CLASSICAL ELECTRODYNAMICS AS A DISTRIBUTION THEORY

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ABSTRACT. In this paper a precise distribution-theoretic formalism for the description of point charges interacting through a classical electromagnetic field is given. The distribution solutions of the Maxwell equations are shown to reduce to the Lienard–Wiechert fields, which are summable over the whole of space-time. It is not possible to obtain directly a unique field intensity, on the world lines, from these distribution solutions. A brief analysis shows that analytic continuation methods also do not give a unique field intensity on the world lines. The energy tensor for the field is a distribution which can only be rigorously defined in terms of the motions of the charges. Thus only the value of the field energy or momentum residing in any finite region of space-time can be given any meaning. Equations of motion for the charges may be derived from the field energy tensor, using the principle of conservation of total energy and momentum for the system. These equations agree with those obtained previously by Dirac (**2**). Also any canonical formalism in which the field potentials and the particle variables enter in an equivalent manner is not possible. The difficulty of the definition of products of distributions in the stress tensor does not occur.

1. *Introduction.* The main purpose of this paper is to obtain a mathematically consistent formalism for a system composed of charges interacting through their classical electromagnetic fields. These charges may be considered as either point charges or as continuous distributions of charge. It is important to determine such a formalism in order to exhibit the precise limitations on physical concepts defined in the formalism. The limitations associated with field and field energy are discussed in §7. Further, a mathematically correct quantized field theory can only be obtained by quantizing a mathematically correct classical field theory.

No fundamental difficulties, such as self-energy divergences, arise in the classical theory when the charges are continuously distributed in space. The divergence difficulties arise when it is attempted to apply the methods used for continuous charge distributions to systems of point charges. A consistent divergence-free classical theory of interacting point charges has already been obtained by Riesz (**8**) and Fremberg (**4**), who used the method of analytic continuation, which was shown equivalent to the λ-limiting method. However, the continuation method hides the essential mathematical structure involved (see §4 for complete discussion). Further, the extension of the continuation method to quantum field theory by Nilsson (**7**) did not eliminate high-order divergences in the self-energy terms, and also introduced covariance difficulties.

The direct approach to a system containing point charges may be effected by the method of distributions, as developed by Schwartz (**9**). Any discussion of the problem essentially involves the δ-function, and hence the use of distribution theory is essential to ensure that results are obtained from the physical concepts by rigorous mathematical reasoning. A distribution-theoretic treatment of classical electrodynamics is elaborated here.

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The main results of the paper are as follows:

(i) The retarded distribution solution of the field equations is the retarded Lienard–Wiechert field. The field intensity is only defined *almost everywhere* in space-time. In particular, the self-field on a point charge is not defined uniquely. This disagreement with the results of the analytic continuation method is resolved when it is noted that analytic continuation has been applied too freely. The precise application of continuation again only gives a field intensity defined almost everywhere in space-time.

This lack of definition, on any set of measure zero, of the solution of the Maxwell equations (1) and (4) of § 2 will always arise, unless further conditions on the solution are specified. Thus, however equation (1) is solved, the equations of motion for the point charges cannot be obtained from it without further conditions.

(ii) The field energy tensor is a distribution, so only averages of this tensor over finite space-time regions have meaning. A rigorous definition of the energy tensor would appear to require a rigorous definition of the products of distributions. That this cannot be given has been shown by Schwartz (10). It is shown in this paper that the energy-tensor distribution may be rigorously constructed by the use of the field strengths, if these are known in terms of the motions of the point charges.

(iii) The principle of conservation of total energy and momentum, together with the energy tensor for the field of a single point charge, determine the self-force on the point charge uniquely, and hence the equations of motion for the charge. These equations of motion agree with those obtained by Dirac (2) and Fremberg (4). Similarly, the equations of motion of a system of point charges are determined from the associated energy-tensor distribution for the field, together with energy-momentum conservation.

(iv) The usual canonical formalism for a system composed of a field and point charges is obtained by considering the field variables and particle variables as independent. However, the above results show that it is not possible to preserve this independence of dynamical variables when the Hamiltonian for the system is required in a rigorous form.

(v) A general formalism in which the actual structure of the charges is arbitrary, i.e. which applies to particles of positive or zero radius, can be completely given by means of distribution theory. This formalism, as stated in (iv) above, does not allow of a simple canonical formulation.

Owing to the apparent lack of any rigorous classical Hamiltonian formalism, the canonical quantization method cannot be rigorously defined. Further, the quantized field equations, including interaction, are not well defined, owing to the existence of products of distributions in the interaction terms. Thus a distribution-theoretic divergence-free theory of interacting quantized fields is not immediately evident.

In the appendix a brief discussion of the notations used in this paper is given.

2. *Maxwell equations for point sources.* Relativistic notation will be used throughout the paper, with the velocity of light taken to be unity. Greek suffixes will take the possible values 0, 1, 2, 3. The contravariant components of a vector will be written K_μ, with covariant components K^μ. These components are connected by the metric tensor $g^{\mu\nu}$, which has components $g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$, while all the other components of $g^{\mu\nu}$ vanish.

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The Maxwell equations for the electromagnetic potentials A_μ, and the field strengths $F_{\mu\nu}$, in terms of the current density vector j_μ, are as follows:

$$\Box A_\mu = \frac{\partial^2}{\partial x_\nu \partial x^\nu} A_\mu = 4\pi j_\mu, \tag{1}$$

$$\frac{\partial A_\mu}{\partial x_\mu} = 0, \tag{2}$$

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}. \tag{3}$$

For a system of n point charges, let the charges move along given world lines specified by the parametric equations $z_\mu = z_\mu^{(i)}(s_i)$ for the ith charge, where the $z_\mu^{(i)}$ are given functions of the proper times s_i, and are sufficiently differentiable functions of their arguments. Then

$$j_\mu(\mathbf{x}) = \sum_{i=1}^{n} e_i \int \delta(\mathbf{x} - \mathbf{z}(s_i)) \frac{dz_\mu^{(i)}}{ds_i} ds_i, \tag{4}$$

where $\delta(\mathbf{x})$ is the δ-function associated with the origin and e_i is the charge of the ith particle. Since $\delta(\mathbf{x})$ is not a function but a distribution, in the sense of Schwartz (9), the current density vector must be considered as a distribution, which has its support equal to the world lines L_i $(i = 1, 2, \ldots, n)$ of the n charges, and for any function $\phi \in \mathscr{D}$

$$j_\mu \phi = \sum_{i=1}^{n} e_i \int_{L_i} \phi(\mathbf{z})\, dz_\mu. \tag{5}$$

The charge-conservation equation is satisfied in a distribution sense, since any $\phi \in \mathscr{D}$ vanishes outside a finite set, so that

$$\frac{\partial j_\mu}{\partial x_\mu} \phi = -\sum_{i=1}^{n} e_i \int_{L_i} \frac{\partial \phi}{\partial z_\mu} dz_\mu = 0. \tag{6}$$

Hence Maxwell's equations, associated with point sources, are distribution equations, and the fields $F_{\mu\nu}$ and potentials A_μ are distributions. This formulation of the Maxwell equations as distribution equations will be correct, moreover, for an arbitrary current density vector, i.e. when j_μ is a well-defined function, or a distribution. Schwartz has pointed out ((9), I, Introduction, pp. 7 and 127) that not all the solutions of a general inhomogeneous hyperbolic partial differential equation are twice-differentiable functions. Some of the solutions must be considered as distributions, and the equation considered as a distribution equation. In the remainder of this paper, results will be obtained only for point sources, but these results evidently hold for extended sources (when j_μ is a suitably well-behaved function).

The equations (1) and (2) may be written in the composition form

$$\Box * A_\mu = 4\pi j_\mu, \tag{7}$$

$$\frac{\partial}{\partial x_\mu} * A_\mu = 0. \tag{8}$$

The next step is to solve the system of equations (5), (7) and (8), and investigate the properties of the distribution solutions.

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3. *Solutions of the Maxwell equations.* The solutions of the hyperbolic differential equation (7) are determined with the aid of the distributions Z satisfying

$$\Box * Z = \delta_{(0)}, \tag{9}$$

where $\delta_{(0)}$ is the δ-distribution associated with the origin in space-time. These solutions were considered by Riesz (8), and given their distribution form by Schwartz ((9), I, p. 50). They have been completely investigated by Methe (6). There are two elementary solutions $Z_2, \check{Z}_2$ (in the notation of Schwartz) which are invariant under Lorentz transformations which do not reverse time. Z_2 has support equal to the future light cone with vertex at the origin, i.e. $x_0 \geqslant 0$, $x_\mu x^\mu = 0$, while $\check{Z}_2$ is obtained from Z_2 by reflexion in the origin of coordinates, and thus has support equal to the past light cone with vertex at the origin. ($Z_2, \check{Z}_2$ are the invariant distributions usually written $\Delta_{\text{ret.}}, \Delta_{\text{adv.}}$ in the literature.) $Z_2 * j_\mu$ may be defined as an element of $\mathscr{D}'$, provided the L_i have time-like asymptotes at $x_0 = -\infty$. If this is satisfied, the retarded solution of equation (7) is

$$A_\mu^{\text{ret.}} = 4\pi Z_2 * j_\mu. \tag{10}$$

That (10) satisfies equations (7) and (8) follows from equations (6) and (9), and from the associative and commutative properties of the composition product:

$$\begin{aligned} \Box * A_\mu^{\text{ret.}} &= 4\pi \Box * (Z_2 * j_\mu) \\ &= 4\pi \delta_{(0)} * j_\mu = 4\pi j_\mu, \end{aligned} \tag{11}$$

$$\frac{\partial}{\partial x_\mu} * A_\mu^{\text{ret.}} = 4\pi Z_2 * \frac{\partial j_\mu}{\partial x_\mu} = 0. \tag{12}$$

Similarly, if the L_i have time-like asymptotes at $x_0 = +\infty$, the advanced solution of equations (5), (7) and (8) is

$$A_\mu^{\text{adv.}} = 4\pi \check{Z}_2 * j_\mu. \tag{13}$$

$A_\mu^{\text{ret.}}$ and $A_\mu^{\text{adv.}}$ are distributions $\in \mathscr{D}'$.

4. *Definition of the field intensity.* The field distribution $F_{\mu\nu}$, determined by equations (3) and (10), is a distribution which may not reduce to a function of the space-time argument $\mathbf{x}$. The field intensity at the point $\mathbf{x}$ cannot be defined immediately in terms of this distribution. However, if this distribution reduces locally to a function $F_{\mu\nu}(\mathbf{x})$, so that

$$\int F_{\mu\nu} \,.\, \phi(\mathbf{x})\, d^4\mathbf{x} = \int_\Omega F_{\mu\nu}(\mathbf{x})\, \phi(\mathbf{x})\, d^4\mathbf{x}, \tag{14}$$

then the field intensity at the point $\mathbf{x}$ in the four-dimensional volume Ω is defined to be $F_{\mu\nu}(\mathbf{x})$. Further, the Lorentz force on a particle, which has charge e, and is moving with the four-velocity $u_\nu(\mathbf{x})$ at the space-time point $\mathbf{x}$ is defined to be $F_{\mu\nu}(\mathbf{x})\, u^\nu(\mathbf{x})$.

It is now necessary to investigate if the field distribution reduces to a function, so that the above definition would enable the field intensity to be determined.

5. *Calculation of the field intensity.* For the case of one charged particle, equation (10) may be written

$$\begin{aligned} A_\mu^{\text{ret.}} \phi &= 4\pi e \int (Z_2)_x \int_L dz_\mu \phi(\mathbf{x}+\mathbf{z})\, d\mathbf{x} \\ &= e \iiint d^3x \int_L dz_\mu \phi(x_i + z_i, |x| + z_0)\, |x|^{-1}, \end{aligned} \tag{15}$$

where $x = (x_1, x_2, x_3)$ and $|x| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$.

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By means of the transformation $x_\mu + z_\mu(s) = y_\mu$, $|x| = x_0$, and the assumption that L extends into the infinite past and future, the right-hand side of equation (15) becomes

$$A_\mu^{\text{ret.}}\phi = e\int d^4\mathbf{y}\, u_\mu(s)\,(x_\mu u^\mu)^{-1}\,\phi(\mathbf{y}). \tag{16}$$

Thus $A_\mu^{\text{ret.}}$ is equal to a function (the retarded Lienard–Wiechert potential) which is differentiable outside L and is summable over L. (Lienard–Wiechert will be abbreviated to L–W in the following.) The field distribution outside L is thus obtained by means of equation (3), regarding the differentiation in this equation to be function differentiation. The definition of the field intensity given in the last section then implies that the associated field intensity outside L is the retarded L–W field. A similar result evidently holds for the advanced field distribution and for the advanced field intensity.

The L–W field intensity is not defined on L. Further, the L–W field may not be summable over L, and so need not be a distribution solution of the field equations. Thus to determine the field intensity on L, it is first necessary to consider the summability of the L–W field over L. To do this, a convenient system of coordinates will now be introduced.

For any point P, with coordinates x_μ, let P_0 be the point on L with coordinates $z_\mu(s)$, where s is chosen so that with $x_\mu - z_\mu(s) = \gamma_\mu$ then

$$\gamma_\mu u^\mu(s) = \gamma_\mu(dz^\mu/ds) = 0. \tag{17}$$

For any $\epsilon > 0$, define $T(\epsilon)$ as the tube $\gamma_\mu\gamma^\mu = -\epsilon$. Then any point P is specified by the coordinates $s, \epsilon, \theta_1, \theta_2$, where s is defined by equation (17), P lies on the tube $T(\epsilon)$, and θ_1, θ_2 are angular variables for points on the three-dimensional surface S. S is the set of points with coordinates x_μ which satisfy the equations $\gamma_\mu\gamma^\mu = -\epsilon$, $\gamma_\mu u^\mu(s) = 0$ for given ϵ and s. The element of surface area $d\sigma$ on $T(\epsilon)$ is $d\sigma = |1 - \gamma_\lambda \dot{u}^\lambda|\, ds\, dS$ (Dirac (2), p. 168), where dS is the element of surface area on the surface S. The element of length normal to $T(\epsilon)$ is $(\frac{1}{2})i\epsilon^{-\frac{1}{2}}\, d\epsilon$.

The expansion of the L–W field in powers of γ_μ and ϵ, as determined by Dirac ((2), p. 164, equation (60)), is

$$\begin{aligned} F_{\mu\nu}^{\text{ret.}}(P) = e(1-\gamma_\lambda\dot{u}^\lambda)^{-\frac{1}{2}}[&-\epsilon^{-\frac{3}{2}}u_\mu\gamma_\nu - (\tfrac{1}{2})\epsilon^{-1}u_\mu u_\nu(1+\gamma_\lambda\dot{u}^\lambda) \\ &+ (\tfrac{1}{8})\epsilon^{-1}\dot{u}^2 u_\mu\gamma_\nu + (\tfrac{1}{2})\epsilon^{-1}\ddot{u}_\mu\gamma_\nu + (\tfrac{2}{3})\ddot{u}_\mu u_\nu]_- + O(\epsilon), \end{aligned} \tag{18}$$

where $[a_\mu b_\nu]_- = a_\mu b_\nu - a_\nu b_\mu$.

Define $I_{\mu\nu}(\epsilon)$ by the equation $I_{\mu\nu}(\epsilon) = \int_{V(\epsilon)} F_{\mu\nu}(x)\, d^4\mathbf{x}$, (19)

where $V(\epsilon)$ is a region outside $T(\epsilon)$, for example, the region between $T(\epsilon)$ and $T(a)$, where a is small enough so that the expansion of equation (18) is valid. Then from equation (18),

$$\begin{aligned} I_{\mu\nu}(\epsilon) = (\tfrac{1}{2})ie\int_\epsilon^a \delta^{-\frac{1}{2}}\, d\delta \int_L ds \int_S dS(1-\gamma_\lambda\dot{u}^\lambda)^{\frac{1}{2}} \\ \times\{[-\delta^{-\frac{3}{2}}u_\mu\gamma_\nu - (\tfrac{1}{2})\delta^{-\frac{1}{2}}\dot{u}_\mu u_\nu(1+\gamma_\lambda\dot{u}^\lambda) + (\tfrac{1}{8})\delta^{-\frac{1}{2}}\dot{u}^2 u_\mu\gamma_\nu + (\tfrac{1}{2})\delta^{-\frac{1}{2}}\ddot{u}_\mu\gamma_\nu]_- + O(1)\}. \end{aligned} \tag{20}$$

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a is also assumed small enough so that the term $(1-\gamma_\lambda \dot{u}^\lambda)^{\frac{1}{2}}$ may be expanded in powers of γ_λ in the integrand on the right-hand side of equation (20). This integrand is then a sum of terms, each containing products of various components of γ_μ. Any term containing an odd number of γ_μ factors gives zero contribution after integration over S. Any term containing an even number of γ_μ factors can be integrated by means of the expressions

$$\int_S dS(\gamma_\mu a^\mu)(\gamma_\nu b^\nu) = (\tfrac{4}{3})\pi\epsilon^2[(a_\mu u^\mu)(b_\mu u^\mu) - (a_\mu b^\mu)], \tag{21}$$

$$\int_S dS\,\gamma_\nu(\gamma_\mu a^\mu) = (\tfrac{4}{3})\pi\epsilon^2[u_\nu(u_\mu a^\mu) - a_\nu], \tag{22}$$

where (a_μ), (b_ν) are any four-vectors which are independent of θ_1, θ_2. The expressions may be obtained by evaluating the left-hand sides in the particular frame of reference in which the particle is at rest at the point $\mathbf{z}(s)$, and then using the Lorentz transformation of the resultant expression to the frame of reference in which the particle has four-velocity $u_\mu(s)$. Then $I_{\mu\nu}(\epsilon)$ reduces to

$$I_{\mu\nu}(\epsilon) = (\tfrac{4}{3})\pi e i\int_\epsilon^a d\delta \int_L ds\,\{[u_\mu \dot{u}_\nu]_- + \delta^{\frac{1}{2}}[\ddot{u}_\mu u_\nu]_- + O(\delta)\}. \tag{23}$$

Thus $I_{\mu\nu}(0)$ exists, so that the L–W field is summable over L. It may be shown in a similar manner that the L–W potential is summable over L.

The L–W field is not differentiable on L. It may easily be shown, using the system of coordinates ϵ, s, θ_1, θ_2, that $\partial F_{\mu\nu}/\partial x_\nu$ vanishes outside L, and that for any $\phi \in \mathscr{D}$,

$$\begin{aligned}\int F_{\mu\nu}(\mathbf{x})(\partial\phi/\partial x_\nu)\,d^4\mathbf{x} &= \lim_{\epsilon\to 0}\int_{T(\epsilon)} F_{\mu\nu}(\mathbf{x})\,\phi(\mathbf{x})\,d\sigma^\nu \\ &= 4\pi\int_L u_\mu(s)\,\phi(s)\,ds.\end{aligned} \tag{24}$$

Thus the retarded L–W field is equal to the retarded field distribution defined by equations (3) and (10). It must be emphasized here that while the retarded field distributions reduce to functions throughout all space-time, the Maxwell field equations cannot be reduced to function equations. Similar results hold for the equality of the advanced L–W field and the advanced field distribution.

From the definition of field intensity of the previous section, it is evident that the field intensity on L cannot be defined without further assumptions. For example, the assumptions made by Dirac ((2), p. 151) enabled the field intensity to be defined in terms of $(F^{\text{ret.}}_{\mu\nu}(\mathbf{x}) - F^{\text{adv.}}_{\mu\nu}(\mathbf{x}))$, which gives a finite field intensity on L. Other assumptions have been shown to give the same value for the field intensity on L (e.g. Feynman and Wheeler (3)).

Thus the self-force on the point charge cannot be derived from the Maxwell equations for the field quantities alone. This result is further borne out by the following considerations. The value of the L–W field is not uniquely defined on a set Ω of measure zero in space-time and may be changed arbitrarily in Ω without altering the fact that the resulting field function will still satisfy the Maxwell distribution equations. In particular, the value of the field intensity on L, as defined by §4, cannot be specified on L by the Maxwell equations alone. Any particular value of the field intensity on

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L may be chosen, according to the extra assumptions imposed. This ambiguity also applies to the value of the field intensity on the world line of any other point charge in the field of the original one. The equations of motion for this point charge are thus not defined uniquely, and here again further assumptions must be introduced to limit this ambiguity.

This ambiguity was removed by the analytic continuation method, which was developed by Riesz (8), and applied to the problem of the self-force on a moving point charge by Fremberg (4). The method involves the continuation with respect to the complex variable α, of the expression $f_\alpha(\mathbf{x}) = (Z_\alpha * j_\mu)_x$. In this expression, Z_α is the quantity $H(\alpha)\,Pf(s^{\alpha-4})$, and $H(\alpha) = 2^{1-\alpha}[\pi\Gamma(\frac{1}{2}\alpha)\,\Gamma(\frac{1}{2}(\alpha-2))]^{-1}$. The continuation is down to the value $\alpha = 2$, and $f_2(\mathbf{x})$ is just the retarded potential of equation (10).

The continuation method requires that j_μ is a well-defined function of its space-time argument. If this is so, then $f_\alpha(\mathbf{x})$ is a well-defined function of $\mathbf{x}$, and for each value of $\mathbf{x}$ is then a regular function of α for some domain D of values of α. The usual methods of analytic continuation on α are then applicable, for each value of $\mathbf{x}$. However, when j_μ is a distribution, in particular, the distribution defined by equation (4), then for some values of α the quantity $f_\alpha(\mathbf{x})$ is not necessarily a function of $\mathbf{x}$. Thus the expression

$$f_\alpha(\mathbf{x}) = \int_L Z_\alpha(\mathbf{x}-\mathbf{z}(s))\,u_\mu(s)\,ds, \tag{25}$$

to which $f_\alpha(\mathbf{x})$ reduces when j_μ is given by equation (4), is not a function of $\mathbf{x}$ when expressed in this form, for $\alpha \leqslant 2$, and so is not an analytic function of α, for each $\mathbf{x}$, when $\alpha \leqslant 2$. It is thus not correct to apply continuation methods directly to this expression.

As an example of how incorrect results might arise, consider the expression, in one dimension,

$$g_\alpha(x) = (\Gamma(\alpha))^{-1}\,Pf(x^{-\alpha-1})_+.$$

$g_\alpha(x)$ is a function of x for all $\alpha < 0$, and is then an analytic function in α for each x. For any $\alpha < -1$, $g_\alpha(0) = 0$. It is incorrect to conclude from this, by continuing on α, that $f_\alpha(0) = 0$ for all values of α. For, in particular, $g_0(x) = \delta_{(0)}$, which is the δ-distribution.

In the case when it is known that $f_\alpha(x)$ is a distribution on x, for values of α in the range considered, it is necessary to form the expression $f_\alpha(\phi) = \int f_\alpha(x)\,\phi(x)\,dx$, where $\phi \in \mathscr{D}$. For each ϕ, $f_\alpha(\phi)$ is an analytic function of α, and so may be continued down to the required value of α. $f_\alpha(\phi)$ is then a continuous linear functional on $\mathscr{D}$, and is thus a distribution $\in \mathscr{D}'$. In particular, it is necessary to form the expression

$$f_\alpha(\phi) = \int_L Z_\alpha(\mathbf{x}-\mathbf{z}(s))\,u_\mu(s)\,\phi(\mathbf{x})\,ds\,d^4\mathbf{x}$$

before continuation on α can be correctly applied down to the value $\alpha = 2$. The expression $f_2(\phi)$ was shown by Fremberg to be $f_2(\phi) = \int A_\mu^{\text{ret.}}(\mathbf{x})\,\phi(\mathbf{x})\,d^4\mathbf{x}$, where $A_\mu^{\text{ret.}}(\mathbf{x})$ is the retarded L–W potential. However, it is not then possible to determine a unique value of the field intensity on L from this, as has been stated previously.

The finite value for the field intensity on L found by Fremberg is thus seen to arise from a too free use of analytic continuation methods. The value obtained by Fremberg

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would seem to agree with that obtained by arbitrarily cancelling the singularities of the right-hand side of equation (18) by using the four-dimensional distributions $Pf(\delta^{-n})$ instead of δ^{-n}.

It is to be noticed that if direct continuation of $f_\alpha(\mathbf{x})$ is required down to the value $\alpha = 2$, it is necessary that $f_2(\mathbf{x})$ be expressed as a function of $\mathbf{x}$, and not in the form of equation (25), when $\alpha = 2$. This essentially nullifies the use of the continuation method in this case, in favour of the direct approach developed at the beginning of this section.

6. *Energy tensor.* In this section it is shown how it is possible to define an energy-tensor distribution for the charges, from which the equations of motion of the charges may be obtained. This energy tensor will thus remove the ambiguity (observed in §5) associated with the self-field of the point charge.

The energy tensor

$$T_{\mu\rho} = (1/4\pi)\,[F_{\mu\nu}F^{\nu}_{\ \rho} + (\tfrac{1}{4})\,g_{\mu\rho}F_{\alpha\beta}F^{\alpha\beta}], \tag{26}$$

where the $F_{\mu\nu}$ are the L–W fields, is not an integrable function near the world lines. This gives rise to an infinite energy of the system of point charges. These infinities arise from the fact that the $F_{\mu\nu}$ are distributions, so that their products which arise in equation (26) are not necessarily defined. That a product of any two distributions cannot be defined in general has been proved by Schwartz (10).

This difficulty in defining the energy tensor for the field was overcome by Dirac by an intuitive method which, however, did not give unique equations of motion. The continuation method also gets round the divergence difficulty, but it has already been shown that this method is not mathematically powerful enough for the correct treatment of the field quantities. This weakness of the continuation method is even more apparent when the energy tensor $T^{(\alpha)}_{\mu\nu}$ is being considered. For $T^{(2)}_{\mu\nu}$ is a distribution which does not reduce to a function, and so distribution theoretic methods must be used in the treatment of $T_{\mu\nu}$.

There is another important reason for developing the theory in a manner in which the mathematical structure is at all times evident. This reason is that the physical concepts used in a theory to describe the behaviour of a physical system depend on the mathematical language used to formulate that theory. Thus, if any 'abuse of language' is committed, it may allow the introduction of physical concepts which seem of use at their conception, but which may cause difficulties if used further.

This situation evidently holds in quantum field theory, associated mainly with the concept of field energy. These difficulties have been essentially taken over from the 'abuses of language' committed in classical field theories, especially electrodynamics. To prevent these 'abuses of language', it is thus essential to examine to what extent the concept of field energy in classical electrodynamics is limited when this quantity is defined, correctly, as a distribution.

Finally, the continuation method has been shown to be less powerful than the general 'partie finie' method developed by Methe (6).

It will be shown below that from equation (26) a unique energy-tensor distribution $\mathfrak{T}_{\mu\rho}$ may be defined. Thus the objection raised by Schwartz against any general definition of the product of two distributions can be overcome in this special case. This can be effected only by the Pf method, as will be explained below.

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To obtain an energy-tensor distribution $\mathfrak{T}_{\mu\rho}$ which will give unique and finite equations of motion, it is necessary to obtain the distribution given by the stress tensor of equation (26) outside the world lines. This is extended to the world lines by expanding $T_{\mu\rho}$ about the distance from each world line and removing the resulting singularities in a Lorentz-invariant manner by means of the Pf symbol.

Thus the stress tensor $\mathfrak{T}_{\mu\rho} = PfT_{\mu\rho}$ is defined by

$$\mathfrak{T}_{\mu\rho}\phi = \lim_{T\to L}\left[\int_{\text{outside } T} T_{\mu\nu}\phi\, d^4\mathbf{x} + I(T, T_{\mu\nu})\phi\right] \tag{27}$$

for a single charge, where T is a tube surrounding L, $\phi \in \mathscr{D}$, and $T_{\mu\nu}$ is defined by equation (26). $I(T, T_{\mu\nu})$ is a distribution which has support equal to L when T does not reduce to L, but $I\phi$ becomes infinite in such a manner that $\mathfrak{T}_{\mu\nu}$ is well defined as T is reduced to L. Thus $-\lim_{T\to L} I(T, T_{\mu\nu})\,\phi$ is the infinite part of $T_{\mu\nu}\phi$, and it may be defined uniquely by the Pf method of Methe (6), in terms of the invariant distance from L. Thus $\mathfrak{T}_{\mu\rho}$ is defined uniquely as a distribution $\in \mathscr{D}'$. The complete definition of $\mathfrak{T}_{\mu\rho}$ will not be needed. All that is required here is to obtain the equations of motion resulting from equation (27).

In the following only a single charge will be considered. The equations thus obtained can be extended in an evident manner to more than one charge. It will now be shown that the energy tensor of equation (27) satisfies the following conditions:

(i) $\partial\mathfrak{T}_{\mu\rho}/\partial x_\rho$ is a distribution with support equal to the world line L of the charge;

(ii) $$\partial\mathfrak{T}_{\mu\rho}/\partial x_\rho\,\phi = -\int_L F_\mu(s)\,\phi(s)\,ds, \tag{28}$$

where $F_\mu(s) = (\frac{2}{3})\,e^2[\ddot{u}_\mu u_\nu]_-\,u^\nu$. $F_\mu(s)$ is interpreted to be the self-force on the charge. The equation of motion for the charge is then taken to be

$$F_\mu(s) = (\tfrac{2}{3})\,e^2(\ddot{u}_\mu u_\nu - \ddot{u}_\nu u_\mu)\,u^\nu = m\dot{u}_\mu. \tag{29}$$

Equation (28) is to be regarded as the equation of conservation of energy.

To show that assumptions (i) and (ii) above are correct, for the energy tensor $\mathfrak{T}_{\mu\rho}$ defined in equation (27), an integration by parts is performed in equation (28). If the support of ϕ does not cut the world line L, then for T close enough to L, so that the support of ϕ is outside T,

$$\frac{\partial\mathfrak{T}_{\mu\rho}}{\partial x_\rho}\phi = -\int T_{\mu\rho}\frac{\partial\phi}{\partial x_\rho}\,d^4\mathbf{x} = \int\frac{\partial T_{\mu\rho}}{\partial x_\rho}\phi\, d^4\mathbf{x} = 0,$$

since $\partial T_{\mu\rho}/\partial x_\rho = 0$ by construction of equation (26), outside the world line. Hence the support of $\partial\mathfrak{T}_{\mu\rho}/\partial x_\rho$ is the line L. Further, for any $\phi \in \mathscr{D}$,

$$\begin{aligned}\frac{\partial\mathfrak{T}_{\mu\rho}}{\partial x_\rho}\phi &= -\lim_{T\to L}\left[\int_{\text{outside } T} T_{\mu\nu}\frac{\partial\phi}{\partial x_\nu}\,d^4\mathbf{x} + I(T, T_{\mu\nu})\frac{\partial\phi}{\partial x_\nu}\right]\\ &= -\lim_{T\to L}\left[-\int_{\text{outside } T}\frac{\partial T_{\mu\nu}}{\partial x_\nu}\phi\, d^4\mathbf{x} + \int_S T_{\mu\nu}\phi\, d\sigma^\nu + I\frac{\partial\phi}{\partial x_\nu}\right]\\ &= -\lim_{T\to L}\left[\int_S T_{\mu\nu}\phi\, d\sigma^\nu + I\frac{\partial\phi}{\partial x_\nu}\right],\end{aligned} \tag{30}$$

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where S is the surface of tube T, $d\sigma$ is the element of area drawn inwards. Taking T of radius ϵ (Dirac (2), p. 167), we have for small ϵ,

$$\lim_{T\to L}\int_S T_{\mu\nu}\phi\, d\sigma^\nu = \lim_{\epsilon\to 0}\int_L (-\tfrac{1}{2}e^2\epsilon^{-1}\dot{u}_\mu + F_\mu(s))\,\phi(s)\,ds. \tag{31}$$

Thus it is required that

$$I(T, T_{\mu\nu})\frac{\partial\phi}{\partial x_\nu} = \frac{1}{\epsilon}\int_L \tfrac{1}{2}e^2\dot{u}_\mu\phi(s)\,ds. \tag{32}$$

$I(T, T_{\mu\nu})$ will satisfy equation (32). For since $\mathfrak{T}_{\mu\rho}$ is defined as a distribution $\epsilon\mathscr{D}'$, $(\partial\mathfrak{T}_{\mu\rho}/\partial x_\rho)\,\phi$ exists for any $\phi\epsilon\mathscr{D}$. Thus a contradiction will arise unless equation (32) is satisfied. Also by the covariant Pf definition, $I(T, T_{\mu\nu})\,(\partial\phi/\partial x_\nu)$ can only depend on linear combinations of $\epsilon^\lambda(\log\epsilon)^\mu\,(\lambda\leqslant 0, \mu\geqslant 0)$. These functions of ϵ form a linearly independent set. Thus the 'partie finie' of $\partial\mathfrak{T}_{\mu\rho}/\partial x_\rho$ is defined uniquely by equation (33) below.

Hence

$$\frac{\partial\mathfrak{T}_{\mu\rho}}{\partial x_\rho}\phi = -\int_L F_\mu(s)\,\phi(s)\,ds, \tag{33}$$

which is equation (28). The extension of the proof to the case of more than one particle is evident, and will not be given here. The equations of motion (29) have thus been derived in a unique manner from the energy tensor defined uniquely by equation (27). Thus the most natural assumption which will give a unique self-force on the point charge is that equation (28) is the equation of conservation of energy and momentum for the system. This self-force agrees with that obtained by Dirac (2) and Fremberg (4).

7. *Discussion.* From §6 a unique energy tensor for the field can be defined so that unique equations of motion may be determined from it. This energy tensor is a distribution which *cannot* be reduced to a function of space-time everywhere, and so the concept of localization of field energy cannot be considered as completely correct. In the covariant approach used so far, the field and energy-tensor distributions have been regarded as having supports in four-dimensional space-time. The field energy-momentum vector $P_\mu(\sigma)$ residing in a given space-like surface σ is usually defined by

$$P_\mu(\sigma) = \int_\sigma \mathfrak{T}_{\mu\nu}d\sigma^\nu. \tag{34}$$

In equation (34), $\mathfrak{T}_{\mu\nu}$ is to be regarded as a function of its space-time argument $\mathbf{x}$. However, the correct definition of the stress tensor is the distribution of equation (27). Thus $P_\mu(\sigma)$ will, in general, not be a functional of σ, but will be a distribution on s, where $\mathbf{z}(s)$ is the point in which σ cuts the world line L. There occurs in $P_\mu(\sigma)$, for example, a term containing $Pf\int_\sigma \delta^{-\frac{5}{2}}u_\mu u_\rho d\sigma^\rho$. Complete compensation of the singularity at $\delta = 0$ can take place only if σ is orthogonal to L at s, so that $P_\mu(\sigma)$ is then a functional of σ. In the general case, this condition will not hold, and only values of the field energy-momentum vector for finite regions of space-time will have any physical meaning. In other words, it is necessary to take an overall space-time approach in a general covariant theory. That this approach is necessary in quantum electrodynamics has been noted by Valatin (11).

Considerations similar to the above arise when the field intensity is considered. This intensity is not determined uniquely on any set of measure zero, so that only averages

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of the field intensity over space-time regions of non-zero measure have any unique physical meaning. For example, in the case of field measurement by the introduction of a charged test body into the field (when the test body is not assumed to disturb the field) the test body must have finite extension in space and time in order that no ambiguity arises in the resultant measurement. This feature is characteristic of any field theory in which the field solutions must be considered as distributions. It appears, for example, in the results of the analysis of field measurements in quantum electrodynamics, as given by Bohr and Rosenfeld (1).

If matter is considered as consisting of point charges only, then unique interactions between the charges, and unique self-actions, can only be obtained by the introduction of an energy-tensor distribution for the system, and the assumption that the principle of conservation of total energy and momentum is satisfied.

The solutions of the equations of motion (29) or their generalization to a system of point charges have been studied by Dirac (2) and Feynman and Wheeler (3), and will not be discussed here.

It is evident that the preceding work can easily be extended to a system of point charges interacting through a classical meson field. This extension of the preceding formalism is achieved by the use of the elementary solution $(\nabla - \lambda\delta)^{*(-1)}$ as defined by Schwartz ((9), II, p. 35).

Another extension of the preceding theory may be obtained by considering the structure of the sources as arbitrary, i.e. point charges or continuously distributed charges may be considered. For example, if a non-local classical field theory is required to have a correct point-particle theory as its limiting case, then it would appear necessary to consider this non-local theory from a distribution-theoretic viewpoint. For the limiting process requires that the form function tends to a δ-function, and this limit-function can only be considered as a distribution.

Apart from obtaining the equations of motion for the charges in agreement with Dirac, this paper shows that the product of distributions which enters into the energy tensor for the classical field can be defined in a rigorous way.

It is of the greatest importance to see if a definition of the energy-tensor distribution can be extended to quantum electrodynamics, and to determine the relation of this approach to the renormalization method. The extension of the above formalism to quantum electrodynamics is not immediately evident, since the fundamental equations for the field operators are coupled non-linear differential equations which have no immediate rigorous definition in terms of distributions. Further, as mentioned in the introduction, a rigorous canonical formalism for the above theory is not immediately evident.

Whilst a rigorous distribution-theoretic approach can be used in the case of free fields, the definition of the interaction between different fields cannot as yet be rigorously given. It is to be hoped that this rigorous definition will be given in the future, so that the renormalization method of quantum field theory will be put on a sound mathematical basis.

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APPENDIX

Since the work of Schwartz is not readily available, it has been thought advisable to include here a description of those parts of distribution theory which are essential to the understanding of the paper.

Definition of distributions. Distributions are defined by means of a function space $\mathscr{D}$. $\mathscr{D}$ is the set of all functions $\phi(\mathbf{x})$ which are indefinitely differentiable and which vanish outside a compact set. The domain of ϕ is here taken to be a general n-dimensional real space, so $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$, where the x_i are all real.

A distribution T is defined to be a continuous linear functional on $\mathscr{D}$. Thus to each $\phi \in \mathscr{D}$, there is associated a complex number $T(\phi)$, satisfying

(i)
$$T(\phi_1 + \phi_2) = T(\phi_1) + T(\phi_2), \tag{A 1}$$
$$T(k\phi) = kT(\phi), \tag{A 2}$$

for any $\phi_1, \phi_2 \in \mathscr{D}$, and any complex number k.

(ii) If $\phi_i(\mathbf{x})\,(i = 1, 2, \ldots)$ is a sequence of functions which all vanish outside a fixed compact K, and converge uniformly to zero, together with all their derivatives, in K, then
$$\lim_{i \to \infty} T(\phi_i) = 0. \tag{A 3}$$

The support of a function $\phi(\mathbf{x})$ is the smallest closed set outside of which it is zero. A distribution T is zero in an open set Ω if $T(\phi)$ is zero whenever the support of $\phi(\in \mathscr{D})$ is contained in Ω. The support of T is the complement of the union of the open sets in which T is zero.

Any function $f(\mathbf{x})$, summable on all compact sets, defines a distribution f by the equation
$$f(\phi) = \int f(\mathbf{x})\, \phi(\mathbf{x})\, d\mathbf{x} \tag{A 4}$$

for any $\phi \in \mathscr{D}$. The support of the distribution f is thus the support of the function $f(\mathbf{x})$. The δ-distribution $\delta_{(0)}$, with support equal to the origin, is defined by
$$\delta_{(0)}(\phi) = \phi(0) \tag{A 5}$$

for any $\phi \in \mathscr{D}$. It is evident that $\delta_{(0)}$ cannot be related to any function by means of an equation of the form (A 4) above.

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It is usual to write $T(\phi)$ in the forms

$$T(\phi) = T\phi = \int T_x \phi(\mathbf{x})\, d\mathbf{x}. \tag{A 6}$$

The set of all distributions is a linear space $\mathscr{D}'$, which is the dual space of $\mathscr{D}$.

Product of distributions. Given any two distributions S, T, it is not possible to define a distribution which reduces to $f(\mathbf{x})\, g(\mathbf{x})$ when S, T reduce to functions $f(\mathbf{x})$, $g(\mathbf{x})$ summable on all compacts. For example, when $n = 1$ if $f(x) = g(x) = x^{-\frac{1}{2}}$, then $f(x)\, g(x) = x^{-1}$, which is not summable near the origin. Thus the product of two distributions cannot be defined in general.

It is, however, possible to define a product when one of S, T is an indefinitely differentiable function $\alpha(\mathbf{x})$. For this case (αT) is defined by

$$(\alpha T)\, \phi = \int T_x \alpha(\mathbf{x})\, \phi(\mathbf{x})\, d\mathbf{x} = T\alpha\phi. \tag{A 7}$$

It has been shown by Schwartz that it is impossible to define associative multiplication, derivation and a δ-element in the same vector space. (For details see (**10**).)

Differentiation of distributions. The partial distribution derivative $\partial T/\partial x_k$ of the distribution $T \in \mathscr{D}'$ is defined as an element of $\mathscr{D}'$ by the equation

$$\frac{\partial T}{\partial x_k}\phi = -T\frac{\partial \phi}{\partial x_k}. \tag{A 8}$$

When T is a function $f(\mathbf{x})$ such that the function derivative $\partial f/\partial x_k$ is summable on all compact sets, then the distribution derivative of f is equal to $\partial f/\partial x_k$ (as seen by integrating the right-hand side of equation (A 8), with $T = f$, by parts). Thus any $T \in \mathscr{D}'$ is indefinitely differentiable. For example, $\partial\delta/\partial x_k$ is defined by

$$\frac{\partial \delta}{\partial x_k}\phi = -\frac{\partial \phi(0)}{\partial x_k}.$$

Convolution products. The convolution product $(S * T)$ of two distributions S, T is defined as an element of $\mathscr{D}'$ as follows:

$$(S * T)\, \phi = \int S_x T_y \phi(\mathbf{x} + \mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \tag{A 9}$$

(for any $\phi \in \mathscr{D}$). The right-hand side of equation (A 9) exists only under suitable conditions on S, T, in particular if one of S, T has compact support.

When S, T reduce to functions f, g, then the convolution product defined by equation (A 9) reduces to $\int f(\mathbf{x} - \mathbf{t})\, g(\mathbf{t})\, d\mathbf{t}$, provided this exists.

A convolution product $(T_1 * T_2 * \ldots * T_N)$ of N distributions $T_1, T_2, \ldots, T_N$ can be defined by a generalization of equation (A 9), provided all the distributions except one have compact supports. This convolution product is then associative and commutative.

In particular $$(\delta_{(0)} * T)\, \phi = \int T_x [\delta_{(0)y} \phi(\mathbf{x} + \mathbf{y})]\, d\mathbf{x}\, d\mathbf{y} = \int T_x \phi(\mathbf{x})\, d\mathbf{x}$$

shows that $$\delta_{(0)} * T = T. \tag{A 10}$$

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In a similar manner it may be shown that

$$\frac{\partial \delta_{(0)}}{\partial x_k} * T = \frac{\partial T}{\partial x_k}. \tag{A 11}$$

Similarly if $$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \ldots - \frac{\partial^2}{\partial x_{n-1}^2},$$

then $$\Box \delta_{(0)} * T = \Box T. \tag{A 12}$$

It is possible to extend the definition of $(S * T)$ to the case when neither S nor T have compact supports. This extension is necessary to define the convolution product $(Z_2 * j_\mu)$ of equation (10), since neither Z_2 nor j_μ have compact supports.

One method of extension was given by Schwartz, by means of the distribution space $\mathscr{D}'_{+\Gamma}$. This is the set of distributions with support contained in a linear translation of the cone Γ (Γ is the set of points $x_0 \geqslant 0$, $x_0^2 - x_1^2 - \ldots - x_{n-1}^2 \geqslant 0$). Then the convolution product of distributions $T_1, T_2, \ldots, T_N$, all $T_i \in \mathscr{D}'_{+\Gamma}$ except at most one, may be defined as an element of $\mathscr{D}'_{+\Gamma}$, and is then associative and commutative. This is true provided that the distribution which does not belong to $\mathscr{D}'_{+\Gamma}$ has support which cuts $K - \Gamma$ (for any compact K) in a compact. By $K - \Gamma$ is meant the set of points $\mathbf{z} = \mathbf{x} - \mathbf{y}$ for $\mathbf{x} \in K$, $\mathbf{y} \in \Gamma$.

A proof of the above statement is given in (9), II, p. 28. However, this does not enable $Z_2 * j_\mu$ to be defined. For while $Z_2 \in \mathscr{D}'_{+\Gamma}$, it is not true that the support of j_μ cuts $K - \Gamma$ in a compact (for any compact K). It will now be shown that $Z_2 * j_\mu$ does exist as an element of $\mathscr{D}'$, provided that the world lines have time-like asymptotes in the infinite past. For simplicity only one particle will be considered.

Since Z_2 and L (the world line of the particle) are defined in an invariant manner, any special Lorentz frame of reference may be chosen in which to define the convolution product. If the convolution product may be defined in one frame of reference, it will be defined in any other frame. The Lorentz frame in which the particle is at rest as $x_0 \to -\infty$, will be chosen. In the product $(j_\mu)_x (Z_2)_y \phi(\mathbf{x} + \mathbf{y})$, the values of $\mathbf{x}$, $\mathbf{y}$ must satisfy $\mathbf{x} \in L$, $\mathbf{y} \in D_+^0$, $\mathbf{x} + \mathbf{y} \in K$. Since $\phi \in \mathscr{D}$, then K is compact. The difficulties in the definition of the convolution product arise from contributions from large values of $\mathbf{x}$ and $\mathbf{y}$. Since $y_0 \geqslant 0$, then large positive values of x_0 cannot contribute. Suppose x_0 can be large and negative. For large negative values of x_0, the values of x_1, x_2, x_3 become constant, and so the values of (y_1, y_2, y_3) become bounded. Since $y \in D_+^0$, this means that the values of y_0 become bounded, and so, since $x_0 + y_0 \in K$, that x_0 is bounded. This gives a contradiction, so that x_0 cannot be very large and negative, and must be bounded. Hence the convolution product exists as an element of $\mathscr{D}'$.

In the case when L has a time-like asymptote as $x_0 \to -\infty$, $\mathbf{x}$ and $\mathbf{y}$ may take arbitrarily large values, so that the convolution product does not exist. No conditions on L, for $x_0 \to +\infty$, are needed to define $(Z_2 * j_\mu)$, so the 'runaway' solutions of the equations of motion cannot be excluded by mathematical arguments. However, if the advanced potential associated with $(\check{Z}_2 * j_\mu)$ is used (as, for example, by Dirac) then the 'runaway' solutions are excluded by the conditions on the world lines of the particles.

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Pseudo-functions and 'partie finie'. Take $n = 1$, and let $g(x)$ be a function summable in the interval $(a+\epsilon, b)$ for any $\epsilon > 0$. Suppose that $g(x)$ can be represented in the form $g(x) = \sum_\nu \frac{A_\nu}{(x-a)^{\lambda_\nu}} + h(x)$, where the λ_ν are complex numbers, none of which is an integer, and $h(x)$ is a function summable in (a, b). Define

$$\int_{a+\epsilon}^{b} g(x)\,dx = I(\epsilon) + F(\epsilon), \tag{A 13}$$

$$I(\epsilon) = \sum_\nu \frac{A_\nu}{\lambda_\nu - 1}\left(\frac{1}{\epsilon}\right)^{\lambda_\nu - 1}, \tag{A 14}$$

$$\lim_{\epsilon\to 0} F(\epsilon) = F = -\sum_\nu \frac{A_\nu}{\lambda_\nu - 1}\left(\frac{1}{b-a}\right)^{\lambda_\nu - 1} + \int_a^b h(x)\,dx. \tag{A 15}$$

Then F is the 'partie finie' of Hadamard ((5), p. 184) associated with the integral $\int_a^b g(x)\,dx$. This is written $F = Pf\int_a^b g(x)\,dx$. If $\phi \in \mathscr{D}$, then $Pf\int_a^b g(x)\,\phi(x)\,dx$ can be defined uniquely, and is a linear continuous form in ϕ. Thus while $g(x)$ may not be summable in the interval (a, b), and thus may not define a distribution by means of (A 4), it does define a distribution (Pfg) by the equation

$$(Pfg)\,\phi = Pf\int_a^b (gx)\,\phi(x)\,dx. \tag{A 16}$$

This distribution (Pfg) is called a 'pseudo-function' by Schwartz ((9), I, pp. 38 et seq.). Thus the term 'partie finie' refers to the finite part of an integral, while the term 'pseudo-function' refers to a distribution defined by means of the 'partie finie'.

(Pfg) can be generalized to the case when some of the λ_ν are integers. In particular, the pseudo-function associated with the function $(x^m)_+ = \begin{cases} x^m & (x > 0) \\ 0 & (x < 0) \end{cases}$ (where m is any complex number) is defined by

$$Pf(x^m)_+\,\phi = \lim_{\epsilon\to 0}\left[\int_\epsilon^\infty x^m\phi(x)\,dx + \phi(0)\frac{\epsilon^{m+1}}{m+1} + \phi'(0)\frac{\epsilon^{m+2}}{m+2} + \ldots + \frac{\phi^{(k)}(0)}{k!}\frac{\epsilon^{m+k+1}}{m+k+1}\right]. \tag{A 17}$$

The number of terms in the bracket depends on m, and if m is a negative integer, the term $\epsilon^0/0$ must be replaced by $\log\epsilon$. Take $n = 3$, and define $s = (x_0^2 - x_1^2 - x_2^2 - x_3^2)^{\frac{1}{2}}$ when $x_0 \geqslant 0$ and s is real, and $s = 0$ otherwise. The distribution (Pfs^m) is defined as follows:

$$(Pfs^m)\,\phi = Pf\int s^m(\mathbf{x})\,\phi(\mathbf{x})\,d\mathbf{x}. \tag{A 18}$$

$(Pfs^m)\,\phi$ is an analytic function of the complex variable m, except for single poles at $m = -2$, and double poles at $m = -4, -6, \ldots$. The precise definition of the 'partie finie' in equation (A 18) was originally determined by Hadamard ((5), pp. 220–30), for the cases that $m = n-2$, and n is odd. This has been extended to general values of m and n by Methe.

Elementary solutions of the hyperbolic equation. The general distribution solution of the wave equation which is invariant under the proper Lorentz group has been given by Methe.

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A distribution T is said to be invariant under the proper Lorentz group if under the Lorentz transformation $\mathbf{x}' = \lambda\mathbf{x}$ or $x'_\mu = a^\nu_\mu x_\nu$

$$T_x\phi(\lambda\mathbf{x}) = T_x\phi(\mathbf{x}). \tag{A 19}$$

The invariant elementary solutions $Z_2, \check{Z}_2$, used in § 3, have been given by Riesz (8) and Schwartz (9), using the method of analytic continuation on m. Methe has also determined the elementary solutions directly by means of the 'partie finie' method. However, Methe showed that while the continuation method and the direct 'partie finie' method give the same elementary solutions, not all invariant distributions, which have support equal to the surface D^0_+ of the future light cone (and similarly the past light cone D^0_-), can be specified by the analytic continuation method. Thus the use of the direct 'partie finie' definition of distributions associated with functions which are non-summable in the neighbourhood of the surface of the light cone, as developed by Methe, is more powerful than the analytic continuation method developed by Riesz.

The particular invariant elementary solution Z_2 is defined directly as follows:

$$Z_2\phi = \frac{1}{4\pi}\int_{D^0_+} dS\,\phi(\mathbf{x}) = \frac{1}{4\pi}\int d^3x\,|\,x\,|^{-1}\,\phi(x_i, |\,x\,|), \tag{A 20}$$

where D^0_+ is the surface of the future light cone Γ, with surface area

$$dS, \quad x = (x_1, x_2, x_3) \quad \text{and} \quad |\,x\,| = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}.$$

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