Probability in the Engineering and Informational Sciences, **29**, 2015, 329–343.
doi:10.1017/S0269964815000029

RELATIONS BETWEEN STOCHASTIC ORDERINGS AND GENERALIZED STOCHASTIC PRECEDENCE

EMILIO DE SANTIS, FABIO FANTOZZI AND FABIO SPIZZICHINO
Department of Mathematics, University La Sapienza
Rome
E-mails: desantis@mat.uniroma1.it; fantozzi@mat.uniroma1.it; spizzich@mat.uniroma1.it

The concept of *stochastic precedence* between two real-valued random variables has often emerged in different applied frameworks. In this paper, we analyze several aspects of a more general, and completely natural, concept of stochastic precedence that also had appeared in the literature. In particular, we study the relations with the notions of stochastic ordering. Such a study leads us to introducing some special classes of bivariate copulas. Motivations for our study can arise from different fields. In particular, we consider the frame of Target-Based Approach in decisions under risk. This approach has been mainly developed under the assumption of stochastic independence between "Prospects" and "Targets". Our analysis concerns the case of stochastic dependence.

1. INTRODUCTION

Let X_1 and X_2 be two real random variables defined on a same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will denote by F the joint distribution function and by G_1, G_2 their marginal distribution functions, respectively. For the sake of notational simplicity, we will initially concentrate our attention on the case when G_1, G_2 belong to $\mathcal{G}$, the class the probability distribution functions on the real line, that are continuous and strictly increasing in the domain where they are positive and smaller than one. As we shall see later, we can also consider more general cases, but the present restriction allows us to simplify the formulation and the proofs of our results. In order to account for some cases of interest with $\mathbb{P}(X_1 = X_2) > 0$, we will not assume that the distribution function F is absolutely continuous.

The random variable X_1 is said to *stochastically precede* X_2 if $\mathbb{P}(X_1 \leq X_2) \geq 1/2$, written $X_1 \preceq_{\mathrm{sp}} X_2$. The interest of this concept for applications has been pointed out several times in the literature (see in particular [1,3,13]). We recall the reader's attention on the fact that stochastic precedence is not transitive, and then it does not define a partial order (as, on the contrary, stochastic orderings do). However, it can be considered in some cases as an interesting condition, possibly alternative to the usual *stochastic ordering* $X_1 \preceq_{\mathrm{st}} X_2$, defined by the condition $G_1(t) \geq G_2(t)$, $\forall t \in \mathbb{R}$; see [19].

When X_1, X_2 are independent the implication $X_1 \preceq_{\mathrm{st}} X_2 \Rightarrow X_1 \preceq_{\mathrm{sp}} X_2$ holds (see [1]). It is also easy to find several other examples of bivariate probability models where the same implication holds. For instance the condition $X_1 \preceq_{\mathrm{st}} X_2$ even entails $\mathbb{P}(X_1 \leq X_2) = 1$

© Cambridge University Press 2015 0269-9648/15 $25.00

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

when X_1, X_2 are *comonotonic* (see e.g., [15]), that is, when $\mathbb{P}(X_2 = G_2^{-1}(G_1(X_1))) = 1$, with $G_1, G_2 \in \mathcal{G}$. On the other hand, cases of stochastic dependence can be found where the implication $X_1 \preceq_{\text{st}} X_2 \Rightarrow X_1 \preceq_{\text{sp}} X_2$ fails. A couple of examples will be presented in Section 3; see also Proposition 7 in Section 2. On the other hand, the frame of words' occurrences produces, in a natural way, examples in the same direction; see, for example, [7]. This circumstance leads to some paradoxes, especially in the context of economic decisions, as it was early pointed out, for example since 1954 by Savage (see [16]).

In this paper, we analyze a generalized concept of stochastic precedence, that have been considered in the literature (see in particular [21], see also [17]).

DEFINITION 1. *For a given $\gamma \in [0,1]$, we say that X_1 stochastically precedes X_2 at level γ if $\mathbb{P}(X_1 \le X_2) \ge \gamma$. This will be written $X_1 \preceq_{\text{sp}}^{(\gamma)} X_2$.*

Several arguments along the paper will be based on the concept of bivariate copula (see, e.g., [10,15]) and the class of all bivariate copulas will be denoted by $\mathcal{C}$. For a given bivariate distribution function F with margins G_1, G_2, a *connecting copula* is any distribution function $C : [0,1]^2 \to [0,1]$ such that

$$F(x_1, x_2) = C(G_1(x_1), G_2(x_2)). \tag{1}$$

It is well known (see, e.g., [15]) that the connecting copula is unique when G_1 and G_2 are continuous.

We will use the notation

$$A := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \le x_2\}, \tag{2}$$

so that we write

$$\mathbb{P}(X_1 \le X_2) = \int_A dF(x_1, x_2) = \int_{\mathbb{R}^2} \mathbb{1}_A(x_1, x_2)\, dF(x_1, x_2). \tag{3}$$

For given $G_1, G_2 \in \mathcal{G}$ and $C \in \mathcal{C}$, we also set

$$\eta(C, G_1, G_2) := \mathbb{P}(X_1 \le X_2), \tag{4}$$

where X_1 and X_2 are random variables with distributions G_1 and G_2 respectively, and connecting copula C. Thus the condition $X_1 \preceq_{\text{sp}}^{(\gamma)} X_2$ can also be written

$$\eta(C, G_1, G_2) \ge \gamma.$$

Suppose now that X_1, X_2 satisfy the condition $X_1 \preceq_{\text{st}} X_2$. As a main purpose of this paper we give a lower bound for the probability $\mathbb{P}(X_1 \le X_2)$ in terms of the stochastic dependence between X_1 and X_2 or, more precisely, in terms of conditions on the integral $\int_{A \cap [0,1]^2} dC$. More specifically, we will analyze different aspects of the special classes of bivariate copulas, defined as follows.

DEFINITION 2. *For $\gamma \in [0,1]$, we denote by $\mathcal{L}_\gamma$ the class of all copulas $C \in \mathcal{C}$ such that*

$$\eta(C, G_1, G_2) \ge \gamma \tag{5}$$

for all $G_1, G_2 \in \mathcal{G}$ with $G_1 \preceq_{\text{st}} G_2$.

Concerning the role of the concept of copula in our study, we point out the following simple facts. Consider the random variables $X_1' = \phi(X_1)$ and $X_2' = \phi(X_2)$ where $\phi : \mathbb{R} \to \mathbb{R}$

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

is a strictly increasing function. Thus $X_1' \preceq_{\mathrm{st}} X_2'$ if and only if $X_1 \preceq_{\mathrm{st}} X_2$, and $X_1' \preceq_{\mathrm{sp}}^{(\gamma)} X_2'$ if and only if $X_1 \preceq_{\mathrm{sp}}^{(\gamma)} X_2$. At the same time the pair (X_1', X_2') also admits the same connecting copula of (X_1, X_2). See also Remark 2.1 below.

The arguments treated in this paper can reveal of interest in different applied fields. Motivations for this study, in particular, had arisen for us in the field of Target-Based Approach (TBA) to utility theory. A further source of interest has been the comparisons among waiting times to occurrences of *words* in random sequences of *letters* from an alphabet; see [7,8]. Apart from the general economic context that we already mentioned above, further applications can arise for example, in the fields of reliability and in the comparison of pools obtained by two opposite coalitions in political elections.

More precisely the structure of the paper is as follows. In Section 2, we analyze the main aspects of the classes $\mathcal{L}_\gamma$ and present a related characterization. Some further basic properties will be detailed in Section 3, where a few examples will be also presented. Finally, in Section 4, we will briefly review Target-Based utilities, pointing out the relations with our work, in the case of stochastic dependence between targets and prospects. Connections with the field of times to words' occurrences will be discussed in a subsequent note.

2. A CHARACTERIZATION OF THE CLASS $\mathcal{L}_\gamma$

This section will be devoted to providing a characterization of the classes $\mathcal{L}_\gamma$ (see Theorems 5 and 6) along with related discussions. We start by detailing a few basic properties of the quantities $\eta(C, G_1, G_2)$, for $G_1, G_2 \in \mathcal{G}$ and $C \in \mathcal{C}$. In view of the condition $G_1, G_2 \in \mathcal{G}$ we can use the change of variables $u = G_1(x_1)$, $v = G_2(x_2)$. Thus we can rephrase formula (3) according to the following result.

PROPOSITION 1. *For given $G_1, G_2 \in \mathcal{G}$ and $C \in \mathcal{C}$, one has*

$$\eta(C, G_1, G_2) = \int_{[0,1]^2} \mathbb{1}_A(G_1^{-1}(u), G_2^{-1}(v))\, dC(u, v). \tag{6}$$

The use of next Proposition is twofold: it will be applied both for characterizing the classes $\mathcal{L}_\gamma$ and for establishing lower and upper bounds on the quantity $\eta(C, G_1, G_2)$.

PROPOSITION 2. *Let $G_1, G_1', G_2, G_2' \in \mathcal{G}$. Then*

$$G_2 \preceq_{\mathrm{st}} G_2' \Rightarrow \eta(C, G_1, G_2) \leq \eta(C, G_1, G_2'); \tag{7}$$

$$G_1 \preceq_{\mathrm{st}} G_1' \Rightarrow \eta(C, G_1, G_2) \geq \eta(C, G_1', G_2). \tag{8}$$

PROOF: We only prove the implication (7), since the proof of (8) is analogous. By hypothesis, and since $G_1, G_2' \in \mathcal{G}$, for each $x \in (0, 1)$, one has

$$G_2^{-1}(x) \leq G_2'^{-1}(x).$$

Therefore

$$(G_1^{-1}(x), G_2^{-1}(x)) \in A \Rightarrow (G_1^{-1}(x), G_2'^{-1}(x)) \in A.$$

Hence, the proof can be concluded by recalling (6). ∎

From Proposition 2, in particular we get

$$\eta(C, G, G) \leq \eta(C, G', G) \quad \text{and} \quad \eta(C, G, G) \leq \eta(C, G, G''), \tag{9}$$

for any choice of $G, G', G'' \in \mathcal{G}$ such that $G' \preceq_{\mathrm{st}} G \preceq_{\mathrm{st}} G''$.

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

A basic fact in the analysis of the classes $\mathcal{L}_\gamma$ is that the quantities of the form $\eta(C, G, G)$ only depend on the copula C. More formally we state the following result.

PROPOSITION 3. *For any pair of distribution functions $G', G'' \in \mathcal{G}$, one has*

$$\eta(C, G', G') = \eta(C, G'', G''), \tag{10}$$

PROOF: By recalling (6) one obtains

$$\int_{[0,1]^2} \mathbb{1}_A(G'^{-1}(u), G'^{-1}(v))\, dC(u,v) = \int_{[0,1]^2} \mathbb{1}_A(G''^{-1}(u), G''^{-1}(v))\, dC(u,v) \tag{11}$$

because

$$\mathbb{1}_A(G'^{-1}(u), G'^{-1}(v)) = \mathbb{1}_A(G''^{-1}(u), G''^{-1}(v)) = \mathbb{1}_A(u,v),$$

so equality in (10) is proved. ∎

Remark 2.1. Notice that the following more general consequence of formula (6) also holds:

$$\eta(C, G_1', G_2') = \eta(C, G_1'', G_2'')$$

whenever $G_i''(x) = G_i'(\phi(x))$, $i = 1, 2$, for some continuous and strictly increasing $\phi : \mathbb{R} \to \mathbb{R}$.

As a consequence of Proposition 3, for $G \in \mathcal{G}$, we can introduce the symbol

$$\eta(C) := \eta(C, G, G), \tag{12}$$

and, by letting $G_1 = G_2 = G$ in (6), write

$$\eta(C) = \int_{A \cap [0,1]^2} dC. \tag{13}$$

From Proposition 2 and from inequalities in (9), we obtain

PROPOSITION 4. *For $G_1, G_2 \in \mathcal{G}$ the following implication holds*

$$G_1 \preceq_{\mathrm{st}} G_2 \Rightarrow \eta(C) \leq \eta(C, G_1, G_2).$$

We then see that the quantity $\eta(C)$ characterizes the classes $\mathcal{L}_\gamma$, $0 \leq \gamma \leq 1$, in fact we can state the following result.

THEOREM 5. *For any given $\gamma \in [0,1]$, $C \in \mathcal{L}_\gamma$ if and only if $\eta(C) \geq \gamma$.*

We thus have

$$\mathcal{L}_\gamma = \{C \in \mathcal{C} : \eta(C) \geq \gamma\} \tag{14}$$

and we can also write

$$\eta(C) = \inf_{G_1, G_2 \in \mathcal{G}} \{\eta(C, G_1, G_2) : G_1 \preceq_{\mathrm{st}} G_2\}. \tag{15}$$

In other words, the infimum in formula (15) is a minimum and it is attained when $G_1 = G_2 \in \mathcal{G}$.

Remark 2.2. Theorem 5 concerns the possible values of the probability $\mathbb{P}(X \leq Y)$ for a fixed copula C and for different pairs of marginal distributions G_1, G_2 with $G_1 \preceq_{\mathrm{st}} G_2$. In an economical setting, a "dual" result is pointed out in the recent paper [11], where the

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

range of possible values of $\mathbb{P}(X \leq Y)$ is concerned, for a fixed pair (G_1, G_2) and for different copulas C. For general results in this direction, from a mathematical viewpoint, see [20].

We notice furthermore that the definition of $\eta(C, G_1, G_2)$ can be extended to the case when $G_1, G_2 \in D(\mathbb{R})$, the space of all distribution functions on $\mathbb{R}$. The class $\mathcal{G}$ has however a special role in the present setting, as it is shown by the following result.

THEOREM 6. *Let $C \in \mathcal{C}$, $G, H \in D(\mathbb{R})$ with $G \preceq_{\mathrm{st}} H$, then $\eta(C) \leq \eta(C, G, H)$.*

PROOF: Consider two sequences $(G_n : n \in \mathbb{N})$, $(H_n : n \in \mathbb{N})$ such that $G_n, H_n \in \mathcal{G}$ and $G_n \xrightarrow{w} G$, $H_n \xrightarrow{w} H$. By applying Theorem 2 in [18], we obtain that $C(G_n, H_n) \xrightarrow{w} C(G, H)$.

Take the new sequence $(\tilde{H}_n : n \in \mathbb{N})$ where $\tilde{H}_n(x) := \min\{G_n(x), H_n(x)\}$. We notice that $\tilde{H}_n \in \mathcal{G}$, moreover $G_n \preceq_{\mathrm{st}} \tilde{H}_n$ and $\tilde{H}_n \xrightarrow{w} H$. This implies $C(G_n, \tilde{H}_n) \xrightarrow{w} C(G, H)$.

Now, using the standard characterization of weak convergence on separable spaces (see [2] Theorem 6.3 p. 67),

$$\limsup_{n\to\infty} \int_B d\tilde{F}_n \leq \int_B dF,$$

for any closed set $B \subset \mathbb{R}^2$, where $F = C(G, H)$ and $\tilde{F}_n = C(G_n, \tilde{H}_n)$. Taking the closed set A defined in (2) one has

$$\eta(C) \leq \limsup_{n\to\infty} \int_A d\tilde{F}_n \leq \int_A dF = \eta(C, G, H). \tag{16}$$

■

Remark 2.3. Theorem 6 shows that the minimum of $\eta(C, G, H)$, for $G, H \in D(\mathbb{R})$, is attained at (C, L, L), for any $L \in \mathcal{G}$. This result allows us to replace the class $\mathcal{G}$ with $D(\mathbb{R})$ in the expression of $\mathcal{L}_\gamma$ given in (15). We notice furthermore that one can have $\eta(C, G', G') \neq \eta(C, G'', G'')$ when G', G'' are in $D(\mathbb{R})$.

Concerning the classes $\mathcal{L}_\gamma$, we also define

$$\mathcal{B}_\gamma := \{C \in \mathcal{C} \mid \eta(C) = \gamma\}, \tag{17}$$

so that

$$\mathcal{L}_\gamma = \bigcup_{\gamma' \geq \gamma} \mathcal{B}_{\gamma'}.$$

We now show that the classes $\mathcal{B}_\gamma$, with $\gamma \in [0, 1]$, are all non-empty. Several natural examples might be produced to this purpose. We fix attention on a simple example built in terms of the random variables X_1, $X_2^{(\gamma)}$ defined as follows. On the probability space $([0, 1], \mathcal{B}[0, 1], \lambda)$, where λ denotes the Lebesgue measure, we take

$$X_1(\omega) = \omega \quad \text{and} \quad X_2^{(\gamma)}(\omega) = \begin{cases} \omega + 1 - \gamma & \text{if } \omega \in [0, \gamma], \\ \omega - \gamma & \text{if } \omega \in (\gamma, 1]. \end{cases} \tag{18}$$

As it happens for X_1, also the distribution of $X_2^{(\gamma)}$ is uniform in $[0, 1]$ for any $\gamma \in [0, 1]$ and the connecting copula of $X_1, X_2^{(\gamma)}$, that is then uniquely determined, will be denoted by C_γ.

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

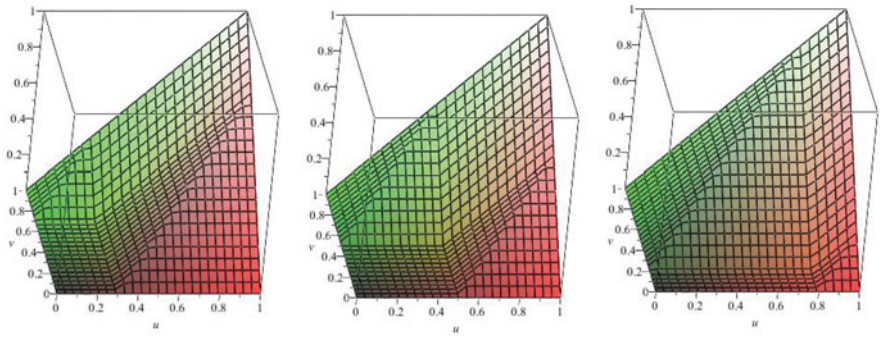


FIGURE 1. (Color online) Copulas from the family C_γ with $\gamma = 0.3, 0.5, 0.8$, respectively.

PROPOSITION 7. *For any* $\gamma \in (0,1]$, *one has*

(i) $C_\gamma \in \mathcal{B}_\gamma$.
(ii) $C_\gamma(u,v) = \min\{u, v, \max\{u-\gamma, 0\} + \max\{v+\gamma-1, 0\}\}$.

PROOF: (i) First we notice that $\mathbb{P}(X_1 \le X_2^{(\gamma)}) = \gamma$. In fact

$$\mathbb{P}(X_1 \le X_2^{(\gamma)}) = \mathbb{P}(X_1 \le X_1 + 1 - \gamma, X_1 \le \gamma) + \mathbb{P}(X_1 \le X_1 - \gamma, X_1 > \gamma) = \gamma.$$

Whence, $\eta(C_\gamma) = \mathbb{P}(X_1 \le X_2^{(\gamma)}) = \gamma$, since both the distributions of $X_1, X_2^{(\gamma)}$ belong to $\mathcal{G}$.
(ii) For $x_1, x_2 \in [0,1]$ we can write

$$\begin{aligned} F_{X_1, X_2^{(\gamma)}}(x_1, x_2) &:= \mathbb{P}(X_1 \le x_1, X_2^{(\gamma)} \le x_2) \\ &= \mathbb{P}(X_1 \le x_1, X_1 + 1 - \gamma \le x_2,\ X_1 \le \gamma) \\ &\quad + \mathbb{P}(X_1 \le x_1, X_1 \le x_2 + \gamma,\ X_1 > \gamma) \\ &= \mathbb{P}(X_1 \le \min\{x_1, x_2 + \gamma - 1, \gamma\}) + \mathbb{P}(\gamma < X_1 \le \min\{x_1, x_2 + \gamma\}) \\ &= \max\{\min\{x_1, x_2 + \gamma - 1, \gamma\}, 0\} + \max\{\min\{x_1, x_2 + \gamma\} - \gamma, 0\} \\ &= \min\{x_1, x_2, \max\{x_1 - \gamma, 0\} + \max\{x_2 + \gamma - 1, 0\}\}. \end{aligned}$$

Since both the marginal distributions of X_1 and $X_2^{(\gamma)}$ are uniform, it follows that

$$C_\gamma(u,v) = \min\{u, v, \max\{u-\gamma, 0\} + \max\{v+\gamma-1, 0\}\}.$$

■

The copulas C_γ have also been considered for different purposes in the literature (see, e.g., [9] and references therein). We point out that the identity $\eta(C_\gamma) = \gamma$ (for $\gamma \in (0,1]$) could also have been obtained directly from formula (13). In this special case, the computation of $\mathbb{P}(X_1 \le X_2)$ is however straightforward.

As an immediate consequence of Proposition 7 we have that $\mathcal{L}_{\gamma'}$ is strictly contained in $\mathcal{L}_\gamma$ for any $0 \le \gamma < \gamma' \le 1$. We notice furthermore that $\mathcal{L}_0 = \mathcal{C}$ and $\mathcal{L}_1 = \{C \in \mathcal{C} : \int_{A \cap [0,1]^2} dC = 1\} \ne \emptyset$.

Graphs of C_γ for different values of γ are provided in Figure 1.

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

3. FURTHER PROPERTIES OF $\mathcal{L}_\gamma$ AND EXAMPLES

We start this section by analyzing further properties of the classes $\mathcal{L}_\gamma$ that can also shed light on the relations between stochastic precedence and stochastic orderings. First we notice that the previous Definition 2 has been formulated in terms of the usual stochastic ordering $\preceq_{\mathrm{st}}$. However, similar results can also be obtained for other important concepts of stochastic ordering that have been considered in the literature (such as the *hazard rate*, the *likelihood ratio*, and the *mean residual life* orderings, see [19]).

Let us fix, in fact, a stochastic ordering $\preceq_*$ different from $\preceq_{\mathrm{st}}$. Definition 2 can be modified by replacing therein $\preceq_{\mathrm{st}}$ with $\preceq_*$ and this operation leads us to a new class of copulas that we can denote by $\mathcal{L}_\gamma^{(*)}$. More precisely, we set

$$\mathcal{L}_\gamma^{(*)} := \{C \in \mathcal{C} : \eta(C, G_1, G_2) \geq \gamma,\ \forall\, G_1, G_2 \in \mathcal{G}\ s.t.\ G_1 \preceq_* G_2\} \tag{19}$$

or equivalently

$$\mathcal{L}_\gamma^{(*)} = \{C \in \mathcal{C} : \eta^*(C) \geq \gamma\}, \tag{20}$$

where

$$\eta^*(C) := \inf_{G_1, G_2 \in \mathcal{G}} \{\eta(C, G_1, G_2) : G_1 \preceq_* G_2\}. \tag{21}$$

For given $\gamma \in (0, 1)$, one might wonder about possible relations between $\mathcal{L}_\gamma^{(*)}$ and $\mathcal{L}_\gamma$. Actually one has the following result, which will be formulated for binary relations (not necessarily stochastic orderings) over the space $D(\mathbb{R})$.

PROPOSITION 8. *Let $\preceq_*$ be a relation satisfying*

(a) *for any $G \in D(\mathbb{R})$ one has $G \preceq_* G$;*

(b) *for any $G_1, G_2 \in D(\mathbb{R})$ with $G_1 \preceq_* G_2$ one has $G_1 \preceq_{\mathrm{st}} G_2$.*

Then $\mathcal{L}_\gamma = \mathcal{L}_\gamma^{(*)}$.

PROOF: In view of (b), one has $\eta(C) \leq \eta^*(C)$. In fact, both the quantities $\eta(C)$ and $\eta^*(C)$ are obtained as an infimum of the same functional and, compared with η, the quantity η^* is an infimum computed on a smaller set.

Due to (a), however, $\eta(C)$ and $\eta^*(C)$ are both obtained, in (15) and (21) respectively, as minima attained on a same point (G, G). We can then conclude that $\mathcal{L}_\gamma^{(*)} = \mathcal{L}_\gamma$. ∎

Concerning Proposition 8 we notice that, for example, the hazard rate and the likelihood ratio orderings, $\preceq_{\mathrm{hr}}$ and $\preceq_{\mathrm{lr}}$, both satisfy conditions (a) and (b).

As far as applied problems are dealt with, it can be relevant to remark that imposing stochastic orderings stronger than $\preceq_{\mathrm{st}}$ does not necessarily increase the level of stochastic precedence.

For the sake of notational simplicity we come back to considering the usual stochastic ordering $\preceq_{\mathrm{st}}$ and the classes $\mathcal{L}_\gamma$.

For what follows, it is now convenient also to consider the quantities $\xi(C, G_1, G_2)$ and $\xi(C)$ defined as follows:

$$\xi(C, G_1, G_2) := \mathbb{P}(X_1 = X_2), \tag{22}$$

$$\xi(C) := \xi(C, G, G), \tag{23}$$

where X_1 and X_2 are random variables with distributions $G_1, G_2 \in \mathcal{G}$, respectively, and connecting copula C.

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

For a given bivariate model we have considered so far the quantities $\eta(C)$ with C denoting the connecting copula. In what follows, we point out the relations among $\eta(C)$, $\eta(\widehat{C})$, $\eta(C^t)$, where $\widehat{C}$ and C^t denote the *survival copula* and the *transposed copula* of C, respectively. The transposed copula C^t is defined by

$$C^t(u,v) := C(v,u), \tag{24}$$

so that if C is the connecting copula of the pair (X_1, X_2), then C^t is the copula of the pair (X_2, X_1). Whence, if X_1 and X_2 have the same distribution $G \in \mathcal{G}$, then

$$\eta(C^t) = \mathbb{P}(X_2 \leq X_1).$$

On the other hand, the notion of survival copula of the pair (X_1, X_2), which comes out as natural when considering pairs of non-negative random variables, is defined by the equation

$$\overline{F}_{X_1,X_2}(x_1,x_2) = \widehat{C}\left(\overline{G}_1(x_1), \overline{G}_2(x_2)\right), \tag{25}$$

with $\overline{G}_1$ and $\overline{G}_2$ respectively denoting the marginal survival functions:

$$\overline{G}_1(x_1) = \mathbb{P}(X_1 > x_1), \quad \overline{G}_2(x_2) = \mathbb{P}(X_2 > x_2).$$

The relationship between the survival copula $\widehat{C}$ of (X_1, X_2) and the connecting copula C is given by (see [15])

$$\widehat{C}(u,v) = u + v - 1 + C(1-u, 1-v). \tag{26}$$

The following result shows the relations tying the different quantities $\eta(C)$, $\eta(\widehat{C})$, $\eta(C^t)$. The proof is easy and will be omitted.

Proposition 9. *Let $C \in \mathcal{C}$. The following relation holds:*

$$\eta(\widehat{C}) = \eta(C^t) = 1 - \eta(C) + \xi(C).$$

A basic property of the classes $\mathcal{L}_\gamma$ and $\mathcal{B}_\gamma$ is given by the following result.

Proposition 10. *For $\gamma \in [0,1]$, the classes $\mathcal{L}_\gamma$, $\mathcal{L}_\gamma^c = \mathcal{C} \setminus \mathcal{L}_\gamma$, and $\mathcal{B}_\gamma$ are convex.*

Proof: We consider two bivariate copulas $C_1, C_2 \in \mathcal{L}_\gamma$ and a convex combination of them, that is, take $\alpha \in (0,1)$ and $C := \alpha C_1 + (1-\alpha)C_2$. We show that $C \in \mathcal{L}_\gamma$, indeed

$$\eta(C) = \int_A dC(u,v) = \alpha \int_A dC_1(u,v) + (1-\alpha) \int_A dC_2(u,v) = \alpha\eta(C_1) + (1-\alpha)\eta(C_2).$$

Since $\eta(C_1), \eta(C_2)$ are larger or equal than γ then $\eta(C) \geq \gamma$, whence $\mathcal{L}_\gamma$ is convex. Now one can use the same argument in order to show that $\mathcal{L}_\gamma^c$ and $\mathcal{B}_\gamma$ are convex as well. ∎

An immediate application of Proposition 10 concerns the case when, given a random parameter Θ, all the connecting copulas of the conditional distributions of (X_1, X_2) given Θ belong to a same class $\mathcal{L}_\gamma$. Proposition 10 in fact, guarantees that the copula of (X_1, X_2) belongs to $\mathcal{L}_\gamma$ as well.

Some aspects of the definitions and results given so far will be demonstrated here by presenting a few examples. We notice that, as shown by Proposition 7, the condition $\preceq_{\mathrm{st}}$ does not imply $\preceq_{\mathrm{sp}}^{(\gamma)}$, with $\gamma \in (0,1)$. For the special case $\gamma = 1/2$ we now present an example of applied interest.

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

Example 1. Let X, Y be two non-negative random variables, where Y has an exponential density $f_Y(y)$ with failure rate λ and where stochastic dependence between X and Y is described by a "load-sharing" dynamic model as follows: conditionally on $(Y = y)$, the failure rate of X amounts to $\alpha = 1$ for $t < y$ and to β for $t > y$. We assume $1 < \lambda < \beta < 1 + \lambda$. This position gives rise to a jointly absolutely continuous distribution for which we can consider

$$\mathbb{P}(X > x|Y = y) := \int_x^{+\infty} f_{X,Y}(t, y)dt,$$

$f_{X,Y}$ denoting the joint density of X, Y. As to the survival function of X, for any fixed value $x > 0$, we can argue as follows.

$$\begin{aligned}
\overline{F}_X(x) := \mathbb{P}(X > x) &= \int_0^{+\infty} \mathbb{P}(X > x|Y = y) f_Y(y)dy \\
&= \int_0^x \mathbb{P}(X > x|Y = y) f_Y(y)dy + \int_x^{+\infty} \mathbb{P}(X > x|Y = y) f_Y(y)dy \\
&= \int_0^x \mathbb{P}(X > y|Y = y)\mathbb{P}(X > x|Y = y, X > y) f_Y(y)dy + \int_x^{+\infty} e^{-x} f_Y(y)dy \\
&= \left(1 - \frac{\lambda}{1 + \lambda - \beta}\right) e^{-(1+\lambda)x} + \frac{\lambda}{1 + \lambda - \beta} e^{-\beta x} \leq e^{-\lambda x}, \quad \forall x > 0.
\end{aligned}$$

We can then conclude that $X \preceq_{\text{st}} Y$. On the other hand, the same position gives also rise to $\mathbb{P}(X \leq Y) = 1/(1 + \lambda) < 1/2$.

Further examples in the same direction can for example, arise in the *change-point* models studied in [6].

Next example shows that for three random variables T, X', X'', the implication $T \preceq_{\text{st}} X' \preceq_{\text{st}} X'' \Rightarrow \mathbb{P}(T \leq X'') \leq \mathbb{P}(T \leq X')$ can fail when the connecting copulas of (T, X') and (T, X'') are different.

Example 2. Let $Y_1, \ldots, Y_5$ be i.i.d. random variables, with a continuous distribution and defined on a same probability space, and set

$$T = \min(Y_1, Y_2),\ X' = \max(Y_1, Y_2), \quad X'' = \max(Y_3, Y_4, Y_5).$$

Thus $X' \preceq_{\text{st}} X''$, but $\mathbb{P}(T \leq X') = 1$ and $\mathbb{P}(T \leq X'') < 1$.

Remark 3.1. For some special types of copulas C, the computation of $\eta(C, G_1, G_2)$ can be carried out directly, in terms of probabilistic arguments, provided the distributions G_1, G_2 belong to some appropriate class. For such copulas, Proposition 2 can be used to directly obtain inequalities for $\eta(C, H_1, H_2)$ even if H_1, H_2 do not belong to $\mathcal{G}$, provided, for example, that $H_1 \preceq_{\text{st}} G_1$, $G_2 \preceq_{\text{st}} H_2$ and $G_1, G_2 \in \mathcal{G}$.

This circumstance in particular manifests for the models considered in the subsequent examples. Next example will be devoted to bivariate Gaussian models, that is, to a relevant case of symmetric copulas.

Example 3. **Gaussian copulas.**

The family of bivariate Gaussian copulas (see, e.g., [15]) is parameterized by the correlation coefficient $\rho \in (-1, 1)$. The corresponding copula $C^{(\rho)}$ is absolutely continuous and symmetric. Thus $\eta(C^{(\rho)}) = 1/2$ and it does not depend on ρ. For fixed pairs of distributions

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

G_1, G_2, on the contrary, the quantity $\eta(C^{(\rho)}, G_1, G_2)$ does actually depend on ρ, besides on G_1 and G_2. This class provides the most direct instance of the situation outlined in the above Remark 3.1. The value for $\eta(C^{(\rho)}, G_1, G_2)$ is in fact immediately obtained when G_1, G_2 are Gaussian. Let X_1, X_2 denote Gaussian random variables with connecting copula $C^{(\rho)}$ and parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$. Since the random variable $Z = X_1 - X_2$ is distributed according to $\mathcal{N}(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)$ we can write

$$\eta(C^{(\rho)}, G_1, G_2) = \mathbb{P}(Z \leq 0) = \Phi\left(\frac{\mu_2 - \mu_1}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}\right). \tag{27}$$

We recall that, when $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for $i = 1, 2$, the necessary and sufficient condition for $X_1 \preceq_{\text{st}} X_2$ is $\mu_1 \leq \mu_2$ and $\sigma_1 = \sigma_2$ (see, e.g., [1]). In other words, for G_1, G_2 Gaussian, $G_1 \preceq_{\text{st}} G_2$ means $X_1 \preceq_{\text{sp}} X_2$ and $\sigma_1 = \sigma_2$. By using formula (27), with $\sigma_1 = \sigma_2 = \sigma$, we have

$$\eta(C^{(\rho)}, G_1, G_2) = \Phi\left(\frac{\mu_2 - \mu_1}{\sigma\sqrt{2(1-\rho)}}\right). \tag{28}$$

Thus $G_1 \preceq_{\text{st}} G_2 \Rightarrow \eta(C^{(\rho)}, G_1, G_2) \geq 1/2$, as shown by Proposition 4 and Theorem 6. We notice that $\eta(C^{(\rho)}, G_1, G_2)$ is an increasing function of ρ.

Proposition 2 can be extended to obtain, for example, that

$$\eta(C^{(\rho)}, G_1, G_2) \leq \eta(C^{(\rho)}, H_1, H_2),$$

when $H_1 \preceq_{\text{st}} G_1$ and $G_2 \preceq_{\text{st}} H_2$, for $G_1, G_2 \in \mathcal{G}$ and $H_1, H_2 \notin \mathcal{G}$. We then can give inequalities for $\eta(C^{(\rho)}, H_1, H_2)$ in terms of (27), provided that H_1, H_2 are suitably comparable in the $\preceq_{\text{st}}$ sense with Gaussian distributions.

In the cases when $\xi(C) > 0$, we should obviously distinguish between computations of $\mathbb{P}(X_1 \leq X_2)$ and $\mathbb{P}(X_1 < X_2)$, where C is the connecting copula of X_1, X_2. A remarkable case when this circumstance happens is considered in the following example.

Example 4. **Marshall–Olkin Models**

We consider the Marshall–Olkin copulas (see, e.g., [10,12,15]), namely those whose expression is the following:

$$\widehat{C}^{(\alpha_1,\alpha_2)}(u, v) := u\, v\, \min\{u^{-\alpha_1}, v^{-\alpha_2}\}$$

for $0 < \alpha_i < 1$, $i = 1, 2$. We notice that the Marshall–Olkin copula has a singular part that is concentrated on the curve $u^{\alpha_1} = v^{\alpha_2}$ (see also Figure 2). Actually the measure of such a singular component is given by

$$\frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2}.$$

As for the computation of $\eta(\widehat{C}^{(\alpha_1,\alpha_2)})$ we use the expression in (13). By separately considering the curve $u^{\alpha_1} = v^{\alpha_2}$ and the domains where $\widehat{C}^{(\alpha_1,\alpha_2)}$ is absolutely continuous, we obtain

$$\eta(\widehat{C}^{(\alpha_1,\alpha_2)}) = \frac{1}{2 - \alpha_1 \wedge \alpha_2}\left(1 - \frac{(\alpha_1 - \alpha_1 \wedge \alpha_2)(\alpha_1 \wedge \alpha_2)}{\alpha_1 - \alpha_2}\right).$$

Consider now the copulas

$$C^{(\alpha_1,\alpha_2)}(u, v) := \widehat{C}^{(\alpha_1,\alpha_2)}(1 - u, 1 - v) + u + v - 1.$$

We will see that the value of $\eta(C^{(\alpha_1,\alpha_2)}, G_1, G_2)$ directly follows from probabilistic arguments, provided G_1, G_2 are exponential distributions with appropriate parameters. Let in

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

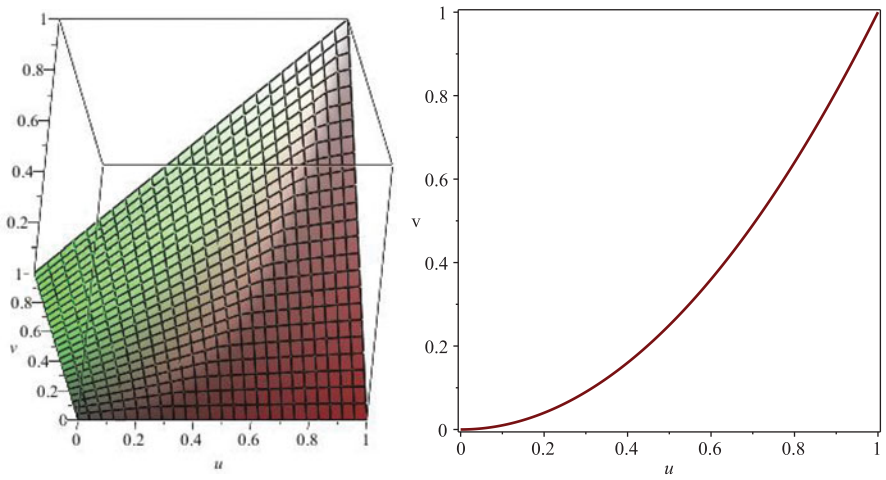


FIGURE 2. (color online) Marshall–Olkin Copula (left) and graph of $u^{\alpha_1} = v^{\alpha_2}$ (right). Special case $\alpha_1 = 0.4$, $\alpha_2 = 0.2$.

fact V, W and Z be three random variables, independent and exponentially distributed with parameters $\mu_1 = 1/\alpha_1 - 1$, $\mu_2 = 1/\alpha_2 - 1$ and $\mu = 1$, respectively. The new random variables

$$X_1 := V \wedge Z, \quad X_2 := W \wedge Z,$$

have survival copula $\widehat{C}^{(\alpha_1,\alpha_2)}$, connecting copula $C^{(\alpha_1,\alpha_2)}$, and exponential distributions $G_1^{(\alpha_1)}$ and $G_2^{(\alpha_2)}$, with parameters $1/\alpha_1$ and $1/\alpha_2$, respectively. We now proceed with the computation of

$$\eta(C^{(\alpha_1,\alpha_2)}, G_1^{(\alpha_1)}, G_2^{(\alpha_2)}) = \mathbb{P}(X_1 \leq X_2).$$

We can write

$$\begin{aligned}\xi(C^{(\alpha_1,\alpha_2)}, G_1^{(\alpha_1)}, G_2^{(\alpha_2)}) &= \mathbb{P}(X_1 = X_2) = \mathbb{P}(Z \leq V \wedge W) \\ &= \frac{1}{\mu_1 + \mu_2 + 1} = \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2},\end{aligned}$$

furthermore

$$\mathbb{P}(X_1 < X_2) = \mathbb{P}(V < W \wedge Z) = \frac{(1-\alpha_1)\,\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2},$$

and finally we obtain

$$\mathbb{P}(X_1 \leq X_2) = \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2}.$$

Then

$$\eta(C^{(\alpha_1,\alpha_2)}, G_1^{(\alpha_1)}, G_2^{(\alpha_2)}) = \frac{\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2}.$$

We now conclude this section with an example showing an extreme case in the direction of Remark 3.1.

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

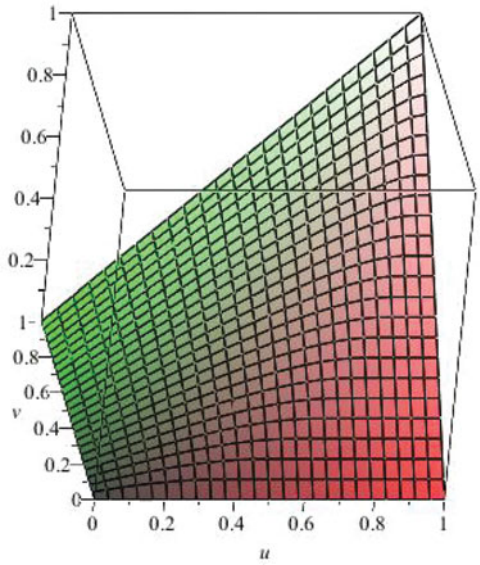


FIGURE 3. (color online) Ordered statistic Copula K.

Example 5. **Copulas of order statistics.**

Let A, B be two i.i.d. random variables with distribution function $G \in \mathcal{G}$ and denote by X_1, X_2 their order statistics, namely $X_1 = \min\{A, B\}, X_2 = \max\{A, B\}$. The distributions of X_1, X_2 depend on G and are respectively given by

$$F_1^{(G)}(x_1) = \mathbb{P}(\min\{X_1, X_2\} \leq x_1) = 2G(x_1) - G(x_1)^2,$$
$$F_2^{(G)}(x_2) = \mathbb{P}(\max\{X_1, X_2\} \leq x_2) = G(x_2)^2\,.$$

The connecting copula of (X_1, X_2), represented in Figure 3, is given by

$$K(u,v) = \begin{cases} 2(1-(1-u)^{1/2})v^{1/2} - (1-(1-u)^{1/2})^2, & \text{if } v \geq (1-(1-u)^{1/2})^2, \\ v, & \text{otherwise.} \end{cases}$$

We have, by definition,

$$\eta(K, F_1^{(G)}, F_2^{(G)}) = 1,$$

and it does not depend on G. We notice, on the other hand, that the computation of $\eta(K) = \eta(K, G, G)$, with $G \in \mathcal{G}$, is to be carried out explicitly, since the pair (G, G) can never appear as the pair of marginal distributions of order statistics. By recalling (6) one obtains

$$\eta(K) = \int_{[0,1]^2} \frac{\mathbf{1}_A(u,v)}{2\sqrt{v}\sqrt{1-u}}\, dv\, du = 2 - \frac{\pi}{2} < \frac{1}{2}.$$

We can extend this example to the case when the connecting copula of A, B is a copula D different from the product copula Π, but still A and B are identically distributed according to a distribution function G. In this case, the connecting copula K of X_1, X_2 depends on D, but again it does not depend on G (see [14] page 478).

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

4. THE TARGET-BASED APPROACH TO DECISIONS UNDER RISK AND THE CLASSES $\mathcal{L}_\gamma$

In this section, we will look at the arguments of the previous sections in the perspective of one-attribute decisions problems under risk and, more in particular, of the related *TBA*. In such problems, a *risky prospect* (or *lottery*) X is nothing else than a real random variable representing, say, the random amount of wealth obtained by an investor as the consequence of an action or of an economic investment. The investor (or Decision Maker) I is supposed to choose one out of many different actions by evaluating and comparing the different probability distributions corresponding to any single prospect. This choice is implemented on the basis of I's attitudes toward risk.

As very well known, in such a frame, the *Expected Utility Principle* first suggests that I describe her/his own attitudes by means of a utility function U ($U : \mathbb{R} \to \mathbb{R}$) and consequently prescribes that any prospect X (with its probability distribution F_X) be evaluated in terms of the expected-utility

$$\mathbb{E}(U(X)) = \int_{\mathbb{R}} U(x) F_X(dx).$$

In the same frame, the *TBA* is based on a different principle. The TBA assumes in fact that the exclusive interest of the investor I, in the use of the amount of wealth X, is concentrated on the possibility of "buying" a specific good (a house, a car, a block of shares of a stock, etc.). The price of such good is a random variable T (the *target*), with a probability distribution F_T. Whence I is for first supposed to specify the target T as a way to describe his/her own attitude with respect to risk. Then I will evaluate any single prospect X in terms of the probability $\mathbb{P}(T \leq X)$. The best prospect will be the one that maximizes $\mathbb{P}(T \leq X)$. Such an approach was proposed by Bordley, LiCalzi, and Castagnoli (see [4,5]). Some related ideas were already around in the economic literature in the past and other interesting developments appeared in the subsequent years, especially for what concerns the multi-attribute setting. Generally T and X may in fact also be vectors.

Here we concentrate attention on the single-attribute case where (T, X) are pairs of real-valued random variables. It is clear then that the objects of central interest in the TBA are, for a fixed target T, the probabilities $\mathbb{P}(T \leq X)$ and the analysis developed in the previous sections can reveal of interest. We assume the existence of regular conditional distributions. In particular, we assume that, for any prospect X, we can determine $v_T^{(X)}(x) := \mathbb{P}(T \leq x | X = x)$, so that we can write

$$\mathbb{P}(T \leq X) = \int_{\mathbb{R}} v_T^{(X)}(x)\, dF_X(x). \tag{29}$$

Before continuing it is useful to look at the special case when X and T are stochastically independent. We can thus write

$$\mathbb{P}(T \leq X) = \int_{\mathbb{R}} v_T^{(X)}(x)\, dF_X(x) = \int_{\mathbb{R}} F_T(x)\, dF_X(x).$$

We notice that, in such a case, $\mathbb{P}(T \leq X)$ can be seen as the expected value of a utility: by considering $U = F_T$ as the utility function, we have

$$\mathbb{E}(U(X)) = \int_{\mathbb{R}} U(x)\, dF_X(x) = \int_{\mathbb{R}} F_T(x)\, dF_X(x) = \mathbb{P}(T \leq X).$$

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

Under the condition of independence, any bounded and right-continuous utility function can thus be seen as the distribution function of a target T, and vice versa. Such an approach gives rise to easily understandable and practically useful interpretations of several notions of utility theory. TBA however becomes, in a sense, more general than the expected utility approach by allowing for stochastic dependence between targets and prospects. In fact the TBA considers more general decision rules, if we admit the possibility of some correlation between the target and the prospects. If X and T are not independent, $v_T^{(X)}(x)$ does not coincide anymore with the distribution function $F_T(x)$ and the integral in the r.h.s. of Eq. (29) cannot be seen as the expected value of a utility, in that the function $v_T^{(X)}(x)$ may turn out to be non-increasing. For further discussion see again [4,5].

We now briefly summarize the arguments of Sections 2 and 3 in the perspective of a decision problem where, for a fixed target T, we aim to rank two different prospects with marginal distributions G_{X_1}, G_{X_2} and with connecting copulas C_1, C_2, corresponding to the pairs (T, X_1) and (T, X_2), respectively.

In the case of independence, a prospect X_2 should be obviously preferred to a prospect X_1 if $X_1 \preceq_{\mathrm{st}} X_2$. In the case of dependence, on the contrary, this comparison is not sufficient anymore. In fact the choice of a prospect X should be based not only on the corresponding distribution F_X, but also on the connecting copula of the pair (T, X).

For fixed C, the quantity $\eta(C, G_T, G_X) = \mathbb{P}(T \leq X)$ is equal to the quantity $\eta(C)$ for all pairs such that $G_T = G_X = G$ with G belonging to $\mathcal{G}$ (see Proposition 3). For $G_T \neq G_X$, the implication $T \preceq_{\mathrm{st}} X \Rightarrow \mathbb{P}(T \leq X) \geq \gamma$ does not necessarily hold (see Proposition 7 Example 1). For two different prospects X_1, X_2, Proposition 2 guarantees that, when $C_1 = C_2 = C$, the condition $G_T \preceq_{\mathrm{st}} G_{X_1} \preceq_{\mathrm{st}} G_{X_2}$ implies $\mathbb{P}(T \leq X_1) = \eta(C, G_T, G_{X_1}) \leq \eta(C, G_T, G_{X_2}) = \mathbb{P}(T \leq X_2)$. As shown by Example 2, when $C_1 \neq C_2$, we can have both the conditions $\eta(C_1, G_T, G_{X_1}) > \eta(C_2, G_T, G_{X_2})$ and $G_T \preceq_{\mathrm{st}} G_{X_1} \preceq_{\mathrm{st}} G_{X_2}$ $(G_{X_1} \neq G_{X_2})$. Concerning the quantities $\eta(C_1, G_T, G_{X_1})$ and $\eta(C_2, G_T, G_{X_2})$, Theorems 5 and 6 show that, for $G_T \preceq_{\mathrm{st}} G_{X_i}$ $(i = 1, 2)$,

$$\mathbb{P}(T \leq X_i) = \eta(C_i, G_T, G_{X_i}) \geq \eta(C_i).$$

Finally, let us consider the case when the only available information about C_1 and C_2 is that $\eta(C_i) \geq \gamma_i$ (i.e., that C_i belongs to the class $\mathcal{L}_{\gamma_i}$). Then a rough and conservative choice between X_1 and X_2 suggests to select X_i with the larger value of γ_i, provided $G_{X_1} \preceq_{\mathrm{st}} G_{X_2}$ or that X_1, X_2 are nearly identically distributed.

All these apparently paradoxical results suggest that the criteria for selection of random variables based only on stochastic orderings are not suitable enough for decision-making problems, such as those described by the TBA, when dependance among variables is present. We have shown, in fact, that the usual stochastic orderings can give results in disagreement with the expected utility concepts expressed by TBA. Furthermore, we explicitly provided examples in which the choice of a prospect which is "better" in the stochastic sense may give worse results in the utility context.

In order to describe his preferences to the best, a DM adopting the Target-Based model will then also need to take into account properties of dependence of the random variables involved in his choices, trough the study of their connecting copulas. To this purpose a deeper analysis of properties of asymmetry for copulas in the classes $\mathcal{L}_\gamma$ can be of interest.

Acknowledgments

The authors like to thank Marco Scarsini (Luiss, Rome) for useful comments. Partially supported by Project "Aspetti di dipendenza stocastica e rischio in modelli probabilistici," University Sapienza, 2014.

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

References

1. Arcones, M.A., Kvam, P.H. & Samaniego, F.J. (2002). Nonparametric estimation of a distribution subject to a stochastic precedence constraint. *Journal of American Statistical Association* 97(457): 170–182.
2. Billingsley, P. (2009). *Convergence of Probability Measures*. Wiley Series in Probability and Statistics. New York: Wiley.
3. Boland, P.J., Singh, H. & Cukic, B. (2004). The stochastic precedence ordering with applications in sampling and testing. *Journal of Applied Probabability* 41(1): 73–82.
4. Bordley, R. & LiCalzi, M. (2000). Decision analysis using targets instead of utility functions. *Decisions in Economics and Finance* 23(1): 53–74.
5. Castagnoli, E. & LiCalzi, M. (1996). Expected utility without utility. *Theory and Decision* 41(3): 281–301.
6. De Santis, E. & Spizzichino, F. (2004). Change-point models and conditionally pure birth processes: An inequality on the stochastic intensity. *Journal of Applied Probability* 41(4): pp. 939–952.
7. De Santis, E. & Spizzichino, F. (2012). First occurrence of a word among the elements of a finite dictionary in random sequences of letters. *Electronic Journal of Probability* 17: 1–9.
8. De Santis, E. & Spizzichino, F. (2012). Stochastic comparisons between first-passage times for Markov chains. arXiv:1210.1116 [math.PR], pp. 1–18.
9. Durante, F., Klement, E.P., Sempi, C. & Úbeda-Flores, M. (2010). Measures of non-exchangeability for bivariate random vectors. *Statistical Papers* 51(3): 687–699.
10. Joe, H. (1997) *Multivariate Models and Multivariate Dependence Concepts*. New York: Springer.
11. Kamihigashi, T. & Stachurski, J. (2014). Partial stochastic dominance. Technical Report.
12. Muliere, P. & Scarsini, M. (1987). Characterization of a Marshall–Olkin type class of distributions. *Annals of the Institute of Statistical Mathematics* 39(2): 429–441.
13. Navarro, J. & Rubio, R. (2010). Comparisons of coherent systems using stochastic precedence. *TEST* 19: 469–486.
14. Navarro, J. & Spizzichino, F. (2010). On the relationships between copulas of order statistics and marginal distributions. *Statistics & probability Letters* 80(5): 473–479.
15. Nelsen, R. (2006). *An Introduction to Copulas*. Springer Series in Statistics. New York: Springer.
16. Savage, L.J. (1954). *The Foundations of Statistics*. New York: Wiley.
17. Scarsini, M. (1985). A note on bernoulli's principle and probability dominance. *Journal of Optimization Theory and Applications* 47(1): 109–113.
18. Sempi, C. (2004). Convergence of copulas: critical remarks. *Radov Matematicki* 12(2): 241–249.
19. Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic Orders*. Springer Series in Statistics. New York: Springer.
20. Strassen, V. (1965). The existence of probability measures with given marginals. *The Annals of Mathematical Statistics* 36: 423–439.
21. Wrather, C. & Yu, P. (1982). Probability dominance in random outcomes. *Journal of Optimization Theory and Applications* 36(3): 315–334.

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press

https://doi.org/10.1017/S0269964815000029 Published online by Cambridge University Press