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ORDER STATISTICS FROM HETEROGENOUS NEGATIVE BINOMIAL RANDOM VARIABLES

MAOCHAO XU
Department of Mathematics
Illinois State University,
Normal, IL
E-mail: mxu2@ilstu.edu

TAIZHONG HU
Department of Statistics and Finance
School of Management
University of Science and Technology of China
Hefei, Anhui 230026, People's Republic of China
E-mail: thu@ustc.edu.cn

In this article, we study the order statistics from heterogenous negative binomial random variables. Sufficient conditions are provided for comparing the extreme order statistics according to the usual stochastic order. For the special case of geometric distribution, a sufficient condition is established for comparing order statistics in the sense of multivariate stochastic order. Applications in the Poisson–Gamma shock model and redundant systems have been described as well.

1. INTRODUCTION

The negative binomial distribution is one of the most fundamental distributions in statistics and has found widely applications in engineering, game theory, regression, population selection, and reliability theory. It is also closely related to many well-known distributions such as geometric, binomial, Poisson and Gamma distributions.

In the literature, order statistics from binomial random variables have received much attention due to their interesting applications. Weiss [15] discussed the operation of systems with redundant and failure-independent elements that perform a given task repetitively at discrete intervals, which could be modeled as a special case of negative binomial distribution: geometric distribution. This model has also been

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mentioned by Lundberg [6] to study the aircraft wing fatigue failure. Margolin and Winokur [9] studied the order statistics from geometric distribution in the context of inverse sampling. Young [16] derived the recurrence relations for probability generating functions and moments of distributions of order statistics from negative binomial distributions. Young [17] further discussed order statistics from negative binomial distributions under a slippage configuration. Recently, Jeske and Blessinger [4] analyzed the maximum order statistics from heterogeneous geometric distributions and gave approximation formulas for expected value and variance of the maximum order statistics for the large sample size. They pointed out that the maximum order statistics could be used in the real-world engineering models, such as wireless broadcast transmission systems.

Although order statistics from negative binomial random variables have many interesting applications due to the complicated distribution forms of discrete order statistics, less attention has been paid to the heterogenous effect on the distribution properties of order statistics compared to the continuous random variables; see Kochar and Xu [5] and Mao and Hu [8] and references therein. The purpose of this research is to further study this topic. First, we give the following two examples to illustrate the possible applications of this research.

Poisson–Gamma Shock Model. Assume that a system suffers shocks from n different sources. A shock from source i arrives according to a Poisson process with random parameter Λ_i, which has the Gamma distribution with parameter (α_i, β_i) for $i = 1, \ldots, n$. The system fails when the number of shocks from any source exceeds some threshold number. Hence, the safety of the system could be monitored by recording the maximum number of shocks from different sources. Let $X_1(t), \ldots, X_n(t)$ be the number of shocks from n different sources during a time period t. Then we are interested in the following statistic:

$$X_{n:n}(t) = \max\{X_1(t), \ldots, X_n(t)\}.$$

It is well known that

$$\mathsf{P}(X_i(t) = x) = \binom{\alpha_i + x - 1}{x} \left(\frac{\beta_i}{\beta_i + t}\right)^{\alpha_i} \left(1 - \frac{\beta_i}{\beta_i + t}\right)^{x}, \quad x = 0, 1, \ldots,$$

for $i = 1, \ldots, n$. Hence, $X_i(t)$ is negative binomial with parameters $k_i = \alpha_i$ and $p_i = \beta_i/(\beta_i + t)$.

Redundant Systems. Suppose a system consists of n operating positions with position i having k_i components for $i = 1, \ldots, n$. The system operates at discrete times (say daily) and performs a given task repetitively. For each position, only one operates during a cycle. The operating element at position i in the system has the same probability p_i of failure in a cycle. When a component in a given position fails, it remains inoperative and one of the remaining components in that position becomes operational on the next cycle. Such system has been discussed in Young [16]. Now, let us consider two systems. System 1 fails

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at cycle k if one position has failed at the first time at cycle k. System 2 fails at cycle l if all positions have failed at cycle l. It is easy to see that the lifetime of system 1 can be modeled as minimum order statistics from negative binomial distributions whereas the lifetime of system 2 can be modeled as maximum order statistics from negative binomial distributions. If each position has only one component, system failure is defined to be failure at cycle k when r or more positions have failed at cycle k for the first time. Hence, the system's life can be modeled as the order statistics from geometric distributions. Such system has been considered by Weiss [15].

Let X be a negative binomial with parameter $(k, p) \in (0, +\infty) \times (0, 1)$. If k is a positive integer, then

$$\mathsf{P}(X = x) = \binom{k + x - 1}{x} p^k (1 - p)^x$$

is the probability that $k + x$ independent trials are required to get k successes when p is the probability of success at each trial. It has been shown in Patil [11] that

$$\mathsf{P}(X \leq r) = B(p; k, r + 1) = I_p(k, r + 1), \quad r = 0, 1, \ldots,$$

where

$$I_p(m, n) = \frac{1}{B(m, n)} \int_0^p u^{m-1} (1 - u)^{n-1} \, du$$

is the incomplete beta function and $B(\cdot; k, r + 1)$ denotes the distribution function of a beta distribution with parameter $(k, r + 1)$.

Let $X_1, \ldots, X_n$ be a sequence of independent negative binomial random variables with respective parameters $(k_1, p_1), \ldots, (k_n, p_n)$ and let $Y_1, \ldots, Y_n$ be another sequence of independent negative binomial random variables with respective parameters $(k_1^*, p_1^*), \ldots, (k_n^*, p_n^*)$. In Section 3 it will be shown that if $k_i = k_i^* = k$ for $i = 1, \ldots, n$, then

$$(p_1, \ldots, p_n) \overset{\mathrm{p}}{\succeq} (p_1^*, \ldots, p_n^*) \Longrightarrow X_{n:n} \geq_{\mathrm{st}} Y_{n:n}.$$

For the smallest order statistics from negative binomial random variables with $k_i = k_i^* = k$ for each i, it holds that if $k \geq 1$, then

$$(1 - p_1, \ldots, 1 - p_n) \overset{\mathrm{p}}{\succeq} (1 - p_1^*, \ldots, 1 - p_n^*) \Longrightarrow X_{1:n} \leq_{\mathrm{st}} Y_{1:n},$$

and if $k \in (0, 1]$, then

$$(\log(1 - p_1^*), \ldots, \log(1 - p_n^*)) \preceq_{\mathrm{w}} (\log(1 - p_1), \ldots, \log(1 - p_n)) \Longrightarrow Y_{1:n} \leq_{st} X_{1:n}.$$

In Section 3 we also consider the case of $p_i = p_i^* = p$ for each i and prove that

$$(k_1, \ldots, k_n) \succeq_{\mathrm{w}} [\overset{\mathrm{w}}{\succeq}] (k_1^*, \ldots, k_n^*) \Longrightarrow X_{n:n} \geq_{\mathrm{st}} Y_{n:n} \, [X_{1:n} \leq_{\mathrm{st}} Y_{1:n}].$$

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For the special case of geometric distributions, it is shown in Section 4 that,

$$(\ln(1-p_1),\ldots,\ln(1-p_n)) \succeq_{\mathrm{w}} (\ln(1-p_1^*),\ldots,\ln(1-p_n^*))$$

implies

$$(X_{1:n},\ldots,X_{n:n}) \geq_{\mathrm{st}} (Y_{1:n},\ldots,Y_{n:n}),$$

which delivers a parallel result to Theorem 3.4 of Proschan and Sethuraman [12]. The definitions of the stochastic orders and majorization orders used in this section will be recalled in Section 2.

Throughout the article, the terms "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing," respectively.

2. PRELIMINARIES

In this section we recall some stochastic orders and majorization orders, which will be used in the sequel.

Assume random variables X and Y have distribution functions F and G, respectively, and survival functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, respectively.

DEFINITION 2.1: *X is said to be greater than Y in the usual stochastic order, denoted by $X \geq_{\mathrm{st}} Y$, if $\overline{F}(x) \geq \overline{G}(x)$ for all x.*

The usual stochastic order has the following multivariate version.

DEFINITION 2.2: *The random vector $\boldsymbol{X} = (X_1,\ldots,X_n)$ is said to be greater than another random vector $\boldsymbol{Y} = (Y_1,\ldots,Y_n)$, denoted by $\boldsymbol{X} \geq_{\mathrm{st}} \boldsymbol{Y}$, according to the multivariate usual stochastic order if $\mathsf{E}[\phi(\boldsymbol{X})] \geq \mathsf{E}[\phi(\boldsymbol{Y})]$ for all increasing functions ϕ.*

It is known that the multivariate stochastic order implies the componentwise stochastic order. For more details on the univariate and multivariate stochastic orders, please refer to the excellent book by Shaked and Shanthikumar [14].

The concepts of majorization orders will also be used in our discussion. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.

DEFINITION 2.3: *For real vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x}$ is said to be*

- *majorized by $\boldsymbol{y}$, denoted by $\boldsymbol{x} \preceq_{\mathrm{m}} \boldsymbol{y}$, if $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$ and $\sum_{i=1}^j x_{(i)} \geq \sum_{i=1}^j y_{(i)}$ for $j = 1,\ldots,n-1$;*
- *weakly supermajorized by $\boldsymbol{y}$, denoted by $\boldsymbol{x} \overset{\mathrm{w}}{\preceq} \boldsymbol{y}$, if $\sum_{i=1}^j x_{(i)} \geq \sum_{i=1}^j y_{(i)}$ for $j = 1,\ldots,n$;*
- *weakly submajorized by $\boldsymbol{y}$, denoted by $\boldsymbol{x} \preceq_{\mathrm{w}} \boldsymbol{y}$, if $\sum_{i=j}^n x_{(i)} \leq \sum_{i=j}^n y_{(i)}$ for $j = 1,\ldots,n$.*

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Another interesting weaker order related to the majorization order is the p-order.

DEFINITION 2.4: *A vector* $\boldsymbol{x}$ *in* $\mathbb{R}_+^n \equiv [0,\infty)^n$ *is said to be p-smaller than another vector* $\boldsymbol{y}$ *in* $\mathbb{R}_+^n$, *denoted by* $\boldsymbol{x} \overset{\mathrm{p}}{\preceq} \boldsymbol{y}$, *if*

$$\prod_{i=1}^{j} x_{(i)} \geq \prod_{i=1}^{j} y_{(i)}, \qquad j = 1, \ldots, n$$

or, equivalently,

$$(\ln(x_1), \ldots, \ln(x_n)) \overset{\mathrm{w}}{\preceq} (\ln(y_1), \ldots, \ln(y_n)).$$

For extensive and comprehensive details on the theory of majorization orders and their applications, please refer to the excellent book by Marshall and Olkin [10].

3. MAIN RESULTS

Before we state and prove the main results of this section, we need the following three lemmas.

LEMMA 3.1 (Marshall and Olkin [**10**, p. 57]): *Let* $D \subseteq \mathbb{R}^n$ *be a symmetric set and let* $\phi : D \mapsto \mathbb{R}$ *be continuously differentiable. Then* ϕ *is Schur-convex on* D *if and only if* ϕ *is symmetric on* D *and*

$$(x_i - x_j)\left(\frac{\partial \phi(\boldsymbol{x})}{\partial x_i} - \frac{\partial \phi(\boldsymbol{x})}{\partial x_j}\right) \geq 0$$

for all $i \neq j$ *and* $\boldsymbol{x} \in D$.

LEMMA 3.2 (Marshall and Olkin [**10**, p. 59]): *A real-valued function* ϕ *defined on* $\mathbb{R}^n$ *satisfies*

$$\boldsymbol{x} \succeq_{\mathrm{w}} [\overset{\mathrm{w}}{\succeq}] \boldsymbol{y} \Longrightarrow \phi(\boldsymbol{x}) \geq \phi(\boldsymbol{y})$$

if and only if ϕ *is increasing [decreasing] and Schur-convex on* $\mathbb{R}^n$.

LEMMA 3.3:

(a) For $k > 0$ *and* $r > 0$, *we have*

$$\ell_1(s) = \frac{e^{-ks}(1 - e^{-s})^r}{\int_0^{e^{-s}} u^{k-1}(1-u)^r \, du}$$

is increasing in $s \in \mathbb{R}_+$.

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(b) *For $k \geq 1$ and $r > 0$,*

$$\ell_2(s) = \frac{e^{-(r+1)s}(1-e^{-s})^{k-1}}{\int_{1-e^{-s}}^{1} u^{k-1}(1-u)^r du}$$

is increasing in $s \in \mathbb{R}_+$. If $k \in (0,1]$ and $r > 0$, then $\ell_2(s)$ is decreasing in $s \in \mathbb{R}_+$.

PROOF:

(a) Assume that a random variable Z has the survival function given by

$$\overline{H}(s) = \frac{1}{B(k, r+1)} \int_0^{e^{-s}} u^{k-1}(1-u)^r \, du, \quad s \in \mathbb{R}_+.$$

It is easy to see that its hazard rate function can be represented as $\ell_1(s)$. Hence, it is enough for us to prove that Z has the increasing hazard rate property; that is, $\overline{H}(s)$ is log-concave in s. Note that the density function of Z is

$$h(s) = \frac{1}{B(k, r+1)} e^{-ks}(1-e^{-s})^r, \qquad s \in \mathbb{R}_+.$$

Since

$$\log h(s) = r \log(1-e^{-s}) - ks - \log B(k, r+1)$$

is concave in $s \in \mathbb{R}_+$, it follows from Barlow and Proschan ([1], p. 77) that $\overline{H}(s)$ is log-concave in $s \in \mathbb{R}_+$.

(b) It follows by a argument similar to part (a) by using the property that the log-concavity [log-convexity] of a density function implies the log-concavity [log-convexity] of the corresponding survival function (see Sengupta and Nanda [13]).

■

THEOREM 3.4: *Let $X_1, \ldots, X_n$ be a sequence of independent negative binomial random variables with respective parameters $(k, p_1), \ldots, (k, p_n)$ and let $Y_1, \ldots, Y_n$ be another sequence of independent negative binomial random variables with respective parameters $(k, p_1^*), \ldots, (k, p_n^*)$, where $k > 0$. Then,*

$$(p_1, \ldots, p_n) \overset{\text{p}}{\geq} (p_1^*, \ldots, p_n^*) \Longrightarrow X_{n:n} \geq_{\text{st}} Y_{n:n}.$$

PROOF: The distribution function of $X_{n:n}$ can be written as

$$F_{n:n}(r) = \prod_{i=1}^{n} I_{p_i}(k, r+1), \qquad r = 0, 1, \ldots.$$

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Similarly, the distribution function of $Y_{n:n}$ can be written as

$$G_{n:n}(r) = \prod_{i=1}^{n} I_{p_i^*}(k, r+1), \qquad r = 0, 1, \ldots.$$

Now, making the transform

$$p_i = e^{s_i}, \qquad p_i^* = e^{s_i^*}$$

according to Lemma 3.2, it is enough for us to prove that

$$\Psi(\mathbf{s}) = \Psi(s_1, \ldots, s_n) = \prod_{i=1}^{n} \int_0^{e^{s_i}} u^{k-1}(1-u)^r \, du$$

is increasing and Schur-concave in $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{R}_-^n$. The increasing property is obvious. Note that

$$\frac{\partial \Psi(\mathbf{s})}{\partial s_i} = \Psi(\mathbf{s}) \frac{e^{ks_i}(1-e^{s_i})^r}{\int_0^{e^{s_i}} u^{k-1}(1-u)^r \, du},$$

and for $i \neq j$,

$$\frac{1}{\Psi(\mathbf{s})}\left(\frac{\partial \Psi(\mathbf{s})}{\partial s_i} - \frac{\partial \Psi(\mathbf{s})}{\partial s_j}\right) = \frac{e^{ks_i}(1-e^{s_i})^r}{\int_0^{e^{s_i}} u^{k-1}(1-u)^r \, du} - \frac{e^{ks_j}(1-e^{s_j})^r}{\int_0^{e^{s_j}} u^{k-1}(1-u)^r \, du}.$$

Now, using Lemma 3.3(a), it follows that

$$(s_i - s_j)\left(\frac{\partial \Psi(\mathbf{s})}{s_i} - \frac{\partial \Psi(\mathbf{s})}{s_j}\right) \leq 0, \quad \forall\, i \neq j.$$

Hence, by Lemma 3.1, $\Psi(\mathbf{s})$ is Schur-concave. So, the proof is complete. ■

Remark: Under the special case $k_1 = \cdots = k_n = 1$, it has been shown in Mao and Hu [8] that

$$(p_1, \ldots, p_n) \overset{\text{p}}{\geq} (p_1^*, \ldots, p_n^*) \Rightarrow X_{n:n} \geq_{\text{st}} Y_{n:n}.$$

They also showed that the other order statistics might not hold under the above condition.

Next, we will provide a sufficient condition for comparing the smallest order statistics.

THEOREM 3.5: *Let $X_1, \ldots, X_n$ be a sequence of independent negative binomial random variables with respective parameters $(k, p_1), \ldots, (k, p_n)$ and let $Y_1, \ldots, Y_n$ be another sequence of independent negative binomial random variables with respective parameters $(k, p_1^*), \ldots, (k, p_n^*)$.*

(a) If $k \geq 1$, then

$$(1-p_1, \ldots, 1-p_n) \overset{\text{p}}{\geq} (1-p_1^*, \ldots, 1-p_n^*) \Longrightarrow X_{1:n} \leq_{\text{st}} Y_{1:n}.$$

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(b) If $k \in (0, 1]$, *then*

$$(\ln(1-p_1), \ldots, \ln(1-p_n)) \succeq_{\mathrm{w}} (\ln(1-p_1^*), \ldots, \ln(1-p_n^*))$$
$$\Longrightarrow X_{1:n} \geq_{\mathrm{st}} Y_{1:n}.$$

PROOF: We give the proof of part (a) only; the proof of part (b) is similar. The survival function of $X_{1:n}$ can be written as

$$\overline{F}_{1:n}(r) = \prod_{i=1}^{n} \left[1 - I_{p_i}(k, r+1)\right], \qquad r = 0, 1, \ldots.$$

Similarly, the survival function of $Y_{1:n}$ can be written as

$$\overline{G}_{1:n}(r) = \prod_{i=1}^{n} \left[1 - I_{p_i^*}(k, r+1)\right], \qquad r = 0, 1, \ldots.$$

Now, making the transform

$$p_i = 1 - e^{s_i}, \qquad p_i^* = 1 - e^{s_i^*}$$

according to Lemma 3.2, it is enough for us to prove that

$$\Psi(\mathbf{s}) = \Psi(s_1, \ldots, s_n) = \prod_{i=1}^{n} \int_{1-e^{s_i}}^{1} u^{k-1}(1-u)^r \, du$$

is increasing and Schur-concave in $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{R}_-^n$. The increasing property is obvious. Note that

$$\frac{\partial \Psi(\mathbf{s})}{\partial s_i} = \Psi(\mathbf{s}) \frac{e^{(r+1)s_i}(1-e^{s_i})^{k-1}}{\int_{1-e^{s_i}}^{1} u^{k-1}(1-u)^r \, du},$$

and for $i \neq j$,

$$\frac{1}{\Psi(\mathbf{s})}\left(\frac{\partial \Psi(\mathbf{s})}{\partial s_i} - \frac{\partial \Psi(\mathbf{s})}{\partial s_j}\right) = \frac{e^{(r+1)s_i}(1-e^{s_i})^{k-1}}{\int_{1-e^{s_i}}^{1} u^{k-1}(1-u)^r \, du} - \frac{e^{(r+1)s_j}(1-e^{s_j})^{k-1}}{\int_{1-e^{s_j}}^{1} u^{k-1}(1-u)^r \, du}.$$

Now, using Lemmas 3.3(b) and 3.1, it follows that

$$(s_i - s_j)\left(\frac{\partial \Psi(\mathbf{s})}{\partial s_i} - \frac{\partial \Psi(\mathbf{s})}{\partial s_j}\right) \leq 0, \qquad \forall\, i \neq j;$$

that is, $\Psi(\mathbf{s})$ is Schur-concave. Hence, the proof is complete. ■

The following result is a direct consequence of Theorem 3.5(b) by applying Theorem 2 in Ma [7].

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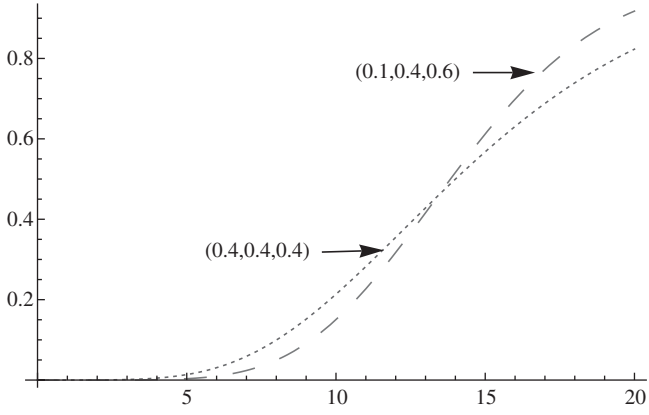


FIGURE 1. The distribution functions of $X_{2:3}$ from negative binomial random variables with parameters $(p_1, p_2, p_3) = (0.1, 0.4, 0.6)$ and $(k_1, k_2, k_3) = (10, 10, 10)$ and of $Y_{2:3}$ from negative binomial random variables with $(p_1, p_2, p_3) = (0.4, 0.4, 0.4)$ and $(k_1, k_2, k_3) = (10, 10, 10)$.

COROLLARY 3.6: *Let $X_1, \ldots, X_n$ be a sequence of independent negative binomial random variables with respective parameters $(k, p_1), \ldots, (k, p_n)$ and let $Y_1, \ldots, Y_n$ be another sequence of independent and identically distributed* (iid) *negative binomial random variables with common parameters (k, p). If $k \in (0, 1]$, then*

$$p \geq 1 - \sqrt[n]{\prod_{i=1}^{n}(1 - p_i)} \Longrightarrow X_{r:n} \geq_{\text{st}} Y_{r:n}, \quad r = 1, \ldots, n.$$

One may wonder whether Corollary 3.6 is also true for the negative binomial random variables for $k_i > 1, i = 1, \ldots, n$. Unfortunately, the following example gives a negative answer.

EXAMPLE 3.7: Let X_1, X_2, and X_3 be independent negative binomial random variables with parameters $(p_1, p_2, p_3) = (0.1, 0.4, 0.6)$ and $(k_1, k_2, k_3) = (10, 10, 10)$. Then

$$1 - \sqrt[3]{\prod_{i=1}^{3}(1 - p_i)} = 0.4.$$

Now, assume that Y_1, Y_2, and Y_3 be are i.i.d. negative binomial random variables with parameters $p = 0.4$ and $k = 10$. Then we plot the distribution functions of $X_{2:3}$ and $Y_{2:3}$ in Figure 1 by Mathematica 7. It is seen that there is a crossing for both functions. Hence, Corollary 3.6 is not true for the general negative binomial case.

The following interesting result reveals how the variability between the numbers of components affects the lifetimes of redundant systems described in the section 1.

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THEOREM 3.8: *Let $X_1, \ldots, X_n$ be a sequence of independent negative binomial random variables with respective parameters $(k_1, p), \ldots, (k_n, p)$ and let $Y_1, \ldots, Y_n$ be other sequence of independent negative binomial random variables with respective parameters $(k_1^*, p), \ldots, (k_n^*, p)$, where $k_i > 0$ and $k_i^* > 0$ for each i. Then*

(a) $(k_1, \ldots, k_n) \succeq_{\mathrm{w}} (k_1^*, \ldots, k_n^*) \Longrightarrow X_{n:n} \geq_{\mathrm{st}} Y_{n:n}$,

(b) $(k_1, \ldots, k_n) \overset{\mathrm{w}}{\succeq} (k_1^*, \ldots, k_n^*) \Longrightarrow X_{1:n} \leq_{\mathrm{st}} Y_{1:n}$.

PROOF:

(a) The distribution function of $X_{n:n}$ can be rewritten as

$$F_{n:n}(r) = \prod_{i=1}^{n} B(p; k_i, r+1), \quad r = 0, 1, \ldots.$$

Similarly, the distribution function of $Y_{n:n}$ can be written as

$$G_{n:n}(r) = \prod_{i=1}^{n} B(p; k_i^*, r+1), \quad r = 0, 1, \ldots.$$

Now, it is enough for us to prove

$$\prod_{i=1}^{n} B(p; k_i, r+1) \leq \prod_{i=1}^{n} B(p; k_i^*, r+1),$$

which can be equivalently expressed as

$$\sum_{i=1}^{n} [-\ln B(p; k_i, r+1)] \geq \sum_{i=1}^{n} [-\ln B(p; k_i^*, r+1)].$$

It has been proved in Das Gupta and Sarkar [2] (see also Finner and Roter [3]) that $B(p; k_i, r+1)$ is log-concave in k_i for $k_i > 0$ and $r \geq 0$. Because $B(p; k_i, r+1)$ is decreasing in k_i, it follows from Lemma 3.2 that

$$(k_1, \ldots, k_n) \succeq_{\mathrm{w}} (k_1^*, \ldots, k_n^*)$$

implies that

$$\sum_{i=1}^{n} [-\ln B(p; k_i, r+1)] \geq \sum_{i=1}^{n} [-\ln B(p; k_i^*, r+1)], \quad r = 0, 1, \ldots.$$

Hence, the required result follows.

(b) The survival function of $X_{1:n}$ can be expressed as

$$\overline{F}_{1:n}(r) = \prod_{i=1}^{n} \bar{B}(p; k_i, r+1), \quad r = 0, 1, \ldots,$$

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where $\bar{B}(p; k_i, r+1) = 1 - B(p_i; k_i, r+1)$. Similarly, the survival function of $Y_{1:n}$ can be rewritten as

$$\overline{G}_{1:n}(r+1) = \prod_{i=1}^{n} \bar{B}(p; k_i^*, r+1), \quad r = 0, 1, \ldots.$$

Hence, it is sufficient to prove that

$$\sum_{i=1}^{n} [-\ln \bar{B}(p; k_i, r+1)] \geq \sum_{i=1}^{n} [-\ln \bar{B}(p; k_i^*, r+1)], \quad r = 0, 1, \ldots.$$

Corollary 4.6 of Finner and Roter [3] shows that $\bar{B}(p; k_i, r+1)$ is log-concave in k_i for $k_i > 0$ and $r \geq 0$. Observing that $\bar{B}(p; k_i, r+1)$ is increasing in k_i, and using Lemma 3.2 again, it follows that

$$(k_1, \ldots, k_n) \overset{\text{w}}{\geq} (k_1^*, \ldots, k_n^*) \Longrightarrow \sum_{i=1}^{n} [-\ln \bar{B}(p; k_i, r+1)] \geq \sum_{i=1}^{n} [-\ln \bar{B}(p; k_i^*, r+1)].$$

Hence, the proof is complete. ■

4. GEOMETRIC DISTRIBUTION

In this section we develop a parallel result to Theorem 3.4 of Proschan and Sethuraman [12] for the geometric distribution.

THEOREM 4.1: *Let $X_1, \ldots, X_n$ be a sequence of independent geometric random variables with respective parameters $p_1, \ldots, p_n$ and let $Y_1, \ldots, Y_n$ be another sequence of independent geometric random variables with respective parameters $p_1^*, \ldots, p_n^*$. Then*

$$(\ln(1-p_1), \ldots, \ln(1-p_n)) \succeq_{\text{w}} (\ln(1-p_1^*), \ldots, \ln(1-p_n^*))$$

implies

$$(X_{1:n}, \ldots, X_{n:n}) \succeq_{\text{st}} (Y_{1:n}, \ldots, Y_{n:n}).$$

PROOF: First let us prove the case of $n = 2$. According to Theorem 4.13 of Barlow and Proschan [1, p. 148], it is sufficient to verify three conditions:

(a) $X_{1:2} \geq_{\text{st}} Y_{1:2}$;

(b) $[X_{2:2} | X_{1:2} = r'] \geq_{\text{st}} [X_{2:2} | X_{1:2} = r]$ for $r' \geq r$;

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(c) $[X_{2:2}|X_{1:2}=r] \geq_{\mathrm{st}} [Y_{2:2}|Y_{1:2}=r]$ for $r=0,1,\ldots$.

Note that the survival function of $X_{1:2}$ is

$$\mathsf{P}(X_{1:2}\geq k)=(1-p_1)^k(1-p_2)^k, \quad k=0,1,\ldots.$$

Hence, (a) follows immediately from the assumption.

The conditional distribution of $[X_{2:2}|X_{1:2}=r]$, for $r=0,1,\ldots$ is given by

$$\begin{aligned}\mathsf{P}(X_{2:2}\leq x|X_{1:2}=r)&=\frac{\sum_{s=r}^{x}\mathsf{P}(X_{1:2}=r,X_{2:2}=s)}{\mathsf{P}(X_{1:2}=r)}\\&=\frac{\mathsf{P}(X_1=r,X_2=r)+\sum_{s=r+1}^{x}\mathsf{P}(X_{1:2}=r,X_{2:2}=s)}{\mathsf{P}(X_{1:2}=r)}.\end{aligned}$$

As proved in Margolin and Winokur [9], $X_{1:2}$ has a geometric distribution with parameter $1-(1-p_1)(1-p_2)$. Hence,

$$\mathsf{P}(X_{1:2}=r)=[1-(1-p_1)(1-p_2)][(1-p_1)(1-p_2)]^r, \quad r=0,1,\ldots.$$

Note that for $s>r$,

$$\begin{aligned}\mathsf{P}(X_{1:2}=r,X_{2:2}=s)&=\mathsf{P}(X_{1:2}=r,X_{2:2}=s,X_1<X_2)\\&\quad+\mathsf{P}(X_{1:2}=r,X_{2:2}=s,X_1>X_2)\\&=\mathsf{P}(X_1=r,X_2=s)+\mathsf{P}(X_1=s,X_2=r)\\&=p_1p_2(1-p_1)^r(1-p_2)^s+p_1p_2(1-p_1)^s(1-p_2)^r.\end{aligned}$$

Now, after some simplifications,

$$\begin{aligned}\sum_{s=r+1}^{x}\mathsf{P}(X_{1:2}=r,X_{2:2}=s)&=p_1(1-p_1)^r(1-p_2)^{r+1}+p_2(1-p_1)^{r+1}(1-p_2)^r\\&\quad-p_1(1-p_1)^r(1-p_2)^{x+1}-p_2(1-p_1)^{x+1}(1-p_2)^r.\end{aligned}$$

So,

$$\mathsf{P}(X_{2:2}\leq x|X_{1:2}=r)=1-\frac{p_1(1-p_2)^{x-r+1}+p_2(1-p_1)^{x-r+1}}{p_1+p_2-p_1p_2}.$$

It is easy to see that $\mathsf{P}(X_{2:2}\leq x|X_{1:2}=r)$ is decreasing in r. Hence, (b) follows.

To prove (c), observe that

$$(\ln(1-p_1),\ldots,\ln(1-p_n))\succeq_{\mathrm{w}}(\ln(1-p_1^*),\ldots,\ln(1-p_n^*))$$

implies

$$(p_1,\ldots,p_n)\overset{\mathrm{w}}{\succeq}(p_1^*,\ldots,p_n^*);$$

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see Theorem 5.A.2 in Marshall and Olkin [10]. It is sufficient to prove that for fixed (x, r),

$$\Delta(p_1, p_2) \equiv \mathsf{P}(X_{2:2} \le x | X_{1:2} = r)$$

is increasing and Schur-concave with respect to (p_1, p_2) by Lemma 3.2. Now, taking the derivative with respect p_1, after collecting terms, we have

$$(p_1 + p_2 - p_1p_2)^2 \frac{\partial \Delta(p_1, p_2)}{\partial p_1} = (x - r + 1)p_2(p_1 + p_2 - p_1p_2)(1 - p_1)^{x-r} + p_2(1 - p_1)^{x-r+1}(1 - p_2) - p_2(1 - p_2)^{x-r+1}.$$

Hence,

$$(p_1 + p_2 - p_1p_2)^2 \left[\frac{\partial \Delta(p_1, p_2)}{\partial p_1} - \frac{\partial \Delta(p_1, p_2)}{\partial p_2} \right] = \Delta_1 + \Delta_2,$$

where

$$\Delta_1 = (x - r + 1)(p_1 + p_2 - p_1p_2)[p_2(1 - p_1)^{x-r} - p_1(1 - p_2)^{x-r}],$$
$$\Delta_2 = (1 - p_1)^{x-r+1}[p_1 + p_2 - p_2^2] - (1 - p_2)^{x-r+1}[p_1 + p_2 - p_1^2].$$

Using Lemma 3.1, it is enough to prove that

$$(p_1 - p_2) \left[\frac{\partial \Delta(p_1, p_2)}{\partial p_1} - \frac{\partial \Delta(p_1, p_2)}{\partial p_2} \right] \le 0. \tag{4.1}$$

Without loss of generality, assume that $p_1 > p_2$. It is easy to see that $\Delta_1 \le 0$. Additionally,

$$(1 - p_1)[p_1 + p_2 - p_2^2] \le (1 - p_2)[p_1 + p_2 - p_1^2],$$

which implies $\Delta_2 \le 0$. This means that (4.1) holds. Thus, the required result follows immediately.

For the case of $n \ge 3$, one may easily follow the similar arguments used in Theorem 3.4 of Proschan and Sethuraman [12] to prove it. ■

The following result is a direct consequence of Theorem 4.1.

COROLLARY 4.2: *Let $X_1, \ldots, X_n$ be a sequence of independent geometric random variables with respective parameters $p_1, \ldots, p_n$ and let $Y_1, \ldots, Y_n$ be another sequence of i.i.d. geometric random variables with a common parameter p. Then*

$$p \ge 1 - \sqrt[n]{\prod_{i=1}^{n}(1 - p_i)} \implies (X_{1:n}, \ldots, X_{n:n}) \ge_{\text{st}} (Y_{1:n}, \ldots, Y_{n:n}).$$

Corollary 3.6 for $k = 1$ is also a direct consequence of Corollary 4.2.

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