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Proofs, denotational semantics and observational equivalences in Multiplicative Linear Logic

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We study full completeness and syntactical separability of *MLL* proof nets with the mix rule. The general method we use consists of first addressing these two questions in the less restrictive framework of proof structures, and then adapting the results to proof nets. At the level of proof structures, we find a semantical characterisation of their interpretations in relational semantics, and define an observational equivalence that is proved to be the equivalence induced by cut elimination. Hence, we obtain a semantical characterisation (in coherent spaces) and an observational equivalence for the proof nets with the mix rule.

1. Introduction

In the *proofs-as-programs* paradigm a logical proof is a computational process that has to make the information implicit in the cut rules explicit.

Such a process can be described from several points of view. From a syntax perspective, a proof can be seen as a graph and the cut-reduction rules as graph rewriting rules. From a denotational semantics perspective, a proof can be interpreted as a function invariant under cut reduction. Finally, from an observational semantics perspective, a proof can be identified with its behaviour with respect to a set of contexts and values.

Each of these points of view provides an equivalence between proofs. Broadly speaking, two proofs are: syntactically equivalent when they are reducible to the same graph; equivalent in a denotational semantics when they are associated with the same invariant; and observationally equivalent when they are given the same value for any context they are put in.

The question of comparing such points of view and their induced equivalences is a crucial one in modern logic.

When comparing a syntax with a denotational semantics we (mainly) deal with two questions: injectivity (or faithfullness) and surjectivity (or full completeness). A denotational semantics is injective for a syntax if it interprets syntactically non-equivalent proofs with different objects. The question of injectivity has been addressed for linear logic in Tortora de Falco (2003). On the other hand, a semantics is fully complete for a

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syntax when it is possible to characterise those objects that are interpretations of proofs. The first full completeness result in linear logic was due to Abramsky and Jagadeesan and given in Abramsky and Jagadeesan (1994).

When comparing a syntax with an observational semantics we deal with the so-called separation problem. A syntax enjoys separation with respect to an observational semantics if for any two syntactically non-equivalent proofs there exists a context in which they behave differently. Initially, separation was stated in pure λ-calculus by Morris in Morris (1968) as a corollary of Böhm's theorem (Böhm 1968): if t, t' are two distinct $\beta\eta$-normal terms, then there exists a context $C\,[\,]$ such that $C\,[t]$ is normalisable but $C\,[t']$ is not. This kind of question was later studied by Statman for the simply typed λ-calculus in Statman (1983). In a more proof-theoretical framework, separation is a key property of Girard's ludics (Girard 2001). Some work on separation has also been done in linear logic. The first was presented in Mascari and Pedicini (1994), in the framework of pure proof nets, while in the typed case, work by Matsuoka (Matsuoka 2005) deals with the separation of the implicational multiplicative linear logic fragment.

In the present paper we study all these questions in a simple case: the multiplicative fragment of linear logic (MLL). Our syntactical objects will be the proof nets, the denotational semantics will be relational and coherent semantics, and we define two observational equivalences between MLL proofs.

The set of proof nets is a subset of a wider set of graphs: the set of proof structures. More precisely, the proof nets are those proof structures that correspond to correct proofs. The very importance of proof structures is that cut reduction is directly defined on them, so we can speak of proof structures reducing to the same graph, as well as of denotational interpretations or observational behaviours of proof structures.

Our general method is to compare syntax, denotational semantics and observational semantics, first at the level of proof structures, with fewer constraints, and later to adapt the results to proof nets.

In Section 2 we recall the definition of proof structures and in Section 3 the definition of relational semantics. The main result of Section 3 is the semantical characterisation of those sets that are interpretations of proof structures (Theorem 3.3). In Section 6 we use this result and a theorem from Retoré (1997) – Theorem 6.2 in the current paper – to deduce an alternative proof (from that given in Tan (1997)) of the full completeness of coherent semantics with respect to the proof nets of MLL with mix (Corollary 6.1).

In Section 4 we introduce an observational equivalence between proof structures (Definition 4.1). The main result of this section is the separation theorem for MLL proof structures (Theorem 4.1). As corollaries, we prove that the defined observational equivalence coincides with the equivalence induced by cut-elimination (Corollary 4.1) and that such an equivalence is a maximal congruence between proof structures (Corollary 4.2). In Section 7 we weaken the observational equivalence of Definition 4.1 by reducing the admissible contexts (Definition 7.1) and prove that with this weaker equivalence the separation of MLL does not hold (Proposition 7.1).

For simplicity, we restrict ourselves to the case of a unique pair of atomic formulas X and $X^{\perp}$, however, all the results in the present paper still hold in the general case, but with slightly more technical definitions.

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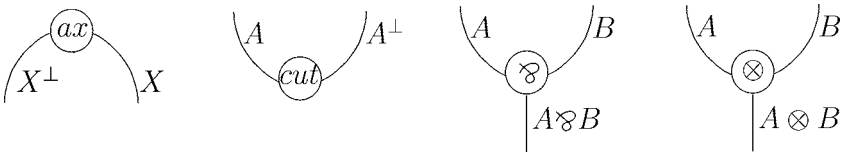


Fig. 1. Multiplicative links.

2. Proof structures

In this section we review *MLL* proof structures and *MLL* cut-reduction rules. The **formulas** of *MLL* are:

$$F ::= X \mid X^\perp \mid F \parr F \mid F \otimes F .$$

As always, we set $(A \parr B)^\perp = A^\perp \otimes B^\perp$ and $(A \otimes B)^\perp = A^\perp \parr B^\perp$. We use capital Greek letters $\Sigma, \Pi, \ldots$ to denote the sets of formulas, and we write $A_1 \odot \ldots \odot A_{n-1} \odot A_n$ for $A_1 \odot (\ldots \odot (A_{n-1} \odot A_n) \ldots)$, where $\odot$ is $\parr$ or $\otimes$.

Proof structures are oriented graphs (which may be empty) whose nodes are called **links** and whose edges are labelled by formulas of linear logic. When drawing a proof structure, we represent edges oriented up-down so that we may speak of moving *downwards* or *upwards* in the graph. Links are defined with both an arity (the number of incident edges, called the **premises of the link**) and a coarity (the number of emergent edges, called the **conclusions of the link**). *MLL* links are as follows (see Figure 1):

1 The **axiom** (*ax*-link), which has two conclusions labelled by dual atomic formulas, but no premise.
2 The **cut** (*cut*-link), which has two premises labelled by dual formulas but no conclusion.
3 The **par** ($\parr$-link), which has two ordered premises and one conclusion. If the left premise is labelled by the formula A and the right premise is labelled by the formula B, then the conclusion is labelled by the formula $A \parr B$.
4 The **tensor** ($\otimes$-link), which has two ordered premises and one conclusion. If the left premise is labelled by the formula A and the right premise is labelled by the formula B, then the conclusion is labelled by the formula $A \otimes B$.

Each edge is the conclusion of a unique link and the premise of at most one link. Edges that are not the premise of any link are the **conclusions** of the proof structure. If π is a proof structure with conclusions $C_1, \ldots, C_n$, we define the **closure of π with conclusion $\mathbf{C_1} \parr \ldots \parr \mathbf{C_n}$** as the proof structure obtained from π by adding the necessary $\parr$-links below $C_1, \ldots, C_n$.

Proof structures are denoted by lower-case Greek letters, $\pi, \sigma, \tau, \ldots$; the edges by letters from the beginning of the Latin alphabet, $a, b, c \ldots$; and the links by letters from the middle of the Latin alphabet, $l, m, n, o \ldots$. We write $a : A$ if a is an edge labelled by the formula A.

Note that axioms introduce atomic formulas only: this is a common way of avoiding the η-expansion rule (see, for example, Tortora de Falco (2003)).

The notion of congruent equivalence, which was defined by Girard in Girard (1991), is defined in the framework of proof structures as follows.

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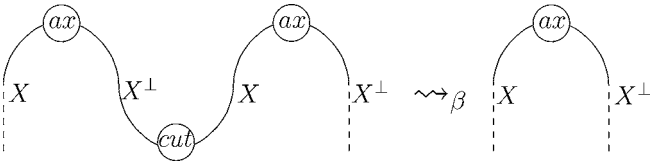


Fig. 2. Axiom cut reduction 1.

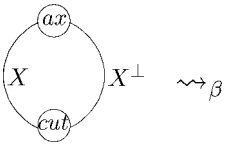


Fig. 3. Axiom cut reduction 2.

Definition 2.1. An equivalence $\equiv$ between proof structures is a **congruence** (or **is congruent**) when for all proof structures π_1, π_2, if $\pi_1 \equiv \pi_2$, then π_1 and π_2 have the same conclusions, and whenever π_1' and π_2' have been obtained from π_1 and π_2 by adding the same links, $\pi_1' \equiv \pi_2'$.

The *cut*-link defines the composition between proof structures: if π and σ are two proof structures with conclusions Π, A and $\Sigma, A^\perp$, respectively, the **composition** of π and σ on $A, A^\perp$, which is denoted by $[\pi, \sigma]_{\mathbf{A},\mathbf{A}^\perp}$, is the proof structure with conclusions Π, Σ obtained by joining π and σ with a new *cut*-link with premises A and $A^\perp$. We omit the indexes $_{A,A^\perp}$ when the premises of the cut are obvious.

A proof structure without cuts is called **cut-free**. The MLL **cut-reduction rules** are graph rewriting rules that modify a proof structure π to give a proof structure π' with the same conclusions as π. We use $\pi \rightsquigarrow_\beta \pi'$ to denote the cut-reduction relation between π and π', recalling the β-reduction of λ-calculus.

If l is a cut in a proof structure, the premises of l may be labelled either by dual atomic formulas X and $X^\perp$, or by dual compound formulas $A \parr B$ and $A^\perp \otimes B^\perp$. The cut-reduction rule for l is as follows:

— If l has premises labelled by dual atomic formulas X and $X^\perp$, let m be the axiom of which a conclusion is the premise of l labelled by X and let n be the axiom of which a conclusion is the premise of l labelled by $X^\perp$. If $m \neq n$, then l is reduced by erasing l, m, n and the l premises, and then linking the remaining m, n conclusions through a new axiom link (Figure 2). If $m = n$, then l is reduced by simply erasing l, m and the l premises (Figure 3).

— If l has premises labelled by dual compound formulas $A \parr B$ and $A^\perp \otimes B^\perp$, let m be the par whose conclusion is the premise of l labelled by $A \parr B$ and n be the tensor whose conclusion is the premise of l labelled by $A^\perp \otimes B^\perp$ (remember that compound formulas do not label conclusions of axioms). Let a, b (respectively, a', b') be the left and right premises of m (respectively, of n). Then l is reduced by simply erasing l, m, n and l premises, and then linking a, a' and b, b', respectively, by two new cuts (Figure 4).

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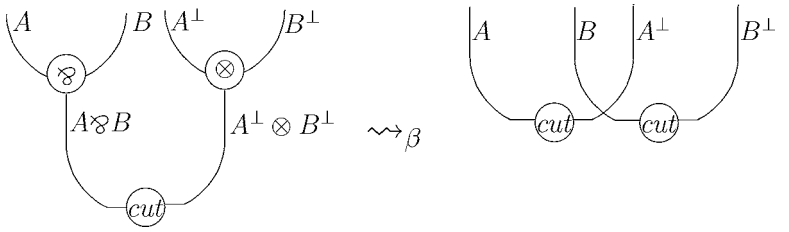


Fig. 4. ⅋/⊗ cut reduction.

The reduction in Figure 3 is perhaps unusual, indeed it has a dubious logical meaning. However, we are not working at the logic level: we are looking at the reduction rules simply as rewriting rules for proof structures. In Section 5 we will upgrade to proof nets, and the links, as well as the reduction rules, will acquire a logical meaning. In particular, proof nets do not allow *'vicious cuts'* such as the cut between the two conclusions of an axiom.

The reflexive and transitive closure of $\rightsquigarrow_\beta$ is denoted by $\rightarrow_\beta$. The symmetric closure of $\rightarrow_\beta$ is denoted by $=_\beta$ and called **β-equivalence**.

It is already known that $\rightarrow_\beta$ enjoys the properties of confluence and strong normalisation[†].

Theorem 2.1 (Confluence). For every proof structure π_1, π_2 and π_3 such that $\pi_1 \rightarrow_\beta \pi_2$ and $\pi_1 \rightarrow_\beta \pi_3$, there is a proof structure π_4 such that $\pi_2 \rightarrow_\beta \pi_4$ and $\pi_3 \rightarrow_\beta \pi_4$.

Theorem 2.2 (Strong normalisation). For every proof structure π, there is no infinite sequence of proof structures $\pi_0, \pi_1, \pi_2, \ldots$ such that $\pi_0 = \pi$ and $\pi_i \rightsquigarrow_\beta \pi_{i+1}$.

Confluence and strong normalisation ensure that in each equivalence class of $=_\beta$ there is one and only one cut-free proof structure. Note that the only cut-free proof structure without conclusions is the empty graph, so all proof structures without conclusions are reduced to the empty graph.

It is well known that the conclusions of a cut-free proof structure determine it up to the axioms: a cut-free proof structure with conclusions $C_1, \ldots, C_n$ is the forest of the n syntax trees of the formulas $C_1, \ldots, C_n$ and a set of axioms linking such forest leaves in pairs.

3. Relational semantics

Relational semantics associates sets with *MLL* formulas and relations that are invariant under cut reduction with proof structures.

[†] Actually, Girard proves in Girard (1987) that $\rightarrow_\beta$ is confluent and strong normalising for proof nets, but these proofs can be extended straightforwardly to the general case of proof structures.

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Let $\mathscr{X}$ be a set. A **relational model on** $\mathscr{X}$ ($\mathfrak{Rel}^{\mathscr{X}}$) associates sets with formulas as follows:

— $\mathscr{X}$ is associated with the atomic formulas $X, X^{\perp}$.
— If $\mathscr{A}$ and $\mathscr{B}$ are associated with A and B, respectively, then $\mathscr{A} \times \mathscr{B}$ is associated with $A \parr B$ and $A \otimes B$.

We use letters from the end of the Latin alphabet, $\ldots$, u, v, x, y, z, to denote the elements of sets, and letters from the start of the Greek alphabet, α, β, $\ldots$, to denote subsets.

For each proof structure π, we define the **interpretation of** π **in** $\mathfrak{Rel}^{\mathscr{X}}$, denoted by $[\![\pi]\!]_{\mathscr{X}}$, though the index $_{\mathscr{X}}$ will be omitted if the set associated with the atomic formulas is obvious.

If π has no conclusions, $[\![\pi]\!]$ is undefined. Otherwise, if $c_1 : C_1, \ldots, c_n : C_n$ are the conclusions of π, then $[\![\pi]\!]$ is a subset of $\mathscr{C}_1 \times \ldots \times \mathscr{C}_n$, which we define using the notion of an experiment. Experiments were introduced by Girard in Girard (1987), and studied extensively in Tortora de Falco (2000).

Definition 3.1 (Experiment). A $\mathfrak{Rel}^{\mathscr{X}}$ **experiment** e **on a proof structure** π, denoted by $e : \pi$, is a function associating with every edge $a : A$ of π an element of $\mathscr{A}$ such that the following conditions are respected:

Axiom: if a, b are the conclusions of an ax-link, then $e(a) = e(b)$.
Cut: if a, b are the premises of a cut-link, then $e(a) = e(b)$.
Multiplicative: if c is the conclusion of a $\parr$- or $\otimes$-link with premises a and b, then $e(c) =< e(a), e(b) >$.

Experiments can be viewed as π edge decorations either from axioms to conclusions or from conclusions to axioms: a multiplicative condition determines an experiment by either assigning values to the axioms if the cut condition is satisfied, or by assigning values to the conclusions and to the cuts of π if the axiom condition is satisfied.

Let π be a proof structure with conclusions $c_1 : C_1, \ldots, c_n : C_n$ and $e : \pi$ be an experiment. Then the **result of e** is the element $< e(c_1), \ldots, e(c_n) >$ of $\mathscr{C}_1 \times \ldots \times \mathscr{C}_n$. The interpretation of π in $\mathfrak{Rel}^{\mathscr{X}}$ is the set of the results of all experiments on π:

$$[\![\pi]\!]_{\mathscr{X}} = \{< e(c_1), \ldots, e(c_n) > \mid e \text{ is a } \mathfrak{Rel}^{\mathscr{X}} \text{ experiment on } \pi\}.$$

For each formula C we have, on the one hand, the proof structures with conclusion C, and, on the other hand, the subsets of $\mathscr{C}$, with $[\![\ ]\!]_{\mathscr{X}}$ a function from the proof structures to the subsets of $\mathscr{C}$. The following theorems are well known[†].

Theorem 3.1 (Soundness of $[\![\ ]\!]_{\mathscr{X}}$**).** For all proof structures π, π', we have $\pi =_{\beta} \pi'$ implies $[\![\pi]\!]_{\mathscr{X}} = [\![\pi']\!]_{\mathscr{X}}$.

Theorem 3.2 (Injectivity of $[\![\ ]\!]_{\mathscr{X}}$**).** If $\mathscr{X}$ is infinite, then for all proof structures π, π', we have $[\![\pi]\!]_{\mathscr{X}} = [\![\pi']\!]_{\mathscr{X}}$ implies $\pi =_{\beta} \pi'$.

[†] In fact, the proofs of semantical soundness and injectivity in Girard (1987) and Tortora de Falco (2003)), respectively, are for the more restricted case of proof nets, but these proofs can be extended straightforwardly to the general case of proof structures.

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The main result of this section is to characterise those subsets of $\mathscr{C}$, called complete subsets, that are the interpretations of proof structures with conclusion C (Theorem 3.3). In this way, $[\![\ ]\!]_{\mathscr{X}}$ becomes a bijection between the cut-free proof structures with conclusion C and the complete subsets of $\mathscr{C}$.

To achieve Theorem 3.3, we start from the proof of the injectivity of $[\![\ ]\!]_{\mathscr{X}}$. Let π be a cut-free proof structure with conclusion C, we have already noted that π can be presented as a set of axioms linking the leaves of the syntax tree of C. The proof of the injectivity of $[\![\ ]\!]_{\mathscr{X}}$ mainly uses the fact that there exists $u \in [\![\pi]\!]_{\mathscr{X}}$ that codes all the pairs of dual leaves linked by an axiom of π. Indeed such an element u is the result of an injective experiment, which is defined as follows in Tortora de Falco (2000).

Definition 3.2. Let π be a cut-free proof structure and $e : \pi$ be an experiment. e is **injective** when for all formulas A and edges a, a' labelled by A, we have $e(a) \neq e(a')$.

Note that if $\mathscr{X}$ is infinite, any cut-free proof structure has injective experiments: to see this, simply take an injective assignment of values to the axioms of the proof structure.

The results of injective experiments are the most informative points of $\mathscr{C}$. We define a pre-order $\geq$ on the elements of $\mathscr{C}$ (Definition 3.3) measuring how much information on proof structures is coded by an element. As expected, the results of injective experiments are maximal among the (balanced, see Definition 3.4) elements of $\mathscr{C}$. Conversely, in Lemma 3.1 we prove that all the maximals among the (balanced) elements of $\mathscr{A}$ are results of injective experiments.

In Lemma 3.2 we prove that for every proof structure π, the set $[\![\pi]\!]$ has the shape $\{v | u \geq v\}$, where u is the result of an injective experiment on π. Thus, we define the complete subsets of $\mathscr{C}$ as those subsets of the form $\{v | u \geq v\}$ for a maximal u among the (balanced) elements of $\mathscr{C}$. In this way we get a characterisation of those subsets of $\mathscr{C}$ that are interpretations of proof structures (Theorem 3.3).

An element u of a set $\mathscr{C}$ is a sequence of elements of the basic set $\mathscr{X}$ and the symbols $<, >$. We call the elements of $\mathscr{X}$ that are in u the **atoms** of u. Note that any element u in $\mathscr{C}$ defines a labelling of the edges of the syntax tree of C; in particular, the atoms of u will label the leaves of such a tree. An occurrence of an atom x in u is a **positive occurrence** if it labels a subformula X of C, and a **negative occurrence** if it labels a subformula $X^{\perp}$ of C.

Having given two elements $x, y \in \mathscr{X}$, we define $u[y/x]$ as the element of $\mathscr{C}$ obtained from u by substituting y for each occurrence of x. As always, we extend the definition to simultaneous substitutions $u[y_1/x_1, \ldots, y_n/x_n]$.

Definition 3.3. Let C be an MLL formula, $\mathscr{C}$ its associated set and $u, u' \in \mathscr{C}$. We write $\mathbf{u} \geq \mathbf{u}'$ if there is a substitution $[y_1/x_1, \ldots, y_n/x_n]$ such that $u[y_1/x_1, \ldots, y_n/x_n] = u'$. We set $\mathbf{u} \approx \mathbf{u}'$ if $u \geq u'$ and $u' \geq u$.

In general, $\approx$ identifies the results of the experiments differing only by a renaming of the values appointed to the conclusions of the axioms.

The following definition allows us to remove from $\mathscr{C}$ those elements that cannot be in the interpretation of a proof structure.

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Definition 3.4. An element $u \in \mathscr{C}$ is **balanced** if for every atom the number of its positive occurrences in u is equal to the number of its negative occurrences.

The property of being balanced is stable under substitution: if u is a balanced element, then $u[y_1/x_1, \ldots, y_n/x_n]$ is balanced for every substitution $[y_1/x_1, \ldots, y_n/x_n]$.

The pre-order $\succeq$ tells us how informative the elements of $\mathscr{C}$ are. The results of injective experiments are balanced and maximal among the balanced elements of $\mathscr{C}$. We prove the converse in the next lemma.

Lemma 3.1. Let $\mathscr{X}$ be an infinite set, $\mathscr{C}$ be a set associated with a formula C in $\mathfrak{Rel}^{\mathscr{X}}$. Let $u \in \mathscr{C}$ be a balanced element that is maximal among the balanced elements of $\mathscr{C}$. Then there is a cut-free closed proof structure π^u with conclusion C and an injective experiment $e^u : \pi^u$ such that the result of e^u is u.

Proof. From the C tree we get π^u up to the axioms. Since u is balanced and maximal among the balanced elements and $\mathscr{X}$ is infinite, each atom x of u has exactly one positive and one negative occurrence in u, hence each atom x defines a pair of leaves $X, X^{\perp}$ of the C tree. We get π^u by linking such pairs with axioms.

Clearly, u is the result of the injective experiment on π^u that takes the value x on the pair of edges of type $X, X^{\perp}$ associated with x in u. □

Lemma 3.2. Let $\mathscr{X}$ be an infinite set and π be a closed cut-free proof structure with conclusion C. There is a balanced element u in $[\![\pi]\!]_{\mathscr{X}}$ that is maximal among the balanced elements of $\mathscr{C}$. Moreover, for any such balanced and maximal u, $[\![\pi]\!]_{\mathscr{X}} = \{v | u \succeq v\}$.

Proof. Since $\mathscr{X}$ is infinite, there are injective experiments on π. Let $e : \pi$ be an injective experiment, and u be its result. Clearly, u is balanced and maximal among the balanced elements of $\mathscr{C}$. Now, take any such u.

Let $a_1, \ldots, a_n$ be the conclusions of type X of the axioms of π. Let $e' : \pi$ be an experiment and v its result. Clearly, $v = u[e'(a_1)/e(a_1), \ldots, e'(a_n)/e(a_n)]$, so $u \succeq v$.

Conversely, if we let $v \in \mathscr{C}$ be such that $u \succeq v$, then there exists a substitution $[y_1/e(a_1), \ldots, y_n/e(a_n)]$ such that $v = u[y_1/e(a_1), \ldots, y_n/e(a_n)]$. Let e' be the experiment such that $e'(a_1) = y_1, \ldots, e'(a_n) = y_n$. Clearly, e' has v as result. □

Definition 3.5. A subset $\alpha \subset \mathscr{C}$ is **complete** if there is a balanced element $u \in \alpha$ that is maximal among the balanced elements of $\mathscr{C}$ and

$$\alpha = \{v | u \succeq v\}.$$

Theorem 3.3. Let $\mathscr{X}$ be an infinite set. Let C be an *MLL* formula and $\mathscr{C}$ be its interpretation in $\mathfrak{Rel}^{\mathscr{X}}$. A subset of $\mathscr{C}$ is the interpretation of a closed proof structure with conclusion C if and only if it is complete.

Proof. Let α be a complete set of $\mathscr{C}$. By its definition, there is a balanced element $u \in \alpha$ that is maximal among the balanced elements of $\mathscr{C}$ and $\alpha = \{v | u \succeq v\}$. By Lemma 3.1 there is a proof structure π^u and an injective experiment $e^u : \pi^u$ such that the result of e^u is u. By Lemma 3.2, $[\![\pi^u]\!] = \alpha$.

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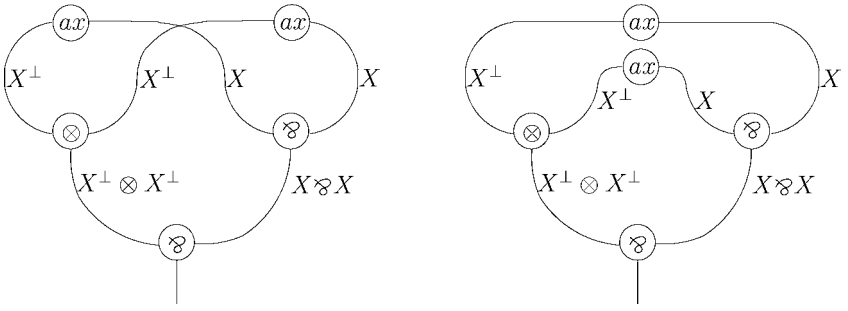


Fig. 5. Observational values $\mho$ and Ω.

Conversely, let π be a closed proof structure with conclusion C. Then, by Lemma 3.2, $[\![\pi]\!]$ is complete. □

4. Observational equivalence of proof structures

In Definition 4.1 we introduce an observational equivalence $\sim_{\mathbb{B}}$ between proof structures. The main result of this section is Theorem 4.1, from which we get that $=_\beta$ and $\sim_{\mathbb{B}}$ are the same equivalence (Corollary 4.1) and that such an equivalence is a maximal congruence (Corollary 4.2).

We choose as observational values the only two cut-free proof structures with conclusion $(X^\perp \otimes X^\perp) \parr (X \parr X)$ (Figure 5). We use $\mathbb{B}$ to denote $(X^\perp \otimes X^\perp) \parr (X \parr X)$, and $\mho$ and Ω, respectively, to denote the two cut-free proof structures with conclusion $\mathbb{B}$.

A **proper axiom with conclusions** $\mathbf{C_1}, \ldots, \mathbf{C_n}$ is a link without premises but with n conclusions labelled by $C_1, \ldots, C_n$, respectively. A **context of type** $\mathbf{C_1}, \ldots, \mathbf{C_n}$ is a proof structure with conclusion $\mathbb{B}$ where proper axioms with conclusions $C_1, \ldots, C_n$ can occur. We use $C[\,]$ to denote a context.

Let π be a proof structure with conclusions $C_1, \ldots, C_n$ and $C[\,]$ be a context of the same type. We use $C[\pi]$ to denote the proof structure with conclusion $\mathbb{B}$ obtained from $C[\,]$ substituting π for each occurrence of the proper axiom.

Definition 4.1. Let π_1, π_2 be proof structures with conclusions $C_1, \ldots, C_n$. We say that π_1 and π_2 are **observationally equal** (denoted by $\pi_1 \sim_{\mathbb{B}} \pi_2$) if for all contexts $C[\,]$ of type $C_1, \ldots, C_n$, we have $C[\pi_1] =_\beta C[\pi_2]$.

Clearly, $\sim_{\mathbb{B}}$ is a congruence. We can use Theorem 4.1 to prove that $\sim_{\mathbb{B}}$ and $=_\beta$ are indeed the same equivalence (Corollary 4.1).

Theorem 4.1 (Separation of *MLL*). Let π_1 and π_2 be two closed proof structures with conclusion C. If $\pi_1 \neq_\beta \pi_2$, then there is a proof structure σ with conclusion $C^\perp, \mathbb{B}$, such that $[\sigma, \pi_1] \to_\beta \mho$ and $[\sigma, \pi_2] \to_\beta \Omega$.

Proof. Let π_1, π_2 be two different cut-free proof structures with conclusion C. Let $1, \ldots, 2n$ be an enumeration of the leaves of the syntax tree of C such that the odd numbers enumerate the leaves labelled by X and the even numbers those labelled by $X^\perp$.

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We have already noted that π_1, π_2 can be presented as bijections from the odd to the even numbers of $\{1,\dots,2n\}$. Since $\pi_1 \neq \pi_2$, there is an odd number $o \leqslant 2n$ such that $\pi_1(o) = e$ and $\pi_2(o) = e'$ for $e \neq e'$.

We define the proof structure σ with conclusions $C^\perp, \mathbb{B}$, such that $[\pi_1, \sigma] \to_\beta \mho$ and $[\pi_1, \sigma] \to_\beta \Omega$. The forest of the syntax trees of $C^\perp, \mathbb{B}$ has $2n+4$ leaves. The above enumeration of the leaves of the syntax tree of C induces an enumeration $1,\dots,2n, 2n+1,\dots,2n+4$ of the leaves of the forest such that:

— The odd (respectively, even) numbers in $\{1,\dots,2n\}$ enumerate the leaves labelled by $X^\perp$ (respectively, X) above $C^\perp$.
— The odd (respectively, even) numbers in $\{2n+1,\dots,2n+4\}$ enumerate the leaves labelled by X (respectively, $X^\perp$) above $\mathbb{B}$.

In particular, note that e, e' are now associated with leaves labelled by X above $C^\perp$, and o with a leaf labelled by $X^\perp$ above $C^\perp$, finally $2n+1$ and $2n+3$ (respectively, $2n+2$ and $2n+4$) are the two leaves labelled by X (respectively, $X^\perp$) above $\mathbb{B}$.

σ is any bijection between the leaves labelled by X and those labelled by $X^\perp$ such that $\sigma(o) = 2n+1$, $\sigma(e) = 2n+2$ and $\sigma(e') = 2n+4$. Clearly, we have $[\sigma, \pi_1] \to_\beta \mho$ and $[\sigma, \pi_2] \to_\beta \Omega$. □

Corollary 4.1 (Equality of $\sim_{\mathbb{B}}$ and $=_\beta$). Let π_1 and π_2 be two proof structures with the same conclusions. Then $\pi_1 \sim_{\mathbb{B}} \pi_2$ iff $\pi_1 =_\beta \pi_2$.

Proof. Let π_1 and π_2 be two proof structures with the same conclusions, we may suppose π_1, π_2 closed, since both $\sim_{\mathbb{B}}$ and $=_\beta$ are congruences. By the confluence of $=_\beta$, if $\pi_1 =_\beta \pi_2$, then $\pi_1 \sim_{\mathbb{B}} \pi_2$, and the converse holds by Theorem 4.1. □

Corollary 4.2 (Maximality of $=_\beta$). Let $\equiv$ be a congruence that contains $=_\beta$. Then either $\equiv$ is equal to $=_\beta$ or $\equiv$ collapses.

Proof. Let $\equiv$ be a congruence containing $=_\beta$ and let us suppose that there are two distinct proof structures π_1, π_2 such that $\pi_1 \equiv \pi_2$ but $\pi_1 \neq_\beta \pi_2$. We prove $\tau_1 \equiv \tau_2$ for every proof structure τ_1, τ_2 with the same conclusions.

Since $\equiv$ is a congruence, we can assume that π_1 and π_2 are closed with the same conclusion C. Since $\pi_1 \neq_\beta \pi_2$, by Theorem 4.1 there is a proof structure σ with conclusions $C^\perp, \mathbb{B}$ such that $[\pi_1, \sigma] \to_\beta \mho$ and $[\pi_2, \sigma] \to_\beta \Omega$. By the congruence of $\equiv$, we deduce $[\pi_1, \sigma] \equiv [\pi_2, \sigma]$, and hence $\mho \equiv \Omega$.

Let τ_1, τ_2 be two distinct proof structures with the same conclusions. We will prove that $\tau_1 \equiv \tau_2$. Since $\equiv$ is a congruent extension of $=_\beta$, we can suppose τ_1, τ_2 to be cut-free and with only one conclusion D. Let $1,\dots,2n$ be an enumeration of the leaves of the syntax tree of D such that the odd numbers enumerate the leaves labelled by X and the even numbers the leaves labelled by $X^\perp$. Since D has at least two distinct cut-free proof structures (that is, τ_1, τ_2), D has at least two occurrences of X and two of $X^\perp$, that is, $n \geqslant 2$.

We have already seen that τ_1, τ_2 can be presented as bijections from the odd to the even numbers of $\{1,\dots,2n\}$. Since $\tau_1 \neq \tau_2$, there is an odd number $o \leqslant 2n$ such that $\tau_1(o) \neq \tau_2(o)$. We now choose o minimal and let $\tau_1(o) = e$, $\tau_2(o) = e'$ and $\tau_2^{-1}(e) = o'$. By the minimality of o, we have $o < o'$.

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Now, we define a proof structure σ with conclusions $D^{\perp}, D, \mathbb{B}^{\perp}$. The forest of the syntax trees of such conclusions has $2n + 2n + 4$ leaves. The above enumeration of the leaves of the syntax tree of D induces an enumeration $1, \ldots, 2n, 2n+1, \ldots, 4n, 4n+1, \ldots, 4n+4$ of the forest leaves such that:

— The odd (respectively, even) numbers in $\{1, \ldots, 2n\}$ enumerate the leaves labelled by $X^{\perp}$ (respectively, X) above $D^{\perp}$.
— The odd (respectively, even) numbers in $\{2n+1, \ldots, 4n\}$ enumerate the leaves labelled by X (respectively, $X^{\perp}$) above D.
— The odd (respectively, even) numbers in $\{4n+1, \ldots, 4n+4\}$ enumerate the leaves labelled by $X^{\perp}$ (respectively, X) of the tree of $\mathbb{B}^{\perp}$.

In particular, note that e and e' are associated with leaves labelled by X above $D^{\perp}$, while $2n + e$ and $2n + e'$ are associated with leaves labelled by $X^{\perp}$ above D, and, finally, $4n + 1$ and $4n + 3$ (respectively, $4n + 2$ and $4n + 4$) are the two leaves labelled by X (respectively, $X^{\perp}$) above $\mathbb{B}$.

We set $\sigma(e) = 4n + 2$, $\sigma(e') = 4n + 4$, $\sigma(2n + e) = 4n + 1$, $\sigma(2n + e') = 4n + 3$, and, for all the others, $i \leqslant 2n$, $\sigma(i) = 2n + i$.

The peculiarity of σ is that the action of $[\sigma, \mathfrak{V}]$ is the identity, while the action of $[\sigma, \Omega]$ is to flip e and e'. More precisely, for any proof structure π with conclusion D, we have $[[\sigma, \mathfrak{V}], \pi] \rightarrow_{\beta} \pi$, while $[[\sigma, \Omega], \pi] \rightarrow_{\beta} \pi'$, where π' is obtained from π by flipping e and e'. Moreover, by the congruence of $\equiv$ and the fact that $\mathfrak{V} \equiv \Omega$, we have $[\sigma, \mathfrak{V}] \equiv [\sigma, \Omega]$.

Now, by induction on $2n - o$, we prove that $\tau_1 \equiv \tau_2$:

— If $2n - o = 1$, then $o = 2n - 1$ and $o' = 2n$, and $\tau_1(o') = e'$. As we have already noted, $[[\sigma, \mathfrak{V}], \tau_1] \rightarrow_{\beta} \tau_1$ and $[[\sigma, \Omega], \tau_1] \rightarrow_{\beta} \tau_2$. Since $[\sigma, \mathfrak{V}] \equiv [\sigma, \Omega]$, we get $\tau_1 \equiv \tau_2$.
— If $2n - o > 1$. As we have already noted, $[[\sigma, \mathfrak{V}], \tau_1] \rightarrow_{\beta} \tau_1$ and $[[\sigma, \Omega], \tau_1] \rightarrow_{\beta} \tau_3$, where τ_3 is obtained from τ_1 by flipping e and e'. In particular, $\tau_3(e') = o$, so τ_2 and τ_3 differ at most on an $o'' > o$, and thus, by the induction hypothesis, $\tau_3 \equiv \tau_2$. Therefore, $\tau_1 \equiv [[\sigma, \mathfrak{V}], \tau_1] \equiv [[\sigma, \Omega], \tau_1] \equiv \tau_3 \equiv \tau_2$. □

The relational semantics defines a congruence $\equiv_{\mathfrak{Rel}}$ between proof structures, which means $\pi_1 \equiv_{\mathfrak{Rel}} \pi_2$ if for all $\mathscr{X}$, $[\![\pi_1]\!]_{\mathscr{X}} = [\![\pi_2]\!]_{\mathscr{X}}$. By the soundness of the relational semantics we know that $=_{\beta} \subseteq \equiv_{\mathfrak{Rel}}$. Now, by Corollary 4.2, we get the converse, *viz.* $\equiv_{\mathfrak{Rel}} \subseteq =_{\beta}$, that is, we have an alternative proof of the injectivity of the relational semantics to that given in Tortora de Falco (2003).

5. Proof nets

It is well known that *MLL* **sequent calculus** is defined by the following rules (Girard 1987):

$$\frac{}{\vdash X, X^{\perp}}\ ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\ cut$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes$$

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which can be extended by the addition of the mix rule:

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\ mix$$

In this paper we consider the sequent calculus extended with the mix rule.

The MLL sequent calculus proofs can be translated into proof structures: there is a function, called **desequentialisation**, that associates with a sequent proof P a proof structure $(P)^\bullet$. $(P)^\bullet$ is defined by induction on P (see Girard (1987)):

— If P is an axiom with conclusions $X, X^\perp$, then $(P)^\bullet$ is an axiom link with conclusions $X, X^\perp$.
— If P ends in a $\parr$-rule, having as premise the subproof P', then $(P)^\bullet$ is obtained by adding to $(P')^\bullet$ the $\parr$-link corresponding to the $\parr$-rule.
— If P ends in a $\otimes$-rule (respectively, *cut*-rule), with premises the subproofs P' and P'', then $(P)^\bullet$ is obtained by connecting $(P')^\bullet$ and $(P'')^\bullet$ by means of the $\otimes$-link (respectively, *cut*-link) corresponding to the $\otimes$-rule (respectively, *cut*-rule).
— If P ends in a *mix*-rule, with premises the subproofs P' and P'', then $(P)^\bullet$ is obtained by taking the disjoint union of $(P')^\bullet$ and $(P'')^\bullet$.

A **proof net** is a proof structure associated with a sequent calculus proof. A unique proof net can be associated with several sequent calculus proofs: it yields an abstract representation of the sequent proofs modulo inessential commutation of rules (see Bellin and de Wiele (1995)). Note that both semantical injectivity and syntactical separability can be studied in linear logic thanks to such abstractions from inessential commutations.

There are many geometrical characterisations of MLL proof nets, which are called **correctness criteria** (for example, those given in Girard (1987) or Danos and Regnier (1989)). We recall here the criterion given in Danos and Regnier (1989).

A **correctness graph** of a proof structure π is a subgraph of π that is obtained by erasing one premise for each $\parr$-link. A proof structure is **correct** if all its correctness graphs are acyclic. The following theorem is proved in Danos and Regnier (1989).

Theorem 5.1. A proof structure is correct if and only if it is a proof net.

In Section 6 we recall a result by Retoré (Retoré 1997) that provides a semantical characterisation of MLL correctness by means of coherent semantics. Such a result, together with Theorem 3.3, proves the correspondence between proof nets and complete cliques (Corollary 6.1).

6. Coherent semantics

A **coherent space** $\mathscr{X}$ is a couple $(|\mathscr{X}|, \frown)$, where $|\mathscr{X}|$ is a set, called the **web of** $\mathscr{X}$, and $\frown$ is a binary relation in $|\mathscr{X}|$ that is reflexive and symmetric, called the **coherence of** $\mathscr{X}$. A **clique of** $\mathscr{X}$ is a subset of $|\mathscr{X}|$ whose elements are pairwise coherent.

We will write $x \frown y\ [\mathscr{X}]$ if we want to show explicitly which coherent space $\frown$ refers to. We introduce the following notation, which is well known in the framework of coherent

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spaces:

— $x \frown y$, if $x \smile y$ and $x \neq y$.
— $x \precsim y$, if not $x \frown y$.
— $x \prec y$, if not $x \smile y$.

Note that we may define a coherent space by specifying its web and one among its relations $\smile$, $\precsim$, $\frown$, $\prec$.

A coherent space is identified with a graph whose vertex set is $|\mathscr{X}|$ and whose edges set is the extension of $\smile$.

Let $\mathscr{X}$ be a coherent space. A **coherent model on** $\mathscr{X}$ ($\mathfrak{Coh}^{\mathscr{X}}$) associates coherent spaces with MLL formulas, which are defined by induction on the formulas as follows:

— With X it associates $\mathscr{X}$.
— With $A^{\perp}$ it associates $\mathscr{A}^{\perp}$ defined as follows: $|\mathscr{A}^{\perp}| = |\mathscr{A}|$, the coherence of $\mathscr{A}^{\perp}$ is the incoherence of $\mathscr{A}$, that is, $x \smile y\ [\mathscr{A}^{\perp}]$ iff $x \precsim y\ [\mathscr{A}]$.
— With $A \otimes B$ it associates $\mathscr{A} \otimes \mathscr{B}$ defined as follows: $|\mathscr{A} \otimes \mathscr{B}| = |\mathscr{A}| \times |\mathscr{B}|$ and $< a, b > \smile < a', b' >\ [\mathscr{A} \otimes \mathscr{B}]$ iff $a \smile a'\ [\mathscr{A}]$ and $b \smile b'\ [\mathscr{B}]$.

Of course, the space $\mathscr{A} \parr \mathscr{B}$ is defined by $(\mathscr{A}^{\perp} \otimes \mathscr{B}^{\perp})^{\perp}$.

Note that the web associated with a formula A by $\mathfrak{Coh}^{\mathscr{X}}$ is precisely the interpretation of A in $\mathfrak{Rel}^{|\mathscr{X}|}$.

Let π be a proof structure with conclusions $c_1 : C_1, \ldots, c_n : C_n$. The **interpretation** of π in $\mathfrak{Coh}^{\mathscr{X}}$ is a subset of $|\mathscr{C}_1 \parr \ldots \parr \mathscr{C}_n|$, denoted by $[\![\pi]\!]_{\mathscr{X}}$, where the index $_{\mathscr{X}}$ is omitted when the coherent space associated with X is obvious.

$[\![\pi]\!]$ is defined in exactly the same way as in the relational semantics (see Section 3). We have the same definitions concerning the **experiment** e on a proof structure π, its **result**, and the interpretation $[\![\pi]\!]$. The relational interpretation of π differs from the coherent one only in the presence of exponentials: if π is an MLL proof structure, $[\![\pi]\!]_{\mathfrak{Rel}} = [\![\pi]\!]_{\mathfrak{Coh}}$.

What we achieve by introducing coherence is that the set $[\![\pi]\!]_{\mathscr{X}}$ may or may not be a clique. Girard proves in Girard (1987) that if π is a proof net, then $[\![\pi]\!]_{\mathscr{X}}$ is a clique (Theorem 6.1). Retoré proves the converse for the cut-free proof nets (Theorem 6.2), hence the correctness of a cut-free proof structure corresponds to the pairwise coherence of the results of its experiments.

Theorem 6.1. If π is a proof net, then $[\![\pi]\!]_{\mathscr{X}}$ is a clique.

Theorem 6.2. Let π be a cut-free proof structure and $\mathscr{X}$ be a coherent space with at least x, y, z, such that $x \frown y$ and $x \prec z$. If $[\![\pi]\!]_{\mathscr{X}}$ is a clique, then π is a proof net.

Since the web of a coherent space is a set, we can introduce the pre-order $\geq$ (Definition 3.3) and the notion of a **complete subset** (Definition 3.5) on webs in exactly the same way as we did with the relational semantics.

If $\mathscr{A}$ is a coherent space, a **complete clique** of $\mathscr{A}$ is a complete subset of $|\mathscr{A}|$ that is a clique.

Corollary 6.1. Let $\mathscr{X}$ be a coherent space whose web is infinite and with x, y, z, such that $x \frown y$ and $x \prec z$. Let C be an MLL formula and $\mathscr{C}$ be its interpretation in $\mathfrak{Coh}^{\mathscr{X}}$.

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A subset of $\mathscr{C}$ is the interpretation of a closed proof net with conclusion C if and only if it is a complete clique.

Proof. Let α be a complete clique of $\mathscr{C}$. Since α is complete, by Theorem 3.3, there is a closed cut-free proof structure π with conclusion C such that $[\![\pi]\!] = \alpha$. Since α is a clique, by Theorem 6.2, π is a proof net.

Conversely, if π is a proof net, by Theorem 6.1, $[\![\pi]\!]$ is a clique, and by Theorem 3.3, $[\![\pi]\!]$ is complete. □

7. Observational equivalence of proof nets

The observational equivalence $\sim_{\mathbb{B}}$ (Definition 4.1) depends on the proof structures behaviours within all possible contexts. In this section we would like to restrict the observations purely to the correct contexts.

First, we note that the only two proof structures $\mathfrak{V}$ and Ω with conclusion $\mathbb{B}$ (Figure 5) are correct, therefore we can keep them as observational values. Then we extend the correctness criterion to contexts. A **correctness graph** of a context is a subgraph obtained by erasing one premise for each $\parr$-link. A context **is correct** if all of its correctness graphs are acyclic.

Definition 7.1. Let π_1, π_2 be two proof nets with conclusions $C_1, \dots, C_n$. We say that π_1 and π_2 are **observationally weak equal** ($\pi_1 \sim^w_{\mathbb{B}} \pi_2$) if for all the correct contexts $C[\,]$ of type $C_1, \dots, C_n$, we have $C[\pi_1] =_\beta C[\pi_2]$.

Clearly, $=_\beta \subseteq \sim^w_{\mathbb{B}}$. The main result of this section is Proposition 7.1, which states that $=_\beta \subsetneq \sim^w_{\mathbb{B}}$: there are proof nets that are observationally weak equal but not β-equivalent (hence, they are not observationally equal either).

Such a result does not clash with Corollary 4.2, which states that $=_\beta$ is a maximal congruence. This means that $\sim^w_{\mathbb{B}}$ is not a congruence when extended to proof structures. Indeed, $\sim^w_{\mathbb{B}}$ is defined only between proof nets and not between proof structures in general, therefore, if π_1 and π_2 are two observationally weak equal proof nets, and π_1' and π_2' have been obtained from π_1 and π_2 by adding the same links, then $\pi_1' \nsim^w_{\mathbb{B}} \pi_2'$ could happen simply because π_1' and π_2' are not correct.

Note that, in general, a context can be quite complex, namely the proper axioms might be whenever and wherever we want them. Before attacking Proposition 7.1, it is thus convenient to restrain our observations to the simplest contexts, which are the proof nets themselves.

Lemma 7.1 (Context lemma). Let π_1 and π_2 be two proof nets with conclusions $C_1, \dots, C_n$. Let π_1^* and π_2^* be the two closures of π_1, π_2 with conclusion $C_1 \parr \dots \parr C_n$. Then $\pi_1 \nsim^w_{\mathbb{B}} \pi_2$ iff there is a proof net σ with conclusions $C_1^\perp \otimes \dots \otimes C_n^\perp, \mathbb{B}$, such that $[\pi_1^*, \sigma] \neq_\beta [\pi_2^*, \sigma]$.

Proof. The 'if' part is immediate. For the converse, let π_1 and π_2 be two proof nets with same conclusions $C_1, \dots, C_n$ such that $\pi_1 \nsim_{\mathbb{B}} \pi_2$. We prove that there is a proof net σ with conclusions $C_1^\perp \otimes \dots \otimes C_n^\perp, \mathbb{B}$, such that $[\pi_1^*, \sigma] \neq_\beta [\pi_2^*, \sigma]$.

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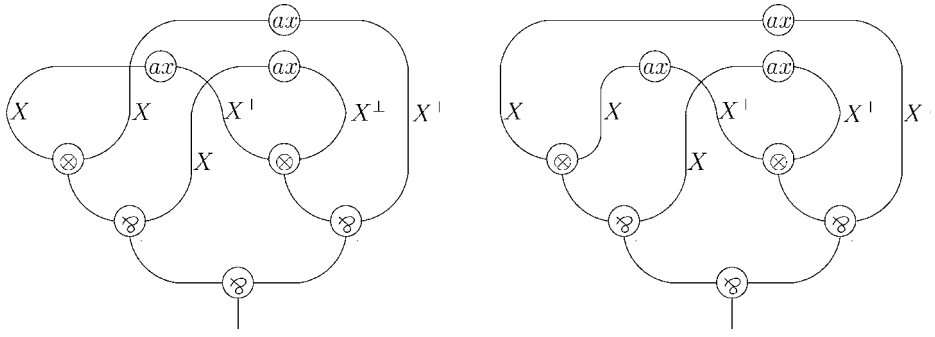


Fig. 6. Example of proof nets π_1, π_2 with conclusion $((X \otimes X)\parr X)\parr(X^\perp \otimes X^\perp)\parr X^\perp$.

By Definition 7.1, there is a correct context $C\,[\,]$ such that $C\,[\pi_1] \neq_\beta C\,[\pi_2]$. We use $1,\ldots,k$ to enumerate the occurrences of the proper axiom in $C\,[\,]$. For each $i \leqslant k$, let σ_i be the proof net obtained from $C\,[\,]$ by substituting π_1 for the occurrences $1,\ldots,i$ of the proper axiom and π_2 for the occurrences $i+1,\ldots,k$. Clearly, $\sigma_0 = C\,[\pi_2] \neq_\beta C\,[\pi_1] = \sigma_k$, so there is an i such that $\sigma_i \neq_\beta \sigma_{i+1}$. σ is obtained from $C\,[\,]$ in two steps. First we substitute π_1 for the occurrences $1,\ldots,i$ of the proper axiom in $C\,[\,]$ and π_2 for the occurrences $i+2,\ldots,k$. Then we substitute the $i+1$-th occurrence of the proper axiom with the set of the n axioms with conclusions $C_1^\perp, C_1, \ldots, C_n^\perp, C_n$, respectively, and we link the conclusions $C_1^\perp,\ldots,C_n^\perp$ with tensors to get a unique conclusion $C_1^\perp \otimes \ldots \otimes C_n^\perp$.

Clearly, σ is correct, and, moreover, $[\pi_1^*, \sigma] =_\beta \sigma_i \neq_\beta \sigma_{i+1} =_\beta [\pi_2^*, \sigma]$. □

Now we prove that $\sim^w_{\mathbb{B}}$ is a strict extension of $=_\beta$.

Proposition 7.1. There are proof nets π_1, π_2 such that $\pi_1 \neq_\beta \pi_2$ and $\pi_1 \sim^w_{\mathbb{B}} \pi_2$.

Proof. Let C be the formula $((X \otimes X)\parr X)\parr(X^\perp \otimes X^\perp)\parr X^\perp$, and π_1, π_2 be any two different cut-free proof nets with conclusion C (take, for example, those in Figure 6).

Let us suppose that $\pi_1 \nsim^w_{\mathbb{B}} \pi_2$ and prove the resulting absurdity. By Lemma 7.1, there is a proof net σ with conclusions $C^\perp, \mathbb{B}$ such that $[\pi_1, \sigma] \neq_\beta [\pi_2, \sigma]$. Since $\mathfrak{V}$ and Ω are the only two cut-free proof nets with conclusion $\mathbb{B}$, we may suppose $[\pi_1, \sigma] \to_\beta \mathfrak{V}$ and $[\pi_2, \sigma] \to_\beta \Omega$.

Let $\mathscr{X}$ be a coherent space with $x, y, z \in |\mathscr{X}|$ such that $x \frown y$, $x \smile z$ and $y \smile z$: we will prove that $[\![\sigma]\!]_{\mathscr{X}}$ is not a clique, thereby contradicting Theorem 6.1.

Note that $\langle\langle x,z\rangle, \langle x,z\rangle\rangle \in [\![\mathfrak{V}]\!]$ and $\langle\langle x,z\rangle, \langle z,x\rangle\rangle \in [\![\Omega]\!]$, therefore there are $u \in [\![\pi_1]\!]$ and $v \in [\![\pi_2]\!]$ such that $\langle u, \langle\langle x,z\rangle, \langle x,z\rangle\rangle\rangle$, $\langle v, \langle\langle z,x\rangle, \langle z,x\rangle\rangle\rangle \in [\![\sigma]\!]$.

By Theorem 6.1, $[\![\pi_1]\!]$ and $[\![\pi_2]\!]$ are complete cliques, thus for all $u', v' \in |\mathscr{C}|$, such that $u' \preceq u$ (respectively, $v' \preceq v$), $u' \in [\![\pi_1]\!]$ (respectively, $v' \in [\![\pi_2]\!]$). In particular, let $w_1,\ldots,w_n$ be the atoms different from z and x in u and v. We define $u' = u[x/w_1,\ldots,x/w_n]$ (respectively, $v' = v[x/w_1,\ldots,x/w_n]$). Since $u' \preceq u$ (respectively, $v' \preceq v$) , $u' \in [\![\pi_1]\!]$ (respectively, $v' \in [\![\pi_2]\!]$). Moreover, since $[\![\sigma]\!]$ is a complete clique too and we have $\langle u', \langle\langle x,z\rangle, \langle x,z\rangle\rangle\rangle \preceq \langle u, \langle\langle x,z\rangle, \langle x,z\rangle\rangle\rangle$ (respectively, $\langle v', \langle\langle x,z\rangle, \langle z,x\rangle\rangle\rangle \preceq \langle v, \langle\langle x,z\rangle, \langle z,x\rangle\rangle\rangle$), we have that $\langle u', \langle\langle x,z\rangle, \langle x,z\rangle\rangle\rangle$, $\langle v', \langle\langle x,z\rangle, \langle z,x\rangle\rangle\rangle \in [\![\sigma]\!]$.

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Now let us look at the atom a (respectively, b) of u' (respectively, v') corresponding to the bold occurrence of X in $C^{\perp} = ((X^{\perp} \parr X^{\perp}) \otimes X^{\perp}) \otimes (X \parr X) \otimes \mathbf{X}$.

If $a = x$ and $b = z$ (or, conversely, $a = z$, $b = x$), then $a \smile b\,[X]$, which implies $u' \smile v'\ \left[C^{\perp}\right]$ by the definition of the coherent spaces associated with $C^{\perp}$. Moreover, $\langle\langle x, z\rangle, \langle x, z\rangle\rangle \smile \langle\langle x, z\rangle, \langle z, x\rangle\rangle$ [$\mathbb{B}$], by the definition of the coherent spaces associated with $\mathbb{B} = (X^{\perp} \otimes X^{\perp}) \parr (X \parr X)$. Thus, $\langle u', \langle\langle x, z\rangle, \langle x, z\rangle\rangle\rangle \smile \langle v', \langle\langle x, z\rangle, \langle z, x\rangle\rangle\rangle\ \left[C^{\perp} \parr \mathbb{B}\right]$, that is, $[\![\sigma]\!]$ is not a clique.

If $a = b$, we assume $a, b = x$ (the case $a, b = z$ being similar). In this case we consider $u'' = u'\left[y/z\right]$ and $v'' = v'\left[z/x, x/z\right]$. Since $\langle u', \langle\langle x, z\rangle, \langle x, z\rangle\rangle\rangle \approx \langle u'', \langle\langle x, y\rangle, \langle x, y\rangle\rangle\rangle$ (respectively, $\langle v', \langle\langle x, z\rangle, \langle z, x\rangle\rangle\rangle \approx \langle v'', \langle\langle z, x\rangle, \langle x, z\rangle\rangle\rangle$), we deduce $\langle u'', \langle\langle x, y\rangle, \langle x, y\rangle\rangle\rangle$, $\langle v'', \langle\langle z, x\rangle, \langle x, z\rangle\rangle\rangle \in [\![\sigma]\!]$. Since $x \smile z$ [X], we infer $u'' \smile v''\ \left[C^{\perp}\right]$ by the definition of the coherent spaces associated with $C^{\perp}$. Moreover, $\langle\langle x, y\rangle, \langle x, y\rangle\rangle \smile \langle\langle z, x\rangle, \langle x, z\rangle\rangle$ [$\mathbb{B}$] by the definition of the coherent spaces associated with $\mathbb{B}$. Thus, $\langle u'', \langle\langle x, y\rangle, \langle x, y\rangle\rangle\rangle \smile \langle v'', \langle\langle z, x\rangle, \langle x, z\rangle\rangle\rangle\ \left[C^{\perp} \parr \mathbb{B}\right]$, that is, $[\![\sigma]\!]$ is not a clique. □

We conclude this section with some remarks on the above proposition.

The failure of the equality between $=_{\beta}$ and $\sim^{w}_{\mathbb{B}}$ does not depend on the formula $\mathbb{B}$ chosen as the type for the observational values. Indeed, for any formula A, we may use $\sim^{w}_{A}$ to denote the observational weak equivalence defined by looking at the correct contexts with conclusion A instead of $\mathbb{B}$, and still get $=_{\beta} \subsetneq \sim^{w}_{A}$.

In the simple typed λ-calculus we can prove a separation theorem (analogous to Theorem 4.1) only if we substitute the atom X with more complex formulas (see Statman (1983) and Joly (2000)). One might thus think that Proposition 7.1 derives from the fact that we have not allowed the substitution of the atom X in Definition 7.1. However, this is not the case. In fact, the atom substitution is useful in the presence of exponentials (as in λ-calculus), but is useless in a linear framework (like MLL). Indeed, the proof of Proposition 7.1 can be easily extended to the case in which we allow the substitution of X with more complex MLL formulas.

The failure of the equality between $=_{\beta}$ and $\sim^{w}_{\mathbb{B}}$ is actually due to the lack of garbage collectors among the correct contexts. Proof structures have garbage collectors (the cyclic cuts, erased by β-reduction), so we can prove Theorem 4.1, but the proof nets (which have to be correct) have no garbage collectors. For example, recall the proof nets π_1, π_2 in Figure 6: π_1 and π_2 are separable by the non-correct proof structure in Figure 7, in fact $[\pi_1, \sigma] \rightarrow_{\beta} \mho$ and $[\pi_2, \sigma] \rightarrow_{\beta} \Omega$. Note that during the reductions of $[\pi_1, \sigma]$ and $[\pi_2, \sigma]$ we meet cyclic cuts.

In this framework there is an interesting result by Matsuoka in Matsuoka (2005) dealing with the intuitionistic multiplicative linear logic fragment (which corresponds to the linear λ-calculus with pairing). The author notes that such a fragment has correct garbage collectors, and from this he proves a separation theorem.

8. Concluding remarks

We have dealt with different equivalences between MLL proofs: the β-equivalence $=_{\beta}$, the equivalences $\equiv_{\mathfrak{Rel}}$ and $\equiv_{\mathfrak{Coh}}$, which are induced by the relational semantics, coherent

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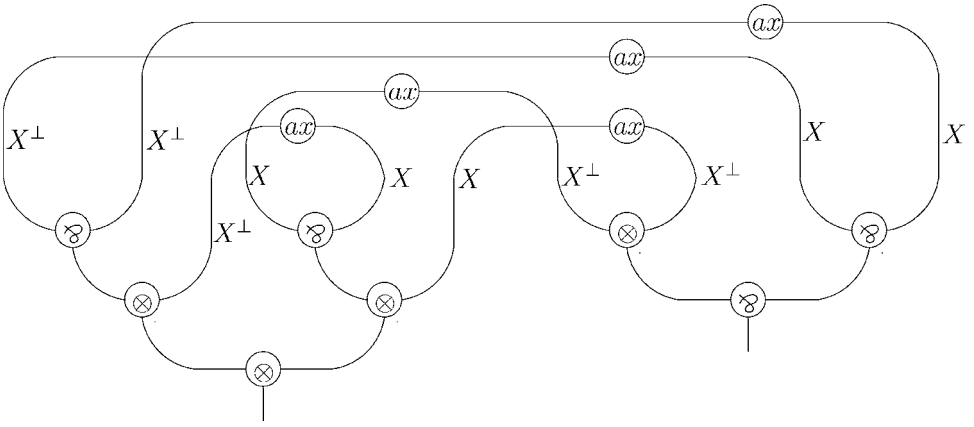


Fig. 7. Non-correct proof structure σ with conclusion $C^\perp, \mathbb{B}$.

semantics and the observational equivalences $\sim_{\mathbb{B}}$ and $\sim^{w}_{\mathbb{B}}$, respectively. By well-known inclusions, we see that $=_\beta \subseteq \equiv_{\mathfrak{Rel}} = \equiv_{\mathfrak{Coh}}$ and $=_\beta \subseteq \sim_{\mathbb{B}} \subseteq \sim^{w}_{\mathbb{B}}$. Theorem 4.1 and Proposition 7.1 complete our knowledge as follows:

$$=_\beta \;=\; \equiv_{\mathfrak{Rel}} \;=\; \equiv_{\mathfrak{Coh}} \;=\; \sim_{\mathbb{B}} \;\subsetneq\; \sim^{w}_{\mathbb{B}}$$

Moreover, Theorem 3.3 (respectively, Corollary 6.1) links the cut-free proof structures (respectively, proof nets) with the complete sets (respectively, complete cliques) of the relational (respectively, coherent) semantics.

We have set our analysis in the multiplicative fragment of linear logic, where the proof nets syntax is simple and clear. Our aim is to extend such an analysis to wider fragments of linear logic. In particular:

MALL. A proof nets syntax for the multiplicative additive linear logic fragment (*MALL*) was introduced in Hughes and van Glabbeek (2003). Such a syntax has provided, for the first time since the beginning of linear logic in 1986, an abstract representation for *MALL* cut-free proofs modulo inessential rule commutation. An open problem is to find semantical notions corresponding to such *MALL* cut-free proof nets, as the analogue of the complete cliques for *MLL* cut-free proof nets.

Blute, Hamano and Scott have actually proved (Blute *et al.* 2005) that hypercoherent semantics provides a semantical notion for the cut-free *MALL* proof nets introduced in Girard (1996). Unfortunately, Girard's proof nets fail to be canonical with respect to *MALL* sequent calculus: a host of such cut-free proof nets can correspond to the same cut-free *MALL* sequent proof. Nevertheless, hypercoherent semantics might allow a characterisation for Hughes and Van Glabbeek's proof nets too.

MELL. All the above-defined equivalences become very complex in the multiplicative exponential linear logic fragment (*MELL*). Tortora de Falco has shown several counter-examples to the coherent semantics injectivity for *MELL* proof nets (Tortora de Falco 2003).

From the observational point of view, David and Py have shown (David and Py 2001) a counter-example to the separability of $\lambda\mu$-calculus. From such a counter-example one

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can define, using the translation described in Laurent (2003), a pair of cut-free proof nets of the polarised fragment of $MELL$ that are observationally indistinguishable (see Pagani (2006)). Up to now, the $MELL$ proof nets syntax has been unable to provide canonical representatives for either the observational equivalence or for the equivalence induced by the coherent semantics. The injectivity of relational semantics for $MELL$ proof nets is still an open problem.

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