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PROBLEMS AND SOLUTIONS

PROBLEMS

01.3.1. *A Necessary and Sufficient Condition for the Equivalence of Affine Unbiased Estimators*, proposed by Geert Dhaene. Consider the standard linear regression model $y = X\beta + u$, where X has full column rank. Show that the OLS residuals are zero if and only if all affine unbiased estimators of β coincide.

01.3.2. *Conditional and Unconditional Correlatedness and Heteroskedasticity*, proposed by Stanislav Anatolyev. For jointly stationary scalar processes e_t and z_t, define Ω_t to be the σ-field generated by $z_t, z_{t-1}, z_{t-2}, \ldots$ and let e_t have conditional mean zero relative to Ω_t, i.e., $E[e_t|\Omega_t] = 0$. Show that

(a) It is possible that $E[e_t e_{t-1}|\Omega_t] \neq 0$ almost everywhere but $E[e_t e_{t-1}] = 0$. That is, an unconditionally uncorrelated process may be conditionally serially correlated.
(b) It is possible that $E[e_t e_{t-1}|\Omega_t]$ is almost surely constant but $E[e_t^2|\Omega_t]$ is not constant on a set of positive measure or vice versa. That is, a process may be conditionally homoskedastic in covariance but conditionally heteroskedastic in variance or vice versa.
(c) When $E[e_t e_{t-1}|\Omega_t]$ and $E[e_t^2|\Omega_t]$ are measurable with respect to the σ-field generated by some scalar random variable ζ_t in which both are monotonic, it is possible that $E[e_t e_{t-1}|\Omega_t]$ is increasing but $E[e_t^2|\Omega_t]$ is decreasing in ζ_t or vice versa. That is, the direction of conditional heteroskedasticity in variance and that in covariance may be opposite.

SOLUTIONS

00.3.1. *Sampling Distribution of OLS under a Logit Model*—Solution[1] proposed by Badi H. Baltagi.

(a) From the logit model with $\beta = 1$, we get

y_t	u_t	Probability
0	$-\Lambda(x_t)$	$1 - \Lambda(x_t)$
1	$1 - \Lambda(x_t)$	$\Lambda(x_t)$

with $E(u_t) = 0$ and $\text{var}(u_t) = \Lambda(x_t)[1 - \Lambda(x_t)]$. Now $\hat{\beta}_{ols} = \sum x_t y_t / \sum x_t^2 = (y_1 + 2y_2)/5$. Therefore,

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y_1	y_2	$\hat{\beta}_{ols}$	Probability
0	0	0	$1/((1+e)(1+e^2))$
0	1	0.4	$e^2/((1+e)(1+e^2))$
1	0	0.2	$e/((1+e)(1+e^2))$
1	1	0.6	$e^3/((1+e)(1+e^2))$

It is easy to verify that $E(\hat{\beta}_{ols}) = (e + 2e^2 + 3e^3)/(5(1+e)(1+e^2)) = 0.4985$.

(b) The OLS fitted values are given by $\hat{y}_t = x_t\hat{\beta}_{ols}$ and the residuals by $\hat{u}_t = y_t - \hat{y}_t$.

$\hat{y}_1$	$\hat{y}_2$	$\hat{u}_1$	$\hat{u}_2$	$\sum\hat{u}_t^2$	Probability
0	0	0	0	0	$1/((1+e)(1+e^2))$
0.4	0.8	−0.4	0.2	0.2	$e^2/((1+e)(1+e^2))$
0.2	0.4	0.8	−0.4	0.8	$e/((1+e)(1+e^2))$
0.6	1.2	0.4	−0.2	0.2	$e^3/((1+e)(1+e^2))$

The MSE of the OLS regression is given by $s^2 = \sum\hat{u}_t^2/(n-1) = \sum\hat{u}_t^2$, because the sample size $n = 2$. Hence $\widehat{\text{var}}(\hat{\beta}_{ols}) = s^2/\sum x_t^2$ and $E[\widehat{\text{var}}(\hat{\beta}_{ols})] = E(s^2/5) = (4e + e^2 + e^3)/(25(1+e)(1+e^2)) = 0.0492$. But $\hat{\beta}_{ols} = (y_1 + 2y_2)/5 = [\Lambda(t) + 2\Lambda(2) + u_1 + 2u_2]/5$ with $E(\hat{\beta}_{ols}) = [\Lambda(1) + 2\Lambda(2)]/5$ and $\text{var}(\hat{\beta}_{ols}) = \text{var}[(u_1 + 2u_2)/5] = [\text{var}(u_1) + 4\text{var}(u_2)]/25 = e/(25(1+e)^2) + 4e^2/(25(1+e^2)^2) = 0.02466$. Hence, on average, the estimated variance of $\hat{\beta}_{ols}$ is almost double the true $\text{var}(\hat{\beta}_{ols})$.

NOTE

1. An alternative solution has been proposed by Xing Ming.

00.3.2. *A Second-Degree Matrix Equation*—Solution,[1] proposed by Götz Trenkler. Obviously, $\mathbf{X} = \mathbf{I}$ and $\mathbf{X} = -\mathbf{I}$ solve the equation $\mathbf{XAX}' = \mathbf{A}$ for all $\mathbf{A}$. We now show that there are no other solutions. Choosing $\mathbf{A} = \mathbf{e}_i\,\mathbf{e}_i'$, where $\mathbf{e}_i$ is the ith unit vector, we see that $\mathbf{X} = (x_{ij})$ must be a diagonal matrix with elements $x_{ii} = \pm 1$. Assume that for $j \neq k$ we have $x_{jj} = 1$ and $x_{kk} = -1$ and let $\mathbf{u}_j$ and $\mathbf{u}_k$ be nonzero eigenvectors (e.g., unit vectors) belonging to the eigenvalues $x_{jj} = 1$ and $x_{kk} = -1$, respectively. Then $\mathbf{X}\mathbf{u}_j\,\mathbf{u}_k'\mathbf{X}' = \mathbf{u}_j(-\mathbf{u}_k') = -\mathbf{u}_j\,\mathbf{u}_k'$. On the other hand we must have $\mathbf{X}\mathbf{u}_j\,\mathbf{u}_k'\mathbf{X}' = \mathbf{u}_j\,\mathbf{u}_k'$. This is a contradiction. Thus $x_{ii} = 1$ for all i or $x_{ii} = -1$ for all i.

We note that the requirement that $\boldsymbol{X}$ has to be nonsingular is superfluous.

NOTE

1. Excellent solutions have been independently proposed by Xing Ming, Steve Lawford and Karim Abadir, and Heinz Neudecker, the poser of the problem. In the last two proofs, Steve Lawford, Karim Abadir and Heinz Neudecker noted that vectorizing $XAX' = A$ one obtains $((X \otimes X) - I)a = 0$, where $a = \text{vec}\,A$; choosing a in turn equal to a set of linearly independent vectors implies $X \otimes X = I$. The proofs then proceeded as in the preceding discussion.

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