i = r+1, \quad (X,H) \in \mathscr{A}; \\ &\text{if} \quad i = r, \quad\quad (X,H) \in \mathscr{B}; \\ &\text{if} \quad i = r-1, \quad (X,H) \in \mathscr{C}. \end{aligned}$$

Assuming this lemma for the moment, we shall prove the Theorem.

Proof of the Theorem. (1·1) If V is an effective curve on X such that $(V.K_X) \geqslant 0$, it follows that $(K_X + (r+1)H.V) > 0$ by ampleness of H. Assume that $K_X + (r+1)H$ is not nef. By (0·1) we may find an extremal rational curve C such that

$$(K_X + (r+1)H.C) < 0.$$

But $(H.C) \geqslant 1$ by ampleness of H and $(K_X.C) \geqslant -r-1$ since C is an extremal rational curve. We have reached a contradiction, which shows that $K_X + (r+1)H$ is nef. By (0·2) $K_X + (r+1)H$ is semi-ample.

(1·2) Using (1·1) we may apply the Lemma with $i = r+1$. Since $(X,H) \notin \mathscr{A}$, ϕ_{r+1} is generically finite by (*c*), and hence it is finite, by (*b*). Therefore $K_X + (r+1)H$ is ample.

(1·3) By (0·2) it is enough to prove that $K_X + rH$ is nef. Assuming the contrary and using (0·1) as above, we may find an extremal rational curve C such that

(1) $(K_X + rH.C) < 0$.

If we let $a =: (H.C)$, $b =: -(K_X.C)$, we have $a \geqslant 1$, $b \leqslant r+1$; by (1·2) we get

(2) $(K_X + (r+1)H.C) > 0$.

Comparing (1) and (2), it follows that

(3) $ra < b < (r+1)a$.

Thus we must have $a \geqslant 2$ and we get $2r \leqslant ar < b \leqslant r+1$, which is impossible for $r \geqslant 1$.

(1·4) This is, like (1·2), a consequence of (1·3) and assertions (*b*) and (*c*) of the Lemma.

(1·5) Assume, as in the proof of (1·3), that $(X,H) \notin \mathscr{A} \cup \mathscr{B}$ and $K_X + (r-1)H$ is not nef. We shall prove that $X \simeq \mathbb{P}^2$, $H \in |\mathscr{O}(2)|$. Using (0·1) we may find an extremal rational curve C such that, if we let $a = (H.C)$, $b = -(K_X.C)$, it follows as above that

$$(r-1)a < b < ra.$$

Therefore $a \geqslant 2$ and we get $2(r-1) \leqslant (r-1)a < b \leqslant r+1$.

This implies $r = 2$, $a = 2$, $b = 3$. Now, using the explicit description of extremal rational curves in the case of surfaces (see [**20**], theorem 2·1), it follows that $X \simeq \mathbb{P}^2$. Alternatively, consider the divisor $H_1 =: K_X + 2H$. H_1 is ample, by (1·4). Since $(K_X + 3H_1.C) = 0$, it follows by (1·2) that $X \simeq \mathbb{P}^2$, $H_1 \in |\mathscr{O}(1)|$, so $H \in |\mathscr{O}(2)|$.

(1·6) The proof is similar to that of (1·4), using (1·5), parts (*b*) and (*c*) of the Lemma and (0·17).

(1·7) Assume that $(X,H) \notin \mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$ and consider the reduction (X',H'). Then $(X',H') \notin \mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$, see (0·17). Suppose that $K' + (r-2)H'$ is not nef. Using (0·1) we find an extremal rational curve C such that

(4) $(K' + (r-2)H'.C) < 0$.

If we let $a = (H'.C)$, $b = -(K'.C)$, so that $a \geqslant 1$, $b \leqslant r+1$, it follows by (1·6) that

(5) $(K' + (r-1)H'.C) > 0$.

Therefore, using (4) and (5), we get

(6) $(r-2)a < b < (r-1)a$.

This gives $a \geqslant 2$ and

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(7) $2(r-2) \leqslant (r-2)a < b \leqslant r+1$.

Thus, one of the following holds, by (6) and (7):

(i) $r = 3, a = 3, b = 4$;
(ii) $r = 3, a = 2, b = 3$;
(iii) $r = 4, a = 2, b = 5$.

In case (i), it follows by (0·5) that $\mathrm{Pic}(X') \simeq \mathbb{Z}$. If we consider the divisor

$$H_1 =: -K' - H',$$

we have $(H_1.C) = 1$, so H_1 is ample since $\mathrm{Pic}(X') \simeq \mathbb{Z}$. We have $(K' + 4H_1.C) = 0$; therefore, by (1·2), $X' \simeq \mathbb{P}^3$, $H_1 \in |\mathcal{O}(1)|$; hence $H' \in |\mathcal{O}(3)|$.

If we are in case (ii), we let $H_1 =: K' + 2H'$. By (1·6) H_1 is ample and we have $(K' + 3H_1.C) = 0$, so $(X', H_1) \in \mathscr{A} \cup \mathscr{B}$ by (1·4). Clearly we cannot have $X' \simeq \mathbb{P}^3$, $H_1 \in |\mathcal{O}(1)|$. We are left with two possibilities: either $X' \simeq Q^3$, $H_1 \in |\mathcal{O}(1)|$, so $H' \in |\mathcal{O}(2)|$, or (X', H') is a scroll over a curve, so X' is a $\mathbb{P}^2$-bundle and H' induces $\mathcal{O}(2)$ on each fibre. Finally, assume that we are in case (iii). Then $H_1 =: K' + 3H'$ is ample by (1·6). Since $(K' + 5H_1.C) = 0$, it follows by (1·2) that $X' \simeq \mathbb{P}^4$, $H_1 \in |\mathcal{O}(1)|$; hence $H' \in |\mathcal{O}(2)|$.

Thus we have proved that either $(X', H') \in \mathscr{D}$ or $K' + (r-2)H'$ is nef, and hence semi-ample by (0·2). The proof of the Theorem is complete modulo the key lemma above.

Proof of the Lemma. (*a*) Assume that $\dim Y < r$ and denote by F a general (hence smooth) fibre of ϕ_i. We have $m(K_X + iH)|_F \sim 0$; hence

(8) $K_F + iH|_F \approx 0$.

The Kodaira vanishing theorem gives

(9) $H^j(\mathcal{O}_F(K_F + nH|_F)) = 0$ for $j > 0$ and $n > 0$.

Using (8) we get

(10) $H^0(\mathcal{O}_F(K_F + nH|_F)) = 0$ if $n \leqslant i-1$.

Consider the polynomial $P(n) = \chi(\mathcal{O}_F(K_F + nH|_F))$. By (9) and (10) it follows that $P(n) = 0$ for $1 \leqslant n \leqslant i-1$. We get

$$\dim F = \deg P \geqslant i-1.$$

Since $\dim F = r - \dim Y$, it follows that $\dim Y \leqslant r - i + 1$.

(*b*) If $V \in \overline{NE}(X)$, $(K_X + iH.V) \geqslant 0$ since we assumed $K_X + iH$ nef. Moreover, $(K_X + iH.V) > 0$ if $(K_X.V) \geqslant 0$, by ampleness of H. As we supposed that ϕ_i is not finite, there is an effective curve V such that $(K_X + iH.V) = 0$. Therefore, by (0·1), we may find an extremal rational curve C such that $(K_X + iH.C) = 0$. If we let $a = (H.C)$, $b = -(K_X.C)$, we get

$$ai = b \leqslant r+1.$$

Since by hypothesis $i \geqslant r-1$, one of the following holds:

(i) $r = 2$;
(ii) $r = 3, a = 2$;
(iii) $r \geqslant 3, a = 1$.

In case (i) the conclusion follows easily from the explicit description of extremal rational curves in case of surfaces (see [**20**], theorem 2·1). Case (ii) does not occur. Indeed, we would have $b = 4$, so $\mathrm{Pic}(X) \simeq \mathbb{Z}$ by (0·5) and in this case ϕ_2 would be finite. In what follows we shall be concerned with the remaining case $r \geqslant 3$, $a = 1$, hence $b = i$. If C' is any effective curve belonging to R, we may write $C' \approx \alpha C$ for some

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positive rational number α. Since we have $a = 1$, it follows that α is a natural number. Therefore we have $b = \min\{-(K_X.C')|C' \text{ an effective curve of } R\}$. Denote by $E \subset X$ the locus of points of effective curves belonging to R. Since $b = i \geqslant r-1$, it follows from (0·4) that we have $\dim E \geqslant r-1$. As ϕ_i is generically finite, we must have $E \neq X$, hence $\dim E = r-1$. It follows that E is an irreducible divisor (see [**14**], proposition 5·5). Moreover, the argument in [**21**], lemma 3·3, shows that we have $c =: -(E.C) > 0$. The morphism $f = \mathrm{cont}_R$ (see 0·3) is an isomorphism outside E and

$$s =: \dim f(E) < r-1.$$

Our aim is to prove that E is an exceptional divisor, see (0·11). We first show that $s = 0$, so E is contracted by f to a point. Suppose the morphism f is given by the complete linear system $|D|$. Denote by B the intersection of s generic members of $|D|$ (so $B = X$ if $s = 0$). By Bertini's theorem, B is smooth and connected. Let f' denote the restriction of f to E and let F be a general fibre of f'. Note that F is a (reduced) connected component of $G =: B \cap E$. We want first to prove that

(11) $H^j(\mathcal{O}_G(-nH|_G)) = 0$ for $j < \dim G = r-s-1$ and $n \geqslant c$.

To prove (11), consider the exact sequence

$$0 \to \mathcal{O}_B(-G-nH|_B) \to \mathcal{O}_B(-nH|_B) \to \mathcal{O}_G(-nH|_G) \to 0.$$

By Kodaira vanishing we have

(12) $H^j(\mathcal{O}_B(-nH|_B)) = 0$ for $j < \dim B = r-s$ and $n > 0$.

On the other hand, we claim that the divisor $G+nH|_B$ on B is nef and big if $n \geqslant c$. Indeed, by ampleness of $H|_B$, we have $(G+nH|_B.V) > 0$ for any effective curve $V \subset B$ which is not contained in G. If $V \subset G$, it follows that $[V] \in R$ since V is contracted by f. Recalling that $(E.C) = -c$, we get $(G+nH|_B.C) \geqslant 0$ if $n \geqslant c$, so $(G+nH|_B.V) \geqslant 0$ if $[V] \in R$. Therefore $(G+nH|_B.V) \geqslant 0$ for any effective curve $V \subset B$. Moreover, $G+nH|_B$ is also big, since $H|_B$ is ample and G is effective. Hence we deduce, using Kawamata–Viehweg vanishing (see [**13**] or [**27**]), that

(13) $H^j(\mathcal{O}_B(-G-nH|_B)) = 0$ for $j < r-s$ and $n \geqslant c$.

By (12), (13) and the exact sequence we get (11). Then (since F is a reduced, connected component of G) we also have

(14) $H^j(\mathcal{O}_F(-nH|_F)) = 0$ for $j < r-s-1$ and $n \geqslant c$.

On the other hand,

(15) $K_F \sim (K_X+E)|_F \approx (-b-c)H|_F$.

Indeed, by the properties of f, any effective curve on F belongs to R. Therefore, if D_1 and D_2 are two divisors on X such that $(D_1.C) = (D_2.C)$ we get $D_1|_F \approx D_2|_F$ and the relation (15) follows.

Using Serre duality and (15) we obtain

(16) $H^{r-s-1}(\mathcal{O}_F(-nH|_F)) = H^0(\mathcal{O}_F(K_F+nH|_F)) = 0$ for $n \leqslant b+c-1$.

Consider now the polynomial $P(n) =: \chi(\mathcal{O}_F(nH|_F))$. By (14) and (16) it follows that

(17) $P(-n) = 0$ for $c \leqslant n \leqslant b+c-1$.

This gives

$$r-s-1 = \dim F = \deg P \geqslant b = i \geqslant r-1; \quad \text{hence} \quad s = 0 \quad \text{and} \quad b = i = r-1.$$

Therefore we have $F = E$ and moreover

(18) $\chi(\mathcal{O}_E(-nH|_E)) = 0$ for $c \leqslant n \leqslant c+b-1 = c+\dim E-1$.

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If we let $d =: (H|_E)^{r-1} > 0$, the relation (18) gives

$$(19)\quad \chi(\mathcal{O}_E(nH|_E)) = \frac{d}{(r-1)!}(n+c)(n+c+1)\dots(n+c+r-2).$$

Next we want to prove that $\chi(\mathcal{O}_E) = 1$. Indeed, since $K_E \approx (-c-r+1)H|_E$ by (15), we get (using [15], chapter II, § 2, theorem 1)

$$(20)\quad \chi(\mathcal{O}_E) = \chi(\mathcal{O}_E(K_E + (c+r-1)H|_E)).$$

But, by duality and (14), it follows that

$$(21)\quad H^j(\mathcal{O}_E(K_E + (c+r-1)H|_E)) = H^{r-j-1}(\mathcal{O}_E(-(c+r-1)H|_E)) = 0 \quad \text{if} \quad j > 0.$$

Using (20) and (21) we get

$$(22)\quad \chi(\mathcal{O}_E) = h^0(\mathcal{O}_E(K_E + (c+r-1)H|_E)), \text{ so } \chi(\mathcal{O}_E) = 0 \text{ or } 1$$

since $K_E + (c+r-1)H|_E \approx 0$.

By (19) $\chi(\mathcal{O}_E) \neq 0$. It follows that

$$(23)\quad \chi(\mathcal{O}_E) = 1.$$

Using this relation, (19) gives $d = c = 1$ and

$$(24)\quad \chi(\mathcal{O}_E(nH|_E)) = \frac{1}{(r-1)!}(n+1)(n+2)\dots(n+r-1).$$

Now, by Serre duality, the relations (14) and (24) give

$$\begin{aligned}(25)\quad h^0(\mathcal{O}_E(K_E + (r+1)H|_E)) &= h^{r-1}(\mathcal{O}_E(-(r+1)H|_E)) \\ &= (-1)^{r-1}\chi(\mathcal{O}_E(-(r+1)H|_E)) \\ &= (-1)^{r-1}(-1)^{r-1}r \\ &= r.\end{aligned}$$

Since $c = 1$, (15) gives $K_E \approx -rH|_E$, so $K_E + (r+1)H|_E \approx H|_E$. Therefore,

$$K_E + (r+1)H|_E$$

is ample, $(K_E + (r+1)H|_E)^{r-1} = (H|_E)^{r-1} = d = 1$ and $h^0(\mathcal{O}_E(K_E + (r+1)H|_E)) = r$ by (25). It follows by (0·7) that $E \simeq \mathbb{P}^{r-1}$ and $H|_E \in |\mathcal{O}(1)|$. Since C is a line and

$$(E.C) = -c = -1,$$

we get $\mathcal{O}_X(E) \otimes \mathcal{O}_E \simeq \mathcal{O}(-1)$, so E is an exceptional divisor, as we wanted.

(c) Assume that $\dim Y < r$. If $i = r+1$, Y is a point by the first assertion of the Lemma and we get $K_X + (r+1)H \approx 0$. By (0·6) and (0·9) we get $(X, H) \in \mathscr{A}$.

If $i = r$, $\dim Y \leqslant 1$ by statement (a) of the Lemma. If Y is a point, using (0·6) and (0·10) as above we get $X \simeq Q^r$, $H \in |\mathcal{O}(1)|$. If Y is a (smooth) curve, ϕ_r is flat. If F denotes a smooth fibre of ϕ_r, we get $F \simeq \mathbb{P}^{r-1}$, $H|_F \in |\mathcal{O}(1)|$ as above. Now, if E is an arbitrary (closed) fibre, it follows that $(H^{r-1}.E) = 1$ by flatness of ϕ_r. In particular E is irreducible and generically reduced, hence reduced because it is Cohen–Macaulay. By semicontinuity we get

$$h^0(\mathcal{O}_E(H|_E)) \geqslant h^0(\mathcal{O}_F(H|_F)) = r.$$

It follows that $E \simeq \mathbb{P}^{r-1}$, $H|_E \in |\mathcal{O}(1)|$, by (0·7), so ϕ_r is a scroll over a curve. Thus $(X, H) \in \mathscr{B}$ if $i = r$.

Assume now that $i = r-1$. Again by the first part of the Lemma we get $\dim Y \leqslant 2$. If Y is a point, using (0·6) we get that (X, H) is a Del Pezzo manifold, see (0·15). We are left with the cases when Y is a curve or a surface. First of all, if $r = 2$ and Y is a curve, (X, H) is easily seen to be a hyperquadric fibration, with reduced, possibly

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reducible, fibres. Assume that $r \geqslant 3$. In order to have better control on the special fibres we replace ϕ_{r-1} by a contraction of an extremal ray. Indeed, using (0·1), we may find an extremal rational curve C with $(K_X + (r-1)H.C) = 0$. We again let $a = (H.C)$, $b = -(K_X.C)$ and we get

$$a(r-1) = b \leqslant r+1.$$

Therefore, either $a = 1$, or $r = 3$, $a = 2$, $b = 4$. This last possibility is absurd since by (0·5) we would have $\mathrm{Pic}(X) \simeq \mathbb{Z}$, so ϕ_2 would be constant. So we have $a = 1, b = r-1$. Consider the extremal ray R generated by C. If R is not nef, reasoning exactly as in the proof of part (b) of the Lemma, we may find an exceptional divisor on X. Therefore, after replacing (X, H) by its reduction (X', H'), we may assume that R is nef. Consider the morphism $f =: \mathrm{cont}_R$. For a general fibre F of f, we get as in the proof of (15):

$$K_F \sim K_{X'}|_F \approx (1-r)H'|_F.$$

Using this and Kodaira vanishing it follows as before that

$$\chi(\mathcal{O}_F(K_F + nH'|_F)) = 0 \quad \text{for} \quad 1 \leqslant n \leqslant r-2; \quad \text{hence} \quad \dim F \geqslant r-2.$$

Since $K' + (r-1)H' \approx 0$, $\dim F < r$ or, equivalently, $\dim f(X') > 0$. Assume that $\dim F = r-1$, hence $f(X')$ is a (smooth) curve. Then f is flat and a simple argument shows that it has reduced, irreducible fibres (see [**21**], p. 185). It follows by (0·10) that we have $F \simeq Q^{r-1}$, $H'|_F \in |\mathcal{O}(1)|$ for a smooth (closed) fibre F of f. For an arbitrary fibre E we get $(H'^{r-1}.E) = 2$ by flatness and $h^0(\mathcal{O}_E(H'|_E)) \geqslant h^0(\mathcal{O}_F(H'|_F)) = r+1$ by semi-continuity. Therefore E is also a hyperquadric by (0·8) and (X', H') is a hyperquadric fibration. Actually, since $r \geqslant 3$, we must have $X = X'$ (see the proof of (0·17)).

Finally, assume that $\dim F = r-2$ and denote by S the image of f, which is a normal surface. We claim that f is equidimensional and S is smooth. Indeed, let $s \in S$ be a closed point and denote by E the fibre over s. Consider the embedding of X' given by $|mH'|$ for $m \gg 0$. Let $\tilde{S}$ be the smooth surface got by intersecting $r-2$ general members of $|mH'|$. We first prove that $\dim E = r-2$. Assume that $\dim E = r-1$; since $(E.R) = 0$, it follows, as in the proof of (15), that

(26) $E|_E \approx 0$.

If we denote by $\tilde{f}$ the restriction of f to $\tilde{S}$, we see that $\tilde{f}$ contracts the curve $V =: \tilde{S} \cap E$ to a point. Therefore, using (26), we get

$$0 > (V^2)_{\tilde{S}} = m^{r-2}(H'^{r-2}.E^2)_X = m^{r-2}(H'|_E^{r-2}.E|_E)_E = 0.$$

This contradiction shows that $\dim E = r-2$; hence f is equidimensional. Therefore, by construction of $\tilde{S}$, we may assume that $\tilde{S} \cap E'$ is zero-dimensional and reduced, where E' denotes E with reduced structure. On the other hand, for a general fibre F, we get using (0·9) $F \simeq \mathbb{P}^{r-2}$, $H'|_F \in |\mathcal{O}(1)|$. Therefore, $\tilde{f}$ has degree m^{r-2}; since the number of points in $\tilde{S} \cap E'$ is $m^{r-2}(E'.H'^{r-2}) \geqslant m^{r-2}$ and S is normal, it follows by a well-known criterion that $\tilde{f}$ is étale over s. Therefore S is smooth at s since $\tilde{S}$ is smooth. Now, since both S and X' are smooth and f is equidimensional, it follows that f is flat. Therefore $(H'^{r-2}.F) = 1$ for any fibre F. It follows that F is irreducible and generically reduced, hence reduced since it is Cohen–Macaulay. We may now deduce exactly as before, using semicontinuity and (0·7), that $F \simeq \mathbb{P}^{r-2}$, $H'|_F \in |\mathcal{O}(1)|$. Thus the reduction (X', H') is a scroll over a surface. But in this case (see (0·17)) we must have $X = X'$. This ends the proof of our lemma.

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Section 2

This section is devoted to a couple of applications of the main result. In the following four corollaries we consider polarized pairs (X, H), where X is a threefold and H is an *effective*, *smooth*, ample divisor on X.

The first application is an improvement of a result due to Sommese (see [**24**], theorem 2·4). In case H is very ample it was proved in [**11**], theorem II.

COROLLARY 1. *Consider a polarized pair (X, H) such that H is a smooth, birationally ruled surface. Then $(X, H) \in \mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$ or there is a reduction $(X', H') \in \mathscr{D}$.*

Proof. By the adjunction formula $(K_X + H)|_H \sim K_H$. Since H is ruled, K_H, hence also $K_X + H$, is not nef. The result follows from (1·7).

The second result is due to L. Bădescu, who proved it by direct arguments (see [**1**], theorem 5, [**2**], theorem 1 and theorem 3, and [**3**]).

COROLLARY 2 (Bădescu). *Let (X, H) be a polarized pair such that H is a smooth geometrically ruled surface. Then (X, H) is either a scroll over a curve, or, in case $H \simeq \mathbb{P}^1 \times \mathbb{P}^1$, there are two further possibilities: $X \simeq \mathbb{P}^3, H \in |\mathscr{O}(2)|$ and $X \simeq Q^3, H \in |\mathscr{O}(1)|$.*

Proof. By Corollary 1, $(X, H) \in \mathscr{B} \cup \mathscr{C}$ or there is a reduction $(X', H') \in \mathscr{D}$. We shall prove that either $(X, H) \in \mathscr{B}$ or $X \simeq \mathbb{P}^3$, $H \in |\mathscr{O}(2)|$. Assume that $(X, H) \in \mathscr{C}$. If (X, H) is a Del Pezzo threefold, it follows that H is a Del Pezzo surface. But, as is well-known, the only geometrically ruled Del Pezzo surfaces are $\mathbb{P}^1 \times \mathbb{P}^1$ and the projective plane blown up at a point, denoted F_1. Using the classification of Del Pezzo threefolds (see [**12**] or [**8**]), it follows that the case $H \simeq F_1$ is impossible, while for $H \simeq \mathbb{P}^1 \times \mathbb{P}^1$ we get $X \simeq \mathbb{P}^3$, $H \in |\mathscr{O}(2)|$. Next we prove that (X, H) cannot be a hyperquadric fibration. Indeed, in this case, as in Bădescu's original approach, by Lefschetz's theorem on hyperplane sections we would have $\mathrm{Pic}(X) \simeq \mathrm{Pic}(H)$ via restriction. Therefore, if Q denotes a general fibre of the hyperquadric fibration and $F =: Q \cap H$, we may find an invertible sheaf on X, say $\mathscr{O}_X(D)$, such that $(D|_H . F) = 1$. This leads to a contradiction, since we have

$$1 = (D|_H . F)_H = (D . H . Q)_X = (D|_Q . H|_Q)_Q$$

and the last integer has to be even.

Assume now that (X, H) is a scroll over a surface. We shall first prove that this is possible only if $H \simeq F_1$. Indeed, recall that a geometrically ruled surface is a minimal model, unless it is isomorphic to F_1. Now, if $f: X \to S$ is a morphism making X a scroll over the surface S, the restriction of f to H is birational, hence an isomorphism, unless $H \simeq F_1$ and $S \simeq \mathbb{P}^2$, when it is the blowing up of a point. As before, we must have $\mathrm{Pic}(X) \simeq \mathrm{Pic}(H) \simeq \mathrm{Pic}(S)$, if $H \not\simeq F_1$. But this is clearly absurd since X is a $\mathbb{P}^1$-bundle over S. Assume now that $f: X \to \mathbb{P}^2$ gives the scroll structure and f restricted to H is the blowing up of a point. Denote by $C \subset H$ the exceptional divisor of the blowing up. In this case we shall see that $X \simeq \mathbb{P}^1 \times \mathbb{P}^2$, $H \in |\mathscr{O}(1, 1)|$, so X is a scroll over $\mathbb{P}^1$, too. If F denotes a fibre of the ruling of H, $\mathscr{O}_H(C)$ and $\mathscr{O}_H(F)$ form a basis for $\mathrm{Pic}(H)$ (see [**9**], chapter V, § 2). Write $H|_H \sim aC + bF$, with $a > 0$, $b > a$ (see [**9**], loc. cit.). If $P \simeq \mathbb{P}^1$ is a fibre of f, we may write

$$-2 = (K_X . P)_X = (K_X|_H . C)_H = (K_H - H|_H . C)_H = a - b - 1,$$

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using the fact that C is a fibre of f, and the formula $K_H \sim -2C - 3F$ (see [9], loc. cit.). It follows that $b = a+1$. If E denotes the inverse image by f of a line in $\mathbb{P}^2$, we get

$$(H^3) = a(a+2), \quad (H^2.E) = a+1, \quad (H.E^2) = 1, \quad (E^3) = 0.$$

Since $\mathrm{Pic}(X) \simeq \mathrm{Pic}(H)$, we may find an invertible sheaf on X, say $\mathcal{O}_X(D)$, such that $D|_H = F$. We easily deduce $D \sim H - aE$.

Consider now the exact sequence

$$0 \to \mathcal{O}_X(-aE) \to \mathcal{O}_X(D) \to \mathcal{O}_H(F) \to 0.$$

Since we have $H^1(X, \mathcal{O}_X(-aE)) = H^1(\mathbb{P}^2, \mathcal{O}(-a)) = 0$, we get that $|D|$ is a pencil. Since two distinct members of $|D|$ can meet only outside H, their intersection is finite, hence empty. It follows that $(D^3) = (H-aE)^3 = 0$ and this gives $a = 1$. Now it is very easy to see that $X \simeq \mathbb{P}^1 \times \mathbb{P}^2$, $H \in |\mathcal{O}(1,1)|$.

Assume now that we have a reduction $(X', H') \in \mathscr{D}$. We shall prove that this is impossible. Since H is minimal unless $H \simeq F_1$, it follows that either $X = X'$ or $H' \simeq \mathbb{P}^2$. This last case is absurd since (as is well-known, see for instance [7], corollary 3·10) we would have $X' \simeq \mathbb{P}^3$, $H' \in |\mathcal{O}(1)|$ and this forces $X = X'$. Therefore we have $(X, H) \in \mathscr{D}$. The cases $X \simeq \mathbb{P}^3$, $H \in |\mathcal{O}(3)|$ and $X \simeq Q^3$, $H \in |\mathcal{O}(2)|$ are not possible since H is geometrically ruled. It remains to exclude the case when X is a $\mathbb{P}^2$-bundle and H induces $\mathcal{O}(2)$ on each fibre. This is done by the same kind of argument as in the case of hyperquadric fibrations. The proof of Corollary 2 is complete.

The next application is the main result from [**24**].

COROLLARY 3 (Sommese). *Let (X, H) be a polarized pair such that H is a smooth non-ruled surface. Then either (X, H) is a scroll over a surface, or there is a reduction (X', H') such that H' is a minimal model.*

Proof. Assume H to be non-minimal. Since $(K_X + H)|_H \sim K_H$, it follows that $K_X + H$ is not nef. By (1·7), either $(X, H) \in \mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$, or there is a reduction (X', H') such that $(X', H') \in \mathscr{D}$ or H' is minimal. Since we assumed H to be non-ruled, $(X, H) \in \mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$ is possible only if (X, H) is a scroll over a surface. Finally, $(X', H') \in \mathscr{D}$ is impossible since H' is non-ruled.

Our main application in dimension three is the following result, which coupled with Corollary 1 describes completely the threefolds supporting a smooth surface which is not of general type as an ample divisor.

COROLLARY 4. *Let (X, H) be a polarized pair such that H is a smooth surface and denote by $\kappa(H)$ its Kodaira dimension. If $\kappa(H) = 0$, (X, H) is either a scroll over a surface, or there is a reduction (X', H') such that H' is a K3 surface and, consequently, X' is a Fano manifold with $H' = -K'$. If $\kappa(H) = 1$, either (X, H) is a scroll over a surface, or there is a reduction (X', H') and a morphism $g: X' \to C$ onto some smooth curve C, which is a Del Pezzo fibration, i.e. $H'|_F \sim -K_F$ for any smooth fibre F of g.*

Proof. Assume that (X, H) is not a scroll over a surface. If $\kappa(H) = 0$, by Corollary 3, we may find a reduction (X', H') such that H' is minimal. By the classification of minimal surfaces of Kodaira dimension zero, H' may be abelian, K3, Enriques or hyperelliptic. But one may prove (see for instance [2]) using Lefschetz's theorem that an abelian, Enriques or hyperelliptic surface cannot be an ample divisor on a smooth threefold.

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Thus we are left with the case when H' is $K3$. It follows that $H'+K' \sim 0$ by Lefschetz's theorem. Assume that $\kappa(H) = 1$. Using (1·7) as in the proof of Corollary 3, we may find a reduction (X', H') such that $K'+H'$ is semi-ample (in particular H' is minimal). Let g be the map associated to $|m(K'+H')|$ for $m \gg 0$. Since H' is an elliptic surface, the restriction of g to H' gives a (pluricanonical) map onto some curve and the rest is clear.

To put the preceding corollary into perspective, we mention the following rather general (and not difficult) fact.

PROPOSITION 5. *Let (X, H) be a polarized pair with $\dim X = r \geqslant 2$. Assume that H is smooth and $\kappa(H) < r-1$. Then $\kappa(X) = -\infty$.*

Proof. Assume that $|mK_X| \neq \varnothing$ for some $m > 0$ and let $E \in |mK_X|$. Write $E = aH + E'$, with $a \geqslant 0$ and $H \notin \operatorname{supp}(E')$. Since nH is very ample for $n \gg 0$, we may find $n \gg 0$ and $D \in |nE|$ such that $H \notin \operatorname{supp}(D)$. Since we have $(D+nmH)|_H \sim nmK_H$, it follows that $|nmK_H|$ is very ample outside $H \cap \operatorname{supp}(D)$, so $\kappa(H) = r-1$. This contradiction proves the proposition.

In the sequel, we consider polarized pairs (X, H) with $\dim X = r \geqslant 1$. Write the Hilbert polynomial of the pair (X, H) as

$$\chi(\mathcal{O}_X(nH)) = \sum_{i=0}^{r} a_i \binom{n+i-1}{i};$$

we define the *sectional genus* g of the pair (X, H) by

$$g = 1 - a_{r-1}.$$

It is not difficult to prove that the following relation is true:

(27) $2g-2 = (K_X + (r-1)H.H^{r-1})$.

LEMMA 6. *Assume that (X, H) is a scroll over a curve C. Then the sectional genus of (X, H) equals the genus of C.*

Proof. Let $f\colon X \to C$ be the morphism making (X, H) a scroll. As is well-known, we may find a very ample divisor $\bar{H}$ on X such that $\bar{H}|_F \in |\mathcal{O}(1)|$ for any fibre F of f. It follows easily by induction on $\dim X = r$ that the sectional genus of the pair $(X, \bar{H})$ is equal to the genus of C. Therefore, it will be enough, using (27), to prove the following:

(28) $(K_X + (r-1)H.H^{r-1}) = (K_X + (r-1)\bar{H}.\bar{H}^{r-1})$.

Since $\bar{H}|_F \in |\mathcal{O}(1)|$ and $H|_F \in |\mathcal{O}(1)|$ for any fibre F of f, there is some divisor D on C such that

(29) $\bar{H} \approx H + f^*(D)$.

Moreover, by a well-known formula, we have

(30) $K_X \approx -rH + f^*(E)$, for some divisor E on C.

Using (29) and (30) we get after some computations

$$(K_X + (r-1)\bar{H}.\bar{H}^{r-1}) = -(H^r) + (H^{r-1}.f^*(E)) = (K_X + (r-1)H.H^{r-1}).$$

Thus (28) is proved and we are done.

The next results were proved by Beltrametti and Palleschi in the case of three-folds [4]. Using their idea, we extend them to arbitrary dimension (see also [10], [11] for the case where H is very ample).

LEMMA 7. *For any polarized pair (X, H), the sectional genus g is non-negative.*

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Proof. By (1·5) we may assume that either $(K_X+(r-1)H.H^{r-1}) \geqslant 0$ (so $g \geqslant 1$ by (27)) or $(X,H) \in \mathscr{A} \cup \mathscr{B}'$. If $(X,H) \in \mathscr{A} \cup \mathscr{B}'$ it follows that $g = 0$ unless (X,H) is a scroll over a curve, when the previous lemma applies.

COROLLARY 8 (cf. [4], proposition 3·1 and [10], proposition 2·3). *Let (X,H) be a polarized pair with $g = 0$. Then one of the following holds:*

$X \simeq \mathbb{P}^r$, $H \in |\mathscr{O}(1)|$, or $X \simeq Q^r$, $H \in |\mathscr{O}(1)|$, or $X \simeq \mathbb{P}^2$, $H \in |\mathscr{O}(2)|$, or (X,H) is a scroll over $\mathbb{P}^1$.

Proof. We argue as in the proof of Lemma 7, using also Lemma 6.

COROLLARY 9 (cf. [4], proposition 3·2, and [10], proposition 2·6). *Let (X,H) be a polarized pair with $g = 1$. Then (X,H) is either a Del Pezzo manifold or a scroll over an elliptic curve.*

Proof. If $K_X+(r-1)H$ is not nef, as in the proof of the preceding Corollary, it follows that (X,H) is a scroll over an elliptic curve. If $K_X+(r-1)H$ is nef, it has to be trivial since $(K_X+(r-1)H.H^{r-1}) = 0$ (see [15], chapter I, § 4, proposition 3, and (0·6)).

COROLLARY 10 (cf. [4], theorem 3·3, and [11], theorem 1). *Let (X,H) be a polarized pair with $g \geqslant 2$ and $2g-2 \leqslant (H^r)$. Then (X,H) is of one of the following types:*

(2·1) *$r = 2$ and either X is birationally ruled or $K_X \approx 0$; $r \geqslant 3$ and either:*

(2·2) *(X,H) is a scroll over a curve or a surface;*

(2·3) *(X,H) is a hyperquadric fibration;*

(2·4) *There is a reduction (X',H') such that $(X',H') \in \mathscr{D}$;*

(2·5) $K_X+(r-2)H \sim 0$.

Proof. Assume that $r = 2$. Since $2g-2 = (H^2)+(H.K_X)$, the hypothesis gives $(H.K_X) \leqslant 0$. Thus either $|mK_X| = \varnothing$ for any $m \geqslant 1$ and X is ruled by Enriques' criterion, or, if $|mK_X| \neq \varnothing$ for some $m \geqslant 1$, it follows that $K_X \approx 0$. Assume $r \geqslant 3$.

Using (1·7) we deduce that either (X,H) is as stated in (2·2)–(2·4), or $K'+(r-2)H'$ is nef. On the other hand, by direct computation, we find that (X,H) and (X',H') have the same sectional genus; moreover, $(H^r) \leqslant (H'^r)$, with equality only if $X = X'$. Using this and the hypothesis $2g-2 \leqslant (H^r)$ we get

$$(31)\quad 2g-2 = (K'+(r-1)H'.H'^{r-1}) \leqslant (H^r) \leqslant (H'^r).$$

This implies

$$(K'+(r-2)H'.H'^{r-1}) \leqslant 0.$$

Assuming now $K'+(r-2)H'$ to be nef, by [15], chapter I, § 4, proposition 3, and (0·6), we get

$$K'+(r-2)H' \sim 0.$$

Using (31) it follows that $(H^r) = (H'^r)$, so $X = X'$ and we are in case (2·5).

The final application is due to Lanteri and Palleschi [18].

COROLLARY 11 (cf. [5]). *Let $f\colon X \to Q^r$ be a finite morphism from a manifold X to the smooth r-dimensional hyperquadric. Suppose that f is not an isomorphism. Then the ramification divisor R is ample, unless $r = 2$, X is a scroll and R is a sum of fibres.*

Proof. Let H be a divisor in $|f^*(\mathscr{O}_Q(1))|$ and assume that R is not ample. Since $R \in |K_X+rH|$, it follows by (1·4) that $(X,H) \in \mathscr{A} \cup \mathscr{B}$. Since $(H^r) = 2.\deg f$, we may assume that (X,H) is a scroll over a curve. The restriction of f to a fibre of the scroll gives a $\mathbb{P}^{r-1}$ embedded in Q^r as a linear space. This implies $r = 2$ and, in this case, K_X+2H is a combination of fibres.

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