Ergod. Th. & Dynam. Sys. (2006), **26**, 923–936
doi:10.1017/S0143385705000581
© *2006 Cambridge University Press*
Printed in the United Kingdom

Projective hyperbolicity and fixed points

THÉRÈSE VIVIER
150 avenue Armand Guillebaud, 92160 Antony, France
(e-mail: therese.vivier@polytechnique.org)

(*Received* 1 *October* 2004 *and accepted in revised form* 9 *October* 2005)

Abstract. Generalizing a three-dimensional result of Doering, we prove that a robustly transitive vector field defined on a compact manifold of any dimension greater than three admits no fixed point.

1. *Introduction*

In this paper, we shall denote by M a smooth compact boundaryless Riemannian manifold of dimension $n \geqslant 3$ and by $\chi^1(M)$ the set of C^1-vector fields defined on M. We shall denote by X an element of $\chi^1(M)$, by φ the flow associated to the vector field X and by D_zX (respectively $T_z\varphi_t$) the differential of X (respectively the differential of the flow map φ_t) at the point z. Recall that, for any real t and any point $x \in M$, the point $\varphi_t(x)$ is the image of the point x by the time t of the flow φ. The vector field X is *transitive* if it admits an orbit dense in the whole manifold and X is *robustly transitive* if there is a C^1-neighborhood $\mathcal{U} \ni X$ of transitive vector fields. More generally, an invariant set $\Lambda \subset M$ is *transitive* (for the vector field X) if it contains a point whose positive orbit is dense in Λ. Such a set is *robustly transitive* if there are a neighborhood $\mathcal{U} \subset \chi^1(M)$ of X and a neighborhood $U \subset \Lambda$ such that both following assumptions hold:

- Λ is the maximal invariant set in $\bar{U}$ for the vector field X;
- for any vector field $Y \in \mathcal{U}$, the maximal invariant set in $\bar{U}$ is transitive.

Basic examples of transitive flows are given by irrational flows on tori; however, these flows are not robustly transitive. Examples of robustly transitive flows, such as geodesic flows on surfaces with -1 curvature and suspensions of robustly transitive diffeomorphisms, are also known, but none of them admit a fixed point. However, some attractors, such as the Lorenz attractor, are known to be robustly transitive attractors and to contain a hyperbolic singularity. Of course, the preceding definition for robustly transitive vector fields is not compatible with the existence of attractors (because the existence of an attractor would forbid the density of any orbit in the whole manifold). However, the existence of such robustly transitive attractors raised the following question: is there any robustly transitive vector field admitting a fixed point?

The answer has been known to be negative in the three-dimensional case since Doering published the following theorem in [**1**], based on a result of Mañé for diffeomorphisms (see [**2**]).

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

THEOREM. (Doering) *Any robustly transitive vector field on a three-dimensional manifold is Anosov.*

THEOREM. (Mañé) *Any robustly transitive diffeomorphism on a two-dimensional manifold is Anosov.*

This last result does not hold in higher dimensions (see Shub's counterexample in dimension four in [**3**] and Mañé's counterexample in dimension three in [**4**]). The aim of our study was therefore to get a better understanding of what the dynamics around a fixed point could be like for a robustly transitive vector field in the n-dimensional case, with $n \geqslant 4$. We thus obtained the following main result.

THEOREM 1. *If $X \in \chi^1(M)$ is a robustly transitive vector field, then X admits no fixed point.*

As we shall see, the first part of our proof (see Proposition 2.1) can be regarded as a good generalization for Doering's result in higher dimensions. The key notion is a form of projective hyperbolicity which, as we shall prove, is satisfied by any robustly transitive vector field (Proposition 2.1), but also turns out to be incompatible with the existence of a fixed point (Proposition 2.2).

We recently learned that Li *et al* have found another proof for Theorem 1 (see [**5**]), by generalizing the notions of linear Poincaré flow and of dominated splitting of the normal bundle in order to include the points of $\mathrm{Sing}(X)$ in their study.

2. *Definitions*

Consider $X \in \chi^1(M)$. A fixed point $z \in M$ is *hyperbolic* if the following assumption holds:

$$\text{for all } \lambda \in \mathrm{Spec}(T_z\varphi_1), \quad |\lambda| \neq 1.$$

This enables us to distinguish three different classes of hyperbolic fixed points for vector fields: the *sources* (the norms of all eigenvalues of $T_z\varphi_1$ are all strictly larger than 1), the *sinks* (the norms of all eigenvalues of $T_z\varphi_1$ are all strictly smaller than 1) and the *saddles* (only some of the eigenvalues have norm strictly larger than 1, while the others have norm strictly smaller than 1). Note that a transitive vector field cannot admit a sink nor a source (for the existence of such fixed points would forbid the existence of an orbit dense in the whole manifold M).

Consider a regular point $p \in M$. By assumption, $X(p) \neq 0$; hence, we can define a subspace $N_p = X(p)^\perp$ in the tangent space T_pM. We thus obtain the so-called *normal bundle* $N \subset TM$ defined over the set of regular points $M \setminus \mathrm{Sing}(X)$. Define now the *linear Poincaré flow*. For any real t and any regular point p, we define the time t of the linear Poincaré flow by

$$P_t(p) = \pi_{|N_{\varphi_t(p)}} \circ T_p\varphi_{t|N_p} : N_p \to N_{\varphi_t(p)},$$

where $\pi_{|N_{\varphi_t(p)}} : T_{\varphi_t(p)}M \to N_{\varphi_t(p)}$ is the projection along the direction of the flow $X(\varphi_t(p))$. This diffeomorphism coincides with the differential of the Poincaré application defined from a hypersurface of M whose tangent space at the point p is N_p, to a hypersurface of M whose tangent space at the point $\varphi_t(p)$ is $N_{\varphi_t(p)}$; hence the name.

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

By definition, the normal bundle N is invariant by the linear Poincaré flow. Note that the above-defined linear Poincaré flow depends on the considered metric.

The C^1-vector field X, being defined on a compact manifold, exhibits a constant $K > 0$ such that

$$K = \max_{x \in M,\ t \in [-1,1]} \|T_x \varphi_t\|.$$

As composing a linear application by an orthogonal projection only decreases the norm of the considered application, we get the following remark.

Remark 2.1. For all $p \in M \setminus \mathrm{Sing}(X)$, and all $t \in [-1, 1]$, $\|P_t(p)\| \leqslant K$.

Definition 2.1. A *splitting* of the normal bundle N (of dimension $n-1$) over an invariant set $\Lambda \subset M \setminus \mathrm{Sing}(X)$ is a pair (N^s, N^u) of non-trivial subbundles defined over Λ, of constant dimensions ($i \in [1, \ldots, n-2]$, $j = n-1-i$), such that, for any point $p \in \Lambda$, we have $N_p = N_p^s \oplus N_p^u$. Such a splitting is *invariant* by the linear Poincaré flow if, for any time t and any point $p \in \Lambda$, the following property holds:

$$\begin{cases} P_t(p)(N_p^s) = N_{\varphi_t(p)}^s, \\ P_t(p)(N_p^u) = N_{\varphi_t(p)}^u. \end{cases}$$

We are now ready to introduce the notion of projective hyperbolicity announced in the introduction.

Definition 2.2. A splitting $N = N^s \oplus N^u$ of the normal bundle over an invariant set $\Lambda \subset M \setminus \mathrm{Sing}(X)$ is said to be *dominated* for the linear Poincaré flow, and then denoted $N^s \prec N^u$, if the following two assumptions hold:

(1) it is an invariant splitting for the linear Poincaré flow;

(2) there exist two constants $C > 1$ and $T > 0$ such that, for any point $p \in \Lambda$, for any pair of non-zero vectors $(v^s, v^u) \in N_p^s \times N_p^u$:

$$\frac{|P_t(p)v^s|}{|v^s|} \leqslant \left(\frac{1}{C}\right)^t \frac{|P_t(p)v^u|}{|v^u|} \quad \text{for all } t \geqslant T,$$

or, which is strictly equivalent, there exists an integer ℓ such that, for any point $p \in \Lambda$ and any pair of non-zero vectors $(v^s, v^u) \in N_p^s \times N_p^u$, the following inequality holds:

$$\frac{|P_t(p)v^s|}{|v^s|} < \frac{1}{2}\frac{|P_t(p)v^u|}{|v^u|} \quad \text{for all } t \geqslant \ell.$$

In this case, the splitting is an *ℓ-dominated splitting*, denoted by $N^s \prec_\ell N^u$.

Remark 2.2. Note that the notion of domination of an invariant splitting of the normal bundle over an invariant set of regular points for the linear Poincaré flow does not depend on the considered norm, as long as it is equivalent to the one induced by the Riemannian metric.

In [**6**, Lemma 1.4], Bonatti *et al* proved in the diffeomorphism case that, whenever a dominated splitting of the tangent bundle exists over an invariant set Λ, it can be extended over the (invariant) closed set $\overline{\Lambda}$. The same reasoning holds for vector fields.

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

Remark 2.3. Let $\Lambda \subset M \setminus \mathrm{Sing}(X)$ be an invariant set such that the normal bundle admits a dominated splitting over Λ for the linear Poincaré flow. Then the normal bundle admits a dominated splitting over $\overline{\Lambda} \setminus \mathrm{Sing}(X)$ for the linear Poincaré flow.

LEMMA 2.1. *Assume $N^s \oplus N^u$ is a dominated splitting of the normal bundle for the linear Poincaré flow over $M \setminus \mathrm{Sing}(X)$. Then there exists a constant $\zeta > 0$ such that, for any point $p \in M \setminus \mathrm{Sing}(X)$, for any pair of unitary vectors $(v^s, v^u) \in N^s_p \times N^u_p$,*

$$|v^s - v^u| > \zeta.$$

In other words, the angle between the subbundles N^s and N^u is uniformly bounded by below.

Proof. Fix $\ell > 0$ such that $N^s \oplus N^u$ is an ℓ-dominated splitting of the normal bundle for the linear Poincaré flow over $M \setminus \mathrm{Sing}(X)$. By definition, for any point $p \in M \setminus \mathrm{Sing}(X)$, for any pair of unitary vectors $(v^s, v^u) \in N^s_p \times N^u_p$,

$$\frac{|P_\ell(p)v^s|}{|P_\ell(p)v^u|} < \frac{1}{2}. \tag{1}$$

Assume by contradiction that there exists a sequence of points $(x_n)_{n\in\mathbb{N}} \in (M \setminus \mathrm{Sing}(X))^{\mathbb{N}}$ and a sequence of pairs of unitary vectors $(v^s_n, v^u_n) \in N^s_{x_n} \times N^u_{x_n}$ such that $|v^s_n - v^u_n| \xrightarrow[n\to+\infty]{} 0$. This assertion, put together with equation (1), is in contradiction with Remark 2.1. □

Consider a dominated splitting $N = N^s \oplus N^u$ over an invariant set $\Lambda \subset M \setminus \mathrm{Sing}(X)$. Then, for any point $p \in \Lambda$, since any vector $v \in N_p$ admits by assumption a unique splitting $v = v^s \oplus v^u$ with $(v^s, v^u) \in N^s_p \times N^u_p$, we can define a norm on the normal bundle over Λ as follows:

$$|v|' = \max\{|v^s|, |v^u|\}.$$

As a direct corollary of the previous lemma (together with the triangular inequality), we know that, for any point $p \in \Lambda$ and for any vector $v \in N_p$, $\zeta|v|' < |v| < 2|v|'$, we have the following.

COROLLARY 2.1. *The norm $|\ |'$ associated to the dominated splitting $N^s \oplus N^u$ over Λ is equivalent to the norm induced by the Riemannian metric on the normal bundle over Λ.*

We shall now prove Theorem 1 in two steps.

PROPOSITION 2.1. *If X is a robustly transitive vector field, then the normal bundle N admits a dominated splitting for the linear Poincaré flow over $M \setminus \mathrm{Sing}(X)$.*

PROPOSITION 2.2. *Assume that X is a robustly transitive vector field such that $\mathrm{Sing}(X)$ is non-empty. Then there exist arbitrarily C^1-close vector fields Y such that their normal bundle admits no dominated splitting for their associated linear Poincaré flow over $M \setminus \mathrm{Sing}(Y)$.*

Notice that, given a robustly transitive vector field X, any vector field Y C^1-sufficiently close to X is also robustly transitive by definition. Thus, Proposition 2.1 and Proposition 2.2 lead to the announced Theorem 1.

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

3. *Robust transitivity and dominated splitting*

Bonatti, Díaz and Pujals proved [**6**, Theorem 2] that any C^1-robustly transitive diffeomorphism admits a dominated splitting of the tangent bundle. Their proof, based on the highly-technical notion of *transitions*, can be reinterpreted in terms of vector fields, but this requires tricky technical arguments, essentially due to the fact that their notion of transitions fails to be purely algebraic. More recently, new arguments developed in [**7**] enabled us to prove the following statement (see [**7**, Corollary 2.22]).

PROPOSITION 3.1. *Recall that M is a compact manifold. Consider $X \in \chi^1(M)$. Then for any $\epsilon > 0$ there are two integers ℓ and n such that, for any periodic point x of period $p(x) \geqslant n$:*

(1) *either there is an l-dominated splitting of the normal bundle for the linear Poincaré flow P over the orbit of x;*

(2) *or, for any neighborhood U of the orbit of x, there exists a C^1 ϵ-perturbation Y of X coinciding with X outside U and along the orbit of x, and such that the linear Poincaré flow $P^Y_{p(x)}(x)$ (associated to the vector field Y) has all eigenvalues real and with the same modulus; furthermore, this modulus can be chosen different from* 1*, and thus the orbit of x is a source or a sink for Y.*

Let us show how to deduce Proposition 2.1 from the previous statement.

Proof of Proposition 2.1. Since the existence of a source or a sink for a vector field arbitrarily C^1-close to X is incompatible with the assumption of X being robustly transitive, we deduce from the previous proposition that, for any robustly transitive vector fields, there exists an ℓ-dominated splitting of the normal bundle for the linear Poincaré flow over the set of periodic points with large periods. This set being dense in M for a residual subset in the space of robustly transitive vector fields, we deduce from Remark 2.3 that Proposition 2.1 holds. □

The following stronger statement is also proved in [**7**, Corollary 2.23].

PROPOSITION 3.2. *Consider $X \in \chi^1(M)$. Assume that K is a compact invariant set which is robustly transitive. Then the normal bundle admits a dominated splitting over $K \setminus \mathrm{Sing}(X)$ for the linear Poincaré flow of X.*

4. *Dominated splitting and fixed points*

Any vector field $Y \in \chi^1(M)$ admitting a fixed point can be arbitrarily C^1-closely approximated by a vector field $Z \in \chi^1(M)$ admitting a hyperbolic fixed point (see [**8**]). Moreover, if a vector field $Z \in \chi^1(M)$ admits a hyperbolic fixed point, there is a vector field $W \in \chi^1(M)$ arbitrarily C^1-close to Z admitting a hyperbolic fixed point p in the neighborhood of which the vector field W is linear, hence equal to D_pW. This leads us to the following remark.

- If $X \in \chi^1(M)$ admits a fixed point, there is a vector field $Y \in \chi^1(M)$, arbitrarily C^1-close to X, which admits a hyperbolic fixed point in the neighborhood of which Y is linear.

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

Notice that, since any robustly transitive vector field X admits a neighborhood $\mathcal{U} \subset \chi^1(M)$ of transitive vector fields, then any vector field $Y \in \mathcal{U}$ also admits $\mathcal{U}$ as neighborhood of transitive vector fields, hence:

- any vector field $Y \in \mathcal{U}$ is a robustly transitive vector field.

Thanks to both preceding remarks, we get the following lemma.

LEMMA 4.1. *If $X \in \chi^1(M)$ is robustly transitive and admits a fixed point, then X can be arbitrarily closely C^1-approximated by a robustly transitive vector field $Y \in \chi^1(M)$ admitting a hyperbolic fixed point in the neighborhood of which Y is linear.*

Definition 4.1. Consider $X \in \chi^1(M)$ and p a hyperbolic fixed point. Assume there exist smooth local coordinates around p such that X is linear equal to D_pX in these coordinates. We shall then say that p is a *linear hyperbolic fixed point* for X.

Since a robustly transitive vector field Y cannot admit hyperbolic sinks or sources, its hyperbolic fixed points are necessarily saddles.

Proposition 2.2 is therefore a straightforward corollary of the following proposition.

PROPOSITION 4.1. *If $Y \in \chi^1(M)$ admits a linear hyperbolic fixed point of saddle-type, then there is no dominated splitting of the normal fiber (defined over the set of regular points) for the linear Poincaré flow associated to Y.*

As a matter of fact, our reasoning shall imply the following slightly stronger statement.

Remark 4.1. Assume $\Lambda \subset M \setminus \mathrm{Sing}(X)$ is an invariant set such that the normal bundle admits a dominated splitting for the linear Poincaré flow over Λ. Then any linear hyperbolic fixed point of saddle-type p satisfies the following property:

$$p \notin \mathrm{Int}(\overline{\Lambda} \cup \{p\}).$$

This remark, together with Proposition 3.2 implies the following corollary.

COROLLARY 4.1. *Consider $X \in \chi^1(M)$. Assume that Λ is an invariant set which is robustly transitive. Then, any linear hyperbolic fixed point p satisfies the following property:*

$$p \notin \mathrm{Int}(\overline{\Lambda} \cup \{p\}).$$

The rest of this paper will be devoted to the proof of Proposition 4.1. The key tool in the proof of Proposition 4.1 (hence in the proof of Proposition 2.2) is a property concerning the exponential growth of the norm of vectors of the normal bundle whenever this normal bundle admits a dominated splitting for the linear Poincaré flow. More precisely, this property is to be seen as a weakened version of a property satisfied in a more restrictive case (hyperbolic case). We first present this *never-ending growth* property in the hyperbolic case. We then state (and prove) the weakened version satisfied under the assumption of existence of a dominated splitting of the normal bundle (see Lemma 4.4), which we shall use to prove Proposition 4.1.

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

4.1. *The* never-ending growth *property in the hyperbolic case.* All the propositions of this section are classical results that we only state for pedagogical purposes. In particular, we shall not use the proofs written in this section in the rest of the paper: the reader should therefore feel free to skip any part of this section.

Let us first state the definition of a *hyperbolic set of saddle-type*.

Definition 4.2. Consider a vector field $X \in \chi^1(M)$ and a compact invariant set $K \subset M \setminus \mathrm{Sing}(X)$. The set K is of *hyperbolic saddle-type* if there exist a splitting of the normal bundle $N = N^{ss} \oplus N^{uu}$ invariant by the linear Poincaré flow over Λ and two positive constants $C \geqslant 1,\ c < 1$ such that for all $p \in K$, and all $t \geqslant 0$:

- for all $v^s \in N^{ss}_p,\ |P_t(p)v^s| \leqslant Cc^t|v^s|$;
- for all $v^u \in N^{uu}_p,\ |P_{-t}(p)v^u| \leqslant Cc^t|v^u|$.

Remark 4.2. Since the normal bundle over a compact hyperbolic set of saddle-type K admits by definition a dominated splitting $N^{ss} \prec N^{su}$ for the linear Poincaré flow, Lemma 2.1 states that there exists $\zeta > 0$ such that, for any point $p \in K$ and any pair of unitary vectors $(v, w) \in N^{ss} \times N^{uu}$,

$$|v - w| > \zeta.$$

We shall make use of the norm $|\cdot|'$ defined as follows: for all $x \in M \setminus \mathrm{Sing}(X)$, for all $v = v^s \oplus v^u \in N^{ss}_x \oplus N^{uu}_x$,

$$|v|' = \max(|v^s|, |v^u|).$$

(Recall that, thanks to Corollary 2.1, this norm is equivalent to the one induced by the Riemannian metric.)

What we call the *never-ending growth* property is the following: recall that, by definition of a hyperbolic system, we need to wait for a characteristic time τ_0 such that $Cc^{\tau_0} = 1$ for the exponential growth of the norm of vectors of N^{ss} (respectively of N^{uu}) to become negative (respectively positive) under the action of the linear Poincaré flow. If after this time the exponential growth of a vector $v \in N$ reaches a positive critical value, then the asymptotical exponential growth of this vector is known: it is strictly positive (and uniformly bounded away from zero). In other words, once the exponential growth of such a vector reaches a critical level (after a characteristic time), it is positive for ever after some time. More precisely, we have the following.

LEMMA 4.2. *Consider $X \in \chi^1(M)$ and let K be a compact invariant hyperbolic set of saddle-type, characterized by the positive constants $C \geqslant 1$ and $c \in]0, 1[$.*

For any time $t_0 \geqslant 0$ such that $Cc^{t_0} \leqslant 1$, there exists a positive time t_1 such that, for any point $p \in K$ and for any vector $v \in N_p$, if

$$\frac{1}{t_0} \ln\left(\frac{|P_{t_0}v|'}{|v|'}\right) > \frac{1}{t_0} \ln(Cc^{t_0}),$$

then, for any time $t \geqslant t_1$,

$$\frac{1}{t} \ln\left(\frac{|P_t v|'}{|v|'}\right) \geqslant -\frac{1}{2} \ln c.$$

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

Before proving this lemma, let us give an analogous statement independent of the considered norm $|\cdot|$ provided it is equivalent to the one induced by the Riemannian metric (and hence equivalent to the norm $|\cdot|'$).

LEMMA 4.3. *Consider $X \in \chi^1(M)$. Let K be a compact invariant hyperbolic set of saddle-type, characterized by the positive constants $C \geqslant 1$ and $c \in]0,1[$ and let $|\cdot|$ denote a norm equivalent to the one induced by the Riemannian metric on the tangent bundle.*

There exists $\epsilon > 0$ such that, for any time $t_0 \geqslant 0$ verifying $Cc^{t_0} \leqslant 1$, there exists a positive time t_1 such that, for any point $p \in K$ and for any vector $v \in N_p$, if

$$\frac{1}{t_0}\ln\left(\frac{|P_{t_0}v|}{|v|}\right) > \frac{1}{t_0}(\epsilon + \ln(Cc^{t_0})),$$

then, for any time $t \geqslant t_1$,

$$\frac{1}{t}\ln\left(\frac{|P_t v|}{|v|}\right) \geqslant -\frac{1}{2}\ln c - \frac{\epsilon}{t}.$$

Let us first explain how to deduce this statement from Lemma 4.2.

Proof. As we assume the norm $|\cdot|$ to be equivalent to the one induced by the Riemannian metric, it is equivalent to the norm $|\cdot|'$ used in Lemma 4.2. Thus, there exists a constant $\mathcal{C} \geqslant 1$ such that for any point $p \in K$, for any vector $v \in N_p$,

$$\frac{1}{\mathcal{C}}|v| \leqslant |v|' \leqslant \mathcal{C}|v|.$$

Fix $\epsilon = 2\ln\mathcal{C} > 0$.

Then, for any point $p \in K$, for any vector $v \in N_p$, for any time $t_0 \geqslant T$, if we assume

$$\frac{1}{t_0}\ln\left(\frac{|P_{t_0}(p)v|}{|v|}\right) > \frac{1}{t_0}(\epsilon + \ln(Cc^{t_0})),$$

by definition of the constants ϵ and $\mathcal{C}$, we have

$$\frac{1}{t_0}\ln\left(\frac{|P_{t_0}(p)v|'}{|v|'}\right) > \frac{1}{t_0}\ln(Cc^{t_0}).$$

Lemma 4.2 then states that there exists $t_1 > 0$ such that for any time $t \geqslant t_1$,

$$\frac{1}{t}\ln\left(\frac{|P_t(p)v|'}{|v|'}\right) \geqslant -\frac{1}{2}\ln c.$$

This implies that for any time $t \geqslant t_1$,

$$\frac{1}{t}\ln\left(\frac{|P_t(p)v|}{|v|}\right) \geqslant -\frac{1}{2}\ln c - \frac{\epsilon}{t}. \qquad \square$$

We can now turn to the proof of Lemma 4.2.

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

Proof of Lemma 4.2. We shall prove that the announced statement holds for $t_1 = \max\{4t_0,\ t_0 - \ln C/\ln c\}$.

Consider $p \in K$ and $v = v^s + v^u \in N_p^{ss} \oplus N_p^{uu}$.

Notice, first, that

$$\frac{1}{t_0}\ln\left(\frac{|P_{t_0}(p)v^s|}{|v|'}\right) \leqslant \frac{1}{t_0}\ln\left(\frac{|P_{t_0}(p)v^s|}{|v^s|}\right) \leqslant \frac{1}{t_0}\ln(Cc^{t_0}) < 0;$$

thus, any vector satisfying the assumption

$$\frac{1}{t_0}\ln\left(\frac{|P_{t_0}(p)v|'}{|v|'}\right) > \frac{1}{t_0}\ln(Cc^{t_0}) \tag{2}$$

necessarily verifies the equation

$$|P_{t_0}(p)v|' = |P_{t_0}(p)v^u|.$$

Equation (2) thus becomes

$$\frac{1}{t_0}\ln\left(\frac{|P_{t_0}(p)v^u|}{|v|'}\right) > \frac{1}{t_0}\ln(Cc^{t_0}).$$

This equation, together with the inequality $|P_{t_0}(p)v^u| \geqslant 1/(Cc^{t_0})|v^u|$ implies that

$$\frac{1}{t_0}\ln\left(\frac{|v^u|}{|v|'}\right) > 2\left(\ln c + \frac{\ln C}{t_0}\right). \tag{3}$$

Moreover, since $t \geqslant -\ln C/\ln c$ implies that $(Cc^t) \leqslant 1$, we can write for any $t \geqslant -\ln C/\ln c$,

$$\begin{aligned}|P_t(\varphi_{t_0}(p)) \circ P_{t_0}(p)v^s| &\leqslant Cc^t|P_{t_0}(p)v^s| \\ &\leqslant Cc^t|P_{t_0}(p)v^u| \\ &\leqslant (Cc^t)^2|P_t(\varphi_{t_0}(p)) \circ P_{t_0}(p)v^u| \\ &\leqslant |P_t(\varphi_{t_0}(p)) \circ P_{t_0}(p)v^u|.\end{aligned}$$

Hence, for any $t \geqslant t_1$,

$$|P_t(p)v|' = |P_t(p)v^u|. \tag{4}$$

Equations (3) and (4) then imply that, for any $t \geqslant t_1$,

$$\begin{aligned}\frac{1}{t}\ln\left(\frac{|P_t(p)v|'}{|v|'}\right) &= \frac{1}{t}\ln\left(\frac{|P_t(p)v^u|}{|v^u|}\frac{|v^u|}{|v|'}\right) \\ &\geqslant -\frac{\ln C}{t} - \ln c + \frac{t_0}{t}\frac{1}{t_0}\ln\left(\frac{|v^u|}{|v|'}\right) \\ &\geqslant -\frac{\ln C}{t} - \ln c + \frac{t_0}{t}2\left(\ln c + \frac{\ln C}{t_0}\right) \\ &\geqslant \left(-1 + 2\frac{t_0}{t}\right)\ln c \\ &\geqslant -\frac{1}{2}\ln c > 0.\end{aligned}$$

□

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

4.2. *Weaker property in the case of dominated splitting of the normal bundle for the linear Poincaré flow.* The existence of a dominated splitting $N = N^s \oplus N^u$ of the normal bundle for the linear Poincaré flow does not imply that the norm of vectors in N^s (respectively, in N^u) decreases exponentially (respectively, increases exponentially) under the action of the linear Poincaré flow. We only know that the norm of vectors in N^s decreases exponentially *more than* the norm of vectors in N^u. Of course, we will not be able to recover the property of *never-ending growth* as stated in Lemma 4.3 in this case. However, a slightly less restrictive property remains true: if the exponential growth (after a characteristic time) of the norm of such a vector under the action of the linear Poincaré flow reaches a critical value, then the exponential growth of this vector is asymptotically bounded *away from the exponential growth of vectors of* N^s (instead of being asymptotically bounded away from zero). In particular, it is impossible for a vector $v \in N_p$ to have almost the exponential growth of a vector $u \in N_p^u$ during an arbitrarily long time (let us say from time 0 to time t) and then (starting at time t) almost the exponential growth of a vector $w \in N^s_{\varphi_t(p)}$ during an arbitrarily long time. The following lemma states this fact more precisely.

LEMMA 4.4. *Let X be a C^1-vector field such that the normal bundle N admits a dominated splitting for the linear Poincaré flow. Then there exist two constants $\epsilon > 0$ and $T > 0$ such that, for any pair $(r, \rho) \in]T, +\infty[^2$, for any real $t > r$, for any regular point x, and for any vector $v \in N_x$, one of these inequalities holds:*

$$\frac{1}{r}\ln\left(\frac{|P_r(x)v|}{|v|}\right) \leqslant \frac{1}{r}\ln(\mathbf{m}(P_r(x)_{|N_x^u})) - \epsilon, \tag{5}$$

$$\frac{1}{\rho}\ln\left(\frac{|P_\rho(\varphi_t(x)) \circ P_t(x)v|}{|P_t(x)v|}\right) \geqslant \frac{1}{\rho}\ln(\|P_\rho(\varphi_t(x))_{|N^s_{\varphi(t,x)}}\|) + \epsilon. \tag{6}$$

In the last statement and in the following proof, we denote by $\mathbf{m}$ the usual conorm of a linear map, that is,

$$\mathbf{m}(P_r(x)_{|N_x^u}) = \min_{v \in N_x^u,\ v \neq 0} \frac{|P_r(x)v|}{|v|}.$$

Proof. Recall that the existence of dominated splitting of the normal bundle for the linear Poincaré flow does not depend on the considered norm, provided it is equivalent to the one induced by the Riemannian metric (see Remark 2.2). Moreover, it is easy to check that if there exists a norm, equivalent to the one induced by the Riemannian metric, such that the announced result holds, then it also holds for any norm equivalent to the one induced by the Riemannian metric. Thus, it is sufficient to do the proof with the norm $|\cdot|'$ defined as follows: for all $x \in M \setminus \mathrm{Sing}(X)$, and all $v = v^s \oplus v^u \in N_x^s \oplus N_x^u$,

$$|v|' = \max(|v^s|, |v^u|).$$

(Recall that, thanks to Corollary 2.1, this norm is equivalent to the one induced by the Riemannian metric.)

Denote by C and T_0 the constants associated to the domination $N^s \prec N^u$ for the norm $|\ |'$. Let us prove that $\epsilon = \ln(C)$ and $T = T_0$ are the announced constants.

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

CLAIM 4.1. *If inequality (5) is not verified for $r > T$, then*

$$|P_r(x)v|' = |P_r(x)v^u|. \tag{7}$$

Proof. By contradiction, if (7) did not hold, we would get

$$\begin{aligned}\frac{1}{r}\ln\left(\frac{|P_r(x)v|'}{|v|'}\right) &\leqslant \frac{1}{r}\ln\left(\frac{|P_r(x)v^s|}{|v^s|}\right)\\ &\leqslant \frac{1}{r}\ln(\|P_r(x)_{|N_x^s}\|'),\end{aligned}$$

which, together with the negation of inequality (5),

$$\frac{1}{r}\ln\left(\frac{|P_r(x)v|'}{|v|'}\right) > \frac{1}{r}\ln(\mathbf{m}(P_r(x)_{|N_x^u})') - \epsilon,$$

would clearly contradicts the domination assumption $N^s \prec N^u$. □

Thus, if inequality (5) does not hold, we get for any $t \geqslant r$, $|P_t(x)v|' = |P_t(x)v^u|$, and for any $\rho \geqslant 0$, $|P_{t+\rho}(x)v|' = |P_{t+\rho}(x)v^u|$. Inequality (6) is therefore verified for the norm $|\ |'$, since for any $\rho > T_0$,

$$\begin{aligned}\frac{1}{\rho}\ln\left(\frac{|P_{\rho+t}(x)v|'}{|P_t(x)v|'}\right) &= \frac{1}{\rho}\ln\left(\frac{|P_{t+\rho}(x)v^u|}{|P_t(x)v^u|}\right)\\ &\geqslant \frac{1}{\rho}\ln(\mathbf{m}(P_\rho(x)_{|N^u_{\varphi_t(x)}})'),\\ &\geqslant \frac{1}{\rho}\ln(\|P_\rho(x)_{|N^s_{\varphi_t(x)}}\|') + \epsilon.\end{aligned}$$

(The last inequality comes from the domination assumption $N^s \prec N^u$.) □

4.3. *Proof of Proposition 4.1.* We now need only prove that, if the vector field X is linear in the neighborhood of a hyperbolic fixed point of saddle-type, then there exists a vector in the normal bundle contradicting Lemma 4.4.

We shall henceforth restrict our study to a local neighborhood $\mathcal{U}$ of a hyperbolic fixed point p of saddle-type (we can therefore assume that we are working in an open set of $\mathbb{R}^n$), such that X is linear, hence equal to D_pX on $\mathcal{U}$. Since p is a hyperbolic saddle, we are entitled to consider two eigenvalues λ^s and λ^u of D_pX such that:

- $\mathrm{Re}(\lambda^s) = \min\{\mathrm{Re}(\lambda),\ \lambda \in \mathrm{Spec}(D_pX)\} < 0$;
- $\mathrm{Re}(\lambda^u) = \max\{\mathrm{Re}(\lambda),\ \lambda \in \mathrm{Spec}(D_pX)\} > 0$.

Let us denote by E^s (respectively by E^u) the eigenspace of D_pX associated to λ^s (respectively to λ^u). Let Π denote the intersection of $\mathcal{U}$ with the vector space $E^s \oplus E^u$, necessarily invariant since X is linear. We shall denote by j the dimension of the vector space E^s, and by k the dimension of the vector space E^u; thus, $\dim(\Pi) = j + k$.

We shall restrict our study to Π. Consider four constants $\delta, \eta, \theta, \zeta > 0$ and let us define two sets U^s and U^u as follows:

- $U^s = \{x \in (\Pi - E^s),\ d(x, E^s) \leqslant \delta,\ \mathrm{Angle}(X(x), E^s) \leqslant \theta\}$;
- $U^u = \{x \in (\Pi - E^u),\ d(x, E^u) \leqslant \eta,\ \mathrm{Angle}(X(x), E^u) \leqslant \zeta\}$.

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

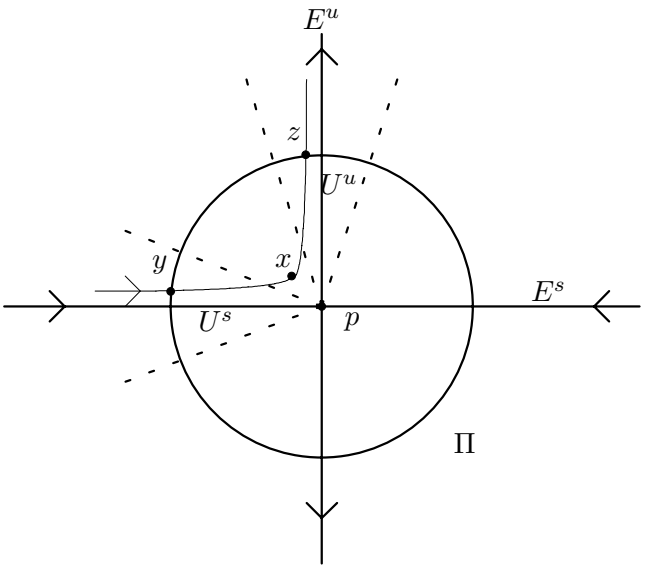


FIGURE 1. Dynamics around a hyperbolic saddle.

We can choose θ and ζ small enough, so that $U^s \cap U^u = \emptyset$. If δ is chosen small enough, we can moreover assume that for all $y \in U^s$, there exists $\tau > 0$, there exists $\varphi_\tau(y) \in U^u$.

Consider a sequence of points $(x_n)_n \in \Pi$ converging to p (see Figure 1). Let us define two sequences of points $(y_n)_n$ and $(z_n)_n$: y_n corresponds to the (chronological) *entrance point* of the flowline of x_n in U^s, and z_n to the (chronological) *exit point* of the flowline of x_n from U^u. We thus write:

- for all n, $y_n = \varphi_{-\tau_n}(x_n) \in U^s$, $\tau_n = \max_{\tau>0}\{\varphi_{-\tau}(x_n) \in U^s\}$;
- for all n, $z_n = \varphi_{+\tau'_n}(x_n) \in U^u$, $\tau' n = \max_{\tau'>0}\{\varphi_{\tau'}(x_n) \in U^u\}$.

Let us denote by r_n (respectively ρ_n) the length of time that x_n spends in U^s (respectively in U^u):

- $r_n = \max_{r>0}\{\varphi_r(y_n) \in U^s\}$;
- $\rho_n = \max_{\rho>0}\{\varphi_{-\rho}(z_n) \in U^u\}$.

It is easy to check that

$$\begin{cases} \lim_{n\to\infty} r_n = +\infty \text{ and } \lim_{n\to\infty} y_n^u = 0, \\ \lim_{n\to\infty} \rho_n = +\infty \text{ and } \lim_{n\to\infty} z_n^s = 0, \end{cases}$$

where $y_n = y_n^s \oplus y_n^u \in E^s \oplus E^u$ and $z_n = z_n^s \oplus z_n^u \in E^s \oplus E^u$. (Indeed, for any $\nu > 0$ the point p does not belong to the closure of the set $\{\varphi_t(x),\ t \geqslant 0,\ x \in U^s \text{ and } x^u \geqslant \nu\}$. The convergence of the sequence $(x_n)_n$ to p therefore implies the convergence of $(y_n^u)_n$ to 0, hence that $(r_n)_n$ tends to $+\infty$.)

Define, for any point $y \in U^s$, a $(k + l - 1)$-plane F_y included in Π as follows:

$$F_y = ((X(y)^\perp \cap E^s) \oplus E^u) \cap \Pi.$$

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

Let us consider, for any point $y \in U^s$, the projection $\pi(y) : N_y \cap \Pi \to F_y$ parallel to the vector field $X(y)$. By definition of U^s, for any point $y \in U^s$, $\text{Angle}(X(y), E^s) \leqslant \theta$.

Therefore, up to a choice of θ small enough,

$$\text{there exists } \epsilon > 0, \text{for all } y \in U^s, \quad \text{Angle}(N_y \cap \Pi, F(y)) < \epsilon.$$

This means that $\pi(y)$ and $\pi(y)^{-1}$ have (uniformly) bounded norms:

$$\text{there exists } C > 1, \text{for all } y \in U^s, \quad \frac{1}{C} \leqslant \|\pi(y)\| \leqslant C,$$

with

$$\lim_{\theta \to 0} C \to 1.$$

Thus, for any $w \in \pi(y_n)^{-1}(E^u)$, we know that $|w|$ has an exponential growth almost equal to the exponential growth of vectors of E^u (that is, almost maximal) during a time r_n.

Indeed, for any point $y \in U^s$, and any time $r > 0$ such that (for all $\rho \in [0, r]$, $\varphi_\rho(y) \in U^s$), we can write: for all $w \in \pi(y_n)^{-1}(E^u)$, there exists $v^u \in E^u$,

$$\begin{aligned}
\frac{|P_r(y)w|}{|w|} &= \frac{|P_r(y) \circ \pi^{-1}(y)v^u|}{|\pi^{-1}(y)v^u|} \\
&\geqslant \frac{1}{C} \frac{|P_r(y) \circ \pi^{-1}(y)v^u|}{|v^u|} \\
&\geqslant \frac{1}{C^2} \frac{|T_y\varphi_r \circ \pi^{-1}(y)v^u|}{|v^u|} \\
&\geqslant \frac{\exp(\lambda^u r)}{C^3}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{r} \ln\left(\frac{|P_r(y)w|}{|w|}\right) &\geqslant \frac{1}{r} \ln(\|T_y\varphi_{r|E^u}\|) - \frac{3}{r} \ln(C) \\
&\geqslant \frac{1}{r} \ln(\|P_r(y)_{|N^u(y)}\|) - \frac{3}{r} \ln(C).
\end{aligned}$$

Thus, there exists an integer $N_1 > 0$ such that, for any integer $n \geqslant N_1$, any vector $w \in \pi(y_n)^{-1}E^u$ satisfies inequality (5). Moreover, the space E^u (hence the space $\pi^{-1}(y_n)E^u$) has dimension l.

Considering the time-reversed flow and the projection $\pi'(z)$ parallel to X defined, for any $z \in U^u$, from $N(z) \cap \Pi$ to $(X(z)^\perp \cap E^u) \oplus E^s$, we get, by a similar method, the existence of an integer $N_2 > 0$, such that, for any integer $n \geqslant N_2$, any vector $w' \in \pi(z_n)'^{-1}(E^s)$ satisfies the inequality (6). However, the space E^s (thus the space $\pi(z_n)'^{-1}E^s$) has dimension k.

Since the space $N_{z_n} \cap \Pi$ has only dimension $k + l - 1$, we infer that, for any integer $n \geqslant \sup(N_1, N_2)$, there exists a vector

$$w \in (P_{-\tau_n - \tau'_n}(z_n) \circ \pi(z_n)'^{-1}E^s) \cap (\pi(y_n)^{-1}E^u) \subset N_{y_n} \cap \Pi$$

which simultaneously satisfies *neither* inequality (5) *nor* (6), which contradicts Lemma 4.4.

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press

REFERENCES

[1] C. I. Doering. Persistently transitive vector fields on three-dimensional manifolds. *Proceedings on Dynamical Systems and Bifurcation Theory* **160** (1987), 59–89.

[2] R. Mañé. An ergodic closing lemma. *Ann. of Math.* (2) **116** (1982), 503–540.

[3] M. Shub. *Topologically Transitive Diffeomorphisms of T^4 (Lecture Notes in Mathematics, 206)*. Springer, 1971, pp. 39–40.

[4] R. Mañé. Contributions to the stability conjecture. *Topology* **17** (1978), 383–396.

[5] M. Li, S. Gan and L. Wen. Robustly transitive singular sets via approach of an extended linear Poincaré flow. *Discrete Contin. Dyn. Syst.* **13**(2) (2005), 239–269.

[6] C. Bonatti, L. J. Díaz and E. R. Pujals. A C^1 generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. *Ann. of Math.* (2) **158** (2003), 355–418.

[7] C. Bonatti, N. Gourmelon and T. Vivier. Perturbations of the derivative along periodic orbits. *Prépublication IMB Dijon* **376** (2004).

[8] J. Palis and W. de Melo. *Geometric Theory of Dynamical Systems*. Springer, Berlin, 1982.

https://doi.org/10.1017/S0143385705000581 Published online by Cambridge University Press