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OPTIMAL PORTFOLIO AND CONSUMPTION MODELS UNDER LOSS AVERSION IN INFINITE TIME HORIZON

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This paper investigates continuous-time optimal portfolio and consumption problems under loss aversion in an infinite time horizon. The investor's goal is to choose the optimal portfolio and consumption policies to maximize total discounted S-shaped utility from consumption. The problems are solved under two different situations respectively for the reference level: exogenous or endogenous. For the case of exogenous reference level, which is independent of the consumption policy, the optimal consumption policy and wealth process are obtained through the martingale method and replicating technique. For the case of endogenous reference level, which is related to the past actual consumption, the optimization problem with stochastic reference level is first transformed into an equivalent optimization problem with zero reference point, the corresponding relationship between them is proved, and then the relevant optimal consumption policy and wealth process are also obtained. When the investment opportunity sets are constants, the closed-form solutions of the portfolio and consumption policies are derived under two different situations respectively.

1. INTRODUCTION

There have been many researches on financial portfolio selection models since Merton [11] started this research in 1969. The traditional financial portfolio selection models are based on the assumption that investors are rational and risk averse. However, in the real world, substantial experimental evidences are inconsistent with this rationality hypothesis. Kahneman and Tversky [6,17] found that investors treat gains and losses differently, they are risk averse on gains and risk seeking on losses, and more sensitive to losses than to gains, and what is more, the investors tend to overweight small probabilities. Because of these things,

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Kahneman and Tversky [6] for the first time proposed the concept of loss aversion in the framework of prospect theory, and later [16] defined it for choice under certainty.

Loss aversion is an important psychological concept. In mathematics, it can be represented by a utility function, which is concave for gains and convex for losses, and steeper for losses than for gains. Loss aversion can explain many phenomena such as the endowment effect (Thaler [15]), the status quo bias (Samuelson and Zeckhauser [13]), and the equity premium puzzle (Benartzi and Thaler [1]), which remain paradoxes in traditional selection theory. Therefore, it has frequently been applied in behavioral finance and received more and more attention. Nevertheless, as far as we know, there are very few literature available that investigated continuous-time portfolio optimization problems under loss aversion although it has been proposed for ages.

Berkelara, Kouwenberg, and Post [2] derived closed-form solutions for the optimal portfolio choice under loss aversion by extending the martingale methodology to allow for pseudoconcave utility functions. Jin and Zhou [5] established a general continuous-time portfolio selection model under the cumulated prospect theory (CPT) for the first time. A systematic approach was developed to overcome the immense difficulties arising from the analytically ill-behaved utility model and probability. Zhang, Jin, and Zhou [18] formulated a continuous-time behavioral portfolio selection model where the losses were constrained by a prespecified upper bound and derived the optimal solution explicitly by solving a concave Choquet minimization problem with an additional upper bound. Mi, Bi, and Zhang [12] investigated a dynamic asset allocation model for loss averse investors in a jump-diffusion financial market, where the optimal wealth process and investment strategy were obtained using martingale approach.

However, all of them supposed that the investor was only concerned with the terminal wealth and did not care about the consumption. What is more, the past literature supposed that the reference level was constant or unrelated to the past wealth. Obviously, such a view does not coincide with the facts. Kahneman, Knetsch, and Thaler [7] and Camerer [4] demonstrated that a person's well-being depends not only on his current consumption, but also on how his current consumption compares to his past consumption. Bowman, Minehart, and Rabin [3] indicated people resist decreasing consumption in response to bad news about future income and this resistance is greater than the resistance to increasing consumption in response to good news. Thus, they incorporated Kahneman and Tversky's prospect theory into the gain–loss utility function, which means that the gain–loss utility function was assumed to be S-shaped. Köszegi and Rabin [9,10] defined the agent's reference level of consumption as her own expectations held in the recent past about future consumption choices. Scharoder and Skiadas [14] considered optimal portfolio selection models in finite horizon $[0, T]$, and showed an isomorphism between the models with utilities incorporating linear habit formation and corresponding models without habit formation. Therefore, based on these studies, we investigate the optimal portfolio and consumption problems under loss aversion in an infinite time horizon. What is more, we derive the feedback-form solutions of the portfolio and consumption policies respectively under two different assumptions about the reference level: exogenous – a non-negative constant and endogenous – a stochastic process, which depend on the past actual consumption.

The remainder of this paper is organized as follows. In Section 2, we formulate the optimal portfolio and consumption model under loss aversion. In Section 3, we solve the optimal portfolio and consumption problem with the exogenous reference level, and give the closed-form solutions of the portfolio and consumption policies when the investment opportunity set is constant. Section 4 solves the corresponding problem with the endogenous reference level, and derives the closed-form solutions of the portfolio and consumption policies for the

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case that the investment opportunity set is constant and two-piece power utility function. Finally, Section 5 concludes this work.

2. PROBLEM FORMULATION

We consider an infinite time horizon $(0,\infty)$. Let $(\Omega,\mathcal{F},\mathcal{P},\{\mathcal{F}_t\}_{t\geq 0})$ be a complete filtered probability space. On this space, we define an n-dimensional standard $\mathcal{F}_t$-adapted Brownian motion $W_t=(W_t^1,\ldots,W_t^n)^T$ with $W_0=0$, where the filtration $\mathcal{F}_t=\sigma\{W_\tau:0\leq\tau\leq t\}$. Here and throughout this paper, A^T denotes the transpose of matrix A.

Suppose that there is a frictionless complete market in which $n+1$ assets are traded continuously. One of the assets $S_0(t)$ is riskless (for example, bank account). Here we suppose its dynamics is governed by:

$$dS_0(t)=r(t)S_0(t)dt,\quad S_0(0)=s_0, \tag{1}$$

where for every finite time $T>0$, the progressively measurable risk-free rate process $r(\cdot)$ satisfies

$$\int_0^T |r(t)|dt<+\infty,\quad \text{a.s.}$$

The remaining assets are risky (for example, stocks), their prices $S_i(t),\ i=1,\ldots,n$, are modeled by Ito processes:

$$dS_i(t)=S_i(t)\left(\mu_i(t)dt+\sum_{j=1}^n\sigma_{ij}(t)dW_t^i\right),\quad S_i(0)=s_i,\quad i=1,\ldots,n, \tag{2}$$

where n-dimensional mean rate of return process $\mu(\cdot)=(\mu_1(\cdot),\ldots,\mu_n(\cdot))^T$ and $n\times n$-matrix-valued volatility process $\sigma(\cdot)=(\sigma_{ij}(\cdot))_{n\times n}$ are progressively measurable, and for every finite time $T>0$, they satisfy respectively:

$$\int_0^T \parallel\mu(t)\parallel dt<+\infty,\ \text{a.s.};\quad \sum_{i=1}^n\sum_{j=1}^n\int_0^T\sigma_{ij}^2(t)dt<+\infty,\quad \text{a.s.}$$

where $\|\cdot\|$ denotes the 2-norm of a vector.

Since the market is assumed to be complete, it is well known that this implies the existence of a unique state price density ξ_t, given by

$$\xi_t=\exp\left(-\int_0^t r(s)ds-\frac{1}{2}\int_0^t\|\kappa(s)\|^2ds-\int_0^t\kappa(s)^TdW_s\right),$$

where $\kappa(t)=\sigma^{-1}(t)(\mu(t)-r(t)I)$ denotes the market price of risk process. We can rewrite the state price density process as

$$d\xi_t=-\xi_t(r(t)dt+\kappa(t)^TdW_t),\quad \xi_0=1.$$

Suppose that X_t is the total wealth of the agent at time t. We require $X_t\geq 0$ for $t\in(0,+\infty)$. Let $\pi_t=(\pi_t^1,\ldots,\pi_t^n)^T$ be the fraction invested on the risky assets and C_t be the consumption rate at time t. Thus, the fraction of $1-\sum_{i=1}^n\pi_t^i$ will be invested on the riskless asset. Then, given initial wealth $x>0$, the wealth process can be written as:

$$dX_t=\left(X_t\left[r(t)+(\mu(t)-r(t)I)^T\pi_t\right]-C_t\right)dt+X_t\pi_t^T\sigma(t)dW_t,\quad X_0=x. \tag{3}$$

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DEFINITION 2.1: *A consumption and portfolio processes pair* (C_t, π_t) *in the infinite planning horizon is admissible at* x*, and write* $(C_t, \pi_t) \in \mathcal{A}(x)$ *if it satisfies*

(1) π_t *and* C_t *are* $\mathcal{F}_t$*-progressively measurable;*

(2) the wealth process X_t *corresponding to* x, C_t, π_t *satisfies*

$$X_t \geq 0,\ 0 \leq t < +\infty, \quad a.s.$$

(3) $\mathbf{E}\int_0^\infty min[0, U(t, C_t)]dt > -\infty$.

In this paper, the investor is assumed to treat gains and losses differently, that is, he is risk averse on gains and risk seeking on losses. In other words, we suppose $U(\cdot)$ is an S-shaped utility function:

$$U(x) = \begin{cases} u_1(x), & \text{if } x > 0, \\ u_2(x), & \text{if } x \leq 0, \end{cases}$$

where $u_1(\cdot) : R^+ \to R^+$ is strictly increasing, strictly concave and twice differential, with $u_1(0) = 0,\ u_1'(0+) = +\infty,\ u_1'(+\infty) = 0$, and $u_2(\cdot) : R^- \to R^-$ is strictly increasing and convex with $u_2(0) = 0,\ u_2'(0-) = +\infty,\ u_2'(-\infty) = 0$.

Our goal is then to choose the optimal portfolio and consumption policies $(C_t, \pi_t) \in \mathcal{A}(x)$ to maximize the expected discounted S-shaped utility from consumption, in other words, to maximize the objection function

$$J(\pi_t, C_t) = \mathbf{E}\int_0^\infty e^{-\rho t} U(C_t - l_t)dt, \tag{4}$$

where $\rho > 0$ is the discount rate, l_t is the reference level of consumption.

Then our problem can be rewritten as the following dynamic optimization problem:

$$\begin{aligned} \max_{(C_t,\pi_t)\in\mathcal{A}(x)} \quad & \mathbf{E}\int_0^\infty e^{-\rho t} U(C_t - l_t)dt \\ s.t. \quad & dX_t = \left(X_t\left[r(t) + (\mu(t) - r(t)I)^T\pi_t\right] - C_t\right)dt + X_t\pi_t^T\sigma(t)dW_t, \\ & X_0 = x, \\ & C_t \geq 0,\ X_t \geq 0, \quad \forall\ t \geq 0. \end{aligned} \tag{5}$$

Using the martingale method, the problem (5) can be transformed into the following equivalent static optimization problem:

$$\begin{aligned} \max_{(C_t,\pi_t)\in\mathcal{A}(x)} \quad & \mathbf{E}\int_0^\infty e^{-\rho t} U(C_t - l_t)dt \\ s.t. \quad & \mathbf{E}\int_0^\infty C_t\xi_t dt = x, \\ & C_t \geq 0. \end{aligned} \tag{6}$$

In the following sections, we solve the problem (6) under two different situations for the reference level: exogenous and endogenous, respectively. First, the reference level is assumed to be exogenous, which is independent of the consumption policy, we derive the optimal portfolio and consumption policies, and next, we study a more general case, that is, suppose the reference level is endogenous, which depends on the past actual consumption, also give the relevant conclusion.

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3. CONSTANT REFERENCE LEVEL

In this section, we suppose the reference level l_t is exogenous, which is irrelevant to the consumption and portfolio policy. What is more, for convenience of calculations, we assume that it is a non-negative constant θ, that is, $l_t = \theta,\ \theta \geq 0$.

In fact, if the reference level is a general $\mathcal{F}_t$-measurable random variable (instead of a constant), then, since the market is complete, by considering $C_t - l_t$ as the new consumption rate, the problem (6) can be reduced to one with the zero reference level. And then repeat the following derivation process, we can easily obtain the corresponding conclusion.

3.1. Optimal Consumption Policy and Wealth Process

In this section, we solve the optimization problem (6) with $l_t = \theta,\ \theta \geq 0$, and derive the optimal consumption policy. In the end, the optimal wealth process is given by means of the replicating technique. The following theorem presents the relevant results.

THEOREM 3.1: *Suppose ξ^* is the solution of equation*

$$u_1((u_1')^{-1}(x)) - x[(u_1')^{-1}(x) + \theta] - u_2(-\theta) = 0,$$

and let

$$\mathcal{X}_\infty(y) = \mathbf{E}\int_0^\infty \xi_t \left[(u_1')^{-1}(ye^{\rho t}\xi_t) + \theta\right] I_{\{ye^{\rho t}\xi_t < \xi^*\}} dt. \tag{7}$$

If $\mathcal{X}_\infty(y) < +\infty$, then the optimal consumption policy of the loss averse investor is

$$C_t^* = \begin{cases} \theta + (u_1')^{-1}(ye^{\rho t}\xi_t), & \text{if } ye^{\rho t}\xi_t \leq \xi^*, \\ 0, & \text{if } ye^{\rho t}\xi_t > \xi^*, \end{cases} \tag{8}$$

and the optimal wealth process X_t^* is

$$X_t^* = \frac{1}{\xi_t}\mathbf{E}\left(\int_t^\infty \xi_s \left[(u_1')^{-1}(ye^{\rho s}\xi_s) + \theta\right] I_{\{ye^{\rho s}\xi_s \leq \xi^*\}} ds | \mathcal{F}_t\right), \tag{9}$$

where $y \geq 0$ satisfies

$$\mathcal{X}_\infty(y) = x.$$

PROOF: Let

$$\tilde{U}(C_t) = e^{-\rho t}U(C_t - \theta) - yC_t\xi_t, \tag{10}$$

we can easily see that if C_t^* satisfies the constraint conditions in (6) and makes $\tilde{U}$ reach the maximum value, then C_t^* is the optimal solution of problem (6).

In fact, $\forall\ C_t$ satisfying the constraint conditions in (6), we have

$$\begin{aligned} &\mathbf{E}\int_0^\infty e^{-\rho t}U(C_t^* - \theta)dt - E\int_0^\infty e^{-\rho t}U(C_t - \theta)dt \\ &= \mathbf{E}\int_0^\infty \left([e^{-\rho t}U(C_t^* - \theta) - yC_t^*\xi_t] - [e^{-\rho t}U(C_t - \theta) - yC_t\xi_t]\right) dt \\ &\geq 0. \end{aligned}$$

Now, we solve the maximization problem (10) using the method of Berkelaar, Kouwenberg, and Post [2].

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If $C_t \le \theta$, $U(C_t - \theta)$ is convex, and (10) reaches the maximum value when $C_{1,t} = 0$ or $C_{2,t} = \theta$. On the other hand, if $C_t > \theta$, $U(C_t - \theta)$ is concave, and there exists a unique optimal solution $C_{3,t}$ satisfying the following conditions:

$$\begin{cases} e^{-\rho t}u_1'(C_{3,t} - \theta) - y\xi_t + \lambda = 0, \\ \lambda C_{3,t} = 0. \end{cases}$$

Thus,

$$C_{3,t} = (u_1')^{-1}(ye^{\rho t}\xi_t) + \theta.$$

Next, all we need to do is to compare the local optimal solutions to determine the global optimal solution. Firstly, because $u_1(x)$ is concave, we have

$$\begin{aligned} \tilde{U}(C_{3,t}) - \tilde{U}(\theta) &= e^{-\rho t}u_1((u_1')^{-1}(ye^{\rho t}\xi_t)) - y((u_1')^{-1}(ye^{\rho t}\xi_t) + \theta)\xi_t + y\theta\xi_t \\ &= e^{-\rho t}u_1[(u_1')^{-1}(ye^{\rho t}\xi_t)] - e^{-\rho t}u_1(0) - y(u_1')^{-1}(ye^{\rho t}\xi_t)\xi_t \\ &= e^{-\rho t}u_1'(\tau)(u_1')^{-1}(ye^{\rho t}\xi_t) - y(u_1')^{-1}(ye^{\rho t}\xi_t)\xi_t \\ &\ge e^{-\rho t}u_1'((u_1')^{-1}(ye^{\rho t}\xi_t))(u_1')^{-1}(ye^{\rho t}\xi_t) - y(u_1')^{-1}(ye^{\rho t}\xi_t)\xi_t \\ &= 0, \end{aligned}$$

where $\tau \in (0, (u_1')^{-1}(ye^{\rho t}\xi_t))$. Hence, $C_{2,t} = \theta$ is never the optimal solution.

And secondly,

$$\begin{aligned} &\tilde{U}(C_{3,t}) - \tilde{U}(0) \\ &= e^{-\rho t}u_1((u_1')^{-1}(ye^{\rho t}\xi_t)) - y[(u_1')^{-1}(ye^{\rho t}\xi_t) + \theta]\xi_t - e^{-\rho t}u_2(-\theta) \\ &= e^{-\rho t}u_1[(u_1')^{-1}(ye^{\rho t}\xi_t)] - e^{-\rho t}u_1(0) - y[(u_1')^{-1}(ye^{\rho t}\xi_t) + \theta]\xi_t - e^{-\rho t}u_2(-\theta) \\ &\ge e^{-\rho t}u_1'((u_1')^{-1}(ye^{\rho t}\xi_t))(u_1')^{-1}(ye^{\rho t}\xi_t) - y[(u_1')^{-1}(ye^{\rho t}\xi_t) + \theta]\xi_t - e^{-\rho t}u_2(-\theta) \\ &= -e^{-\rho t}[y\theta e^{\rho t}\xi_t + u_2(-\theta)]. \end{aligned} \tag{11}$$

Let

$$g(ye^{\rho t}\xi_t) = e^{\rho t}[\tilde{U}(C_{3,t}) - \tilde{U}(0)],$$

that is,

$$g(x) = u_1((u_1')^{-1}(x)) - x[(u_1')^{-1}(x) + \theta] - u_2(-\theta).$$

Evidently though, if $g(ye^{\rho t}\xi_t) > 0$, then $\tilde{U}(C_{3,t}) > \tilde{U}(0)$, that is, $C_{3,t}$ is the optimal solution. Otherwise, 0 is the optimal solution.

On the one hand, it can be easily proved that

$$g'(x) = -\theta - (u_1')^{-1}(x) < 0,$$

that is, $g(x)$ is strictly decreasing, and $g(x) \to -\infty$ when $x \to +\infty$. On the other hand, (11) implies that

$$g(-\frac{u_2(-\theta)}{\theta}) \ge -\theta\frac{-u_2(-\theta)}{\theta} - u_2(-\theta) = 0.$$

Therefore, $g(x)$ has a unique null point in $(-(u_2(-\theta)/\theta), +\infty)$. We denote this null point by ξ^*.

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To sum up, when $ye^{\rho t}\xi_t < \xi^*$, $C_{3,t}$ is the optimal solution, otherwise, 0 is the optimal solution. That is to say, the optimal consumption policy of the loss averse investor is

$$C_t^* = \begin{cases} \theta + (u_1')^{-1}(ye^{\rho t}\xi_t), & \text{if } ye^{\rho t}\xi_t \le \xi^*, \\ 0, & \text{if } ye^{\rho t}\xi_t > \xi^*. \end{cases} \tag{12}$$

According to Theorem 9.11 of Karatzas and Shreve [8] (p. 141), if $\mathcal{X}_\infty(y) < +\infty$, then the optimal wealth process is

$$\begin{aligned} X_t^* &= \frac{1}{\xi_t}\mathbf{E}\left(\int_t^\infty \xi_s C_s^* ds|\mathcal{F}_t\right) \\ &= \frac{1}{\xi_t}\mathbf{E}\left(\int_t^\infty \xi_s \left[(u_1')^{-1}(ye^{\rho s}\xi_s) + \theta\right] I_{\{ye^{\rho s}\xi_s \le \xi^*\}} ds|\mathcal{F}_t\right). \end{aligned}$$

■

Based on the above results, we can see that the investor has two different kinds of behavior in the good states ($ye^{\rho t}\xi_t \le \xi^*$) and the bad states ($ye^{\rho t}\xi_t > \xi^*$) of market. What is more, the size of the good-states region not only relates to the initial wealth x, but also depends on the reference level θ. In the good states of market ($ye^{\rho t}\xi_t \le \xi^*$), the investor consumes more than the reference level, while in the bad states of market ($ye^{\rho t}\xi_t > \xi^*$), he does not consume at all. Since $\mathcal{X}_\infty(y)$ is a monotony decreasing function, that is to say, when x increases, y decreases. Consequently, the good-states region increases. And because $g(x)$ is strictly decreasing and $u_2(x)$ is strictly increasing, when the reference point θ increases, ξ^* decreases, which means that the good-states region decreases. In other words, if the prospective consumption level of the investor is too high or the initial wealth is not enough, bad situation is more likely to happen. In this case, the loss averse investor had better consume nothing at all.

3.2. Optimal Portfolio Policy with the Constant Investment Opportunity Set

In this section, we assume that the investment opportunity set is constant:

$$r(\cdot) =: r,\ \kappa(\cdot) =: \kappa, \quad \frac{1}{2} \parallel \kappa \parallel^2 =: \gamma.$$

We give an equivalent condition for $\mathcal{X}_\infty(y) < +\infty$ and drive the closed-form solutions of the optimal consumption and portfolio policies. And at last, a concrete example is solved to demonstrate the general results obtained in this section.

Firstly, when all the market parameters are constant (matrix), we give an equivalent condition for $\mathcal{X}_\infty(y) < +\infty$, and its expression.

LEMMA 3.2: *The condition*

$$\int_0^1 \eta^{-\delta_2} v(\eta) d\eta < +\infty$$

is equivalent to $\mathcal{X}_\infty(y) < +\infty$, *and under this condition,* $\forall\ y > 0$,

$$\begin{aligned} \mathcal{X}_\infty(y) = \frac{1}{\gamma(\delta_1 - \delta_2)} \Bigg[& y^{\delta_2 - 1} \int_0^{\xi^*} \eta^{-\delta_2} v(\eta) d\eta\ I_{\{y > \xi^*\}} \\ & + \left(y^{\delta_2 - 1} \int_0^y \eta^{-\delta_2} v(\eta) d\eta + y^{\delta_1 - 1} \int_y^{\xi^*} \eta^{-\delta_1} v(\eta) d\eta \right) I_{\{y \le \xi^*\}} \Bigg], \end{aligned}$$

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and

$$\gamma y^2 \mathcal{X}''_\infty(y) + (3\gamma - \tilde{\gamma}) y \mathcal{X}'_\infty(y) - r\mathcal{X}_\infty(y) + v(y) I_{\{y \le \xi^*\}} = 0,$$

where

$$v(x) =: (u_1')^{-1}(x) + \theta, \quad \tilde{\gamma} =: r + \gamma - \rho.$$
$$\delta_i =: \frac{\tilde{\gamma}}{2\gamma} - (-1)^i \frac{\sqrt{4\rho\gamma + \tilde{\gamma}^2}}{2\gamma}, \quad i = 1, 2.$$

PROOF: When all the market parameters are constants (matrix),

$$\xi_t = \exp(-(r+\gamma)t - \kappa^T W_t),$$

thus

$$\begin{aligned}\mathcal{X}_\infty(y) &= E\int_0^\infty \xi_t \left[(u_1')^{-1}(ye^{\rho t}\xi_t) + \theta\right] I_{\{ye^{\rho t}\xi_t \le \xi^*\}} dt \\ &= \int_{-\infty}^{+\infty}\int_0^\infty \exp(-(r+\gamma)t - \kappa^T x) v(y\exp(-\tilde{\gamma}t - \kappa^T x)) \\ &\quad \times \frac{1}{\sqrt{2\pi t}} e^{-(x^2/2t)} I_{\{-\tilde{\gamma}t - \kappa^T x \le \ln \xi^* - \ln y\}} dt dx.\end{aligned}$$

Define

$$z = \tilde{\gamma}t + \kappa^T x,$$

so

$$\begin{aligned}\mathcal{X}_\infty(y) &= \int_{-\infty}^{+\infty}\int_0^\infty e^{-z-\rho t} v(ye^{-z}) \cdot \frac{1}{\sqrt{4\pi\gamma t}} e^{-((z-\tilde{\gamma}t)^2/4\gamma t)} I_{\{z \ge \ln y - \ln \xi^*\}} dt dz \\ &= \int_{-\infty}^{+\infty}\int_0^\infty e^{\tilde{\gamma} - 2\gamma/2\gamma z} v(ye^{-z}) I_{\{z \ge \ln y - \ln \xi^*\}} \cdot \frac{1}{\sqrt{4\pi\gamma t}} e^{-(z^2/4\gamma t) - ((\tilde{\gamma}^2/4\gamma t) + \rho)t} dt dz.\end{aligned}$$

Taking account of the following equality:

$$\int_0^\infty \frac{1}{\sqrt{\pi t}} e^{-(a/4t) - pt} dt = \frac{e^{-\sqrt{ap}}}{\sqrt{p}}, \quad \forall\ a > 0, \quad \forall\ p > 0,$$

we can get

$$\mathcal{X}_\infty(y) = \frac{1}{\sqrt{4\rho\gamma + \tilde{\gamma}^2}} \int_{-\infty}^{+\infty} \exp\left(\frac{\tilde{\gamma} - 2\gamma}{2\gamma} z - \frac{\sqrt{4\rho\gamma + \tilde{\gamma}^2}}{2\gamma} |z|\right) \cdot v(ye^{-z}) I_{\{z \ge \ln y - \ln \xi^*\}} dz.$$

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We can deduce that $\delta_1 > 0$ and $\delta_2 < 0$ from the definition of δ_1 and δ_2. Therefore,

$$\begin{aligned}\mathcal{X}_\infty(y) &= \frac{1}{\gamma(\delta_1-\delta_2)}\left[\int_0^\infty e^{(\delta_2-1)z}v(ye^{-z})I_{\{z\geq \ln y-\ln\xi^*\}}dz\right.\\ &\quad\left.+\int_{-\infty}^0 e^{(\delta_1-1)z}v(ye^{-z})I_{\{z\geq \ln y-\ln\xi^*\}}dz\right]\\ &= \frac{1}{\gamma(\delta_1-\delta_2)}\left[\int_0^y y^{\delta_2-1}\eta^{-\delta_2}v(\eta)I_{\{\eta\leq\xi^*\}}d\eta+\int_y^\infty y^{\delta_1-1}\eta^{-\delta_1}v(\eta)I_{\{\eta\leq\xi^*\}}d\eta\right]\\ &= \frac{1}{\gamma(\delta_1-\delta_2)}\left[y^{\delta_2-1}\int_0^{\xi^*}\eta^{-\delta_2}v(\eta)d\eta\ I_{\{y>\xi^*\}}\right.\\ &\quad\left.+\left(y^{\delta_2-1}\int_0^y\eta^{-\delta_2}v(\eta)d\eta+y^{\delta_1-1}\int_y^{\xi^*}\eta^{-\delta_1}v(\eta)d\eta\right)I_{\{y\leq\xi^*\}}\right].\end{aligned}$$

Since

$$\int_0^1\eta^{-\delta_2}v(\eta)d\eta<+\infty$$

is equivalent to

$$\int_0^\varepsilon\eta^{-\delta_2}v(\eta)d\eta<+\infty,\quad\forall\,\varepsilon>0,$$

we can know that $\mathcal{X}_\infty(y)<+\infty$ if and only if

$$\int_0^1\eta^{-\delta_2}v(\eta)d\eta<+\infty.$$

Take the derivative of the function $\mathcal{X}_\infty(y)$,

$$\begin{aligned}\mathcal{X}'_\infty(y) &= \frac{1}{\gamma(\delta_1-\delta_2)}\left[(\delta_2-1)\int_0^y y^{\delta_2-2}\eta^{-\delta_2}v(\eta)I_{\{\eta\leq\xi^*\}}d\eta\right.\\ &\quad\left.+(\delta_1-1)\int_y^\infty y^{\delta_1-2}\eta^{-\delta_1}v(\eta)I_{\{\eta\leq\xi^*\}}d\eta\right],\end{aligned}$$

and

$$\begin{aligned}\mathcal{X}''_\infty(y) &= \frac{1}{\gamma(\delta_1-\delta_2)}\left[(\delta_2-1)(\delta_2-2)\int_0^y y^{\delta_2-3}\eta^{-\delta_2}v(\eta)I_{\{\eta\leq\xi^*\}}d\eta\right.\\ &\quad+(\delta_1-1)(\delta_1-2)\int_y^\infty y^{\delta_1-3}\eta^{-\delta_1}v(\eta)I_{\{\eta\leq\xi^*\}}d\eta\\ &\quad\left.+(\delta_2-\delta_1)y^{-2}v(\eta)I_{\{\eta\leq\xi^*\}}\right].\end{aligned}$$

Hence, we can obtain that for all $y>0$,

$$\gamma y^2\mathcal{X}''_\infty(y)+(3\gamma-\tilde{\gamma})y\mathcal{X}'_\infty(y)-r\mathcal{X}_\infty(y)+v(y)I_{\{y\leq\xi^*\}}=0.$$

■

And next, in the same way as Corollary 9.15 of Karatzas and Shreve [8](p. 144), we can get the following theorem.

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THEOREM 3.3: *If $\mathcal{X}_\infty(y) < +\infty$, then the optimal wealth process for problem (6) is given by*

$$X_t^* = \mathcal{X}_\infty(ye^{\rho t}\xi_t),$$

and the feedback form solutions for the optimal portfolio and consumption policies are respectively

$$C_t^* = [\theta + (u_1')^{-1}(ye^{\rho t}\xi_t)]I_{\{ye^{\rho t}\xi_t \le \xi^*\}},$$
$$\pi_t^* = -\frac{(\sigma)^{-1}\kappa ye^{\rho t}\xi_t \mathcal{X}_\infty'(ye^{\rho t}\xi_t)}{\mathcal{X}_\infty(ye^{\rho t}\xi_t)},$$

where $y \ge 0$ satisfies $\mathcal{X}_\infty(y) = x$.

At last, we give the following example to demonstrate the effectiveness of our results, and give the analytical expression of the optimal portfolio policy.

Example 3.4: Suppose that $U(\cdot)$ is a two-piece power utility function proposed by Tversky and Kahneman [17]:

$$U(x) = \begin{cases} u_1(x) = x^\alpha, & \text{if } x > 0, \\ u_2(x) = -\eta(-x)^\beta, & \text{if } x \le 0, \end{cases} \tag{13}$$

where $0 < \alpha \le \beta < 1,\ \eta > 1$, η describes the coefficient of loss aversion.

Evidently though,

$$u_1(x) = x^\alpha, \quad u_1'(x) = \alpha x^{\alpha-1}, \quad (u_1')^{-1}(x) = \left(\frac{x}{\alpha}\right)^{\frac{1}{\alpha-1}}.$$

Define ξ^* as the solution of the equation

$$(1-\alpha)\left(\frac{x}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \theta x + \eta\theta^\beta = 0,$$

and

$$\delta_5 = \frac{1}{\alpha-1} - \delta_1 + 1, \quad \delta_6 = \frac{1}{\alpha-1} - \delta_2 + 1,$$

it is obvious that $\delta_5 < 0$, and

$$\int_0^1 \eta^{-\delta_2}\left[(u_1')^{-1}(\eta) + \theta\right] d\eta = \frac{\alpha^{\frac{1}{\alpha-1}}}{\delta_6} + \frac{\theta}{1-\delta_2} < +\infty$$

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for $\delta_6 > 0$. Thus, now let us suppose $\delta_6 > 0$, based on Lemma 3.2, we have

$$\begin{aligned}
&\gamma(\delta_1-\delta_2)\mathcal{X}_\infty(y)\\
&\quad= y^{\delta_2-1}\int_0^{\xi^*}\eta^{-\delta_2}\left[\left(\frac{\eta}{\alpha}\right)^{\frac{1}{\alpha-1}}+\theta\right]I_{\{\xi^*<y\}}d\eta+I_{\{\xi^*\geq y\}}\\
&\qquad\times\left(y^{\delta_2-1}\int_0^{y}\eta^{-\delta_2}\left[\left(\frac{\eta}{\alpha}\right)^{\frac{1}{\alpha-1}}+\theta\right]d\eta+y^{\delta_1-1}\int_y^{\xi^*}\eta^{-\delta_1}\left[\left(\frac{\eta}{\alpha}\right)^{\frac{1}{\alpha-1}}+\theta\right]d\eta\right)\\
&\quad=\left(\frac{\theta}{1-\delta_2}\xi^{*(1-\delta_2)}+\frac{1}{\delta_6\alpha^{\frac{1}{\alpha-1}}}\xi^{*\delta_6}\right)y^{\delta_2-1}I_{\{y>\xi^*\}}+\left(\frac{\theta}{1-\delta_2}-\frac{\theta}{1-\delta_1}\right.\\
&\qquad\left.+\frac{1}{\delta_5\alpha^{\frac{1}{\alpha-1}}}\xi^{*\delta_5}y^{\delta_1-1}+\left(\frac{1}{\delta_6}-\frac{1}{\delta_5}\right)\frac{1}{\alpha^{\frac{1}{\alpha-1}}}y^{\frac{1}{\alpha-1}}+\frac{\theta}{1-\delta_1}\xi^{*(1-\delta_1)}y^{\delta_1-1}\right)I_{\{y\leq\xi^*\}},\\
&\mathcal{X}_\infty(\xi^*)=\frac{1}{\delta_6\alpha^{\frac{1}{\alpha-1}}}\xi^{*\frac{1}{\alpha-1}}+\frac{\theta}{1-\delta_2},
\end{aligned}$$

and

$$\begin{aligned}
\gamma(\delta_1-\delta_2)\chi'_\infty(y)&=\left(\frac{\theta}{1-\delta_2}\xi^{*(1-\delta_2)}+\frac{1}{\delta_6\alpha^{\frac{1}{\alpha-1}}}\xi^{*\delta_6}\right)(\delta_2-1)y^{\delta_2-2}I_{\{y>\xi^*\}}\\
&\quad+\left(\frac{1}{\delta_6}-\frac{1}{\delta_5}\right)\frac{1}{\alpha^{\frac{1}{\alpha-1}}(\alpha-1)}y^{(\frac{1}{\alpha-1})-1}I_{\{y\leq\xi^*\}}\\
&\quad-\left(\theta\xi^{*(1-\delta_1)}+\frac{1-\delta_1}{\delta_5\alpha^{\frac{1}{\alpha-1}}}\xi^{*\delta_5}\right)y^{\delta_1-2}I_{\{y\leq\xi^*\}}.
\end{aligned}$$

Let

$$\begin{aligned}
A&=\left(\frac{1}{\delta_6}-\frac{1}{\delta_5}\right)\frac{1}{\alpha^{\frac{1}{\alpha-1}}(1-\alpha)}>0,\\
B&=\theta\xi^{*(1-\delta_1)}+\frac{1-\delta_1}{\delta_5\alpha^{\frac{1}{\alpha-1}}}\xi^{*\delta_5},\\
C&=\frac{\theta}{1-\delta_2}-\frac{\theta}{1-\delta_1},\\
z&=ye^{\rho t}\xi_t.
\end{aligned}$$

Hence, according to Theorem 3.3, the optimal consumption and portfolio policies are respectively

$$C_t^*=\begin{cases}\theta+\left(\frac{z}{\alpha}\right)^{\frac{1}{\alpha-1}}, & \text{if } z\leq\xi^*,\\ 0, & \text{if } z>\xi^*,\end{cases}\tag{14}$$

and

$$\pi_t^*=\begin{cases}\sigma^{-1}\kappa\dfrac{Az^{\delta_5}+B}{(1-\alpha)Az^{\delta_5}+\frac{1}{1-\delta_1}B+Cz^{1-\delta_1}}, & \text{if } z\leq\xi^*,\\ \sigma^{-1}\kappa(1-\delta_2), & \text{if } z>\xi^*,\end{cases}\tag{15}$$

where $y\geq 0$ satisfies $\mathcal{X}_\infty(y)=x$.

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From the above results, we can easily see that the optimal portfolio policy π_t^* is continuous with respect to y, and it has different forms in different states. In the bad states of market ($ye^{\rho t}\xi_t > \xi^*$), the loss averse investor invests in stocks with a fixed proportion, which is the same as the classical CRRA model, but at this time he consumes nothing at all. While in the good states of market ($ye^{\rho t}\xi_t \leq \xi^*$), the contrary is the case, the loss averse investor consumes like the CRRA investor, meanwhile he should adjust and change incessantly his optimal portfolio policy over time rather than stay the same. Moreover, when ξ_t goes to the two different extreme cases, both of the two optimal portfolio policies go to the optimal portfolio policies of the different CRRA investors respectively. Note that as ξ_t goes to the infinity, the optimal consumption policy C_t^* goes to θ and the fraction invested in stocks π_t^* goes to $\sigma^{-1}\kappa(1-\delta_2)$, which is equivalent to the optimal portfolio policy of the CRRA investor with $\gamma = 1 - 1/(1-\delta_2)$. When ξ_t goes to 0, the optimal consumption policy C_t^* goes to 0 and the fraction invested in stocks π_t^* goes to $\sigma^{-1}\kappa/(1-\alpha)$, which is equivalent to the optimal portfolio policy of the CRRA investor with $\gamma = \alpha$. In addition,

$$\lim_{y\to 0} \pi_t^* = \frac{\sigma^{-1}\kappa}{1-\alpha} < \sigma^{-1}\kappa(1-\delta_2),$$

and

$$\lim_{y\to 0} \mathcal{X}_\infty(y) = +\infty,$$

that is to say, when the loss averse investor has more enough money, the proportion invested in stocks reduces instead of rising. To some extent, this is due to the risk aversion of the investor in this moment.

4. ENDOGENOUS REFERENCE LEVEL

In this section, we suppose that the reference level l_t is a stochastic process, which depends on the past consumption rather than a non-negative constant as it used to be assumed. This is more plausible. Especially, we suppose $l_t = b_t\theta_t$, $b_t \geq 0$, and θ_t has the following form:

$$\theta_t = \int_0^t e^{-\int_s^t a_u du} C_s ds + e^{-\int_0^t a_u du}\bar{\theta}_0,$$

that is,

$$d\theta_t = (C_t - a_t\theta_t)dt,\ \theta_0 = \bar{\theta}_0. \tag{16}$$

where $\bar{\theta}_0 \geq 0$ and $a_t \geq 0$, $\forall\, t \in [0,T]$. This is a definition of reference level, which is related to past actual consumption (see, e.g., Scharoder and Skiadas [14]). We can see that θ_t is the sum total of the past own actual cumulative consumption and the initial reference level of consumption, and b_t measures the importance of this reference level. In this linear case, the optimal portfolio and consumption problem with this stochastic reference level can be mechanically transformed to an equivalent problem with zero reference level. In particular, when $b_t = 0$, the reference level $b_t\theta_t = 0$, then this model becomes the classical risk averse situation. When $b_t = b$ is a nonzero constant, let $\widetilde{\theta}(t) = b\theta_t$ and $\widetilde{\theta}_0 = b\bar{\theta}_0$, then the reference

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level becomes

$$\widetilde{\theta}(t) = b\int_0^t e^{-\int_s^t a_u du} C_s ds + e^{-\int_0^t a_u du}\widetilde{\theta}_0,$$

that is to say,

$$d\widetilde{\theta}(t) = (bC_t - a_t\widetilde{\theta}(t))dt,\ \widetilde{\theta}(0) = \widetilde{\theta}_0. \tag{17}$$

At this time, $\widetilde{\theta}(0) > 0$ may be explained by the initial reference level of consumption based on the early experiences. $b > 0$ measures the importance of past own consumption choices relative to $\widetilde{\theta}_0$. Past consumption choices become relatively more important when b increases. And $a_t \geq 0$ indicates the rate of deprecation of the reference level of consumption $\widetilde{\theta}(t)$, and $\widetilde{\theta}(t)$ adjusts relatively more quickly in response to consumption shocks when a_t increases.

In a more general sense, it is more plausible the reference level obeys the following stochastic differential equation with a bounded memory:

$$\begin{aligned} d\theta_s &= f(s,\theta_s,C_s,Y_s)ds + g(s,\theta_s,C_s,Y_s)dW_s,\quad \forall\ s\in[t,T],\\ Y_s &= \int_{-h}^0 e^{-\rho\tau}C_{t-\tau}d\tau,\ h>0. \end{aligned}$$

This would be discussed in our further work.

4.1. Transformation

In this section, we transform the optimization problem (6) with endogenous reference level to an equivalent optimization problem with zero reference point, and prove the corresponding relationship between them.

We first do the following transformation:

$$\hat{C}_t = C_t - b_t\theta_t, \tag{18}$$

and

$$\hat{\theta}_t(\hat{C}) = \int_0^t e^{-\int_s^t (a_u-b_u)du}\hat{C}_s ds + e^{-\int_0^t (a_u-b_u)du}\bar{\theta}_0 =: \int_0^t \frac{h_t}{h_s}\hat{C}_s ds + h_t\bar{\theta}_0, \tag{19}$$

where

$$h_t = e^{-\int_0^t (a_u-b_u)du}.$$

We can see that

$$\theta_t(C) = \hat{\theta}_t(\hat{C}).$$

In fact, on the one hand, differential $\hat{\theta}_t(\hat{C})$, we have

$$d\hat{\theta}_t(\hat{C}) = [\hat{C}_t + (b_t - a_t)\hat{\theta}_t(\hat{C})]dt,\quad \hat{\theta}_0(\hat{C}) = \bar{\theta}_0,$$

and on the other hand, from Eq. (16), we get that

$$d\theta_t(C) = (C_t - a_t\theta_t(C))dt = [\hat{C}_t + b_t\theta_t(C) - a_t\theta_t(C)]dt,\quad \theta_0(C) = \bar{\theta}_0.$$

Thus, according to the existence and uniqueness theorem for linear ordinary differential equations,

$$\theta_t(C) = \hat{\theta}_t(\hat{C}).$$

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In what follows, we always assume that

$$\mathbf{E}\int_0^\infty \xi_s b_s h_s ds < +\infty,$$

which is easy to be satisfied, then let

$$\zeta_t = \frac{\mathbf{E}\left(\int_t^\infty \xi_s b_s h_s ds|\mathcal{F}_t\right)}{\xi_t h_t}, \quad \hat{\xi}_t = (1+\zeta_t)\xi_t.$$

Now we prove the following theorem, which demonstrates the corresponding relationship between before and after the transformation:

THEOREM 4.1: *If* $\mathbf{E} \mid \int_0^\infty \hat{\xi}_s \hat{C}_s ds \mid < +\infty$, *then* $\mathbf{E} \mid \int_0^\infty \xi_s C_s ds \mid < +\infty$, *and*

$$\xi_t(\theta_t(C) + X_t) = \hat{\xi}_t(\hat{\theta}_t(\hat{C}) + \hat{X}_t),$$

where

$$X_t = \frac{\mathbf{E}\left(\int_t^\infty \xi_s C_s ds|\mathcal{F}_t\right)}{\xi_t}, \quad \hat{X}_t = \frac{\mathbf{E}\left(\int_t^\infty \hat{\xi}_s \hat{C}_s ds|\mathcal{F}_t\right)}{\hat{\xi}_t}.$$

PROOF: Firstly, if $\mathbf{E} \mid \int_0^\infty \hat{\xi}_s \hat{C}_s ds \mid < +\infty$, according to the definitions of ζ_t, $\hat{\xi}_t$ and $\hat{X}_t$, we can get

$$\begin{aligned}\hat{\xi}_t(\hat{\theta}_t(\hat{C}) + \hat{X}_t) &= \xi_t(1+\zeta_t)\hat{\theta}_t(\hat{C}) + \mathbf{E}\left(\int_t^\infty \xi_s(1+\zeta_s)\hat{C}_s ds|\mathcal{F}_t\right) \\ &= \xi_t\theta_t(C) + \mathbf{E}\left(\int_t^\infty \xi_s \hat{C}_s ds|\mathcal{F}_t\right) + h_t^{-1}\hat{\theta}_t(\hat{C})\mathbf{E}\left(\int_t^\infty \xi_s b_s h_s ds|\mathcal{F}_t\right) \\ &\quad + \mathbf{E}\left(\int_t^\infty \hat{C}_s h_s^{-1}\mathbf{E}\left(\int_s^\infty \xi_u b_u h_u du|\mathcal{F}_s\right) ds|\mathcal{F}_t\right).\end{aligned} \tag{20}$$

Thus, based on the definition of conditional expectation,

$$\mathbf{E}\left|\int_0^\infty \hat{C}_s h_s^{-1}\left(\int_s^\infty \xi_u b_u h_u du\right) ds\right| < +\infty.$$

And then, $\forall\ T > 0$, we can use integration by parts to obtain

$$\mathbf{E}\left(\int_t^T \xi_s b_s h_s\left(\int_t^s \hat{C}_u h_u^{-1} du\right) ds|\mathcal{F}_t\right) = \mathbf{E}\left(\int_t^T \hat{C}_s h_s^{-1}\left(\int_s^T \xi_u b_u h_u du\right) ds|\mathcal{F}_t\right).$$

Thus, let $t \to +\infty$, we can obtain

$$\mathbf{E}\left(\int_t^\infty \xi_s b_s h_s\left(\int_t^s \hat{C}_u h_u^{-1} du\right) ds|\mathcal{F}_t\right) = \mathbf{E}\left(\int_t^\infty \hat{C}_s h_s^{-1}\left(\int_s^\infty \xi_u b_u h_u du\right) ds|\mathcal{F}_t\right). \tag{21}$$

In addition, from Eq. (19), evidently though,

$$\hat{\theta}_s(\hat{C}) = \theta_s(C) = h_s h_t^{-1}\hat{\theta}_t(\hat{C}) + h_s\int_t^s h_u^{-1}\hat{C}_u du. \tag{22}$$

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Hence, substitute (21) and (22) into (20), we have

$$\begin{aligned}\hat{\xi}_t(\hat{\theta}_t(\hat{C})+\hat{X}_t) &= \xi_t\theta_t(C)+\mathbf{E}\left(\int_t^\infty \xi_s\hat{C}_s ds|\mathcal{F}_t\right)+h_t^{-1}\hat{\theta}_t(\hat{C})\mathbf{E}\left(\int_t^\infty \xi_s b_s h_s ds|\mathcal{F}_t\right)\\ &\quad+\mathbf{E}\left(\int_t^\infty \xi_s b_s h_s\left(\int_t^s \hat{C}_u h_u^{-1}du\right)ds|\mathcal{F}_t\right)\\ &= \xi_t\theta_t(C)+\mathbf{E}\left(\int_t^\infty \xi_s\hat{C}_s ds|\mathcal{F}_t\right)\\ &\quad+\mathbf{E}\left(\int_t^\infty \xi_s b_s h_s\left(h_t^{-1}\hat{\theta}_t(\hat{C})+\int_t^s \hat{C}_u h_u^{-1}du\right)ds|\mathcal{F}_t\right)\\ &= \xi_t\theta_t(C)+\xi_t X_t.\end{aligned}$$

The above equation implies that

$$\mathbf{E}\left|\int_0^\infty \xi_s C_s ds\right| < +\infty.$$

The proof is completed. ■

Therefore, according to the above theorem, when $t=0$, we have

$$\begin{aligned}\xi_0\theta_0(C)+\xi_0 X_0 &= \bar{\theta}_0+\mathbf{E}\int_0^\infty \xi_s C_s ds\\ &= \bar{\theta}_0+x\\ &= (1+\zeta_0)\bar{\theta}_0+\mathbf{E}\int_0^\infty \hat{\xi}_s\hat{C}_s ds,\end{aligned}$$

so,

$$\mathbf{E}\int_0^\infty \hat{\xi}_s\hat{C}_s ds = x-\zeta_0\bar{\theta}_0 =: \hat{x}.$$

Consequently, the primal problem (6) can be transformed to the following optimization problem:

$$\begin{aligned}\max_{(\hat{C}_t,\pi_t)\in\mathcal{A}(x)} \quad & \mathbf{E}\int_0^\infty e^{-\rho t}U(\hat{C}_t)dt,\\ s.t. \quad & \mathbf{E}\int_0^\infty \hat{C}_t\hat{\xi}_t dt = \hat{x},\\ & \hat{C}_t+b_t\hat{\theta}(\hat{C}_t)\geq 0.\end{aligned} \tag{23}$$

4.2. Optimal Consumption Policy and Wealth Process

In this section, we first solve the optimization problem (23), and then derive the optimal consumption policy and the optimal wealth process of the primal problem (6) using Theorem 4.1 and the replicating technique.

Similar to the proof of Theorem 3.1, it is very easy to prove the following theorem, which presents the relevant results of problem (23).

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THEOREM 4.2: *Let* $\widetilde{C}_t$ *and* ξ_t^* *represent respectively the solutions of equation*

$$\widetilde{C}_t + b_t\hat{\theta}(\widetilde{C}_t) = 0$$

and

$$u_1((u_1')^{-1}(x)) - x(u_1')^{-1}(x) + x\widetilde{C}_t - u_2(\widetilde{C}_t) = 0.$$

Define

$$\begin{aligned}
\hat{\mathcal{X}}_\infty(y) &= \mathbf{E}\int_0^\infty \hat{\xi}_t(u_1')^{-1}(ye^{\rho t}\hat{\xi}_t)I_{\{ye^{\rho t}\hat{\xi}_t \le \xi_t^*\}}dt,\\
\mathcal{Y}_\infty(y) &= \mathbf{E}\int_0^\infty \hat{\xi}_t\widetilde{C}_t I_{\{ye^{\rho t}\hat{\xi}_t > \xi_t^*\}}dt,\\
\mathcal{Z}_\infty(y) &= \hat{\mathcal{X}}_\infty(y) + \mathcal{Y}_\infty(y).
\end{aligned} \tag{24}$$

If $\mathcal{Z}_\infty(y) < +\infty$*, then the optimal policy of the optimization problem* (23) *is*

$$\hat{C}_t^* = \begin{cases} (u_1')^{-1}(ye^{\rho t}\hat{\xi}_t), & \text{if } \ ye^{\rho t}\hat{\xi}_t \le \xi_t^*,\\ \widetilde{C}_t, & \text{if } \ ye^{\rho t}\hat{\xi}_t > \xi_t^*, \end{cases} \tag{25}$$

and the corresponding optimal process $\hat{X}_t^*$ *to* $\hat{C}_t^*$ *is*

$$\hat{X}_t^* = \frac{1}{\hat{\xi}_t}\mathbf{E}\left(\int_t^\infty \hat{\xi}_s\left[(u_1')^{-1}(ye^{\rho s}\hat{\xi}_s)I_{\{ye^{\rho s}\hat{\xi}_s \le \xi_s^*\}} + \widetilde{C}_s I_{\{ye^{\rho s}\hat{\xi}_s > \xi_s^*\}}\right]ds|\mathcal{F}_t\right), \tag{26}$$

where $y \ge 0$ *satisfies*

$$\mathcal{Z}_\infty(y) = \hat{x}.$$

According to Theorems 4.1, 4.2 and 9.11 of Karatzas and Shreve [8](p. 141), it is straightforward to derive the optimal consumption policy and the optimal wealth process of the primal problem (6).

THEOREM 4.3: *Let* $\widetilde{C}_t$*,* ξ_t^* *and* $\mathcal{Z}_\infty(y)$ *be defined as the same as in Theorem 4.2. If* $\mathcal{Z}_\infty(y) < +\infty$*, then the optimal consumption policy of the loss averse investor is*

$$C_t^* = \begin{cases} (u_1')^{-1}(ye^{\rho t}\hat{\xi}_t) + b_t\hat{\theta}_t((u_1')^{-1}(ye^{\rho t}\hat{\xi}_t)), & \text{if } \ ye^{\rho t}\hat{\xi}_t \le \xi_t^*,\\ 0, & \text{if } \ ye^{\rho t}\hat{\xi}_t > \xi_t^*, \end{cases} \tag{27}$$

and the optimal wealth process X_t^* *is*

$$X_t^* = \zeta_t\theta_t(C_t^*) + (1+\zeta_t)\hat{X}_t^*, \tag{28}$$

where $y \ge 0$ *satisfies*

$$\mathcal{Z}_\infty(y) = \hat{x}.$$

Base on Theorem 4.3, we can easily see that similar to the problem with exogenous reference level, the loss averse investor with endogenous reference level has two different kinds of behavior in the good states of market ($ye^{\rho t}\hat{\xi}_t \le \xi_t^*$) and the bad states of market ($ye^{\rho t}\hat{\xi}_t > \xi_t^*$). What is more, the size of the good-states region increases when the initial wealth increases. Only, ξ_t^* is dependent on the time t instead of a constant. In addition, from

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(19), we can know that when a_t increases or b_t decreases, $\widetilde{C}_t$ increases. And because $g(x)$ is strictly decreasing and $u_2(x)$ is strictly increasing, when $\widetilde{C}_t$ increases, ξ_t^* increases, this means that the good-states region increases. That is to say, if the more important the past consumption choices become, or the slower the reference level adjusts relatively in response to consumption shocks, the more likely bad situation is to happen. In addition, when the prospective consumption level of the investor is higher or the initial wealth is less, the bad situation is more likely to occur. In this bad situation, he had better consume nothing at all.

4.3. Optimal Portfolio Policy with the Constant Investment Opportunity Set and Two-Piece Power Utility Function

In this section, we suppose the investment opportunity set and the coefficients related to the reference level are constant:

$$r(\cdot) = r,\ \kappa(\cdot) = \kappa, \quad \| \kappa \|^2 = 2\gamma,$$

and

$$a(\cdot) = a,\ b(\cdot) = b, \quad a + r > b.$$

We first give an equivalent condition for $\mathcal{Z}_\infty(y) < +\infty$, and then derived the closed-form solutions of the optimal consumption and portfolio policies for the primal problem when $U(\cdot)$ is two-piece power utility function.

Now, under the condition that all the market parameters are constants (matrix), we give two following lemmas:

LEMMA 4.4: $\widetilde{C}_t = -b\bar{\theta}_0 e^{-at}$.

PROOF: It is obvious that $b\hat{\theta}(\widetilde{C}_t) + \widetilde{C}_t = 0$. In fact,

$$\begin{aligned} b\hat{\theta}(\widetilde{C}_t) + \widetilde{C}_t &= b\int_0^t e^{-(a-b)(t-s)}(-be^{-as}\bar{\theta}_0)ds + be^{-(a-b)t}\bar{\theta}_0 - be^{-at}\bar{\theta}_0 \\ &= be^{-(a-b)t}\bar{\theta}_0(e^{-bt} - 1) + be^{-(a-b)t}\bar{\theta}_0 - be^{-at}\bar{\theta}_0 \\ &= 0. \end{aligned}$$

And since the equation

$$b\int_0^t e^{-(a-b)(t-s)}y(s)ds + y(t) = 0$$

has no more than one solution, and the solution is $y(t) = 0$.

Consequently, $\widetilde{C}_t = -b\bar{\theta}_0 e^{-at}$ is the unique solution of $b\hat{\theta}(\widetilde{C}_t) + \widetilde{C}_t = 0$. That is the end of the proof. ■

LEMMA 4.5: *When $r + a > b$, we have*

$$\mathbf{E}\left(\int_t^\infty \xi_s b_s h_s ds | \mathcal{F}_t\right) = \frac{b}{r + a - b}\xi_t h_t.$$

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PROOF: The result can be directly obtained by simple calculation. In fact, if $r + a > b$,

$$\begin{aligned}
&\mathbf{E}\left(\int_t^\infty \xi_s b_s h_s ds|\mathcal{F}_t\right)\\
&= b\mathbf{E}\left(\int_t^\infty e^{-(a-b)s-(r+\gamma)s-\kappa^T W_s} ds|\mathcal{F}_t\right)\\
&= be^{-\kappa^T W_t}\int_{-\infty}^{+\infty}\int_t^\infty e^{-(a-b+r+\gamma)s-\kappa^T x}\frac{1}{\sqrt{2\pi(s-t)}}e^{-(x^2/2(s-t))}dsdx\\
&= \frac{b}{r+a-b}\xi_t h_t.
\end{aligned}$$

■

Consequently, from the preceding conclusions we have

$$\zeta_t = \frac{b}{r+a-b},\quad \hat{\xi}_t = \left(1+\frac{b}{r+a-b}\right)\xi_t =: p\xi_t.$$

Review last section,

$$\begin{aligned}
\mathcal{Z}_\infty(y) &= \hat{\mathcal{X}}_\infty(y) + \mathcal{Y}_\infty(y)\\
&= \mathbf{E}\int_0^\infty \hat{\xi}_t (u_1')^{-1}(ye^{\rho t}\hat{\xi}_t) I_{\{ye^{\rho t}\hat{\xi}_t \le \xi_t^*\}} dt + \mathbf{E}\int_0^\infty \hat{\xi}_t \widetilde{C}_t I_{\{ye^{\rho t}\hat{\xi}_t > \xi_t^*\}} dt,
\end{aligned}$$

thus, provided that $\hat{\mathcal{X}}_\infty(y) < +\infty$ and $\mathcal{Y}_\infty(y) < +\infty$, we have $\mathcal{Z}_\infty(y) < +\infty$.

We now give a sufficient condition for $\hat{\mathcal{X}}_\infty(y) < +\infty$, that is, the following lemma:

LEMMA 4.6: *If* $\int_0^1 \eta^{-\delta_2}(u_1')^{-1}(\eta)d\eta < +\infty$, *then* $\hat{\mathcal{X}}_\infty(y) < +\infty$, *where* δ_2 *is defined in Lemma 3.2.*

PROOF: When all the market parameters are constants (matrix),

$$\xi_t = \exp(-(r+\gamma)t - \kappa^T W_t),$$

thus

$$\begin{aligned}
\hat{\mathcal{X}}_\infty(y) &= \mathbf{E}\int_0^\infty \hat{\xi}_t (u_1')^{-1}(ye^{\rho t}\hat{\xi}_t) I_{\{ye^{\rho t}\hat{\xi}_t \le \xi_t^*\}} dt\\
&= \int_{-\infty}^{+\infty}\int_0^\infty p\exp(-(r+\gamma)t - \kappa^T x)(u_1')^{-1}(py\exp(-\tilde{\gamma}t - \kappa^T x))\\
&\quad \cdot \frac{1}{\sqrt{2\pi t}} e^{-(x^2/2t)} I_{\{-\tilde{\gamma}t - \kappa^T x \le \ln \xi_t^* - \ln(py)\}} dtdx.
\end{aligned}$$

where $\tilde{\gamma}$ is defined as the same as in Lemma 3.2. Define

$$z = \tilde{\gamma}t + \kappa^T x,$$

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hence

$$\hat{\mathcal{X}}_\infty(y) = p\int_{-\infty}^{+\infty}\int_0^\infty e^{-z-\rho t}(u_1')^{-1}(pye^{-z})\cdot\frac{1}{\sqrt{4\pi\gamma t}}e^{-((z-\tilde{\gamma}t)^2/4\gamma t)}I_{\{z\geq\ln(py)-\ln\xi_t^*\}}dtdz$$
$$\leq p\int_{-\infty}^{+\infty}\int_0^\infty e^{\frac{\tilde{\gamma}-2\gamma}{2\gamma}z}(u_1')^{-1}(pye^{-z})\cdot\frac{1}{\sqrt{4\pi\gamma t}}e^{-(z^2/4\gamma t)-((\tilde{\gamma}^2/4\gamma)+\rho)t}dtdz.$$

Due to

$$\int_0^\infty\frac{1}{\sqrt{\pi t}}e^{-\frac{a}{4t}-pt}dt = \frac{e^{-\sqrt{ap}}}{\sqrt{p}},\quad\forall\ a>0,\quad\forall\ p>0,$$

we can obtain

$$\hat{\mathcal{X}}_\infty(y)\leq\frac{p}{\sqrt{4\rho\gamma+\tilde{\gamma}^2}}\int_{-\infty}^{+\infty}\exp\left(\frac{\tilde{\gamma}-2\gamma}{2\gamma}z-\frac{\sqrt{4\rho\gamma+\tilde{\gamma}^2}}{2\gamma}|z|\right)\cdot(u_1')^{-1}(pye^{-z})dz.$$

Recall the definitions of δ_1 and δ_2 in Lemma 3.2, and $\delta_1>1$ and $\delta_2<0$, we can get

$$\hat{\mathcal{X}}_\infty(y)\leq\frac{p}{\gamma(\delta_1-\delta_2)}\left[\int_0^\infty e^{(\delta_2-1)z}(u_1')^{-1}(pye^{-z})dz+\int_{-\infty}^0 e^{(\delta_1-1)z}(u_1')^{-1}(pye^{-z})dz\right]$$
$$=\frac{p}{\gamma(\delta_1-\delta_2)}\left[\int_0^{py}(py)^{\delta_2-1}\eta^{-\delta_2}(u_1')^{-1}(\eta)d\eta+\int_{py}^\infty(py)^{\delta_1-1}\eta^{-\delta_1}(u_1')^{-1}(\eta)d\eta\right].$$

And since $(u_1')^{-1}(x)$ is decreasing,

$$\int_{py}^\infty(py)^{\delta_1-1}\eta^{-\delta_1}(u_1')^{-1}(\eta)d\eta\leq\int_{py}^\infty(py)^{\delta_1-1}\eta^{-\delta_1}(u_1')^{-1}(py)d\eta$$
$$=\frac{(u_1')^{-1}(py)}{\delta_1-1}.$$

Obviously,

$$\int_0^1\eta^{-\delta_2}(u_1')^{-1}(\eta)d\eta<+\infty$$

is equivalent to

$$\int_0^\varepsilon\eta^{-\delta_2}(u_1')^{-1}(\eta)d\eta<+\infty,\ \forall\ \varepsilon>0,$$

and so, $\hat{\mathcal{X}}_\infty(y)<+\infty$ if

$$\int_0^1\eta^{-\delta_2}(u_1')^{-1}(\eta)d\eta<+\infty.$$

■

And as similar to Lemma 4.6, for $\mathcal{Y}_\infty(y)$, we can get the following conclusion:

LEMMA 4.7: $\frac{pb\bar{\theta}_0}{\gamma(\delta_3-1)(\delta_4-1)}\leq\mathcal{Y}_\infty(y)<0$, *where*

$$\delta_i=\frac{\tilde{\gamma}}{2\gamma}-(-1)^i\frac{\sqrt{4\gamma(\rho+a)+\tilde{\gamma}^2}}{2\gamma},\quad i=3,\ 4.$$

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For convenience of calculations, in the rest of this section, we suppose $U(\cdot)$ is two-piece power utility function proposed by Tversky and Kahneman [17], as shown in Eq. (13). We will give the closed-form solution of the optimal portfolio policy.

Recall the definitions of δ_5 and δ_6 in Example 3.4, we can see that,

$$\int_0^1 \eta^{-\delta_2}(u_1')^{-1}(\eta)d\eta = \frac{\alpha^{\frac{1}{1-\alpha}}}{\delta_6} < +\infty,$$

for $\delta_6 > 0$. Define ξ_t^* as the solution of equation

$$(1-\alpha)\left(\frac{x}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - b\bar{\theta}_0 e^{-at}x + \eta(b\bar{\theta}_0 e^{-at})^{\beta} = 0,$$

as a consequence, based on the above lemmas, we can obtain the following theorem.

THEOREM 4.8: *If $\delta_6 > 0$, the optimal process $\hat{X}_t^*$ for the problem (23) is given by*

$$\hat{X}_t^* = \left(\frac{py}{\alpha}\right)^{\frac{1}{\alpha-1}} e^{(\kappa^T W_t/(1-\alpha))+rt+((1-2\alpha)\gamma t/(1-\alpha)^2)} \int_t^\infty e^{d_0 s}\phi(d_2(s,t,y))ds - b\bar{\theta}_0 e^{rt} \cdot \int_t^\infty e^{-(r+a)s}\phi(d_3(s,t,y))ds, \tag{29}$$

The optimal wealth process for the primal problem (6) is given by

$$X_t^* = p\hat{X}_t^* + (p-1)\hat{\theta}(\hat{C}_t^*), \tag{30}$$

and the feedback form solutions for the optimal portfolio and consumption policies are respectively

$$\begin{aligned} C_t^* &= \left[b\left(\frac{py}{\alpha}\right)^{\frac{1}{\alpha-1}} \int_0^t e^{(a-b)(s-t)}(e^{\rho s}\xi_s)^{\frac{1}{\alpha-1}}ds \right. \\ &\quad \left. + be^{(b-a)t}\bar{\theta}_0 + \left(\frac{pye^{\rho t}\xi_t}{\alpha}\right)^{\frac{1}{\alpha-1}} \right] I_{\{pye^{\rho t}\xi_t \le \xi_t^*\}}, \\ \pi_t^* &= p\sigma^{-1}\frac{\partial \hat{X}_t^*}{X_t^*\partial W_t}. \end{aligned} \tag{31}$$

where $y \ge 0$ satisfies $\mathcal{Z}_\infty(y) = \hat{x}$,

$$\begin{aligned} d_0 &= \frac{(\rho - \alpha r - \alpha\gamma)}{\alpha - 1} + \frac{\alpha^2\gamma}{(1-\alpha)^2}, \\ d_1(s,t,y) &= (\rho - r - \gamma)s + \ln(py) - \ln\xi_s^* - \kappa^T W_t, \\ d_2(s,t,y) &= -\frac{d_1(s,t,y)}{\sqrt{2\gamma(s-t)}} + \frac{\alpha\sqrt{2\gamma(s-t)}}{1-\alpha}, \\ d_3(s,t,y) &= \frac{d_1(s,t,y)}{\sqrt{2\gamma(s-t)}} + \sqrt{2\gamma(s-t)}, \end{aligned} \tag{32}$$

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$$\frac{\partial \hat{X}_t^*}{\partial W_t} = \left(\frac{py}{\alpha}\right)^{\frac{1}{\alpha-1}} e^{(\kappa^T W_t/(1-\alpha))+rt+((1-2\alpha)\gamma t/(1-\alpha)^2)} \left(\frac{\kappa}{1-\alpha}\int_t^\infty e^{d_0 s}\phi(d_2(s,t,y))ds + \frac{\kappa}{\sqrt{2\gamma}}\int_t^\infty \frac{e^{d_0 s}}{\sqrt{s-t}}\varphi(d_2(s,t,y))ds\right) + \frac{\kappa}{\sqrt{2\gamma}} b\bar{\theta}_0 e^{rt}\int_t^\infty \frac{e^{-(r+a)s}}{\sqrt{s-t}}\varphi(d_3(s,t,y))ds,$$

and $\phi(\cdot)$ denotes the cumulative standard normal distribution function and $\varphi(\cdot)$ denotes the standard normal density function.

Proof: When all the market parameters are constants (matrix),

$$\xi_t = \exp(-(r+\gamma)t - \kappa^T W_t).$$

From the definition of $\hat{X}_t^*$, we have

$$\begin{aligned}
\hat{\xi}_t \hat{X}_t^* &= \mathbf{E}\left(\int_t^\infty \hat{\xi}_s\left[(u_1')^{-1}(ye^{\rho s}\hat{\xi}_s)I_{\{ye^{\rho s}\hat{\xi}_s \le \xi_s^*\}} + \widetilde{C}_s I_{\{ye^{\rho s}\hat{\xi}_s > \xi_s^*\}}\right] ds|\mathcal{F}_t\right) \\
&= p\left(\frac{py}{\alpha}\right)^{\frac{1}{\alpha-1}} e^{\alpha\kappa^T W_t/(1-\alpha)}\mathbf{E}\left(\int_t^\infty e^{\frac{(\rho-\alpha r-\alpha\gamma)s}{\alpha-1}} e^{\frac{-\alpha\kappa^T(W_s-W_t)}{\alpha-1}}\right. \\
&\quad \left.\cdot I_{\{\kappa^T(W_s-W_t)\ge d_1(s,t,y)\}} ds|\mathcal{F}_t\right) - pb\bar{\theta}_0 e^{-\kappa^T W_t} \\
&\quad \cdot \mathbf{E}\left(\int_t^\infty e^{-(r+\gamma+a)s-\kappa^T(W_s-W_t)} I_{\{\kappa^T(W_s-W_t)<d_1(s,t,y)\}} ds|\mathcal{F}_t\right) \\
&= p\left(\frac{py}{\alpha}\right)^{\frac{1}{\alpha-1}} e^{\alpha\kappa^T W_t/(1-\alpha)}\int_{-\infty}^{+\infty}\int_t^\infty e^{\frac{(\rho-\alpha r-\alpha\gamma)s}{\alpha-1}} e^{\alpha x/(1-\alpha)} \\
&\quad \cdot \frac{e^{-(x^2/4\gamma(s-t))}}{\sqrt{4\pi\gamma(s-t)}} I_{\{x\ge d_1(s,t,y)\}} ds dx \\
&\quad - pb\bar{\theta}_0 e^{-\kappa^T W_t}\int_{-\infty}^{+\infty}\int_t^\infty e^{-(r+\gamma+a)s-x}\frac{e^{-(x^2/4\gamma(s-t))}}{\sqrt{4\pi\gamma(s-t)}} I_{\{x<d_1(s,t,y)\}} ds dx.
\end{aligned}$$

It is obvious that $d_0 < 0$ due to the definition of d_0 and $\delta_6 > 0$. Then, from Eq. (32) and the definition of $\phi(\cdot)$, we have

$$\begin{aligned}
\hat{\xi}_t \hat{X}_t^* &= p\left(\frac{py}{\alpha}\right)^{\frac{1}{\alpha-1}} e^{(\alpha\kappa^T W_t/(1-\alpha))-(\alpha^2\gamma t/(1-\alpha)^2)}\int_{-\infty}^{+\infty}\int_t^\infty e^{d_0 s} \\
&\quad \cdot \frac{e^{-((x-(2\alpha\gamma(s-t)/(1-\alpha)))^2/4\gamma(s-t))}}{\sqrt{4\pi\gamma(s-t)}} I_{\{x\ge d_1(s,t,y)\}} ds dx \\
&\quad - pb\bar{\theta}_0 e^{-\kappa^T W_t-\gamma t}\int_{-\infty}^{+\infty}\int_t^\infty e^{-(r+a)s}\frac{e^{-((x+2\gamma(s-t))^2/4\gamma(s-t))}}{\sqrt{4\pi\gamma(s-t)}} I_{\{x<d_1(s,t,y)\}} ds dx \\
&= p\left(\frac{py}{\alpha}\right)^{\frac{1}{\alpha-1}} e^{(\alpha\kappa^T W_t/(1-\alpha))-(\alpha^2\gamma t/(1-\alpha)^2)}\int_t^\infty e^{d_0 s}\phi(d_2(s,t,y))ds \\
&\quad - pb\bar{\theta}_0 e^{-\kappa^T W_t-\gamma t}\int_t^\infty e^{-(r+a)s}\phi(d_3(s,t,y))ds.
\end{aligned}$$

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Thus,

$$\hat{X}_t^* = \left(\frac{py}{\alpha}\right)^{\frac{1}{\alpha-1}} e^{(\kappa^T W_t/1-\alpha)+rt+((1-2\alpha)\gamma t/(1-\alpha)^2)} \int_t^\infty e^{d_0 s}\phi(d_2(s,t,y))ds$$
$$- b\bar{\theta}_0 e^{rt}\int_t^\infty e^{-(r+a)s}\phi(d_3(s,t,y))ds.$$

According to the above theorems and the replicating technique, we can derive the optimal wealth process and the feedback forms of the optimal portfolio and consumption policies for the primal problem (6). Hence, we gain the conclusion. ■

From the above results, we can easily see that the optimal portfolio policy π_t^* with endogenous reference level is continuous with respect to y as the same as the policy with exogenous reference level, but it is a more complex function about t and W_t. In addition,

$$\lim_{y\to 0}\pi_t^* = \frac{p\sigma^{-1}\kappa}{1-\alpha} > \frac{\sigma^{-1}\kappa}{1-\alpha},$$

and

$$\lim_{y\to 0}\hat{\mathcal{X}}_\infty(y) = +\infty.$$

That is to say, when he has more enough money, the loss averse investor with endogenous reference level will invest more percentage in stocks than the one with exogenous reference level.

5. CONCLUSIONS

In this work, we discussed a continuous-time optimal portfolio and consumption model under loss aversion in an infinite time horizon. The investor's goal was to choose the optimal portfolio and consumption policies to maximize the infinite horizon expected discounted S-shaped utility from consumption. We solved the problem with two different scenarios respectively for the reference level in detail. In the first case, the reference level was assumed to be exogenous, which was independent of the consumption policy. Using the martingale method and replicating technique, we obtained the optimal consumption policy and the optimal wealth process. When the investment opportunity set was constant, the closed-form solutions of the portfolio and consumption policies were obtained. In the second case, the reference level was endogenous, which depended on the past consumption. We first transformed the optimization problem with stochastic reference level into an equivalent optimization problem with zero reference point and proved the corresponding relationship between them. And then in the similar way to the first case, using the martingale method and replicating technique, we derived the relevant conclusion. Through comparison and analysis, we found that the two optimal policies of the loss averse investors with two different scenarios for the reference level have a certain similarity, the loss averse investors have two different kinds of behavior in the good states of market and the bad states of market. But the optimal portfolio policy with endogenous reference level is more complex. What is more, when the investor has more enough money, the percentage invested in stocks of the loss averse investor with endogenous reference level is more than the one with exogenous reference level.

However, in this paper, the investment horizon was assumed to be infinite. The optimal portfolio and consumption problem in a finite time horizon with endogenous reference level

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is significant and worth investigating, especially when the reference level depends on the past consumption or wealth. This is the focus of our future research.

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