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Connected Matroids with Symmetric Tutte Polynomials

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We show that connected matroids with symmetric Tutte polynomials are not necessarily self-dual; in fact, we construct matroids that are not self-dual, have arbitrarily high connectivity, and yet have symmetric Tutte polynomials.

1. Introduction

For a matroid M with ground set S, the Tutte polynomial of M is defined as

$$t(M;x,y)=\sum_{A\subseteq S}(x-1)^{r(S)-r(A)}(y-1)^{|A|-r(A)}.$$

It is well known, and easy to show, that for the dual M^* of any matroid M we have $t(M;x,y)=t(M^*;y,x)$. Hence the Tutte polynomial of a self-dual matroid is a symmetric function of x and y. Welsh [7] raised the following natural question: *Are there connected matroids that are not self-dual and yet have symmetric Tutte polynomials?*

To see why this question assumes that M is connected, consider two connected matroids M_1 and M_2 that are not isomorphic, not self-dual, and that satisfy $t(M_1;x,y)=t(M_2;x,y)$; the direct sum $M=M_1\oplus M_2^*$ is not self-dual, yet $t(M;x,y)=t(M;y,x)$.

In this paper we answer Welsh's question by developing a general technique for constructing connected matroids that are not self-dual and yet have symmetric Tutte polynomials; we illustrate this by producing four infinite classes of such matroids. Furthermore, we show that by iterating our general technique we obtain such matroids with arbitrarily high connectivity.

We assume familiarity with basic matroid theory. Our notation and terminology follow [6]. The rank function of a matroid M will be denoted r_M.

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2. Double extensions

In this section, the construction of double extension is introduced and used to produce connected matroids that are not self-dual and yet have symmetric Tutte polynomials. Double extensions are formed from free extensions and free coextensions. See Sections 7.3 and 7.4 in [4] or Section 7.2 in [6] for detailed accounts of free extensions and free coextensions.

Let M be a matroid on S. The *free extension* of M by p, denoted $M+p$, is the matroid on $S \cup p$ whose rank function r_{M+p} is given as follows. For all $X \subseteq S$,

$$r_{M+p}(X) = r_M(X), \tag{2.1}$$

$$r_{M+p}(X \cup p) = \begin{cases} r_M(X)+1, & \text{if } X \text{ is not spanning in } M, \\ r_M(X), & \text{otherwise.} \end{cases} \tag{2.2}$$

The *free coextension* of M by q, denoted $M \times q$, is defined to be $(M^* + q)^*$. Since free coextension is the dual operation of free extension, standard arguments about the rank of the dual give the following expression for $r_{M\times q}$. For any $X \subseteq S$,

$$r_{M\times q}(X) = \begin{cases} r_M(X), & \text{if } X \text{ is independent in } M, \\ r_M(X)+1, & \text{otherwise,} \end{cases} \tag{2.3}$$

$$r_{M\times q}(X \cup q) = r_M(X)+1. \tag{2.4}$$

It follows easily from equations (2.1) and (2.2) that for any matroid M, the map that fixes all elements of the ground set of M and switches p and q is an automorphism of $(M+p)+q$. This observation is used in the proof of the following elementary lemma, which plays a key role in the main results of this section.

Lemma 2.1. *Two matroids are isomorphic if and only if their free extensions are isomorphic.*

Proof. It follows easily from the definition that free extensions of isomorphic matroids are isomorphic. For the converse, assume that M and N have ground sets S and T respectively and that $\phi : S \cup p \to T \cup q$ is an isomorphism of $M+p$ onto $N+q$. If $\phi(p) = q$, then the restriction of ϕ to S gives an isomorphism of M onto N. Thus, assume $\phi(p) = q' \neq q$. Since $M+p$ is the free extension of M by p, it follows that $N+q$ is the free extension of $(N+q)\backslash q'$ by q'. Thus $N+q = ((N\backslash q')+q)+q'$, so the map ψ that fixes all elements of $T-q'$ and switches q and q' is an automorphism of $N+q$. Thus $\psi \circ \phi$ is an isomorphism of $M+p$ onto $N+q$ with $\psi \circ \phi(p) = q$, thereby reducing this case to the case already addressed. □

The key construction of this paper, the double extension, is formed by taking the free extension followed by the free coextension.

Definition. The *double extension* of a matroid M by p, q is $(M+p)\times q$.

The following proposition gives the rank function of the double extension.

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Proposition 2.2. *Suppose that M is a matroid on S and that $N = (M + p) \times q$ is the double extension of M. Then, for any $X \subseteq S$,*

$$r_N(X) = \begin{cases} r_M(X), & \text{if } X \text{ is independent in } M, \\ r_M(X) + 1, & \text{otherwise,} \end{cases}$$

$$r_N(X \cup p) = \begin{cases} r_M(X) + 2, & \text{if } X \text{ is neither independent nor spanning in } M, \\ r_M(X) + 1, & \text{otherwise,} \end{cases}$$

$$r_N(X \cup q) = r_M(X) + 1,$$

$$r_N(X \cup p \cup q) = \begin{cases} r_M(X) + 1, & \text{if } X \text{ is spanning in } M, \\ r_M(X) + 2, & \text{otherwise.} \end{cases}$$

Proof. We treat only the case $X \cup p$ since the arguments in the other cases are similar. If X spans M, then $r_{M+p}(X \cup p) = r_M(X)$; therefore $X \cup p$ is dependent in $M + p$, so

$$r_N(X \cup p) = r_{M+p}(X \cup p) + 1 = r_M(X) + 1$$

by equation (2.3). If X is independent but not spanning in M, then $X \cup p$ is independent in $M + p$, so

$$r_N(X \cup p) = r_{M+p}(X \cup p) = r_M(X) + 1$$

by equation (2.3) again. Finally if X is neither independent nor spanning in M, then $X \cup p$ is not independent in $M + p$, so

$$r_N(X \cup p) = r_{M+p}(X \cup p) + 1 = (r_M(X) + 1) + 1 = r_M(X) + 2.$$ □

Similar arguments show that $(M + p) \times q$ and $(M \times q) + p$ have the same rank function, that is, the operations of free extension and free coextension commute.

Proposition 2.3. *For any matroid M, we have $(M \times q) + p = (M + p) \times q$.* □

Not surprisingly, double extensions are always connected.

Proposition 2.4. *For any matroid M,*

(1) *if M has no loops, then $M + p$ is connected,*
(2) *if M has no isthmuses, then $M \times q$ is connected, and*
(3) *$(M + p) \times q$ is connected.*

Proof. For the first claim, it suffices to show that, for each element x of M, there is a circuit of $M + p$ containing both x and p. Since M has no loops, there is a basis B of M with $x \in B$. Therefore $B \cup p$ is a circuit of $M + p$ containing both x and p. The second statement is the dual of the first. The last statement is true since $M + p$ has no isthmuses. □

The next proposition relates double extensions to the property of being self-dual.

Proposition 2.5. *If $N = (M + p) \times q$, then N is self-dual if and only if M is self-dual.*

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Proof. By Proposition 2.3, $N = (M \times q) + p$. Also, since $N = ((M + p)^* + q)^*$ by the definition of coextensions, we have $N^* = (M + p)^* + q$. Therefore, by Lemma 2.1, N is self-dual if and only if $(M + p)^*$ is isomorphic to $M \times q$, that is, to $(M^* + q)^*$. This last condition is equivalent to the duals, $M + p$ and $M^* + q$, being isomorphic. By Lemma 2.1, this holds if and only if M is isomorphic to M^*, that is, M is self-dual. □

The Tutte polynomials of the free extension and the free coextension of M are determined by $t(M; x, y)$. The next lemma is part of Proposition 4.2 in [2].

Lemma 2.6. *If* $t(M; x, y) = \sum b_{ij} x^i y^j$, *then*

$$t(M + p; x, y) = \sum_j \left(b_{0j} y^{j+1} + \sum_{i>0} b_{ij}(x^i + x^{i-1} + \cdots + x + y) y^j \right)$$

and

$$t(M \times q; x, y) = \sum_i \left(b_{i0} x^{i+1} + \sum_{j>0} b_{ij} x^i (y^j + y^{j-1} + \cdots + y + x) \right).$$

Combining these two equations, one can write the Tutte polynomial of the double extension of M in terms of $t(M; x, y)$.

Corollary 2.7. *If* $t(M; x, y) = \sum_{i,j} b_{ij} x^i y^j$, *then*

$$t((M + p) \times q; x, y) = \sum_{i,j} b_{ij} \left(\frac{x(x^{i+1} - 1)}{x - 1} + \frac{xy(x^i - 1)(y^j - 1)}{(x - 1)(y - 1)} + \frac{y(y^{j+1} - 1)}{y - 1} \right).$$

The next theorem, which summarizes the results above, shows that the double extension is a general tool for producing connected matroids that are not self-dual and yet have symmetric Tutte polynomials.

Theorem 2.8. *Let N be the double extension of M. Then N is connected. If $t(M; x, y) = t(M; y, x)$, then $t(N; x, y) = t(N; y, x)$. Furthermore, N is self-dual if and only if M is self-dual.* □

As discussed in Section 1, if two connected, non-isomorphic matroids M_1 and M_2 are not self-dual and have the same Tutte polynomial, then the direct sum $M = M_1 \oplus M_2^*$ is not self-dual and has a symmetric Tutte polynomial. Therefore, the double extension of M is connected, not self-dual, and has a symmetric Tutte polynomial. Thus, to answer the question of Welsh posed in the introduction, we only need to exhibit connected non-isomorphic matroids that are not self-dual and have the same Tutte polynomials. We mention four classes of examples of connected non-isomorphic matroids that are not self-dual and have the same Tutte polynomials. Each of these classes includes infinitely many matroids. By Theorem 2.8, these classes give rise to four distinct infinite families of matroids, with each of these matroids being connected, not self-dual, and yet having a symmetric Tutte polynomial.

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(1) It is well known that all projective planes of order q have the same Tutte polynomial. Also, for every prime power $q = p^n$ with $n > 1$, other than 4 and 8, there are non-isomorphic projective planes of order q.
(2) Likewise, all affine planes of order q have the same Tutte polynomial. Also, for every prime power $q = p^n$ with $n > 1$, other than 4 and 8, there are non-isomorphic affine planes of order q.
(3) In [5], Dowling showed that for $n \geqslant 3$, rank-n Dowling lattices based on non-isomorphic groups of the same finite order are not isomorphic. Bonin and Miller [1, Proposition 4.1] showed that rank-n Dowling lattices based on groups of the same finite order have the same Tutte polynomial. Analogous results hold for jointless Dowling lattices. For rank 3, these results apply equally well to quasigroups.
(4) The results cited in item (3) can be generalized in the following way [8]. Assume that G_1 and G_2 are non-isomorphic groups of the same finite order and let F_1 and F_2 be modular flats of rank-k in the rank-n Dowling lattices $Q_n(G_1)$ and $Q_n(G_2)$, respectively. If $n-k+1 \geqslant 3$, then the complete principal truncations $T_{F_1}Q_n(G_1)$ and $T_{F_2}Q_n(G_2)$ are not isomorphic and yet have the same Tutte polynomial.

Thus, we have provided infinitely many examples proving the following theorem.

Theorem 2.9. *There are connected matroids N that are not self-dual and for which $t(N; x, y)$ is symmetric.* □

We make two observations about Theorem 2.9. First, the smallest such matroid using the double extension construction has 16 elements and rank 8 (see example 6.2.18 in [3]). Other small examples can be based on graphs (see, *e.g.*, exercise 6.30 in [3]) although the double extension of graphic matroids is not generally graphic. Second, although the matroids N are not self-dual, they are highly symmetric. All invariants that depend only on the Tutte polynomial are shared by N and N^*; these include the numbers of bases, independent sets, and spanning sets, as well as the beta invariant and the characteristic polynomial.

3. Higher connectivity

Propositions 3.1 and 3.4 show that the operation of double extension increases the vertical connectivity and the Tutte connectivity. This allows us to strengthen Theorem 2.9 by replacing the conclusion 'connected' with 'vertical connectivity at least k' and 'Tutte connectivity at least k' for any fixed positive integer k (Theorem 3.5). We start by recalling the notions of vertical connectivity and Tutte connectivity (see [6]).

For a positive integer k, a matroid M on S is *vertically k-separated* if there is a partition (X, Y) of S such that

$$\min\{r_M(X), r_M(Y)\} \geqslant k \tag{3.1}$$

and

$$r_M(X) + r_M(Y) - r(M) \leqslant k - 1. \tag{3.2}$$

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The partition (X, Y) is a *vertical k-separation*. The *vertical connectivity* (or Whitney connectivity) $\kappa(M)$ of M is the least positive integer j such that M is vertically j-separated; if there is no such j, then $\kappa(M)$ is $r(M)$. Note that $\kappa(M) \leqslant r(M)$. Also, if (X, Y) is a vertical k-separation of a matroid M for which $\kappa(M)$ is k, then equality holds in inequality (3.2). In general, a vertical k-separation for which equality holds in inequality (3.2) is an *exact vertical k-separation*.

If in place of inequality (3.1) we require $\min\{|X|, |Y|\} \geqslant k$, then the partition (X, Y) is a *k-separation* of M; if, in addition, equality holds in inequality (3.2), then (X, Y) is an *exact k-separation*. We say M is *k-separated* if it has a k-separation. The *connectivity* (or Tutte connectivity) $\lambda(M)$ of M is the least positive integer j such that M is j-separated; if there is no such j, then $\lambda(M)$ is ∞. Note that any k-separation of a matroid M for which $\lambda(M)$ is k is an exact k-separation.

We first consider the vertical connectivity of the double extension.

Proposition 3.1. *If $N = (M + p) \times q$, then $\kappa(N)$ is $\kappa(M) + 1$ or $\kappa(M) + 2$.*

Proof. First we show $\kappa(N) \leqslant \kappa(M) + 2$. This is trivial if $\kappa(M) \geqslant r(M) - 1$, so assume $\kappa(M) = k < r(M) - 1$. Suppose that (X, Y) is an exact vertical k-separation of M. We may assume $r_M(X) \leqslant r_M(Y)$. By Proposition 2.2, $r_N(X \cup p)$ is either $r_M(X) + 1$ or $r_M(X) + 2$ and $r_N(Y \cup q) = r_M(Y) + 1$. If $r_N(X \cup p) = r_M(X) + 1$, it follows that $(X \cup p, Y \cup q)$ is a vertical $(k + 1)$-separation. Thus assume $r_N(X \cup p) = r_M(X) + 2$. If $r_M(Y) = k$, then, since $k \leqslant r_M(X) \leqslant r_M(Y)$, we get $r_M(X) = k$; together with $r_M(X) + r_M(Y) - r(M) = k - 1$, this gives $k = r(M) - 1$, contrary to the initial assumption. Therefore $r_M(Y) > k$. From this, it follows that $(X \cup p, Y \cup q)$ is a vertical $(k + 2)$-separation of N.

For the other direction, we need only show $\kappa(M) \leqslant \kappa(N) - 1$. This is immediate if $\kappa(N) = r(N)$, so assume $\kappa(N) = k < r(N)$. By Proposition 2.4, we have $k \geqslant 2$.

Consider an exact vertical k-separation of N. First assume that this separation of N is $(X \cup \{p, q\}, Y)$ where $X \subseteq S$ and $Y \subseteq S$. If $X = \emptyset$, it follows that $k = 2$ and $Y = S$; from $r_N(\{p, q\}) + r_N(S) - r(N) = 1$, we get $r_N(S) = r(N) - 1$, so S is independent in M and $\kappa(M) = 1 = \kappa(N) - 1$. Thus, assume $X \neq \emptyset$. It follows from the definition of a vertical separation that the set $X \cup \{p, q\}$ does not span N. Therefore X does not span M, so

$$r_M(X) = r_N(X \cup \{p, q\}) - 2 \leqslant r(N) - 3 = r(M) - 2$$

by Proposition 2.2. Also by Proposition 2.2, $r_M(Y)$ is either $r_N(Y)$ or $r_N(Y) - 1$. If $r_M(Y) = r_N(Y) - 1$, it follows that the partition (X, Y) is a vertical $(k-2)$-separation of M. If $r_M(Y) = r_N(Y)$, then Y is independent in M. Since $r_M(X) \leqslant r(M) - 2$, there is an element $y \in Y$ with $y \notin \mathrm{cl}_M(X)$. Therefore $r_M(X \cup y) = r_M(X) + 1$. Also, $r_M(Y - y) = r_M(Y) - 1$. It follows that the partition $(X \cup y, Y - y)$ is a vertical $(k - 1)$-separation of M.

Lastly, assume that the vertical k-separation of N is $(X \cup p, Y \cup q)$ for some subsets X and Y of S. Note that $r_M(Y) = r_N(Y \cup q) - 1$ and $r_M(X)$ is either $r_N(X \cup p) - 1$ or $r_N(X \cup p) - 2$. If $r_M(X) = r_N(X \cup p) - 1$, it follows that the partition (X, Y) of S is a

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vertical $(k-1)$-separation of M; if $r_M(X) = r_N(X \cup p) - 2$, it follows that (X, Y) is a vertical $(k-2)$-separation of M. □

To address the corresponding result for connectivity, we will use the following two basic results [6, Corollary 8.1.8 and Proposition 8.1.13].

Lemma 3.2. *If a matroid has infinite connectivity, then it is uniform. Furthermore,*

$$\lambda(U_{r,n}) = \begin{cases} r+1, & \text{if } n \geqslant 2r+2, \\ n-r+1, & \text{if } n \leqslant 2r-2, \\ \infty, & \text{otherwise.} \end{cases}$$

Lemma 3.3. *Assume the matroid M has at least $2(n-1)$ elements and that $\lambda(M) \geqslant n$. If e is an element of M, then $\lambda(M\backslash e) \geqslant n-1$ and $\lambda(M/e) \geqslant n-1$.*

Proposition 3.4 is a counterpart of Proposition 3.1 for Tutte connectivity.

Proposition 3.4. *Let $N = (M+p) \times q$.*

(1) *If M is the uniform matroid $U_{r,n}$ where $2r-1 \leqslant n \leqslant 2r+1$, then $\lambda(N) = \infty$.*
(2) *If M is the uniform matroid $U_{r,n}$ where $n \leqslant 2r-2$ or $n \geqslant 2r+2$, then $\lambda(N) = \lambda(M)+1$.*
(3) *If M is not uniform, then $\lambda(N)$ is $\lambda(M)+1$ or $\lambda(M)+2$.*

Proof. Note that M is the uniform matroid $U_{r,n}$ if and only if N is the uniform matroid $U_{r+1,n+2}$. From this and Lemma 3.2, assertions (1) and (2) follow. Thus we focus on assertion (3). An elementary argument shows that since M is not uniform, $\lambda(M)$ is at most $r(M)$. The inequality $\lambda(N) - 2 \leqslant \lambda(M)$ follows immediately from Lemma 3.3 since N has at least $2\lambda(N)$ elements.

We now need only show $\lambda(M) \leqslant \lambda(N) - 1$. This is obvious if $\lambda(N) = r(N)$, so assume $\lambda(N) = k < r(N)$. Since in a k-separation of N, the elements p and q are either in the same block or in different blocks, we consider two cases.

First consider an exact k-separation of N of the form $(X \cup \{p,q\}, Y)$ where X and Y are subsets of S. The case $X = \emptyset$ follows as in the proof of Proposition 3.1, so assume $X \neq \emptyset$. First assume $r_M(X) = r_N(X \cup \{p,q\}) - 1$. By Proposition 2.2, X spans M. Therefore $|X| \geqslant r(M) \geqslant k$. It also follows that $X \cup \{p,q\}$ spans N, so from the equality $r_N(X \cup \{p,q\}) + r_N(Y) - r(N) = k-1$, we get $r_N(Y) = k-1$. Therefore Y is dependent in N, and hence in M, so by Proposition 2.2 we have $r_M(Y) = r_N(Y) - 1$. It follows that (X, Y) is a $(k-1)$-separation of M. Thus, assume $r_M(X) = r_N(X \cup \{p,q\}) - 2$. Therefore X does not span M, so there is an element y in $Y - \mathrm{cl}_M(X)$. If $r_M(Y) = r_N(Y)$, it follows that Y is independent in M and so $(X \cup y, Y - y)$ gives a $(k-1)$-separation of M. If $r_M(Y) = r_N(Y) - 1$, it follows that (X, Y) is a $(k-2)$-separation of M.

Lastly consider a k-separation of N of the form $(X \cup p, Y \cup q)$ where X and Y are subsets of S. Since $r_M(Y) = r_N(Y \cup q) - 1$ and $r_M(X)$ is $r_N(X \cup p) - 2$ or $r_N(X \cup p) - 1$, it follows that $(X - p, Y - q)$ is a $(k-1)$-separation of M. □

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The matroids $U_{1,2} \oplus U_{1,1}$ and $U_{1,2} \oplus U_{1,2}$ show that both options in Proposition 3.1 and both options in assertion (3) of Proposition 3.4 can occur.

Starting with any matroid M such that $t(M; x, y) = t(M; y, x)$, one can construct a chain of double extensions. That is, let $N_0 = M, N_1, N_2, \ldots$ be matroids such that N_i is the double extension of N_{i-1} for each $i > 0$. Then $\kappa(N_i) \geqslant i + \kappa(M) > i$ by Proposition 3.1 and $\lambda(N_i) > i$ by Proposition 3.4; also $t(N_i; x, y) = t(N_i; y, x)$ by Theorem 2.8. Furthermore, for all $i > 0$, the matroid N_i is self-dual if and only if M is self-dual. Using these observations, we get the following theorem which strengthens Theorem 2.9.

Theorem 3.5. *For any positive integer k, there are matroids M that are not self-dual such that $\kappa(M) \geqslant k$ and $\lambda(M) \geqslant k$ and $t(M; x, y) = t(M; y, x)$.* □

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