3

Chromatic polynomials

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This chapter is concerned with chromatic polynomials and other related polynomials: flow polynomials, characteristic polynomials, Tutte polynomials and the Potts model partition function. We describe their basic properties and state some recent results and open problems. We also outline some techniques for working with these polynomials.

1. Introduction

The study of chromatic polynomials of graphs was initiated by Birkhoff [3] in 1912 and continued by Whitney [49], [50] in 1932. Inspired by the four-colour conjecture, Birkhoff and Lewis [4] obtained results concerning the distribution of the real zeros of chromatic polynomials of planar graphs and made the stronger conjecture that they have no real zeros greater than or equal to 4. Their hope was that results from analysis and algebra could be used to prove their stronger conjecture, and hence to deduce that the four-colour conjecture was true. This has not yet occurred: indeed, the four-colour conjecture is now a theorem, but the stronger conjecture of Birkhoff and Lewis remains open. Nevertheless, many beautiful results on chromatic polynomials have been obtained, and many other intriguing questions remain unanswered.

Work on chromatic polynomials has received fresh impetus in recent years from an interaction with mathematical physics. The chromatic polynomial is a specialization of the Potts model partition function, used by mathematical physicists to study phase transitions. A combination of ideas and techniques from graph theory and statistical

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mechanics has led to significant new results on both polynomials. Other recent progress has been made through the use of techniques from algebraic geometry to solve a long-standing open problem concerning the coefficients of chromatic polynomials.

2. Definitions and elementary properties

All graphs considered are finite and (unless stated otherwise) without loops or multiple edges. As usual, we use $G - e$ to denote the graph obtained by deleting an edge e from a graph G and G/e the graph obtained by contracting e. For any positive integer q, the *chromatic polynomial* $P(G, q)$ of G is defined as the number of distinct proper q-colourings of G. It is easy to see that a graph G with n vertices and no edges has chromatic polynomial q^n and that $P(K_n, q) = q(q-1)\dots(q-n+1)$.

Some reduction techniques

The following results give a recursive procedure for calculating chromatic polynomials.

Theorem 2.1 (The deletion/contraction lemma) *Let G be a graph and e be an edge of G. Then*

$$P(G, q) = P(G - e, q) - P(G/e, q).$$

Proof Let $e = vw$. Then $P(G-e, q) = P(G, q) + P(G/e, q)$ for all positive integers q, since $P(G, q)$ counts the number of q-colourings of $G - e$ in which v and w are given different colours and $P(G/e, q)$ counts the number in which v and w are given the same colour. ■

This result can be used recursively to express $P(G, q)$ as a linear combination of chromatic polynomials of graphs with no edges. Alternatively, we can use the 'addition/contraction' form

$$P(G, q) = P(G + e, q) + P(G/e, q),$$

where e is a new edge added between two non-adjacent vertices, to express $P(G, q)$ as a linear combination of chromatic polynomials of complete graphs.

Theorem 2.1 can also be used to deduce many elementary properties of chromatic polynomials. In particular, an easy induction on the number of edges of G gives the following result.

Theorem 2.2 *Let G be a graph with n vertices. Then $P(G, q)$ is a monic polynomial of degree n with zero constant term and with integer coefficients that alternate in sign.*

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The recursive calculation of $P(G, q)$ can often be simplified by using the following factorization result.

Theorem 2.3 *Let G be a graph, and let G_1 and G_2 be subgraphs of G for which $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = K_r$. Then*

$$P(G, q) = \frac{P(G_1, q)P(G_2, q)}{q(q-1)\cdots(q-r+1)}.$$

Proof The vertices of K_r must receive distinct colours in any q-colouring of G. There are $P(K_r, q) = q(q-1)\cdots(q-r+1)$ ways to colour K_r and the symmetry between the colours implies that any such colouring of K_r can be extended to exactly $P(G_1, q)/[q(q-1)\cdots(q-r+1)]$ colourings of G_1 and to $P(G_2, q)/[q(q-1)\cdots(q-r+1)]$ colourings of G_2. The result follows. ■

Our third reduction involves a rather special case.

Theorem 2.4 *Let G be a graph, and let v be a vertex adjacent to all other vertices of G. Then*

$$P(G, q) = qP(G - v, q-1).$$

Proof We have q choices to colour v, and this leaves $q-1$ colours to colour $G - v$. ■

Some examples

A tree T with n vertices has $P(T, q) = q(q-1)^{n-1}$.
This can be deduced directly from the definition – we have q choices for colouring any given vertex, then $q-1$ choices for colouring each of its neighbours, and so on. Alternatively, we can express T as $T_1 \cup T_2$, where $T_1 \cap T_2 = K_1$, and apply Theorem 2.3 and induction.
The cycle C_n with n vertices has $P(C_n, q) = (q-1)^n + (-1)^n(q-1)$.
This can be proved by using the deletion/contraction formula and then applying the above result for trees to $C_n - e$ and induction to C_n/e. Alternatively, we can use the addition/contraction formula, and then apply Theorem 2.3 and induction to $C_n + e = C_{n_1} \cup C_{n_2}$ with $C_{n_1} \cap C_{n_1} = K_2$, and $C_n/e = C_{n_1-1} \cup C_{n_2-1}$ with $C_{n_1-1} \cap C_{n_2-1} = K_1$.
The wheel W_n with n vertices has $P(W_n, q) = q[(q-2)^{n-1} + (-1)^{n-1}(q-2)]$.
This follows from the above result for cycles and Theorem 2.4.

Calculating $P(G, q)$ for an arbitrary graph using Theorems 2.1, 2.3 and 2.4 may take an exponential amount of time. This is to be expected, since it is NP-hard to determine the chromatic number of a graph. Since the chromatic number of G is one more than the largest integer zero of $P(G, q)$, it will also be NP-hard to determine $P(G, q)$. More details on algorithms for calculating chromatic polynomials can be found in Chapter 13.

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The subgraph expansion

Theorem 2.2 tells us that the coefficients of chromatic polynomials are integers. Whitney [49] gave an interpretation for these values.

Theorem 2.5 *Given a graph $G = (V, E)$,*

$$P(G, q) = \sum_{A \subseteq E} (-1)^{|A|} q^{k(A)}, \tag{1}$$

where $k(A)$ is the number of connected components of the graph (V, A). Equivalently, $P(G, q) = \sum_{k=1}^{n} c_k q^k$, where $c_k = c_k^{even}(G) - c_k^{odd}(G)$ and $c_k^{even}(G)$ and $c_k^{odd}(G)$ are respectively the numbers of spanning subgraphs of G with k connected components and with an even and odd number of edges.

Proof Since $P(G, q)$ is the number of colourings of G using colours from $[q] = \{1, 2, \ldots, q\}$, we have

$$P(G, q) = \sum_{\sigma : V \to [q]} \prod_{vw \in E} (1 - \delta_{\sigma(v), \sigma(w)}),$$

where $\delta_{i,j}$ is the Kronecker delta function. We now expand the product as a sum over subsets of E and then reverse the order of summation to obtain

$$\begin{aligned} P(G, q) &= \sum_{\sigma : V \to [q]} \sum_{A \subseteq E} (-1)^{|A|} \prod_{vw \in A} \delta_{\sigma(v), \sigma(w)} \\ &= \sum_{A \subseteq E} (-1)^{|A|} \sum_{\sigma : V \to [q]} \prod_{vw \in A} \delta_{\sigma(v), \sigma(w)} \\ &= \sum_{A \subseteq E} (-1)^{|A|} q^{k(A)}, \end{aligned}$$

where the last equality follows since the product is 1 if σ is constant on each component of (V, A), and is 0 otherwise. ■

Theorem 2.5 explains why $P(G, q)$ is a monic polynomial of degree n with zero constant term. It also implies that the coefficient of q^{n-1} is $-|E|$. The fact that the coefficients of $P(G, q)$ alternate in sign gives a non-trivial relationship between $c_k^{even}(G)$ and $c_k^{odd}(G)$.

3. Log concavity and other inequalities

One of the most celebrated conjectures on chromatic polynomials, that their coefficients are log concave, was made by Read [36] in 1968. This has recently been proved by Huh [23] using techniques from algebraic geometry.

Theorem 3.1 *If $P(G, q) = \sum_{k=1}^{n} c_k q^k$ is the chromatic polynomial of a graph G, then*

$$c_{k-1} c_{k+1} \leq c_k^2$$

for all $2 \leq k \leq n - 1$.

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A related conjecture of Brenti [8], that $P(G,q)$ is log concave for integer values of $q > \chi(G)$, was verified for several families of graphs in [8] but was shown to be false in general by Seymour [38]. Fadnavis [17] has recently shown that the conjecture is true for integer values of q that are large compared to the maximum degree of G.

Theorem 3.2 *Let G be a graph with maximum degree Δ. Then*

$$P(G,q-1)P(G,q+1) \leq P(G,q)^2,$$

for all integers $q \geq 9.767\Delta$.

On the other hand, Bartels and Welsh [1] showed that $P(G,q)$ is convex for integers $q > \chi(G)$.

Theorem 3.3 *Let G be a graph. Then for all integers $q > \chi(G)$,*

$$P(G,q-1) + P(G,q+1) \geq 2P(G,q).$$

Given a graph G on n vertices and a positive integer q, the probability that a random q-colouring of the vertices is a proper q-colouring is $P(G,q)/q^n$. It seems reasonable to expect this probability to increase with q, but Colin McDiarmid showed that this is not true in general (see [1]). Dong [10] showed that this probability increases as long as q is large compared to the order of G.

Theorem 3.4 *For all graphs G on n vertices and all real $q \geq n-1$,*

$$\frac{P(G,q)}{q^n} \leq \frac{P(G,q+1)}{(q+1)^n}.$$

The special case of this result when $q = n-1$ was conjectured by Bartels and Welsh [1]. This conjecture became known as the *shameful conjecture*, because it is equivalent to the 'intuitively obvious' statement that the expected number of colours used in a proper colouring of G is minimized when G has no edges.

Fadnavis [18] has recently used methods from statistical mechanics to show that the probability that a random q-colouring of the vertices of G is a proper q-colouring increases with q if G is claw-free (has no induced $K_{1,3}$), or if q is large compared to the maximum degree of G.

Theorem 3.5 *Let G be a graph with maximum degree Δ and let q be a positive integer. If G is claw-free, or if $q > 400\Delta^{3/2}$, then*

$$\frac{P(G,q)}{q^n} \leq \frac{P(G,q+1)}{(q+1)^n}.$$

4. Chromatic roots

A zero of a chromatic polynomial of a graph G is called a *chromatic root* of G. Birkhoff and Lewis's original paper on chromatic polynomials stimulated research

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into finding intervals on the real line that contain no chromatic roots of given families of graphs. Mathematical physicists on the other hand have been concerned with finding regions of the complex plane for which the Potts model partition function is non-zero, because of implications for the location of phase transitions. Regions free from complex chromatic roots are an important special case.

We can view both areas of research as determining real or complex analytic properties of chromatic polynomials. There has been much less work on determining their algebraic properties. In particular, little is known on the fundamental problem of determining whether or not a given algebraic integer is a chromatic root. We describe the main results and open problems in these areas.

Real roots and the sign of the chromatic polynomial

Results which determine intervals on the real line that are free from chromatic roots have invariably determined the sign of the chromatic polynomial in these intervals. The zero-free intervals for the family of all graphs are determined by the following results from [47], [51] and [24].

Theorem 4.1 *Let G be a graph with n vertices, c components and b blocks, with at least two vertices. Then:*

- *$P(G,q)$ is non-zero with sign $(-1)^n$ for $q \in (-\infty, 0)$, and has a zero of multiplicity c at $q = 0$*
- *$P(G,q)$ is non-zero with sign $(-1)^{n+c}$ for $q \in (0,1)$, and has a zero of multiplicity b at $q = 1$*
- *$P(G,q)$ is non-zero with sign $(-1)^{n+c+b}$ for $q \in (1, \frac{32}{27}]$.*

The first part of this theorem follows immediately from Theorem 2.2. The other parts can be deduced from Theorems 2.1 and 2.3 by using increasingly complicated inductive arguments. A conceptually simpler inductive proof for the last part is given in [29], using the added freedom given by the Potts model partition function. We return to this in Section 5.

The fact that Theorem 4.1 gives a complete list of zero-free intervals follows from a result of Thomassen [43], that the chromatic roots of all graphs form a dense subset of $(\frac{32}{27}, \infty)$. A recent related result of Goldberg and Jerrum [16] implies that it is NP-hard to determine the sign of $P(G,q)$ for all $q \in (\frac{32}{27}, \infty)$ other then $q = 2$. (Note that $P(G,2)$ is positive if G is bipartite and is zero otherwise.)

The families of graphs which demonstrate the aforementioned density and complexity results contain many 2-vertex cuts, and it seems likely that Theorem 4.1 can be extended beyond $q = \frac{32}{27}$ when G is 3-connected. In [24] we conjectured that $P(G,q)$ is non-zero with sign $(-1)^n$ for $q \in (1, \alpha]$ when G is 3-connected, where $\alpha \approx 1.781$ is the chromatic root of $K_{3,4}$ in $(1,2)$. Some evidence in favour of this conjecture is given in [11].

We can also consider other special families of graphs. Results of Birkhoff and Lewis [4] and Woodall [51], [52] give an analogue of Theorem 4.1 for plane

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triangulations which holds throughout the interval $(-\infty, \beta)$, where $\beta \approx 2.546$ is a chromatic root of the octahedron. Thomassen [44] showed that the zero-free interval $(1, \frac{32}{27}]$ can be extended to $(1, \gamma]$ for graphs with a Hamiltonian path, where $\gamma \approx 1.296$ is the real root of $(2-t)^3 = 4(t-1)^2$, and he conjectures [42] that it extends all the way to $(1, 2)$ for graphs with a Hamiltonian cycle. Dong and Koh [13] made the same conjecture for the larger family of *independently* 1*-tough graphs* – that is, graphs G with the property that $G - S$ has at most $|S|$ components for all non-empty independent sets of vertices S.

Results have also been obtained on 'upper zero-free intervals'. The idea is that upper bounds on the chromatic number (and hence on the largest integer chromatic root) of a family of graphs often correspond to similar bounds on the largest real chromatic root for the family. A classical example is the result of Birkhoff and Lewis [4] that the five-colour theorem for planar graphs G can be extended to show that $P(G, q) > 0$ for all $q \in [5, \infty)$, and their conjecture that a similar extension holds for the four-colour theorem. An analogous situation occurs for graphs G with maximum degree Δ. The greedy algorithm implies that $\chi(G) \leq \Delta + 1$, and a result of Dong and Koh [12] gives $P(G, q) > 0$ for all $q \in [5.664\Delta, \infty)$. It is an open problem to decide whether $P(G, q) > 0$ for all $q \in (\Delta, \infty)$, although Woodall [53] has shown that this conclusion holds under a stronger hypothesis.

Theorem 4.2 *If every minor of a graph G has maximum degree at most Δ, then $P(G, q) > 0$ for all $q \in (\Delta, \infty)$.*

Note that upper bounds on the chromatic number do not always give rise to upper zero-free intervals: bipartite graphs have chromatic number 2, but can have arbitrarily large real chromatic roots (see Woodall [51]).

Complex roots

Sokal [40] has shown that there are no zero-free regions in the complex plane for the family of all graphs.

Theorem 4.3 *The chromatic roots of all graphs form a dense subset of the complex plane.*

The family of graphs that he used to prove this result is surprisingly simple. The *generalized theta graph* $\Theta_{s,t}$ consists of two vertices joined by t internally disjoint paths of length s. Sokal showed that, if we vary the parameters s and t, then the chromatic roots of these graphs are dense everywhere in the complex plane, with the possible exception of the disc $|q - 1| < 1$. He then covered this disc with chromatic roots by using the families obtained by adding either one vertex, or two adjacent vertices joined to all to other vertices of $\Theta_{s,t}$, and then applying Theorem 2.4.

As for upper zero-free intervals in the real case, we may hope that an upper bound on the chromatic numbers of a family of graphs might correspond to a similar bound on the largest absolute value of chromatic roots for the same family. The generalized theta graphs demonstrate that this does not occur even for the family of planar graphs.

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Sokal [39] showed, however, that it does occur for graphs with maximum degree Δ, by proving that all their chromatic roots lie in a disc $|q| < C\Delta$, for some universal constant C – this was a long-standing conjecture of Biggs, Damerell and Sands [2]. His proof used techniques from statistical mechanics, which will be outlined below. Sokal's value $C = 7.963$ was improved to $C = 6.907$ by Fernández and Procacci [21], by strengthening one of the key results from statistical mechanics. The aforementioned result of Dong and Koh that the *real* roots are bounded by 5.664Δ was obtained by making a further improvement in the real case.

The following is a gross oversimplification of Sokal's proof technique from [39]. (In particular, Sokal's proof holds for the more general Potts model partition function.) He constructed a weighted graph, called a *polymer gas*, whose vertices (or *polymers*) are the connected subgraphs of G. Each polymer H has weight $(-1)^{|E(H)|}q^{|V(H)|-1}$, and two polymers are joined by an edge whenever their union is connected. The chromatic polynomial of G can be expressed as the independent-set polynomial of this polymer gas: we have $P(G,q) = \sum_S w(S)$, where the sum extends over all independent sets of vertices S in the polymer gas, and $w(S)$ is the product of the weights of the polymers in S. He used the hypothesis on the maximum degree of G to show that the number of connected n-vertex subgraphs containing a fixed vertex of G is at most $c^n\Delta^n$, for some constant c. This gives rise to a bound on the weighted degree of each polymer in the polymer gas, and this in turn enabled him to bound the absolute values of the zeros of its independent-set polynomial by a linear function of $c\Delta$, using powerful results from statistical mechanics (see [20], [32]).

Algebraic properties

Since chromatic polynomials are monic polynomials with integer coefficients, every chromatic root is an algebraic integer. It is easy to see that if α is a chromatic root of a graph G, then so is every *conjugate* of α (that is, every root of the minimum polynomial of α over $\mathbb{Q}$). Combined with Theorem 4.1, this gives us the following necessary condition for an algebraic integer to be a chromatic root.

Theorem 4.4 *If an algebraic integer α is a chromatic root, then no conjugate of α can belong to $(-\infty, \frac{32}{27}] \setminus \{0,1\}$.*

Tutte [46] showed that $|P(G,\tau+1)| \le \tau^{5-n}$ for all plane triangulations with n vertices, where $\tau = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio. This indicates that large plane triangulations may have a chromatic root close to $\tau+1$. On the other hand $\tau+1$ itself cannot be a chromatic root of any graph; this follows from Theorem 4.4, because $2-\tau$ is a conjugate of $\tau+1$ and $2-\tau \in (0,1)$.

It seems unlikely that the necessary condition for an algebraic integer to be a chromatic root given in Theorem 4.4 is also sufficient, but no counter-examples are known. The following two problems are particularly attractive (and frustrating)

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instances of this. The first was suggested by Feng Ming Dong at the Newton Institute Programme on Combinatorics and Statistical Mechanics (CSM) held in Cambridge in 2008; the second came out of conversations between Graham Farr and the author.

Problem *Can a purely imaginary complex number be a chromatic root? In particular, is* $i = \sqrt{-1}$ *a chromatic root?*

Problem *Are* 1, $e^{2\pi i/3}$, $e^{-2\pi i/3}$ *the only chromatic roots of unity?*

The numbers $e^{2\pi i/3}$, $e^{-2\pi i/3}$ are chromatic roots of C_7. The only root of unity that we know cannot be a chromatic root is -1.

We can also search for conditions that are sufficient for an algebraic integer to be a chromatic root. The following conjectures were formulated by a discussion group on the algebraic properties of chromatic roots that took place during the above mentioned CSM Programme (see [5] and [9]).

Conjecture A *If* α *is an algebraic integer, then* $\alpha + p$ *is a chromatic root for some* $p \in \mathbb{N}$.

Conjecture B *If* α *is a chromatic root, then* $\alpha\, p$ *is a chromatic root for all* $p \in \mathbb{N}$.

Conjecture **A** was verified for quadratic integers during the CSM programme (see [9]) and for cubic integers by Bohn [6]. Bohn [7] showed that Conjecture **B** is valid for chromatic roots of generalized theta graphs, and hence holds for a set of algebraic integers that is dense in the complex plane, by the remark after Theorem 4.3.

5. Related polynomials

In this section we define flow polynomials, characteristic polynomials of matroids, Potts model partition functions and Tutte polynomials, and outline their connections with the chromatic polynomial. We allow graphs to contain loops and multiple edges, since such edges have a non-trivial effect on these polynomials. This fact also means that we must keep any multiple edges or loops created by the contraction of an edge (in contrast to the situation for the chromatic polynomial). We refer readers who are unfamiliar with flows in graphs to Chapter 9, and readers unfamiliar with matroids to [34].

Flow polynomials

Tutte [45] defined the *flow polynomial* $F(G, q)$ of a graph G to be the number of distinct nowhere-zero Γ-flows in a fixed orientation of G, for a given abelian group Γ of order q, and showed that this number is the same for all orientations of G and all choices of Γ. Note, however, that the number of nowhere-zero q-flows of G (that is, $\mathbb{Z}$-flows which take the values $\{\pm 1, \pm 2, \cdots, \pm(q-1)\}$) is in general not equal to $F(G, t)$ (see Kochol [31]).

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The flow polynomial is dual to the chromatic polynomial in the following sense. If G is a connected plane graph and G^* is its planar dual, then

$$F(G,q) = q^{-1}P(G^*,q). \tag{2}$$

This identity tells us that results and conjectures on chromatic polynomials of plane graphs can be restated in terms of flow polynomials of plane graphs. In particular, since generalized theta graphs are planar, their duals show that flow roots of planar graphs are dense everywhere in the complex plane, with the possible exception of the disc $|q-1| < 1$ (by the remark following Theorem 4.3).

For non-planar graphs there are still many similarities between the two polynomials: versions of Theorems 2.1, 2.2 and 2.3 hold for flow polynomials (see, for example, [25]); the log concavity of the coefficients of $F(G,t)$ was also established by Huh in [23]; Wakelin [48] obtained an analogue of Theorem 4.1 which determines the sign of $F(G,q)$ for all $q \in (-\infty, \frac{32}{27}]$.

On the other hand, the behaviour of the flow polynomial for all graphs seems to be closer to the behaviour of the chromatic polynomial for the special case of planar graphs. For example, we showed in [26] and [27] that the results of Woodall on the real chromatic roots of plane triangulations mentioned in Section 4 extend to flow polynomials of *all* cubic graphs. A major open problem for flow polynomials is to decide whether there is an upper zero-free interval for all graphs. Welsh conjectured over ten years ago (see [25]) that $(4,\infty)$ is an upper zero-free interval for flow polynomials, and hence that the Birkhoff–Lewis conjecture for chromatic polynomials of plane graphs holds for flow polynomials of all graphs, as long as $q \neq 4$. Royle [37] showed that this is false by constructing graphs with flow roots in $(4,5)$, but offered the modified conjecture that $[5,\infty)$ is an upper zero-free interval. The value $q = 5$ is of special significance since it corresponds to Tutte's 5-flow conjecture. The best-known general upper bound on flow roots, that $F(G,q) > 0$ whenever G is a bridgeless graph on n vertices and $q > 2\log_2 n$, is given in [25].

Characteristic polynomials of matroids

Similarities between the chromatic and flow polynomials for non-planar graphs can be explained by extending the duality identity (2) using matroids. We denote a *matroid* as $M = (E,r)$, where E is the set of elements and r is the rank function. The *characteristic polynomial* of M is

$$C(M,q) = \sum_{A\subseteq E} (-1)^{|A|} q^{r(E)-r(A)}. \tag{3}$$

The *cycle matroid* of a graph $G = (V,E)$ is the matroid $M_G = (E,r_G)$, where $r_G(A) = |V| - k(A)$ for all $A \subseteq E$. Comparing (1) and (3) we see that

$$P(G,q) = q^{k(G)}\, C(M_G,q), \tag{4}$$

where $k(G)$ is the number of connected components of G.

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Given a matroid $M = (E, r)$, its *dual matroid* $M^* = (E, r^*)$ is defined by putting $r^*(A) = |A| - r(E) + r(E \setminus A)$ for all $A \subseteq E$. The *cocycle matroid* M^*_G of a graph G is the dual matroid of its cycle matroid. A similar argument to that used in the derivation of (1) (see, for example, [41]) implies that

$$F(G, q) = \sum_{A \subseteq E} (-1)^{|A|} q^{|E \setminus A| - |V| + k(E \setminus A)}, \tag{5}$$

and (3) now gives

$$F(G, q) = C(M^*_G, q). \tag{6}$$

Equations (4) and (6) imply that chromatic and flow polynomials of graphs are both examples of characteristic polynomials of matroids. Furthermore, when G is a connected plane graph and G^* is its planar dual, we have $M^*_G = M_{G^*}$. Hence (4) and (6) also imply (2).

It follows from the above discussion that any result on the characteristic polynomials of a family of matroids which includes both the cycle and cocycle matroids of graphs implies the same result for chromatic and flow polynomials as special cases. For example, Huh showed in [23] that the coefficients of characteristic polynomials of matroids that are representable over a field of characteristic 0 are log concave, and a version of Theorem 4.1 for characteristic polynomials of all matroids was given in [15] and [29].

A challenging open problem is to decide whether some version of the results of Section 4 on complex roots of graphs of bounded degree holds for flow polynomials – or, better still, for characteristic polynomials of a family of matroids which includes the cycle matroids of graphs of maximum degree Δ and their matroid duals. A conjecture and some preliminary results are given in [30] and [28].

The Potts model partition function

Let $G = (V, E)$ be a graph, and let $\mathbf{w} : E \to \mathbb{C}$ be a weight function on E. The *Potts model partition function*, or *multivariate Tutte polynomial*, of the weighted graph $(G, \mathbf{w})$ is

$$Z(G, q, \mathbf{w}) = \sum_{A \subseteq E} q^{k(A)} w_A, \tag{7}$$

where $w_A = \prod_{e \in A} w_e$ and w_e denotes the weight of the edge e. When all edges have the same weight w, we write $Z(G, q, w)$ for the partition function. Comparing (7) with (1), we see that $P(G, q) = Z(G, q, -1)$. A similar argument to that used in the derivation of (1) gives us the following expression for the partition function when q is a positive integer:

$$Z(G, q, \mathbf{w}) = \sum_{\sigma : V \to [q]} \prod_{e = vw \in E} (1 + w_e \delta_{\sigma(v), \sigma(w)}). \tag{8}$$

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This expression for $Z(G, q, \mathbf{w})$ is similar to the form given in the original paper of Potts [35]; the equivalence of the two expressions is due to Fortuin and Kasteleyn [22].

Equation (8) expresses $Z(G, q, \mathbf{w})$ as a weighted sum over all q-colourings of the vertices of G, where a particular q-colouring σ has weight $\prod(1 + w_e)$ and the product ranges over all edges e that are not properly coloured by σ. This colouring interpretation enables us to extend Theorem 2.1 and a special case of Theorem 2.3 to the partition function, using similar proofs.

Theorem 5.1 *Let $(G, \mathbf{w})$ be a weighted graph and let e be an edge of $G = (V, E)$. Then*

$$Z(G, q, \mathbf{w}) = Z(G - e, q, \mathbf{w}|_{E-e}) + w_e\, Z(G/e, q, \mathbf{w}|_{E-e}).$$

Theorem 5.2 *Let $(G, \mathbf{w})$ be a weighted graph and let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be edge-disjoint subgraphs of G with $|V_1 \cap V_2| = r$ for some $r \in \{0, 1\}$ and with $G_1 \cup G_2 = G$. Then*

$$Z(G, q, \mathbf{w}) = q^{-r} Z(G_1, q, \mathbf{w}|_{E_1})\, Z(G_2, q, \mathbf{w}|_{E_2})\,.$$

(The hypothesis of Theorem 5.2 that the subgraphs are edge-disjoint is needed in the case when $r = 1$ and their common vertex is incident to one or more loops.)

An advantage of introducing edge-weights is that they give rise to new reduction operations. The *parallel reduction operation* replaces two parallel edges by a single edge with the same endpoints. The *series reduction operation* replaces a vertex v of degree 2 and its incident edges by a single edge joining the neighbours of v.

Theorem 5.3 *Let $(G, \mathbf{w})$ be a weighted graph and let e_1, e_2 be edges of G.*

If H is obtained by applying the parallel reduction operation to e_1, e_2, then

$$Z(G, q, \mathbf{w}) = Z(H, q, \mathbf{w}|_{E\setminus\{e_1,e_2\}}, w_{e_1} \| w_{e_2})\,,$$

where $w_{e_1} \| w_{e_2} = w_{e_1} + w_{e_2} + w_{e_1} w_{e_2}$ is the weight of the edge that replaces e_1 and e_2.

If H is obtained by applying the series reduction operation to e_1, e_2, then

$$Z(G, q, \mathbf{w}) = (q + w_{e_1} + w_{e_2}) Z(H, q, \mathbf{w}|_{E\setminus\{e_1,e_2\}}, w_{e_1} \bowtie_q w_{e_2})\,,$$

where $w_{e_1} \bowtie_q w_{e_2} = w_{e_1} w_{e_2} (q + w_{e_1} + w_{e_2})^{-1}$ is the weight of the edge that replaces e_1 and e_2.

We refer the reader to Sokal [41] for a proof of these results. He also gives several impressive applications of how the freedom to change the edge-weights can give simple proofs for $Z(G, q, \mathbf{w})$, and hence give results for $P(G, q)$ that were either new or were obtained by much more difficult proofs. One such example is given below.

The basic idea behind the original proof [24] of the last part of Theorem 4.1 is as follows. We proceed by contradiction. Fix $q \in (1, \frac{32}{27}]$ and suppose that

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$(-1)^{n+c+b}P(G,q) \leq 0$ for some graph $G = (V,E)$. Choose G with $|E|$ as small as possible. Theorem 2.3 implies that G is 2-connected and Theorem 2.1 then tells us that either $G - e$ or G/e is not 2-connected for all $e \in E$. This gives structural information about G which can be refined by several more applications of Theorems 2.1 and 2.3 to show that G belongs to a well-defined family of 2-connected graphs called 'generalized triangles'. The proof is completed by an inductive proof that every generalized triangle G satisfies $(-1)^n P(G,q) > 0$.

The proof given in [29] is much simpler. We choose $q \in (1, \frac{32}{27}]$ and use a stronger inductive hypothesis that $(-1)^{n+c+b}Z(G,q,\mathbf{w}) > 0$ for all $\mathbf{w} : E \to I_q$, for some interval $I_q \subseteq (-\infty, -q/2)$ with $-1 \in I_q$. The interval I_q is chosen to be closed under the parallel and series transformations $(w_1, w_2) \mapsto w_1 \| w_2$ and $(w_1, w_2) \mapsto w_1 \bowtie_q w_2$. We use Theorem 5.2 to reduce to the case when G is 2-connected and loopless, and Theorem 5.3 to reduce to the case when G has minimum degree 3 and no multiple edges. We can then use a result of Oxley [33] to deduce that G has an edge e for which $G-e$ and G/e are both 2-connected. The proof is completed by applying Theorem 5.1 using this edge e. This proof also explains the significance of the number $\frac{32}{27}$; it is the largest value of q for which there exists an interval $I_q \subseteq (-\infty, 0)$ such that $-1 \in I_q$ and I_q is closed under the parallel and series transformations.

The proof in [29] is matroidal and gives the same result for the matroid version of the partition function. It therefore provides a simple proof that $(1, \frac{32}{27}]$ is a zero-free interval for characteristic polynomials of matroids, and hence also for flow polynomials of graphs.

Tutte polynomials

The *Tutte polynomial* of a graph $G = (V,E)$ was defined by Tutte [45] as

$$T(G,x,y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(G)} (y-1)^{|A|-|V|+k(A)}. \tag{9}$$

Comparing (9) with (1) and (5) tells us that

$$P(G,q) = (-1)^{|V|-k(G)}\, q^{k(G)}\, T(1-q,0)$$

and

$$F(G,q) = (-1)^{|E|+|V|+k(G)}\, T(0,1-q)\,,$$

so the chromatic and flow polynomials are both specializations of the Tutte polynomial. Similarly, a comparison of (9) and (7) implies that

$$T(G,x,y) = (x-1)^{-k(G)}(y-1)^{-|V|}\, Z(G,(x-1)(y-1),y-1)$$

and

$$Z(G,q,w) = q^{k(G)}\, w^{|V|-k(G)}\, T(G,1+q/w,1+w),$$

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so the Tutte polynomial and the partition function with constant edge-weights can be interchanged by a simple change of variables. Note, however, that the variables in these polynomials play different roles. In the partition function, q acts as a global variable representing the number of possible colours or 'states' for each vertex and w represents the weight of an edge. The variables x, y in the Tutte polynomial act as a pair of 'dual' global variables. This duality is best illustrated when we extend the definition of the Tutte polynomial to a matroid $M = (E, r)$: we obtain

$$T(M, x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{r^*(E)-r^*(E \setminus A)}, \tag{10}$$

where r^* is the rank function for the dual matroid M^*. This immediately gives $T(M, x, y) = T(M^*, y, x)$ and implies that $T(G, x, y) = T(G^*, y, x)$ for any connected planar graph G with planar dual G^*.

The evaluation of the Tutte polynomial of a graph G at the points in the xy-plane (the so-called *Tutte plane*) gives an attractive picture for viewing all of the above polynomials (and many more): the x-axis corresponds to the chromatic polynomial, the y-axis corresponds to the flow polynomial and the hyperbolas $(x-1)(y-1) = q$ give the Potts model partition function for fixed q and all edge-weights equal to $y-1$ (see Fig. 1).

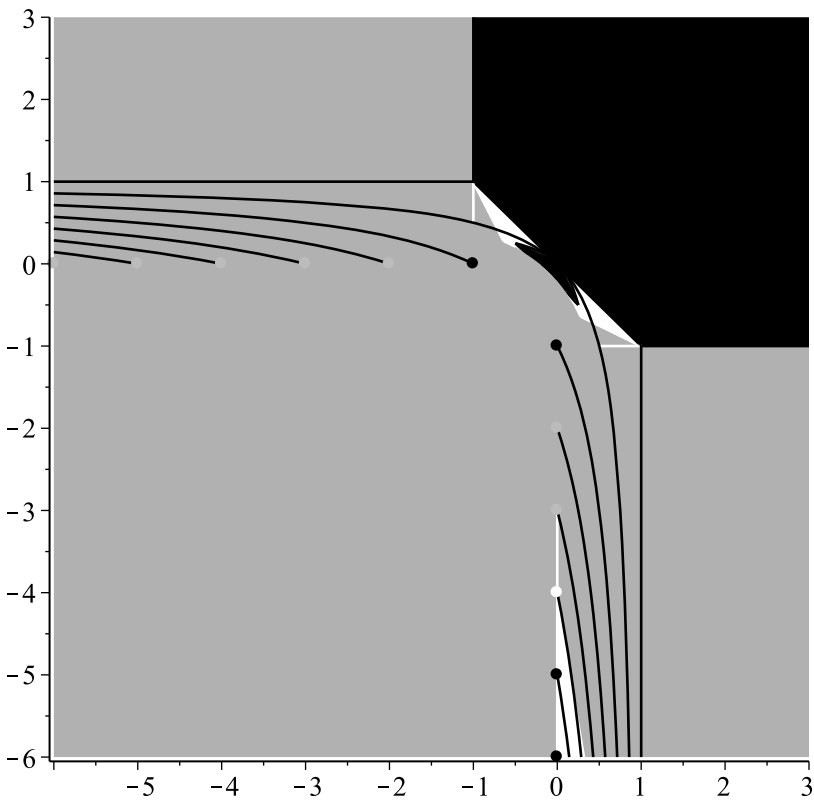


Fig. 1. Determining the sign of $T(x, y)$ in the Tutte plane

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The results of [29] can be applied to give a region in the Tutte plane for which the sign of $T(G, x, y)$ is determined by a function of the numbers of vertices, components and blocks of G. Goldberg and Jerrum [16] have recently obtained a complementary result which identifies large regions in the Tutte plane for which it is NP-hard to determine the sign of $T(G, x, y)$. They also point out that in addition to the tractable region implied by [29], we can use (8) to determine the sign of $T(G, x, y)$ at certain points on the hyperbolas $q = (x-1)(y-1)$ when q is a positive integer.

Computing the sign of the Tutte polynomial is polynomial at black points and is NP-hard at grey points. The complexity at white points is not known: the white point $(0, -4)$ and black point $(0, -5)$ correspond respectively to Tutte's 5-flow conjecture and Seymour's 6-flow theorem; the lower white region is related to the problem of determining whether there is an upper zero-free interval for flow polynomials; the curved boundary between the black and grey regions is given by the hyperbola $(x-1)(y-1) = \frac{32}{27}$; [29, Conjectures 10.1, 10.2] would imply that the black region can be extended to fill most of the two upper white regions.

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We refer the reader to [14], [25] and [37] for more information on chromatic, flow, and characteristic polynomials, to [41] for Potts model partition functions, and to [19] for Tutte polynomials.

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