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Strictly singular and strictly cosingular operators on spaces of continuous functions

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1. *Introduction*

In this paper we will be concerned with studying operators $T: C(K, X) \to Y$ defined on Banach spaces of continuous functions. We will be particularly interested in studying the classes of strictly singular and strictly cosingular operators. In the process, we obtain answers to certain questions recently raised by Bombal and Porras in [**5**]. Specifically, we study Banach space X and Y for which an operator $T: C(K, X) \to Y$ with representing measure m is strictly singular (strictly cosingular) whenever m is strongly bounded and $m(A)$ is strictly singular (strictly cosingular) for each Borel subset A of K. Along the way we establish several results dealing with non-compact operators on continuous function spaces, and we consolidate numerous results concerning extension theorems for operators defined on these same spaces. Also, we join Saab and Saab [**25**] in demonstrating that if l^1 does not embed in X^* then the adjoint T^* of a strongly bounded map must be weakly precompact, thereby presenting an alternative solution to a question raised in [**2**].

Throughout the paper, K will be a compact Hausdorff space, and each of X and Y will be a real (infinite-dimensional) Banach space. The statement that Y embeds in X means that Y is linearly homeomorphic to a closed linear subspace of X. The symbol B_X will represent the closed unit ball of X. If $A \subset X$, then $\bar{A}$ will denote the norm closure of A, $\bar{A}^{\mathrm{w}}$ will denote the weak closure of A, and $[A]$ will denote the closed linear span of A. Further, $C(K, X)$ will be the Banach space (sup norm) of all continuous X-valued functions defined on K, and $T: C(K, X) \to Y$ will be a continuous linear function (= operator) with representing measure $m: \Sigma \to B(X, Y^{**})$, where Σ is the σ-algebra of Borel subsets of K, Y^{**} is the bidual of Y, and $B(X, Y^{**})$ is the Banach space of all operators $L: X \to Y^{**}$. This correspondence between T and m will be denoted by $m \leftrightarrow T$. (We refer the reader to Brooks and Lewis [**8**] or chapter 3 of Dinculeanu [**12**] for a discussion of the Riesz representation theorem in this setting.) We denote the semivariation of the vector measure m by $\tilde{m}$, and we say that m (or T) is strongly bounded (continuous at ϕ) if $(\tilde{m}(A_i)) \to 0$ whenever (A_i) is a pairwise disjoint sequence from Σ. By theorem 4·4 of [**8**], a strongly bounded representing measure takes its values in $B(X, Y)$. Additionally, if m is strongly bounded, then there is a regular probability measure μ on K so that $\tilde{m}(A) \to 0$ as $\mu(A) \to 0$ (see for example lemma 3·1 of [**8**] and the proof of theorem 4, p. 11, of [**11**]). We call this measure μ a control measure for m. We remark that the Riesz representation theorem remains

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basically unchanged in this setting if the domain space is a locally compact Hausdorff space. We refer the reader to [**10**, **11**], or [**8**] for any unexplained notation or terminology.

An operator $T:X\to Y$ is said to be strictly singular if T does not have a bounded inverse on any infinite-dimensional Banach subspace of X, and T is said to be strictly cosingular if $LT:X\to Z$ fails to be a surjection for all infinite-dimensional Banach spaces Z and all operators $L:Y\to Z$. It is not difficult to see that a compact operator is both strictly singular and strictly cosingular. Further, a strictly singular operator $T:X\to Y$ must be unconditionally converging, for by lemma 1, p. 34 of [**19**], T must be an isomorphism on an isomorphic copy of c_0 if T fails to be unconditionally converging. Consequently by Swartz [**26**], a strictly singular operator $m\leftrightarrow T:C(K,X)\to Y$ must be strongly bounded.

In part 2 of the paper cited above, Pelczynski [**20**] proves that the identity mapping $j:c_0\to l^\infty$ is strictly cosingular; clearly this map is not unconditionally converging. For the convenience of the reader, we conclude the introduction with an argument for the strict cosingularity of j based on chapter 6 of Diestel and Uhl [**11**].

Suppose that Z is an infinite-dimensional Banach space and that $L:l^\infty\to Z$ is a bounded linear surjection. Either L is strongly bounded or not. If not, then L is an isomorphism on a copy of l^∞ (e.g. see I·1·18, VI·1·1, and VI·1·3 of Diestel and Uhl [**11**]). Clearly Z is not separable in this case.

If L is strongly bounded, then

$$L(x)=\sum_{n=1}^{\infty}x_n L(e_n),$$

where $x=(x_n)\in c_0$ and e_n is the nth canonical unit vector in c_0. Since the mapping $\phi:[-1,1]^{\mathbb{N}}\to Z$ defined by

$$\phi(a_n)=\sum_{n=1}^{\infty}a_n L(e_n)$$

is continuous ($\Sigma L(e_n)$ being unconditionally convergent), it follows that $L|_{c_0}$ is a compact operator. In both cases, Lj cannot be a surjection.

2. *Non-compact operators on continuous function spaces*

In this section we produce operators $m\leftrightarrow T:C(K,X)\to Y$ so that m is strongly bounded, $m(A)$ is compact for each $A\in\Sigma$, and yet T fails to be compact. (In certain specific instances, we can conclude that T is a surjection.) In the following section we give applications to strictly singular and strictly cosingular operators.

We will repeatedly use the fact that if $T:X\to Y$ is an operator and $\delta>0$ so that $\overline{T(B_X)}\supseteq\delta B_Y$, then there is $\gamma>0$ so that $T(B_X)\supseteq\gamma B_Y$ (e.g. see 22·4 in Jameson [**15**] or lemma B·3·5 in Pietsch [**21**]).

Let Δ be the Cantor set $\{-1,1\}^{\mathbb{N}}$, and let λ be Haar measure on Δ. Let C_{ni}, $1\leqslant i\leqslant 2^n$, denote the dyadic partition at the nth stage, so that, for example $C_{11}=\{(t_n):t_1=1\}$ and $C_{12}=\{(t_n):t_1=-1\}$. Let (r_n) be the sequence of Rademacher functions on Δ, i.e. $r_n(t)=t_n$, for $t\in\Delta$.

THEOREM 2·1. *If Y is an l^p space for $1<p\leqslant\infty$ and $L:X\to Y$ is a non-compact operator, then there is a non-compact operator $T:C(\Delta,X)\to Y$ with representing measure m so that m is strongly bounded and $m(A):X\to Y$ is compact for every $A\in\Sigma$.*

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Proof. First suppose that $L:X\to l^\infty$ is non-compact, and let (x_n) be a sequence in B_X such that no subsequence of $(L(x_n))$ is norm convergent. Let (e_n) denote the canonical unit vectors and (e_j^*) be the sequence of coefficient functionals. By diagonalizing, we may (and do) assume that $(\langle L(x_n), e_j^*\rangle)_{n=1}^\infty$ converges for all j. By choosing a further subsequence, we obtain $\epsilon>0$ so that

$$\|L(x_{2n})-L(x_{2n-1})\|>2\epsilon$$

for all n. It follows that there exist $j\in\{1,2\}$ and $n_1\in\mathbb{N}$ so that $|\langle L(x_j), e_{n_1}^*\rangle|>\epsilon$. Continue this process. Renumbering the terms of the sequences so constructed, we obtain strictly increasing sequences (n_i) and (k_i) of positive integers so that

$$\|\langle L(x_{k_i}), e_{n_i}^*\rangle|>\epsilon, \quad \text{for all } i.$$

For $A\in\Sigma$, define $m(A):X\to l^\infty$ by

$$m(A)(x)=(\langle e_{n_i}^*, L(x)\rangle\int_A r_{n_i}\,d\lambda) \quad (i\in\mathbb{N}).$$

Since $(\int_A r_{n_i}\,d\lambda)\to 0$ for all $A\in\Sigma$, and

$$\{|\langle e_{n_i}^*, L(x)\rangle|: i\in\mathbb{N}, x\in B_X\}$$

is bounded, it is clear that $m(A)\,x\in c_0$ and that $m(A)$ is the norm limit of finite rank operators. Further, it is not difficult to see that $\|m(A)\|\leqslant\|L\|\,\lambda(A)$. Therefore m is a dominated representing measure ([**12**]; [**8**], p. 148]) and is certainly strongly bounded. Let $T:C(\Delta,X)\to c_0\subset l^\infty$ be defined by

$$T(f)=\int_\Delta f\,dm.$$

Next let $f_i=r_{n_i}x_{k_i}$ for each $i\in\mathbb{N}$. Note that

$$\begin{aligned}T(f_j)&=\left(\int_\Delta\langle e_{n_i}^*, L(f_j(t))\rangle\, r_{n_i}(t)\,d\lambda\right)_{i=1}^\infty\\&=\left(\int_\Delta\langle e_{n_i}^*, L(x_{k_j})\rangle\, r_{n_j}(t)\,r_{n_i}(t)\,d\lambda\right)_{i=1}^\infty\\&=\langle e_{n_j}^*, L(x_{k_j})\rangle\, e_{n_j}.\end{aligned}$$

Therefore $\|T(f_i)-T(f_j)\|>\epsilon$ for $i\neq j$, and T is not compact.

Now suppose that $L:X\to l^p$ is not compact for $1<p<\infty$. Choose $\epsilon>0$ and a sequence (x_n) from B_X so that $L(x_n)$ is weakly convergent and $\|L(x_n)-L(x_{n+1})\|>\epsilon$ for all n. Let $w_n=L(x_n)-L(x_{n+1})$ for $n\in\mathbb{N}$: therefore $(w_n)\to 0$ weakly in l^p. Clearly

$$\|w_n\|\leqslant\|L(x_n)\|+\|L(x_{n+1})\|\leqslant 2\|L\|$$

for all n. Now use proposition 1·a·12 of Lindenstrauss and Tzafriri [**18**] to obtain a subsequence of (w_n) which is equivalent to a block basis (b_n) of (e_n), the canonical basis of l^p. For convenience we denote this subsequence by (w_n). Clearly the block basis (b_n) is semi-normalized. Thus it follows $(w_n)\sim(b_n)\sim(e_n)$. Let $J:[\{w_n:n\in\mathbb{N}\}]\to l^p$ be an isomorphism so that $J(w_n)=e_n$ for each $n\in\mathbb{N}$. Further, there is a projection $P:l^p\to[(w_n)]$ (see e.g. p. 53 of [**18**]).

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Now define m on Σ by

$$m(A)(x) = \left(\langle JPL(x), e_n^* \rangle \int_A r_n \, d\lambda\right)_{n=1}^{\infty}.$$

It follows that $m(A): X \to l^p$ is a bounded linear operator, $m(A)$ is again the norm limit of finite rank operators, and $\|m(A)\| \leqslant \|JPL\| \lambda(A)$. Thus m is certainly strongly bounded. Define $T: C(\Delta, X) \to l^p$ by

$$T(f) = \int_\Delta f \, dm.$$

Let $y_n = x_n - x_{n+1}$, and let $f_n = r_n y_n$. It follows that

$$T(f_n) = \langle J(w_n), e_n^* \rangle e_n,$$

and T fails to be compact even as an operator into c_0. ∎

Careful modifications of the construction in the preceding theorem allow us to apply this technique to different spaces. In the next two results, we assume that $L: X \to l^1$ is a non-compact operator. The reader will notice that the proof of Theorem 2·2 and Corollary 2·3 do not use the powerful 'Pelczynski theory' of non-compact operators into l^1 ([**20**]; [**10**], pp. 71–77). Rather, the proof of Theorem 2·2 is quite constructive and follows essentially from the definitions.

THEOREM 2·2. *If $L: X \to l^1$ is non-compact, then there is a continuous linear surjection $T: C(\Delta, X) \to l^1$ with representing measure m so that m is strongly bounded and $m(A): X \to l^1$ is a compact operator for all $A \in \Sigma$.*

Proof. Suppose that $L: X \to l^1$ is non-compact, and let (x_n) be a sequence in B_X so that $(L(x_n))$ does not have a norm convergent subsequence. By passing to a subsequence, we may assume that $(\langle L(x_n), e_j^* \rangle)_{n=1}^{\infty}$ converges for all j. We may also assume that $\epsilon > 0$ and $\|L(x_n) - L(x_{n+1})\| > 2\epsilon$ for all n. Arguing as in Theorem 2·1, we find increasing sequences (m_i), (n_i), and (k_i) of positive integers so that $m_i < n_i < m_{i+1}$ and

$$\|L\| \geqslant \sum_{j=m_i}^{n_i} |\langle L(x_{k_i}), e_j^* \rangle| > \epsilon \quad \text{for each } i.$$

Let

$$y_i = \sum_{i=m_i}^{n_i} \langle L(x_{k_i}), e_j^* \rangle e_j \quad (i \in \mathbb{N}).$$

It is not difficult to see that (y_i) is a basic sequence in l^1, and that (y_i) is equivalent to the canonical unit vector basis, i.e. $(y_i) \sim (e_i)$. Further, $Y = [\{y_i : i \in \mathbb{N}\}]$ is complemented by a projection on l^1. In fact, for each i choose $y_i^* \in l^\infty$ so that

$$y_i^* \in \operatorname{span}\{e_j^* : m_i \leqslant j \leqslant n_i\}, \quad \langle y_i^*, y_i \rangle = 1,$$

and

$$1/\|L\| \leqslant \|y_i^*\| \leqslant 1/\epsilon.$$

Define P by

$$P(x) = \sum_{i=1}^{\infty} y_i^*(x) y_i.$$

(Note that

$$\sum_{i=1}^{\infty} \|y_i^*(x) y_i\| \leqslant \|L\| \sum_{i=1}^{\infty} |y^*(x)| \leqslant (\|L\|/\epsilon)\|x\|.)$$

It follows that P is a projection onto Y.

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Next, we define $m:\Sigma\to B(X,Y)$ by

$$m(A)x=\sum_{s=1}^{\infty}\langle y_s^*,PL(x)\rangle\int_A r_s\,d\lambda.$$

As before, it follows that there is a constant $C>0$ so that $\|m(A)\|\leqslant C\lambda(A)$ for all $A\in\Sigma$. Thus m is a dominated representing measure. Since $(\int_A r_s\,d\lambda)\to 0$ for all $A\in\Sigma$, it follows that $m(A)$ is the norm limit of finite rank operators.

Now let $T:C(\Delta,X)\to Y$ be defined by

$$T(f)=\int_\Delta f\,dm=\sum_{s=1}^{\infty}\left(\int_\Delta\langle y_s^*,PLf(t)\rangle\,r_s(t)\,d\lambda\right)y_s.$$

If we define f_i to be $r_i x_{k_i}$ for $i\in\mathbb{N}$, then $\|f_i\|\leqslant 1$ and

$$\begin{aligned}T(f_i)&=\sum_{s=1}^{\infty}\left(\int_\Delta\langle y_s^*,PL(x_{k_i})\rangle\,r_i r_s\,d\lambda\right)y_s\\&=\langle y_i^*,PL(x_{k_i})\rangle\,y_i=\langle y_i^*,L(x_{k_i})\rangle\,y_i.\\&=y_i^*.\end{aligned}$$

Therefore, if $\gamma=(\gamma_i)\in l^1$ and $\|\gamma\|\leqslant 1$, then $\Sigma\gamma_i f_i$ belongs to the unit ball of $C(\Delta,X)$, and $\sum_{i=1}^{\infty}\gamma_i y_i\in T(B_{C(\Delta,X)})$. Consequently $T:C(\Delta,X)\to Y$ is a surjection.

Finally, let $J:Y\to l^1$ be an isomorphism; clearly JT is a surjection. It is easy to check that $Jm\leftrightarrow JT$ and that Jm is dominated and, consequently, strongly bounded. Obviously $Jm(A)$ is compact for each $A\in\Sigma$. ∎

Corollary 2·3. *If $L:X\to l^1$ is non-compact and Y is any separable Banach space, then there is a continuous surjection $J:C(\Delta,X)\to Y$ with representing measure m so that m is strongly bounded and $m(A):X\to Y$ is compact for $A\in\Sigma$.*

Proof. Suppose that $L:X\to l^1$ is not compact. Applying Theorem 2·2 we obtain a bounded linear surjection $m\leftrightarrow T:C(\Delta,X)\to l^1$ whose representing measure m is strongly bounded and has range in the compact operators from X to l^1. Let $j:l^1\to Y$ be a surjection. Thus, if we set $J=jT:C(\Delta,X)\to Y$, we have a surjection, and its representing measure $jm:\Sigma\to B(X,Y)$ has the required properties. ∎

Theorem 2·4. *If X is a Banach space, Y is a Banach space with an unconditional basis, and $L:X\to Y$ is a bounded linear surjection, then there is a bounded linear surjection $m\leftrightarrow T:C(\Delta,X)\to Y$ so that m is strongly bounded and $m(A)$ is compact for all $A\in\Sigma$.*

Proof. Suppose that X, Y, and L are as in the hypothesis. Let (x_n) be an unconditional basis for Y and let (x_n^*) be the sequence of coefficient functionals. For $t=(t_n)\in\Delta$, we define $Mt:Y\to Y$ by $Mt(x)=\Sigma t_n x_n^*(x)x_n$. Obviously Mt is linear and a simple closed graph argument shows that Mt is continuous. More can be said. If we let

$$\epsilon(y)=\sup\left\{\sum_{n=1}^{\infty}|x_n^*(y)\,y^*(x_n)|:y^*\in B_Y^*\right\},$$

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then a Baire category argument shows that ϵ defines an equivalent norm on Y. Thus it follows that $\{MT : t \in \Delta\}$ is a uniformly bounded family of maps on Y. In fact, $\sup\{\|Mt\| : t \in \Delta\}$ is the unconditional basis constant of (x_n); see [17], p. 18.

Define m on Σ by

$$m(A)\,x = \sum_{n=1}^{\infty} \int_A r_n\, d\lambda\, x_n^*(L(x))\, x_n \quad (x \in X).$$

One may check that $\|m(A)\,x\| \leqslant \epsilon(L(x))\,\lambda(A)$; thus m is a dominated representing measure. Using the ϵ-norm and the fact that $(\int_A r_n\, d\lambda) \to 0$ for each A, it follows that $m(A)$ is the norm limit of finite rank operators. As usual, define $T(f) = \int_\Delta f\, dm$.

We next choose $c > 0$ so that $L(cB_X) \supseteq \{Mt(B_Y) : t \in \Delta\}$, and we choose $N \in \mathbb{N}$ and $y \in B_Y$ so that $x_n^*(y) = 0$ for $n \geqslant N$. Let $t^i \in C_{Ni}$ for $1 \leqslant i \leqslant 2^N$, and let $x^i \in cB_X$ so that $L(x^i) = Mt^i(y)$ for each i. Let $f = \Sigma \chi_{C_{Ni}} x^i$ and consider $T(f)$. Now

$$T(f) = \sum_{i=1}^{2^N} T(\chi_{C_{Ni}} x^i) = \sum_{i=1}^{2^N} \sum_{n=1}^{\infty} \left(\int_{C_{Ni}} r_n\, d\lambda \right) x_n^*(Lx^i)\, x_n.$$

If $n > N$, then

$$\int_{C_{ni}} r_n\, d\lambda = 0 \quad \text{for} \quad 1 \leqslant i \leqslant 2^n.$$

Therefore

$$T(f) = \sum_{i=1}^{2^N} \sum_{n=1}^{N} \left(\int_{C_{Ni}} r_n\, d\lambda \right) x_n^*(Mt^i y)\, x_n.$$

If $1 \leqslant n \leqslant N$, then

$$\left(\int_{C_{Ni}} r_n\, d\lambda \right) x_n^*(Mt^i y)\, x_n = (i/2^N)\, x_n^*(y)\, x_n.$$

Thus

$$T(f) = \sum_{i=1}^{N} x_n^*(y)\, x_n = y, \quad \text{and} \quad \overline{T(cB_{C(\Delta, X)})} \supseteq B_Y. \quad \blacksquare$$

As an almost immediate corollary to the proof of Theorem 2·4, we have the following result.

PROPOSITION 2·5. *If $L : X \to Y$ is an operator, and $L(X)$ contains a subspace W of Y which is isomorphic to c_0, then there is a bounded linear surjection $m \leftrightarrow T : C(\Delta, X) \to W$ so that m is strongly bounded and $m(A) : X \to W$ is compact for all $A \in \Sigma$.*

Proof. Suppose that $L : X \to Y$ is an operator, W is isomorphic to c_0, and $W \subset L(X)$. Let (x_n) be an unconditional basis for W which is equivalent to the canonical basis (e_n) of c_0, and denote the coefficient functionals of (x_n) by (x_n^*). For each n, we obtain a norm preserving extension of x_n^* to all of Y; denote this extension by y_n^*. Define m on Σ by

$$m(A)\,x = \sum_{n=1}^{\infty} \int_A r_n\, d\lambda\, y_n^*(L(x))\, x_n.$$

Since $(y_n^*(L(x)))$ is bounded, $(x_n) \sim (e_n)$, and $(\int_A r_n\, d\lambda) \to 0$, the series converges. Define the operators $\{Mt : t \in \Delta\}$ on W, choose $y \in W$ so that $y_n^*(y) = 0$ for large n, and proceed as in the proof of Theorem 2·4. $\blacksquare$

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COROLLARY 2·6. *If $L:X\to Y$ is not unconditionally converging, then there is a subspace W of Y so that W is isomorphic to c_0, and a surjection $m\leftrightarrow T:C(\Delta,X)\to W$ so that m is strongly bounded and $m(A):X\to W$ is compact for all $A\in\Sigma$.*

We remark that the results of this section show that the net (T_π) of conditional expectation operators studied by Brooks and Lewis in [**9**] fails to converge to T in many instances, even though the representing measure for T is strongly bounded and has values in the ideal of compact operators.

3. *Strictly singular and strictly cosingular operators*

If $Y=c_0$ or $Y=l^1$ and K is an arbitrary compact space, then we characterize the Banach spaces X so that an operator $m\leftrightarrow T:C(K,X)\to Y$ is strictly singular (strictly cosingular) whenever

(i) m is strongly bounded, and

(ii) $m(A):X\to Y$ is strictly singular (strictly cosingular) for each $A\in\Sigma$.

These results provide a partial answer to problem 1 in [**5**].

PROPOSITION 3·1. *If $T:X\to l^1$ is an operator, then the following are equivalent:*

(*a*) *T is compact;*

(*b*) *T is strictly singular;*

(*c*) *T is strictly cosingular.*

Proof. As observed earlier, clearly (*a*) implies both (*b*) and (*c*). Now suppose that T is not compact. Use lemma 1 of [**20**], and let $U:l^1\to l^1$ be an operator so that $UT:X\to l^1$ is a surjection. (By definition, T is not strictly cosingular.) Let $M>0$ and (x_n) be chosen so that $\|x_n\|\leqslant M$ and $UT(x_n)=e_n$ for $n\in\mathbb{N}$. Clearly then (x_n) can have no weak Cauchy subsequence. Therefore by Rosenthal's theorem ([**10**], p. 201), there is a subsequence (x_{n_j}) of (x_n) so that (x_{n_j}) is equivalent to the canonical unit vector basis of l^1. Consequently, the restriction of T is an isomorphism on $[\{x_{n_j}:j\in\mathbb{N}\}]$, and T is not strictly singular. ∎

Our next result makes strong use of a theorem of Elias and Paulette Saab in order to characterize spaces containing complemented copies of l^1 in terms of non-compact maps defined on continuous functions spaces.

PROPOSITION 3·2. *The following are equivalent:*

(i) *l^1 is complemented in X;*

(ii) *there is a compact Hausdorff space K so that l^1 is complemented in $C(K,X)$;*

(iii) *l^1 is complemented in $C(K,X)$ for every compact Hausdorff space K;*

(iv) *there exists a compact Hausdorff space K and a non-compact operator $T:C(K,X)\to l^1$;*

(v) *if Y is an infinite-dimensional Banach space and K is a compact Hausdorff space, then there is a non-compact operator $T:C(K,X)\to Y$;*

(vi) *if Y is an infinite-dimensional Banach space, then there is a non-compact operator $T:C(\Delta,X)\to Y$.*

Proof. The equivalence of (i), (ii), and (iii) is theorem 1 of Saab and Saab [**24**]. Obviously (iii) ⇒ (iv). If $T:C(K,X)\to l^1$ is not compact, lemma 1 of [**20**] guarantees that there is a bounded linear surjection $J:C(K,X)\to l^1$. An appeal to chapter 7 of

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[**10**] shows that l^1 must be complemented in $C(K,X)$. Now suppose that l^1 is complemented in $C(K,X)$, and let $P:C(K,X)\to l^1$ be a projection. If Z is any separable infinite-dimensional subspace of Y and $J:l^1\to Z$ is any bounded linear surjection, then $JP:C(K,X)\to Z\subset Y$ is non-compact; i.e. (iii) $\Rightarrow$ (v). Certainly (v) $\Rightarrow$ (vi). To finish, assume (vi), let $Y=l^1$, and apply [**20**] and [**10**] again to conclude that l^1 is complemented in $C(\Delta,X)$. ∎

We remark that we could have used the material in Section 2 and included in (vi) the additional requirements that the representing measure be strongly bounded and take values in the compact maps from X to Y.

We are now in position to present our solution to problem 1 of [**5**] for $Y=l^1$.

THEOREM 3·3. *The space l^1 is not complemented in X if and only if for each compact Hausdorff space K an operator $m\leftrightarrow T:C(K,X)\to l^1$ is strictly singular (strictly cosingular) whenever*

(i) *m is strongly bounded, and*

(ii) *$m(A):X\to l^1$ is strictly singular (strictly cosingular) for all $A\in\Sigma$.*

Proof. If l^1 is not complemented in X, then, by Proposition 3·2, every operator $m\leftrightarrow T:C(K,X)\to l^1$ must be compact and hence strictly singular and strictly cosingular by 3·1

Conversely, suppose that l^1 is complemented in X. Applying Theorem 2·2 we obtain a surjection

$$m\leftrightarrow T:C(\Delta,X)\to l^1$$

so that m is strongly bonded and $m(A):X\to l^1$ is compact (and therefore strictly singular and strictly cosingular) for each $A\in\Sigma$. Obviously T is not compact, and an application of Proposition 3·1 finishes the proof. ∎

As the following result shows, we can say more in the strictly cosingular case.

THEOREM 3·4. *The following are equivalent*:

(1) *l^1 is complemented in X;*

(2) *for each Banach space Y with a separable infinite-dimensional quotient, there exist a compact Hausdorff space K and an operator $m\leftrightarrow T:C(K,X)\to Y$ so that T is not strictly cosingular but m is strongly bounded and $m(A)$ is strictly cosingular for each A.*

(3) *there exist a compact Hausdorff space K and an operator $m\leftrightarrow T:C(K,X)\to l^1$ so that T is not strictly cosingular but m is strongly bounded and $m(A)$ is strictly cosingular for each A.*

Proof. Suppose that l^1 is complemented in X, $K=\Delta$, and Y is a Banach space with separable infinite-dimensional quotient Z. Let $Q:Y\to Z$ be a quotient map, and let W be a separable subspace of Y so that $Q(W)=Z$. An application of Corollary 2·3 produces

$$m\leftrightarrow T:C(K,X)\to W\subset Y$$

so that m has the required properties and T is a surjection. Since $Q(W)=Z$, T is not strictly cosingular.

It is immediate that (2) $\Rightarrow$ (3). Finally, if $T:C(K,X)\to l^1$ is not strictly cosingular, then T is not compact by Proposition 3·1. An application of Proposition 3·2 shows that (3) $\Rightarrow$ (1). ∎

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COROLLARY 3·5. *The Banach space Y has a separable infinite-dimensional quotient if and only if there is an operator $m \leftrightarrow T : C(\Delta, l^1) \to Y$ whose representing measure is strongly bounded and has its range in the compact operators from l^1 to Y such that T fails to be strictly cosingular.*

The following proposition will be helpful in our study of strictly singular and strictly cosingular maps $T : C(K, X) \to c_0$.

PROPOSITION 3·6. *If $Z = c_0$ and $L : X \to Z$ is an operator, then the following are equivalent:*

(*a*) *L is unconditionally converging;*

(*b*) *L is strictly singular.*

Proof. As was noted in the Introduction, (*b*) always implies (*a*). Now suppose that $L : X \to Z$ is not strictly singular. Let W be an infinite-dimensional subspace of X so that L is an isomorphism on W. Since $L(W)$ is an infinite-dimensional subspace of Z, an application of 2·a·2, p. 53 of [**18**], produces a subspace S of $L(W)$ and an operator P so that S is isomorphic to Z and $P : Z \to S$ is a projection. Therefore there is a subspace R of X so that PL is an isomorphism from R onto S. Since the canonical unit vector basis (e_n) of Z (or of S) is weakly absolutely summable and not unconditionally convergent, PL (and thus L) cannot be unconditionally converging. ∎

Using 3·6 and replacing 'unconditionally converging' with 'strictly singular' in a result of Saab [**23**], we obtain a solution to problem 1 of [**5**] in the strictly singular case for $Y = c_0$.

THEOREM 3·7. *The Banach space c_0 does not embed in X if and only if for each compact Hausdorff space an operator $m \leftrightarrow T : C(K, X) \to c_0$ is strictly singular whenever m is strongly bounded and $m(A) : X \to c_0$ is strictly singular for each A.*

We next turn our attention to a brief investigation of Banach spaces X for which c_0 is a continuous linear image of X. In fact, we obtain the following result. See also the paper 'On Banach spaces whose dual balls are not weak* sequentially compact' by Hagler and Johnson in Volume 28 of the Israel Journal of Mathematics.

PROPOSITION 3·8. *If l^1 embeds in X^* but l^1 does not embed in X, then there is a continuous linear surjection $S : X \to c_0$.*

Before we prove this result, we state and prove the following lemma.

LEMMA 3·9. *If (x_i^*) is a sequence in X^* which is equivalent to the canonical unit vector basis (e_i^*) of l^1, then there is a constant $K > 0$ such that for each $\epsilon > 0$, $n \in \mathbb{N}$, and subsequence $(x_{i_j}^*)$ of (x_i^*), there is a finite set (x_p), $1 \leqslant p \leqslant m$, with $\|x_p\| \leqslant K$ for each p for which $\{x_{i_j}^*(x_p) : 1 \leqslant j \leqslant n, 1 \leqslant p \leqslant m\}$ forms an ϵ-net for the unit ball of* ($\mathbb{R}^n$, sup norm).

Proof. Suppose that $(x_i^*) \sim (e_i^*)$ (the two bases are equivalent), and let $\alpha = (\alpha_j)$ belong to the unit ball of l^∞. Choose $K > 0$ so that there exists $y^{**} \in X^{**}$ with

$$\|y^{**}\| \leqslant K \quad \text{and} \quad y^{**}(x_n^*) = \alpha_n$$

for each n. Now let $\epsilon > 0$, and choose $\alpha_1, \ldots, \alpha_m \in B_{l^\infty}$ so that

$$((\alpha_{ij})) \quad (1 \leqslant j \leqslant n, 1 \leqslant i \leqslant m),$$

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forms an $\epsilon/2$-net for the unit ball of $(\mathbb{R}^n, \text{sup norm})$. Find $y_i^{**} \in X^{**}$ so that

$$\|y_i^{**}\| \leqslant K \quad \text{and} \quad y_i^{**}(x_j^*) = \alpha_{ij} \quad (1 \leqslant i \leqslant m, j \in \mathbb{N}).$$

Now use the w*-denseness of KB_X in KB_X^{**} to find $x_i \in X$ so that $\|x_i\| \leqslant K$ and

$$|\langle x_i, x_j^* \rangle - \langle y_i^{**}, x_j^* \rangle| < \epsilon/2$$

for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. It follows that

$$\{(\langle x_i, x_j^* \rangle) : 1 \leqslant j \leqslant n, 1 \leqslant i \leqslant m\}$$

forms an ϵ-net for the unit ball of $(\mathbb{R}^n, \text{sup norm})$. Finally, if $(x_{i_j}^*)$ is a subsequence of (x_i^*), merely choose $\alpha_1, \ldots, \alpha_m$ in l^∞ so that

$$((\alpha_{pi_j})) \quad (1 \leqslant j \leqslant n, 1 \leqslant p \leqslant m)$$

forms an $\epsilon/2$-net for the ball of $(\mathbb{R}^n, \text{sup norm})$. ∎

Proof of Proposition 3·8. Suppose that l^1 embeds in X^* and l^1 does not embed in X. By the lemma on p. 219 of Diestel [10], there is a w*-null sequence (x_n^*) in X^* so that (x_n^*) is equivalent to the canonical basis (e_n^*) in l^1. Suppose that (x_n^*) is so chosen and that $\|x_n^*\| = 1$ for each n. Let $K > 1$ be a number which satisfies the conclusion of the preceding lemma, and let $n_1 = 1$. Choose $x_1 \in X$ so that $\|x_1\| \leqslant K$ and $x_{n_1}^*(x_1) = 1$. Pick $n_2 > n_1$ so that $|x_n^*(x_1)| < \frac{1}{2}$ for $n \geqslant n_2$. Use the lemma above to select a finite sequence

$$(x_2^i) \quad (1 \leqslant i \leqslant m_2)$$

so that

$$(x_{n_1}^*(x_2^i), x_{n_2}^*(x_2^i)) \quad (1 \leqslant i \leqslant m_2)$$

forms a $\frac{1}{4}$-net for the unit ball of $(\mathbb{R}^2, \text{sup norm})$ and $\|x_2^i\| \leqslant K$ for each i.

At the next step, choose $n_3 > n_2$ so that $|x_n^*(x_2^i)| < \frac{1}{4}$ for each i whenever $n \geqslant n_3$. Applying the lemma again and obtain a finite sequence

$$(x_3^i) \quad (1 \leqslant i \leqslant m_3)$$

so that $\|x_3^i\| \leqslant K$ for each i and

$$(x_{n_j}^*(x_3^i)) \quad (1 \leqslant j \leqslant 3, 1 \leqslant i \leqslant m_3)$$

forms a $(\frac{1}{2})^3$-net for the ball of $(\mathbb{R}^3, \text{sup norm})$.

Continue this process recursively, obtaining an increasing sequence (n_j) of positive integers and a sequence (x_j^i), $j \in \mathbb{N}$, of finite sequences from X so that

(a) $\|x_j^i\| \leqslant K$ for all i and j,

(b) $(x_{n_j}^*(x_j^i))$, $1 \leqslant j \leqslant p$, $1 \leqslant i \leqslant m_p$ forms a $(\frac{1}{2})^p$-net for the unit ball of $(\mathbb{R}^p, \text{sup norm})$ for each p,

(c) $n \geqslant n_{p+1}$ ensures that $|x_n^*(x_p^i)| < (\frac{1}{2})^p$ for all i.

Define S on X by $S(x) = (x_{n_j}^*(x))$. Obviously S is linear, and $S(x) \in c_0$ since (x_n^*) is w*-null. Certainly $\|S\| \leqslant 1$.

Now let $\alpha = (\alpha_n) \in c_0$, $\|\alpha\| \leqslant 1$, and choose $N \in \mathbb{N}$ so that $(\frac{1}{2})^N < \epsilon/2$ and $|\alpha_n| < \epsilon/2$ for $n \geqslant N$. Select x_N^i so that

$$|x_{n_j}^*(x_N^i) - \alpha_j| < \epsilon/2$$

for $j \leqslant N$. If $j > N$, then $n_j \geqslant n_{N+1}$. Thus

$$|x_{n_j}^*(x_N^i)| < (\tfrac{1}{2})^N < \epsilon/2.$$

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Therefore $$|x^*_{n_j}(x^i_n)-\alpha_j| < \epsilon$$

for all j, i.e. $\|S(x^i_N)-\alpha\| \leqslant \epsilon$. Consequently $B_{c_0} \subset \overline{S(KB_X)}$, and S must be a surjection. ∎

COROLLARY 3·10. *The Banach space l^1 embeds in X^* if and only if there exist a subspace Z of X and an operator $L:X \to l^\infty$ so that $L(Z) = c_0$.*

Proof. Suppose that l^1 embeds in X^*. If l^1 does not embed in X, then we use Proposition 3·8. If l^1 does embed in X, then we map l^1 linearly and continuously onto c_0 and use the injectivity of l^∞ (theorem 3, p. 71 of [**10**]) to extend this mapping continuously and linearly to all of X.

Conversely, suppose that Z and L are as above, and let p be a positive number so that $L(pB_Z) \supset B_{c_0}$. Now let $u \in l^1 \subset (l^\infty)^*$ and let $y \in B_{c_0}$ so that $\langle u, y\rangle > (\frac{1}{2})\|u\|$. Choose $z \in B_Z$ such that $L(z) = (1/p)y$. Therefore

$$\|L^*(u)\| \geqslant |\langle L^*(u), z\rangle| = |\langle u, (1/p)y\rangle| \geqslant (\tfrac{1}{2}p)\|u\|,$$

and L^* is an isomorphism on l^1. ∎

Our next proposition is, in a sense, an analogue of lemma 1 of [**20**] for the case where the range space is c_0.

PROPOSITION 3·11. *If $L:X \to c_0$ is an operator, then the following are equivalent:*
(*a*) *L is not strictly cosingular;*
(*b*) *there is an operator $J:c_0 \to c_0$ so that $JL:X \to c_0$ is a surjection.*

Proof. Clearly (*b*) implies (*a*). Conversely, suppose that L is not strictly cosingular, and let Z be an infinite-dimensional (necessarily separable) Banach space and $S:c_0 \to Z$ be an operator so that $SL:X \to Z$ is a surjection. Certainly S cannot be compact. Thus S is not unconditionally converging by the remarks made in the closing paragraphs of the Introduction, and S must be an isomorphism on an isomorphic copy W of c_0 properly contained in c_0. Let $P:Z \to S(W)$ be a projection onto $S(W)$ guaranteed by Sobczyk's theorem ([**10**], p. 71), and let $Q:S(W) \to c_0$ be a surjective isomorphism. The mapping $J = QPS$ satisfies (*b*). ∎

Before presenting our solution to problem 1 of [**5**] in the strictly cosingular case for $Y = c_0$, we need an improvement of a theorem in [**2**]. Specifically, in theorem 2·8 of [**2**] it was demonstrated that if l^1 does not embed in X^*, if X^* has the Radon–Nikodým property, and if $m \leftrightarrow T:C(K,X) \to Y$ is strongly bounded, then T^* must be weakly precompact, i.e. if (y^*_n) is a bounded sequence in Y^*, then $(T^*(y^*_n))$ has a weakly Cauchy subsequence. In the following theorem, we show that the Radon–Nikodým assumption can be omitted. We note that Paulette and Elias Saab [**25**] have also shown that the RNP is not necessary in this theorem. (In fact, using a considerable amount of Choquet theory, they have both removed the RNP assumption and broadened the scope of the theorem.) Our argument is entirely different from the one presented in [**25**]; rather, it is akin to the proof of the theorem in [**24**].

THEOREM 3·12. *If $m \leftrightarrow T:C(K,X) \to Y$ is a strongly bounded operator and l^1 does not embed in X^*, then $T^*:Y^* \to C(K,X)^*$ is weakly precompact.*

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Proof. Suppose that $m \leftrightarrow T: C(K,X) \to Y$ is strongly bounded and that T^* is not weakly precompact. Recalling that T^* takes its values in rcabv (Σ, X^*) and that

$$T^*(y^*) = m_{y^*}: \Sigma \to X^*,$$

we let (y_n^*) be a sequence in B_{Y^*} so that $(m_{y_n^*})$ is equivalent to the canonical basis (e_n^*) of l^1. For convenience, denote $m_{y_n^*}$ by m_n, for $n \in \mathbb{N}$. Without loss of generality, suppose that $\|m_n\| \leqslant 1$ for each n. Since m is strongly bounded, (m_n) is uniformly countably additive. Let

$$\nu' = \sum_{n=1}^{\infty} |m_n|(\cdot)/2^n,$$

and let (K, Σ_0, ν) denote the completion of (K, Σ, ν'). Note that if $n \in \mathbb{N}$, then there is a positive constant c so that $|m_n| \leqslant c\nu$. Let ρ be a lifting of $L^\infty(\nu)$ (see [**12**]). If $f: K \to X^*$ is a function, write $\hat{\rho}(f) = f$ if

(*a*) $\langle f, x\rangle \in L^\infty(\nu)$ for each $x \in X$, and

(*b*) $\rho(\langle f, x\rangle) = \langle f, x\rangle$ for each $x \in X$.

Now use theorem 5, p. 269 of [**12**], to obtain a sequence (g_n) of functions so that

(i) $g_n: K \to X^*$, $\|g_n(\cdot)\|$ is ν-integrable,

(ii) $|m_n|(A) = \int_A \|g_n(\cdot)\|\, d\nu$ for $A \in \Sigma_0$,

(iii) $\int f\, dm_n = \int \langle g_n(t),\ f(t)\rangle\, d\nu(t)$ and consequently $\langle m_n(A), x\rangle = \int_A \langle g_n(t), x\rangle\, d\nu$ for $f \in L^1(|m_n|, X)$, $x \in X$, $A \in \Sigma_0$,

(iv) $\hat{\rho}(g_n) = g_n$,

for each n.

Now let $(a_n)_{n=1}^p$ be a finite sequence of real numbers, and let $\mu = \sum_{n=1}^p a_n m_n$. We use Dinculeanu's theorem again, and let $g: K \to X^*$ be a function which satisfies (i)–(iv) with respect to μ. Let $h = \sum_{n=1}^p a_n g_n$. Then

$$\int_A \langle g(t), x\rangle\, d\nu = \int_A \langle h(t), x\rangle\, d\nu$$

for all A and x; thus $\langle g(\cdot), x\rangle = \langle h(\cdot), x\rangle$ ν-a.e. Since ρ is a (linear) lifting on $L^\infty(\nu)$, it follows that $\hat{\rho}(h) = h$. Combining the facts that $\hat{\rho}(g) = g$, $\hat{\rho}(h) = h$, and $\langle g, x\rangle = \langle h, x\rangle$ ν-a.e. for all x, we conclude that $\langle h, x\rangle = \langle g, x\rangle$ everywhere. Therefore

$$\left\| \sum_{n=1}^p a_n m_n \right\| = \int_K \left\| \sum_{n=1}^p a_n g_n(\cdot) \right\| d\nu$$

for any finite sequence (a_n) of real numbers.

Using the fact that $(m_n) \sim (e_n^*)$, we choose $\alpha > 0$ so that

$$\left\| \sum_{i=1}^p a_i m_i \right\| > \alpha \sum |a_i|,$$

and ϵ so that $0 < \epsilon < \alpha$. Let $\delta > 0$ so that if $\nu(A) < \delta$, then

$$\int_A \|g_n(\cdot)\|\, d\nu = |m_n|(A) < \epsilon$$

for all n (e.g. see Brooks[**7**]). Let $A_n = \{t: \|g_n(t)\| > 1/\delta\}$.

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Since $$\int_K \|g_n(\cdot)\|\, d\nu \leqslant 1,$$

it follows that $\nu(A_n) < \delta$; therefore $|m_n|(A_n) < \epsilon$. Let $\hat{m}_n(B) = m_n(B\backslash A_n)$ for $B \in \Sigma_0$ and $n \in \mathbb{N}$. Note that

$$|\hat{m}_n|(B) = |m_n|(B\backslash A_n)\int_{A\backslash B_n} \|g_n(\cdot)\|\, d\nu \leqslant (1/\delta)\,\nu(B\backslash A_n) \leqslant (1/\delta)\,\nu(B).$$

Also $|\hat{m}_n - m_n|(K) = |m_n|(A_n) < \epsilon$ for all n. Further,

$$\left\|\sum_{n=1}^{p} a_n \hat{m}_n\right\| \geqslant \left\|\sum_{n=1}^{p} a_n m_n\right\| - \sum_{n=1}^{p} |a_n|\,\|m_n - \hat{m}_n\|$$
$$\geqslant \alpha\sum|a_n| - \epsilon\sum|a_n| = (\alpha-\epsilon)\sum|a_n|.$$

This inequality means that $(\hat{m}_n) \sim (e_n^*)$ in l^1.

Next we apply Dinculeanu's theorem to the sequence $(\hat{m}_n)$ and obtain a corresponding sequence $(\hat{g}_n)$ of functions. Of course $(\hat{g}_n)$ will have analogous properties to those of (g_n). Specifically, if (a_n), $1 \leqslant n \leqslant p$, is a finite sequence of real numbers, then

$$\left\|\sum_{n=1}^{p} a_n \hat{m}_n\right\| = \int_K \|\Sigma a_n \hat{g}_n(\cdot)\|\, d\nu$$

and $$|\hat{m}_n|(A) = \int_A \|\hat{g}_n(\cdot)\|\, d\nu \leqslant (1/\delta)\,\nu(A),$$

for $A \in \Sigma_0$. Therefore $\|\hat{g}_n(\cdot)\| \leqslant 1/\delta$ a.e.–ν. Without loss of generality, we assume that $\|g_n(t)\| \leqslant 1/\delta$ for each $t \in K$.

Now let S be the space of all finitely non-zero real sequences, and let (e_n) denote the nth unit vector. For $(a_n) = a \in S$, let $|a|_1 = \Sigma|a_n|$. For $t \in K$, let $|\cdot|_t$ be defined on S by

$$|(a_n)|_t = \|\Sigma a_n \hat{g}_n(t)\|.$$

Note that $|e_i|_t \leqslant 1/\delta$ for all i and t. Also, if $a = (a_n) \in S$, then

$$\int_K |a|_t\, d\nu(t) = \int_K \|a_n \hat{g}_n(t)\|\, d\nu = \|\Sigma a_n \hat{m}_n\| \geqslant (\alpha-\epsilon)\Sigma|a_n|.$$

Define $|\cdot|$ on S by $|a| = \int_K |a|_t\, d\nu(t)$. Therefore

$$(\alpha-\epsilon)\Sigma|a_n| \leqslant |\Sigma a_n e_n| = \int_K \|a_n \hat{g}_n(t)\|\, d\nu(t) \leqslant \int_K (\Sigma|a_n|1/\delta)\, d\nu = (1/\delta)\,\nu(K)\,\Sigma|a_n|.$$

Thus when S is equipped with $|\cdot|$, it follows that (e_n) is equivalent to the canonical basis of l^1. Finally we apply theorem 4 of Bourgain [6] and conclude that there exist a $t \in K$ and a subsequence of (e_n) so that this subsequence is the canonical basis of l^1 for t. (For convenience, we suppose that it is the full sequence.) That is, there are positive numbers p_1 and p_2 and $t \in K$ so that

$$p_1\Sigma|a_n| \leqslant \|\Sigma a_n \hat{g}_n\, d(t)\| \leqslant p_2\Sigma|a_n|$$

for all finite sequences. Therefore $(\hat{g}_n(t))$, $n \in \mathbb{N}$, is equivalent to the usual basis of l^1, and we have a clear contradiction of our hypotheses. ∎

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With this machinery in place, the solution to the strict cosingularity portion of our problem for $Y = c_0$ will be quite straightforward.

THEOREM 3·13. *The Banach space l^1 does not embed in X^* if and only if for every compact Hausdorff space K the operator $m \leftrightarrow T: C(K, X) \to c_0$ is strictly cosingular whenever m is strongly bounded and $m(A): X \to c_0$ is strictly cosingular for each $A \in \Sigma$.*

Proof. Suppose that l^1 does not embed in X^*, that $m \leftrightarrow T: C(K, X) \to c_0$ is strongly bounded, and that T is not strictly cosingular. Apply Proposition 3·9, and let $S: c_0 \to c_0$ be an operator so that $ST: C(K, X) \to c_0$ is a surjection. Thus $(ST)^* = T^*S^*$ is an isomorphic embedding defined on l^1. In particular, $(T^*S^*(e_n^*))$ can have no weakly Cauchy subsequences. Hence T^* cannot be weakly precompact, and we contradict Theorem 3·12.

Conversely, suppose that l^1 does embed in X^*. Apply Corollary 3·10, and let $L: X \to l^\infty$ be an operator and Z be a subspace of X so that $L(Z) = c_0$. An appeal to Proposition 2·5 finishes the proof. ∎

4. *Extension operators and strongly bounded operators*

As usual, let Σ denote the Borel subsets of K, and let $U_X(\Sigma)$ denote the uniform closure of the X-valued simple functions $(S_X(\Sigma))$ with respect to Σ. Certainly $C(K, X)$ is contained isometrically in $U_X(\Sigma)$; further, $U_X(\Sigma)$ embeds isometrically in $C(K, X)^{**}$, e.g. see §2 of [**8**]. It is clear that $\int f dm$ exists for $f \in U_X(\Sigma)$. In fact,

$$\int f dm = T^{**}(f)$$

for $f \in U_X(\Sigma)$ ([**8**], pp. 141–143); therefore this integral defines a natural extension of T to $U_X(\Sigma)$. We follow [**4**] and denote this extension by $\hat{T}$. Several authors have found the study of $\hat{T}$ to be quite helpful. We mention the work of Batt and Berg [**3**], Bombal and Cembranos [**4**], and Bombal and Porras [**5**]. In these papers it has been demonstrated that if m is strongly bounded and T is weakly compact, compact, completely continuous, weakly completely continuous, unconditionally converging, strictly singular, or strictly cosingular, then $\hat{T}$ has the same property. In the first result in this section, we show that all these implications may be dealt with simultaneously. Our argument makes strong use of theorem 1 of [**4**]. If $L: X \to Y$ is an operator and $S \subset X$, then by L_S we shall mean $L_{|S}: S \to Y$.

Definition. A closed operator ideal $\mathcal{O}$ is said to be separably determined provided that, for each pair X and Y of Banach spaces, an operator $L: X \to Y$ belongs to $\mathcal{O}(X, Y)$ if and only if $L_S \in \mathcal{O}(S, Y)$ for each separable subspace S of X.

PROPOSITION 4·1. *If $m \leftrightarrow T: C(K, X) \to Y$ belongs to a separably determined (closed) operator ideal $\mathcal{O}$ and m is strongly bounded, then $\hat{T}: U_X(\Sigma) \to Y$ belongs to $\mathcal{O}$ also.*

Proof. Since $\mathcal{O}$ is separably determined, it clearly suffices to consider $\hat{T}_{|S}$, where S is a separable subspace of $U_X(\Sigma)$. Let $\{f_i : i \in \mathbb{N}\}$ be a dense subset of B_S. Apply Lusin's theorem ([**12**], p. 334) and, for $n \in \mathbb{N}$, let K_n be a compact subset of K so that $f_{i|K_n}$ is continuous for all i and $\tilde{m}(K \backslash K_n) < 1/n$, $n \in \mathbb{N}$. Set

$$H_n = \overline{\text{span}}\{f_{i|K_n} : i \in \mathbb{N}\} \quad (n \in \mathbb{N}).$$

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Let $S_n : H_n \to C(K, X)$ be an isometric extension operator (see [**4**], theorem 1), and define $R_n : S \to H_n \subset C(K_n, X)$ by $R_n(f) = f_{|K_n}$. Obviously $\|R_n\| \leqslant 1$.

Put $T_n = TS_nR_n : S \to Y$, for $n \in \mathbb{N}$. Now suppose that $\|\sum_{i=1}^{p} \alpha_i f_i\| \leqslant 1$. Note that

$$\|T_n(\sum a_i f_i) - \hat{T}(\sum \alpha_i f_i)\| = \left\| \int_K S_n(\sum \alpha_i f_{i|K_n})\, dm - \int_K (\sum \alpha_i f_i)\, dm \right\|$$

$$= \left\| \int_{K_n} S_n(\sum \alpha_i f_{i|K_n})\, dm + \int_{K \setminus K_n} S_n(\sum \alpha_i f_{i|K_n})\, dm - \int_{K_n} \sum \alpha_i f_i\, dm - \int_{K \setminus K_n} \sum \alpha_i f_i\, dm \right\|$$

$$= \left\| \int_{K \setminus K_n} S_n(\sum \alpha_i f_{i|K_n})\, dm - \int_{K \setminus K_n} (\sum \alpha_i f_i)\, dm \right\| \leqslant 2/n.$$

Since $\{f_i : i \in \mathbb{N}\}$ is dense in B_S, it follows that $\|\hat{T}_{|S} - T_n\| \to 0$. Certainly we have $TS_nR_n = T_n \in \mathcal{O}(S, Y)$. Therefore $\hat{T}_{|S} \in \mathcal{O}$ since $\mathcal{O}$ is closed. Thus $\hat{T} \in \mathcal{O}$. ∎

As in [**5**], we have the following corollary.

COROLLARY 4·2. *If $\mathcal{O}$ is any separably determined closed operator ideal, $m \leftrightarrow T : C(K, X) \to Y$ is strongly bounded, and $T \in \mathcal{O}$, then $m(A) \in \mathcal{O}(X, Y)$ for all $A \in \Sigma$.*

Proof. If $A \in \Sigma$ and $A \neq \varnothing$, then $\phi(x) = \chi_A x$ defines an isometric linear embedding of X into $U_X(\Sigma)$, and $m(A)x = \hat{T}\phi(x)$. Since m maps Σ into $B(X, Y)$ in this case ([**8**], theorem 4·4), it follows that $m(A) \in \mathcal{O}(X, Y)$. ∎

It is clear that the weakly compact, compact, unconditionally converging, completely continuous, and weakly completely continuous operator ideals are closed and separately determined. Further, if $L : X \to Y$ does not have a bounded inverse on any separable infinite-dimensional subspace of X, then L cannot have a bounded inverse on any infinite-dimensional subspace of X. (Merely restrict L to a separable infinite-dimensional subspace of this infinite-dimensional space.) Therefore the strictly singular operators also form a separably determined operator ideal.

It is not quite so obvious that the closed ideal of strictly cosingular maps is separably determined. Suppose that $L : X \to Y$ is an operator, $L_{|S} : S \to Y$ is strictly cosingular whenever S is a separable subspace of X, and L fails to be strictly cosingular. Let Z be an infinite-dimensional Banach space and $Q : Y \to Z$ be an operator so that $QL : X \to Z$ is a surjection. Let W be a separable infinite-dimensional subspace of Z, and let $\{w_n : n \in \mathbb{N}\}$ be a dense subset of B_W. Let c be a positive number so that $B_W \subset QL(cB_X)$. For each n choose $x_n \in cB_X$ such that $QL(x_n) = w_n$, and let $S = [\{x_n : n \in \mathbb{N}\}]$. Clearly S is separable, and $\overline{QL(B_X)} \supset \{(1/c)\, w_n : n \in \mathbb{N}\} \supset 1/cB_X$. Therefore there is a number $d > 0$ so that $QL(B_S) \supset dB_W$, and QL_S is a surjection. This contradiction shows that the cosingular maps do form a separably determined operator ideal. It is curious to note that the operators with separable range do form a closed operator ideal, but this class is not separably determined.

The next two results taken together provide an improvement of theorem 1·5 of [**5**].

LEMMA 4·3. *If $m \leftrightarrow T : C(K, X) \to Y$ is a bounded linear operator, then the following are equivalent.*

(i) *$T^{**}(B_{C(K,X)})$ is dense in $T^{**}(B_{U_X(\Sigma)})$;*

(ii) *$m : \Sigma \to B(X, Y)$;*

(iii) *$m_x : \Sigma \to Y$ is countably additive for each $x \in X$.*

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Proof. (i) $\Rightarrow$ (ii). Since $T^{**}(B_{C(K,X)})$ is just the canonical image in Y^{**} of $T(B_{C(K,X)})$, it is clear then that $T^{**}(B_{U_X(\Sigma)})$ must lie in Y. Since $m(A)x = T^{**}(\chi_A x)$ for $A \in \Sigma$ and $x \in X$ by [**8**], §2, (ii) follows.

(ii) $\Rightarrow$ (iii). This implication is contained explicitly in theorem 4·4 of [**8**].

(iii) $\Rightarrow$ (i). See the proof of theorem 5 in Abbott[**1**]. ∎

PROPOSITION 4·4. *If $m \leftrightarrow T: C(K,X) \to Y$ is a bounded linear operator so that $T(B_{C(K,X)})$ is dense in $T^{**}(B_{U_X(\Sigma)})$, then T is strictly cosingular if and only if $\hat{T}$ is strictly cosingular.*

Proof. Suppose that $m \leftrightarrow T: C(K,X) \to Y$ is an operator, $T(B_{C(K,X)})$ is dense in $T^{**}(B_{U_X(\Sigma)})$, and $\hat{T}$ is not strictly cosingular. As noted in the lemma above, $\hat{T}$ maps $U_X(\Sigma)$ into Y. Let Z be an infinite-dimensional Banach space and $Q: Y \to Z$ be an operator so that $Q\hat{T}$ is a surjection. Therefore there is a positive constant c so that $Q\hat{T}(B_{U_X(\Sigma)}) \supset cB_Z$. As a result of the denseness assumption above, $\overline{QT(B_{C(K,X)})} \supset cB_Z$. Consequently, there is a positive number p so that $pB_Z \subset QT(B_{C(K,X)})$, and QT is a surjection. Hence $\hat{T}$ is strictly cosingular whenever T is. The converse implication is clear. ∎

We remark that Lewis[**17**] and Dobrakov[**13**] have given examples of countably additive representing measures which are not strongly bounded. The countable additivity ensures that these measures satisfy the hypothesis of Proposition 4·4.

We conclude with a characterization of strongly bounded maps suggested by the use of Lusin's theorem in Proposition 4·1. We say that a representing measure $m \leftrightarrow T$ satisfies Lusin's condition provided that for each bounded sequence (f_n) from $S_X(\Sigma)$ and each $\epsilon > 0$ there is a compact set $K_\epsilon \subset \bigcup_{n=1}^{\infty}$ support (f_n) so that $f_{n|_{K_\epsilon}}$ is continuous and $\|\int_{K \setminus K_\epsilon} f_n \, dm\| < \epsilon$ for all n. Using the existence of a control measure, one sees that a strongly bounded representing measure must satisfy Lusin's condition.

THEOREM 4·5. *A representing measure $m \leftrightarrow T: C(K,X) \to Y$ is strongly bounded if and only if m satisfies Lusin's condition.*

Proof. Suppose that m satisfies Lusin's condition and that m is not strongly bounded. Therefore there exist a disjoint sequence (A_i) of Borel sets and $\delta > 0$ so that $\tilde{m}(A_i) > \delta$ for each i. Let $z_i^* \in B_Y^*$ so that $|m_{z_i^*}|(A_i) > \delta$ for each i. The sequence $(|m_{z_i^*}|)$ of regular Borel measures fails to be uniformly countably additive. By a theorem of Grothendieck[**14**], we may assume that there is a pairwise disjoint sequence (O_i) of open sets and $\epsilon > 0$ so that $|m_{z_i^*}|(O_i) > 2\epsilon$ for each i. Thus $\tilde{m}(O_i) > 2\epsilon$ for all i.

Given $i \in \mathbb{N}$, let $A_{i1}, \ldots, A_{ik(i)}$ be a partition of O_i and $x_{i1}, \ldots, x_{ik(i)}$ be in B_X so that

$$\|\sum_j m(A_{ij})\, x_{ij}\| > 2\epsilon,$$

and let $f_i = \sum_j \chi_{A_{ij}} x_{ij}$. Therefore $\|\int f_i \, dm\| > 2\epsilon$ for each i. Now apply Lusin's condition and let K_ϵ be a compact set contained in $\bigcup_i$ support (f_i) such that $f_{i|_{K_\epsilon}}$ is continuous and $\|\int_{K \setminus K_\epsilon} f_i \, dm\| < \epsilon$ for each i. Now let $O_1, \ldots, O_p$ be a finite subcover of K_ϵ. Thus

$$\left\| \int_{K \setminus K_\epsilon} f_{p+1} \, dm \right\| = \left\| \int_{O_{p+1}} f_{p+1} \, dm \right\| > \epsilon,$$

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and we have a contradiction, i.e. m must be strongly bounded. As we observed earlier, the converse is straightforward. ∎

We remark that it would be natural in the definition of Lusin's condition to remove the requirement that $K_\varepsilon \subset \bigcup_i \text{support } (f_i)$. We do not know if this is possible.

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